Refining G-structure classifications

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Using G-structure language, a systematic, iterative formalism for computing necessary and sufficient conditions for the existence of N arbitrary linearly independent Killing spinors, in any supergravity, is presented. The key organizational tool is the common isotropy group of the Killing spinors. The formalism is illustrated for configurations in gauged SU(2) supergravity in seven dimensions admitting at least one null Killing spinor, and the possible isotropy groups are shown to be $[SU(2)\ltimes\mathbb{R}^4] \times \mathbb{R}$, $SU(2)$, $\mathbb{R}^5$, or the identity. The constraints associated with the existence of certain additional Killing spinors are computed and used to derive numerous solutions. A discussion of the relevance of the formalism to the complete classification of all supersymmetric configurations in $d = 11$ is given.

I. INTRODUCTION

In recent years, it has been realized that the notion of a G-structure is a powerful tool in classifying supersymmetric geometries in supergravity theories. It has been successfully and fruitfully applied in numerous interesting contexts, for example [1–19]. The strategy is simple. One assumes the existence of at least one Killing spinor. The existence of a Killing spinor is equivalent to the existence of a set of globally defined bilinears, specifying the G-structure, which are invariant under the isotropy group G of the spinor. Applying the Fierz identities to the bilinears allows the deduction of algebraic relations between them. Next, applying the Killing spinor equation to the bilinears determines the intrinsic torsion of the G-structure in terms of the bosonic fields of the theory. If the theory contains any additional fermions, the vanishing of their supersymmetry variations implies additional algebraic relationships between the bosonic fields. When one has derived the complete set of constraints, one substitutes them back into the supersymmetry transformations and shows that they are also sufficient for supersymmetry. One thus arrives at a set of necessary and sufficient conditions for a bosonic configuration to admit at least one Killing spinor. However, the existence of at least one Killing spinor generically implies that some but not all of the equations of motion and Bianchi identities are identically satisfied, so some subset thereof must still be imposed on the most general field configuration consistent with the constraints. The standard G-structure scheme should thus be thought of as providing the most general supersymmetric *Ansatz* for the supergravity in question.

The standard G-structure formalism has proven to be extremely powerful, and useful from the point of view of explicitly constructing new solutions, in simple lower dimensional supergravities with few supercharges. In these simple cases, the computational effort required to arrive at the general *Ansatz* is also fairly minimal.

However, as the dimensionality of spacetime and/or the number of supercharges increases, the constraints implied by the existence of a single Killing spinor become proportionately weaker, and the most general *Ansatz* becomes broader. Though the reduced problem of solving the remaining field equations for the most general supersymmetric *Ansatz* is much simpler than trying to solve the original full set *ab initio*, it is generically hard to do. For example, while it is conceptually beautiful that the most general supersymmetric *Ansatz* for $d = 11$ supergravity may be explicitly computed [9,10], the *Ansatz* is not enormously useful, given current techniques, when it comes to explicitly generating new solutions. Furthermore, calculating the *Ansatz* for more complicated theories using the standard approach requires a lot of (in fact, largely redundant) computation. For a theory such as IIB, the amount of calculation required to derive the *Ansatz* using the standard procedure would be truly gargantuan.

There are thus two related improvements to the standard procedure which are required to fully realize the power of the G-structure formalism. The first, of a purely technical nature, is to give a more efficient means of performing the calculations. The second is to give a systematic formalism for classifying configurations preserving more than one supersymmetry. Some attempts in this direction have been made, for simple supergravities, using a mixture of G-structure and integrability techniques [2,18,19]. However, this approach will not really be viable for more complicated theories; it would be preferable to have a universally applicable formalism which uses G-structure language throughout, as was done (in a purely Riemannian context) in [7]. Requiring a configuration to admit more than one Killing spinor would imply more constraints on the bosonic fields of the theory. Furthermore, it would also imply that more of the field equations and Bianchi identities would be identically satisfied. Thus, a refined G-structure classification scheme would be much more useful from the point of view of explicitly constructing supersymmetric solutions.

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The objective of this paper is to propose a means of implementing these improvements to the standard procedure, illustrated in the context of SU(2) gauged supergravity in seven dimensions, a theory with 16 real supercharges.

The technical novelty employed, which in fact renders a refined classification tractable, is to fully exploit throughout the calculation a point raised in [10]. The point is that any spinor defines a preferred orthonormal basis of spacetime, and in this basis the associated G-structure simplifies dramatically, and the spinor is in fact constant. For example, in the seven-dimensional context of this paper, we will see below that a single null spinor defines an $[\mathrm{SU}(2) \ltimes \mathbb{R}^4] \times \mathbb{R}$ structure. A single pair of null symplectic-majorana spinors (we will work with symplectic-majorana spinors throughout) may be fixed by the projections

$$\Gamma^{12}\epsilon^1 = i\epsilon^2, \qquad \Gamma^{13}\epsilon^1 = -\epsilon^2, \qquad \Gamma^{14}\epsilon^1 = i\epsilon^1,$$
$$\Gamma^5\epsilon^a = \epsilon^a, \tag{1}$$

in the basis

$$ds^2 = -2e^+e^- + \delta_{ij}e^ie^j + (e^5)^2. \tag{2}$$

The four-dimensional Riemannian manifold with metric $\delta_{ij}e^ie^j$ will be referred to as the base. The various bilinears may be trivially computed without having to invoke the full Fierzing machinery, and have constant components in this spacetime basis. This is what was done in [10].

We will take this simplification a stage further. The supergravity we analyze has two fermions, λ and ψ_μ. Consider first the constraints implied by the vanishing of $\delta\lambda$. The standard way of analyzing these would be to contract $\delta\lambda = 0$ with all spinors of the form $\bar{\epsilon}\Gamma^{(n)}$ up to three gamma matrices to form all possible combinations of bilinears, and deduce the implied constraints on the bosonic fields. However, it is much more efficient to first impose the projections (1). Then as we shall see below, $\delta\lambda = 0$ may be cast in the schematic form

$$\delta\lambda = [q + iq^AT^A + q^i\Gamma_i + (r + ir^AT^A)\Gamma^- + r^i\Gamma^-\Gamma_i]\epsilon$$
$$= 0, \tag{3}$$

where the T^A are the Pauli matrices, and we have suppressed symplectic-Majorana indices. In doing this, we shall see below that we are rewriting $\delta\lambda$ manifestly in terms of a basis for spinor space; by linear independence, each of the q, q^A, q^i, r, r^A, r^i must vanish separately. By imposing the projections and decomposing in this fashion, it is very much quicker to derive the constraints. Next consider the Killing spinor equation. The standard procedure for analyzing the differential constraints would be to apply the full Killing spinor equation to each of the bilinears to deduce the various components of the spin connection. Again, this is not efficient. In its preferred

basis, the spinor ϵ has constant components, so in this basis the Killing spinor equation becomes schematically

$$\delta\psi_\mu = \tfrac{1}{4}\omega_{\mu\nu\sigma}\Gamma^{\nu\sigma}\epsilon + \text{fluxes} = 0, \tag{4}$$

and thus yields algebraic relations between the fluxes and the spin connection in the preferred basis. Again we may impose the various defining projections on ϵ to reduce each spacetime component of the Killing spinor equation to the schematic form

$$\delta\psi_\mu = [q_\mu + iq_\mu^AT^A + q_\mu^i\Gamma_i + (r_\mu + ir_\mu^AT^A)\Gamma^-$$
$$+ r_\mu^i\Gamma^-\Gamma_i]\epsilon$$
$$= 0. \tag{5}$$

Requiring by linear independence that each term in each spacetime component of the Killing spinor equation vanishes separately allows one to fix the spin connection in terms of the fluxes, and thus deduce the most general supersymmetric *Ansatz* with a minimum of effort. This streamlined way of computing the constraints has also recently been advocated in [20].

Having sketched the technical innovation used to reduce as much as possible the amount of computation required for the broadest *Ansatz*, we now turn to the question of refining the classification. We will assume the existence of the single null Killing spinor defined by the projections (1), and wish to compute the further constraints on the bosonic fields of the theory for it to admit an arbitrary additional linearly independent Killing spinor. An important observation in organizing the refined classification is the following. Incorporating additional Killing spinors can have one of two effects. Either the existence of an additional Killing spinor implies a further global reduction of the structure group of the frame bundle (so the structure group is reduced from G to some subgroup), or it implies additional restrictions on the intrinsic torsion of the existing G-structure. To illustrate this point, consider the basis of 16 spinors defined by the projections (with no sum on i)

$$\Gamma^{12}\epsilon_{(i)}^1 = i\alpha_{(i)}^1\epsilon_{(i)}^2, \qquad \Gamma^{13}\epsilon_{(i)}^1 = -\alpha_{(i)}^2\epsilon_{(i)}^2,$$
$$\Gamma^{14}\epsilon_{(i)}^1 = i\alpha_{(i)}^3\epsilon_{(i)}^1, \qquad \Gamma^5\epsilon_{(i)}^a = \alpha_{(i)}^4\epsilon_{(i)}^a, \tag{6}$$

for all 16 possible combinations of $\alpha^1, \ldots, \alpha^4 = \pm 1$. All these spinors are null, and they are all constant in the spacetime basis (2). We denote them by $(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$, and also introduce the notation

$$\prod_{j=1}^4 \alpha^j = \gamma, \tag{7}$$

$$\prod_{A=1}^3 \alpha^A = \beta. \tag{8}$$

Our basis spans the space of spinors in the theory, so any

additional Killing spinor, whether timelike or null, must be a linear combination of these spinors; for an arbitrary additional Killing spinor ϵ_K,

$$\epsilon_K = \sum_{i=1}^{N} f_{(i)} \epsilon_{(i)}, \tag{9}$$

where the 16 $f_{(i)}$ are real functions. We may rewrite our basis spinors in terms of our fiducial Killing spinor $\epsilon = (+, +, +, +,)$; the three other basis spinors with $\gamma = 1$, $\beta = 1$ may be written as

$$iT^A\epsilon, \tag{10}$$

since these obey the appropriate projections. Next the four basis spinors with $(\gamma, \beta) = (+, -)$ are given by

$$\Gamma^i\epsilon, \tag{11}$$

while the two sets of four basis spinors with $(\gamma, \beta) = (-, +), (-, -)$ are

$$(\Gamma^-\epsilon, i\Gamma^-T^A\epsilon); \qquad \Gamma^-\Gamma^i\epsilon, \tag{12}$$

respectively. Hence, an arbitrary additional linearly independent Killing spinor is given by

$$\epsilon_K = [\delta + i\delta^A T^A + \delta_i\Gamma^i + \Gamma^-(\theta + i\theta^A T^A + \theta_i\Gamma^i)]\epsilon, \tag{13}$$

and we recognize in Eq. (3) and (5) precisely the decomposition of the supersymmetry variations on the basis. Now, we will compute the common isotropy group of the spinor ϵ_K and the fiducial Killing spinor ϵ for various choices of the 16 functions specifying ϵ_K. We note that basis spinors with $\gamma = 1$ are annihilated by Γ^+:

$$\Gamma^+\epsilon_{(i)} = 0, \qquad \gamma_{(i)} = 1, \tag{14}$$

while basis spinors with $\gamma = -1$ are annihilated by Γ^-,

$$\Gamma^-\epsilon_{(i)} = 0, \qquad \gamma_{(i)} = -1. \tag{15}$$

Furthermore, since β labels minus the chirality of the basis spinors on the four-dimensional base, anti-self-dual linear combinations of the Γ^{ij} annihilate basis spinors with $\beta = 1$; for a two-form $A_{ij}^{(-)}$ which is anti-self-dual on the base but otherwise arbitrary,

$$A_{ij}^{(-)}\Gamma^{ij}\epsilon_{(i)} = 0, \qquad \beta_{(i)} = 1. \tag{16}$$

Throughout this work, for any object with two antisymmetric indices i, j on the base, the superscripts $(+)$ and $(-)$ will denote, respectively, the self-dual and anti-self-dual projections on i, j. Our choice of orientation, together with all other conventions, is given in Appendix A. Next, basis spinors with $\beta = -1$ are annihilated by self-dual linear combinations; for an arbitrary self-dual form $A_{ij}^{(+)}$,

$$A_{ij}^{(+)}\Gamma^{ij}\epsilon_{(i)} = 0, \quad \beta_{(i)} = -1. \tag{17}$$

Therefore, the most general element of the Lie algebra of Spin(1,6) which annihilates the fiducial Killing spinor ϵ is

$$B^A K_{ij}^{(-)A}\Gamma^{ij} + B_i\Gamma^{+i} + B\Gamma^{+5}, \tag{18}$$

which generates the group $[SU(2)\ltimes\mathbb{R}^4] \times \mathbb{R}$; the K^A are a triplet of anti-self-dual forms obeying the quaternionic algebra. However, the three additional Killing spinors with $(\gamma, \beta) = (+, +)$ are also annihilated precisely by (18); all the spinors in the four-dimensional subspace spanned by the $(+, +)$ basis spinors share the same isotropy group. Thus, additional linearly independent Killing spinors of the form

$$\epsilon_K = (\delta + i\delta^A T^A)\epsilon \tag{19}$$

do not imply any further reduction of the structure group. However, they will imply further restrictions on the intrinsic torsion of the $[SU(2)\ltimes\mathbb{R}^4] \times \mathbb{R}$ structure. Since none of the other 12 basis spinors are annihilated by (18), we see that an $[SU(2)\ltimes\mathbb{R}^4] \times \mathbb{R}$ structure is compatible with having one, two, three, or four linearly independent Killing spinors.

Next consider Killing spinors which are linear combinations of $(+, +)$ and $(-, +)$ spinors,

$$\epsilon_K = [\delta + i\delta^A T^A + \Gamma^-(\theta + i\theta^A T^A)]\epsilon, \tag{20}$$

with at least one of the $\theta, \theta^A \neq 0$. An arbitrary $(-, +)$ spinor is annihilated by a different $[SU(2)\ltimes\mathbb{R}^4] \times \mathbb{R} \subset$ Spin(1, 6); the most general Lie algebra element annihilating such a spinor is

$$B^A K_{ij}^{(-)A}\Gamma^{ij} + B_i\Gamma^{-i} + B\Gamma^{-5}. \tag{21}$$

Thus in this case the common isotropy group of the spinors ϵ, ϵ_K is SU(2), generated by

$$B^A K_{ij}^{(-)A}\Gamma^{ij}, \tag{22}$$

and so additional Killing spinors of the form (20) reduce the structure group to SU(2). There are eight linearly independent spinors with this common isotropy group, so given the existence of the fiducial Killing spinor, an SU(2) structure is compatible with the existence of $2, 3, \ldots, 8$ linearly independent Killing spinors. The more Killing spinors there are, the more restrictive the constraints on the torsion will be.

It is clear how to proceed. Next consider a Killing spinor of the form

$$\epsilon_K = (\delta + i\delta^A T^A + \delta_i\Gamma^i)\epsilon, \tag{23}$$

with at least one of the $\delta_i \neq 0$. The $[SU(2)\ltimes\mathbb{R}^4] \times \mathbb{R}$ isotropy group of a $(+, -)$ spinor is generated by

$$B^A J_{ij}^{(+)A}\Gamma^{ij} + B_i\Gamma^{+i} + B\Gamma^{+5}, \tag{24}$$

so in this case the common isotropy group of the spinors ϵ, ϵ_K is $\mathbb{R}^5$, generated by

$$B_i\Gamma^{+i} + B\Gamma^{+5}. \tag{25}$$

As in the SU(2) case, there are eight linearly independent spinors with this common isotropy group, so given the existence of the fiducial Killing spinor, an $\mathbb{R}^5$ structure is compatible with the existence of $2,\ldots,8$ linearly independent Killing spinors.

It is easy to verify that assuming the existence of Killing spinors which are more general linear combinations of the basis spinors than the three cases discussed above imply that the structure group is reduced to the identity. Such a structure, given the existence of the fiducial Killing spinor, is compatible with the existence of $2, 3, \ldots, 16$ linearly independent Killing spinors.

In the refinement of the G-structure classification scheme proposed here, configurations admitting multiple linearly independent Killing spinors are naturally classified according to the structure group, rather than the number of Killing spinors. We have seen how we can have $G = [\mathrm{SU}(2){\ltimes}\mathbb{R}^4] \times \mathbb{R}$ and four linearly independent Killing spinors, or $G = Id$ and only two. This is somewhat different to other approaches, for example generalized holonomy [21–23], where the aim has always been to classify configurations according to the number of preserved supersymmetries. From our perspective, this is rather unnatural; the most significant feature of a supersymmetric configuration in the G-structure formalism is the structure group. Then, for a given structure group, demanding more supersymmetries imposes progressively more severe constraints on the intrinsic torsion.

The most efficient way that we have been able to find of computing the additional constraints implying and implied by the existence of arbitrary additional Killing spinors is the following. Denote the supersymmetry variations $\delta\lambda$, $\delta\psi$ with parameter ϵ_K as $\Delta_\lambda\epsilon_K$, $\mathcal{D}_\mu\epsilon_K$. Since we may write $\epsilon_K = Q\epsilon$, for some Q of the form of Eq. (13), we observe that since ϵ is Killing, ϵ_K is Killing if and only if

$$[\Delta_\lambda, Q]\epsilon = 0, \tag{26}$$

$$[\mathcal{D}_\mu, Q]\epsilon = 0. \tag{27}$$

Having evaluated the commutators, we may as before impose the projections satisfied by ϵ to reduce these expressions to manifest linear combinations of the basis spinors, and by linear independence, each coefficient must vanish separately. The procedure may be iterated at will, for any desired choices of Q, consistent with any desired structure group. For simple supergravities, such as those with eight supercharges, the procedure given here should be easy to employ to perform a fully refined classification, and should involve a modest amount of calculation. For a theory of the complexity of the one studied in this paper, with 16 supercharges, the amount of computation required to perform a complete classification is much larger. This is because the most general additional linearly independent Killing spinor consistent with the existence of an identity structure is parametrized by 16 real functions, and keeping track of all the terms in (26) and (27) is technically involved. Additional Killing spinors consistent with the larger structure groups $[\mathrm{SU}(2){\ltimes}\mathbb{R}^4] \times \mathbb{R}$, SU(2) and $\mathbb{R}^5$ are somewhat easier to handle, since these are parametrized by four, eight, and eight real functions, respectively. Nevertheless, we have not performed the refined classification in its entirety; for the smaller structure groups, we have restricted to some illustrative examples, rather than pursuing the problem in full generality. However, we emphasize that there is no conceptual difficulty in doing so; the calculations involved, while lengthy, are simple and repetitive, involving nothing more than evaluating gamma-matrix commutators and imposing a fixed set of spinor projections.

The traditional approach to finding supersymmetric solutions of supergravity theories has been to make some *Ansatz* for the bosonic fields, and then to use the supersymmetry variations of the fermions to determine if there are any Killing spinors consistent with that *Ansatz*. The G-structure program can be thought of as the exact converse of this; one makes an *Ansatz* for the Killing spinors, and expresses the conditions for a spinorial solution of the supersymmetry variations of that form to exist as a set of constraints on the bosonic fields. The power of the formalism lies in the fact that the spinor *Ansatz* can, if desired, be made completely general, whereby we mean that the constraints on the bosonic fields for N arbitrary linearly independent Killing spinors (for any desired structure group) to exist can be evaluated, albeit with some effort, in any theory. What we have argued above is that the formidably complicated problem of determining the constraints on the bosonic fields for the general multispinor solution of the fermion supersymmetry transformations to exist may naturally be organized into more manageable subproblems, with G-structures providing the organizing principle. One may classify the possible types of multispinor *Ansätze* according to the common isotropy group of the spinors. Then within each class one may deduce the constraints on the intrinsic torsion of the structure implied by the existence of any desired number of arbitrary linearly independent spinors within that class.

The remainder of this paper is organized as follows. In Sec. II we briefly describe SU(2) gauged supergravity in seven dimensions. In Sec. III we perform a streamlined analysis of the constraints implying and implied by the existence of a single null Killing spinor, and introduce coordinates for the problem. In Sec. IV, with a modest assumption of the form of the Yang-Mills fields of the theory introduced for computational convenience, we perform a complete classification of multisupersymmetric configurations admitting a strictly $[\mathrm{SU}(2){\ltimes}\mathbb{R}^4] \times \mathbb{R}$ structure. It is shown that a second Killing spinor with the

same structure group as the first exists if and only if the Yang-Mills fields are truncated to a U(1) subgroup. The existence of a third Killing spinor with the same isotropy group implies and is implied by the vanishing of the Yang-Mills fields, and implies the existence of a fourth Killing spinor. In Sec. V we study configurations with an SU(2) structure; necessary and sufficient conditions for the existence of an additional Killing spinor of a specific form are derived, and also all configurations admitting eight Killing spinors fixed by the same SU(2) are explicitly classified. We illustrate the effect incorporating additional Killing spinors has on the torsion of the SU(2) structure by deriving known AdS_3 solutions of the theory from our *Ansatz*, and generalize these solutions to construct (singular) membrane solutions with AdS_3 world volumes. Configurations with $\mathbb{R}^5$ structures are the topic of Sec. VI. In Sec. VII we discuss identity structures, providing explicit examples. Section VIII contains our conclusions, and a discussion of the applicability of the formalism to the complete classification of supersymmetric configurations in 11 dimensions. In Appendix A we give our conventions, and Appendix B lists a set of projections satisfied by ϵ which we use throughout. In Appendix C we give the components of the spin connection. Appendix D contains the integrability conditions for the theory.

II. THE SUPERGRAVITY

Minimal SU(2) gauged supergravity in seven dimensions was first constructed in [24] but with numerical typos, which were corrected in [25]. In our conventions, the bosonic Lagrangian density for the theory is

$$
e^{-1}\mathcal{L} = \frac{1}{2}R - \frac{1}{24}(G_{\mu\nu\rho\tau})^2 - \frac{1}{2}F_{\mu\nu}{}^a{}_b F^{\mu\nu}{}^b{}_a - \frac{5}{2}(\partial_\mu\phi)^2
$$
$$
+ \frac{h}{18}e^{-2\phi}\epsilon^{\mu\nu\sigma\tau\alpha\beta\gamma}G_{\mu\nu\sigma\tau}A_{\alpha\beta\gamma}
$$
$$
- \frac{1}{24}e^{-\phi}G_{\mu\nu\rho\tau}F_{\kappa\lambda}{}^a{}_b A_\chi{}^b{}_a \epsilon^{\mu\nu\rho\tau\kappa\lambda\chi} - V(\phi).
\tag{28}
$$

Compared to [24] we use the same conventions for the Riemann tensor but Hawking and Ellis conventions for the Ricci tensor and scalar. We have also rescaled $\phi \rightarrow \sqrt{5}\phi$ and the forms by $F \rightarrow \sqrt{2}e^\phi F$, $G \rightarrow \sqrt{2}e^{-2\phi}G$, $A_{(1),(3)} \rightarrow \sqrt{2}A_{(1),(3)}$. The potential is given by

$$
V(\phi) = -60m^2 + 10(m')^2,
\tag{29}
$$

where m is a function of the single scalar field ϕ,

$$
m = -\frac{2}{5}he^{-4\phi} - \frac{1}{10}ge^\phi,
\tag{30}
$$

with g the gauge coupling (we have rescaled the coupling in [24] by $g \rightarrow g/\sqrt{2}$) and h the (constant) topological mass. The supersymmetry variations of the fermions are given by

$$
\delta\lambda^a = \frac{\sqrt{5}}{2}\Gamma^\mu\partial_\mu\phi\epsilon^a + \frac{i}{2\sqrt{5}}\Gamma^{\mu\nu}F_{\mu\nu}{}^a{}_b\epsilon^b
$$
$$
+ \frac{1}{24\sqrt{5}}\Gamma^{\mu\nu\rho\tau}G_{\mu\nu\rho\tau}\epsilon^a - \sqrt{5}m'\epsilon^a,
\tag{31}
$$

$$
\delta\psi_\mu^a = D_\mu\epsilon^a - \frac{i}{10}(\Gamma_\mu{}^{\nu\rho} - 8\delta_\mu^\nu\Gamma^\rho)F_{\nu\rho}{}^a{}_b\epsilon^b
$$
$$
+ \frac{1}{80}\left(\Gamma_\mu{}^{\alpha\beta\gamma\delta} - \frac{8}{3}\delta_\mu^\alpha\Gamma^{\beta\gamma\delta}\right)G_{\alpha\beta\gamma\delta}\epsilon^a
$$
$$
+ m\Gamma_\mu\epsilon^a - igA_{\mu b}^a\epsilon^b,
\tag{32}
$$

and the parameter ϵ^a is a symplectic-Majorana spinor, whose properties are summarized in Appendix A.

Let us introduce the following notation. Let A_p, B_q be p and q forms, respectively. Then

$$
A \lrcorner B_{a_1\ldots a_{q-p}} = \frac{1}{p!}A^{b_1\ldots b_p}B_{b_1\ldots b_p a_1\ldots a_{q-p}}.
\tag{33}
$$

The equations of motion and Bianchi identities are

$$
d(e^{-2\phi}G) = 0,
\tag{34}
$$

$$
d(e^\phi F^A) = g\epsilon^{ABC}e^\phi F^B \wedge A^C,
\tag{35}
$$

$$
P = 5\nabla^2\phi - 4G\lrcorner G + F^A\lrcorner F^A - V' = 0,
\tag{36}
$$

$$
Q = \star[e^{-2\phi}d\star(e^{2\phi}G) - \tfrac{1}{2}F^A \wedge F^A + 8he^{-4\phi}G] = 0,
\tag{37}
$$

$$
R^A = \star[e^\phi d\star(e^{-\phi}F^A) - g\epsilon^{ABC}e^\phi \star F^B \wedge A^C
$$
$$
- 2F^A \wedge G]
$$
$$
= 0,
\tag{38}
$$

$$
E_{\mu\nu} = R_{\mu\nu} - \tfrac{1}{3}(G_{\mu\alpha\beta\gamma}G_\nu{}^{\alpha\beta\gamma} - \tfrac{1}{10}g_{\mu\nu}G_{\alpha\beta\gamma\delta}G^{\alpha\beta\gamma\delta})
$$
$$
- 5\partial_\mu\phi\partial_\nu\phi - (F^A_{\mu\alpha}F_\nu{}^{A\alpha} - \tfrac{1}{10}g_{\mu\nu}F^A_{\alpha\beta}F^{A\alpha\beta})
$$
$$
- \tfrac{2}{5}g_{\mu\nu}V
$$
$$
= 0.
\tag{39}
$$

When $h = 0$, the theory lifts on an S^3 to the Neveu-Schwarz sector in $d = 10$ [26]. When $hg > 0$, it lifts on an S^4 to $d = 11$ [27]. When $h \neq 0$, there is a subtlety in imposing the four-form field equation. The reason is that the three-form $A_{(3)}$, which is massive, would have 20 on-shell degrees of freedom if it satisfied an ordinary second order field equation. However, the three-form in the 7D

supergravity multiplet should have only ten on-shell degrees of freedom. This is achieved by imposing the odd-dimensional self-duality equation [28]:

$$e^{2\phi} \star G - \frac{1}{2}\left(e^{\phi}F^A \wedge A^A - \frac{g}{6}\epsilon^{ABC}A^A \wedge A^B \wedge A^C\right) +$$
$$8hA_{(3)} = 0.$$
(40)

Note that the exterior derivative of this equation is just $\star Q$. Imposing the Bianchi identity and $Q = 0$ fixes $A_{(3)}$ up to an arbitrary closed three-form. The closed three-form is then determined by demanding that $A_{(3)}$ satisfies (40). In the examples given below, we will explicitly impose the Bianchi identity and $Q = 0$, but leave the determination of the closed three-form in $A_{(3)}$ implicit.

III. CONFIGURATIONS ADMITTING AT LEAST ONE NULL KILLING SPINOR

A. The G-structure

It is instructive to verify, using the standard formalism, that a single null spinor in seven dimensions defines an $[SU(2)\ltimes\mathbb{R}^4] \times \mathbb{R}$ structure. It was shown in [19] that we can define the following spinor bilinears:

$$f^{(ab)} = \bar{\epsilon}^a \epsilon^b,$$
(41)

$$\epsilon^{ab}V_\mu = \bar{\epsilon}^a\Gamma_\mu\epsilon^b,$$
(42)

$$\epsilon^{ab}I_{\mu\nu} = \bar{\epsilon}^a\Gamma_{\mu\nu}\epsilon^b,$$
(43)

$$\Omega^{(ab)}_{\mu\nu\rho} = \bar{\epsilon}^a\Gamma_{\mu\nu\rho}\epsilon^b.$$
(44)

From the reality properties of the gamma matrices and the symplectic-Majorana condition, the vector V_μ and the two-form $I_{\mu\nu}$ are seen to be real, while instead the scalars and the three-form can be rewritten as

$$f^a{}_b = -ig^A(T^A)^a{}_b,$$
(45)

$$\Omega^a{}_b = -iX^A(T^A)^a{}_b,$$
(46)

with g^A, $X^A_{\mu\nu\rho}$, $A = 1, 2, 3$ real. $(T^A)^a{}_b = 1/2(\sigma^A)^a{}_b$ are generators of the SU(2) Lie algebra, σ^A being the Pauli matrices, and obey

$$(T^A)^a{}_b(T^B)^b{}_c = \frac{1}{4}\delta^{AB}\delta^a_c + \frac{i}{2}\epsilon^{ABC}(T^C)^a{}_c.$$
(47)

One important consequence of the Fierz identity (266) is that V_μ is either timelike or null:

$$V^2 = -\frac{1}{48}g^A g^A.$$
(48)

The case of a single timelike spinor has been analyzed in a previous work [19]. Here we want to verify the G-structure associated a single null spinor, a spinor such

that $g^A = 0$. From the Fierz identities we may deduce

$$i_V I = 0,$$
(49)

$$I_\lrcorner I = 0,$$
(50)

$$I_{\mu\tau}I^{\nu\tau} = V_\mu V^\nu.$$
(51)

These imply that

$$I = V \wedge K, \qquad K^2 = 1.$$
(52)

Furthermore from the Fierz identities we may deduce the projection

$$K_\mu\Gamma^\mu\epsilon^a = \epsilon^a,$$
(53)

together with

$$i_V X^A = i_K X^A = 0.$$
(54)

Let us introduce a null orthonormal basis where

$$V = -e^+, \qquad K = e^5,$$
(55)

$$ds^2 = -2e^+e^- + \delta_{ij}e^ie^j + (e^5)^2,$$
(56)

$i, j = 1, \ldots, 4$, and we choose an orientation

$$\epsilon^{+-12345} = 1.$$
(57)

Then from the seven-dimensional duality relation for the gamma matrices, (53) becomes a chirality projection in the six Lorentzian dimensions orthogonal to K. It is well known [29] that a pair of symplectic-Majorana-Weyl spinors (or a single Weyl spinor) defines an $SU(2)\ltimes\mathbb{R}^4$ structure in six Lorentzian dimensions, so our seven-dimensional structure is simply $[SU(2)\ltimes\mathbb{R}^4] \times \mathbb{R}$. Now from the results of [5], we may immediately read off the form of the X^A:

$$X^A = 2e^+ \wedge J^A,$$
(58)

$$J^{Ai}{}_j J^{Bj}{}_k = \epsilon^{ABC}J^{Ci}{}_k - \delta^{AB}\delta^i_k.$$
(59)

With our choice of conventions, the J^A are self-dual on the base with orientation

$$\epsilon^{ijkl} = \epsilon^{+-ijkl5}.$$
(60)

The single spinor obeys the projection (53) and

$$J^A_{ij}\Gamma^{ij}\epsilon^a = 8iT^{Aa}_b\epsilon^b.$$
(61)

We may completely fix the spinor by requiring that in addition to (53) it obeys the projections

$$\Gamma^{12}\epsilon^1 = i\epsilon^2, \qquad \Gamma^{13}\epsilon^1 = -\epsilon^2, \qquad \Gamma^{14}\epsilon^1 = i\epsilon^1, \quad (62)$$

so that the J^A take the canonical form

$$J^1 = e^{12} + e^{34}, \qquad J^2 = -e^{13} + e^{24},$$
$$J^3 = e^{14} + e^{23}.$$
(63)

For performing the calculations of the constraints it is very useful to have a set of projections satisfied by ϵ which are implied by the defining projections. We give such a set in Appendix B.

B. Constraints for supersymmetry

Now we solve the constraints $\delta\lambda = \delta\psi_\mu = 0$ for the single spinor ϵ. In order to reduce as much as possible the computation required we will work with $\star G$, the dual of the four-form G. Hopefully without risk of confusion, we also denote the dual by G. Where ambiguity can arise we will explicitly indicate the rank of the form. In terms of the dual G, the supersymmetry variations of the fermions are

$$\sqrt{5}\delta\lambda^a = \frac{5}{2}\partial_\mu\phi\Gamma^\mu\epsilon^a + \frac{i}{2}F^A_{\mu\nu}\Gamma^{\mu\nu}T^{Aa}{}_b\epsilon^b$$
$$- \frac{1}{6}G_{\mu\nu\sigma}\Gamma^{\mu\nu\sigma}\epsilon^a - 5m'\epsilon^a, \tag{64}$$

$$\delta\psi^a_\mu = \nabla_\mu\epsilon^a - \frac{i}{10}(\Gamma_\mu{}^{\nu\sigma} - 8\delta^\nu_\mu\Gamma^\sigma)F^A_{\nu\sigma}T^{Aa}{}_b\epsilon^b$$
$$- \frac{3}{20}G_{\mu\nu\sigma}\Gamma^{\nu\sigma}\epsilon^a + \frac{1}{30}G_{\nu\sigma\tau}\Gamma_\mu{}^{\nu\sigma\tau}\epsilon^a$$
$$- igA^A_\mu T^{Aa}_\beta\epsilon^b + m\Gamma_\mu\epsilon^a. \tag{65}$$

1. Constraints from $\delta\lambda = 0$

Consider first the supersymmetry variation of $\delta\lambda$. We will in fact use two shortcuts from a previous work [19]. The first is that $i_V\partial\phi = 0$. Also, on the assumption that ϵ is Killing, we have that $i_V F^A = 0$. Thus,

$$\partial_-\phi = 0, \tag{66}$$

$$F^A_{-\mu} = 0. \tag{67}$$

An implication of these is that the only place terms with a single Γ^- can arise in $\delta\lambda$ is in $G_{\mu\nu\sigma}\Gamma^{\mu\nu\sigma}$. Also, all terms with a single Γ^+ drop out, since $\Gamma^+\epsilon = 0$. Consider the Γ^- terms. These may easily be computed to be

$$-\Gamma^-[G_{-i5}\Gamma^i + iG_{-ij}J^{Aij}T^A]\epsilon. \tag{68}$$

By linear independence, each of these must vanish separately. Hence,

$$G_{-i5} = 0, \tag{69}$$

$$G^{(+)}_{-ij} = 0. \tag{70}$$

Now all remaining terms in $\delta\lambda = 0$ may be converted into terms containing no gamma matrices or a single gamma matrix on the base. The terms with a single gamma matrix are given by

$$[\tfrac{5}{2}\partial_i\phi + G_{+-i} + \tfrac{1}{6}\epsilon_i{}^{jkl}G_{jkl} + \tfrac{1}{2}F^A_{j5}J^{Aj}{}_i]\Gamma^i\epsilon, \tag{71}$$

and each component of the one form in square brackets must vanish. Finally, the terms involving no gamma matrices are of the form $(A + i\sum_A B^A T^A)\epsilon$, and by linear independence we must have $A = B^A = 0$. This implies that

$$G_{+-5} + \tfrac{5}{2}\partial_5\phi - 5(m + 2he^{-4\phi}) = \tfrac{1}{4}F^A_{ij}J^{Aij}, \tag{72}$$

$$G^{(+)}_{5ij} = -\tfrac{1}{8}\epsilon^{ABC}F^A_{kl}J^{Bkl}J^C_{ij}. \tag{73}$$

This completes the analysis of $\delta\lambda^{(i)} = 0$.

2. Constraints from $\delta\psi_\mu = 0$

Next we turn to the analysis of the Killing spinor equation. We will use another result from [19], namely that, for null Killing spinors,

$$\nabla V = i_V G - d\phi \wedge V - 4he^{-4\phi}I, \tag{74}$$

so that V is a Killing vector and

$$\omega_{\mu\nu-} = G_{\mu\nu-} + (d\phi \wedge e^+)_{\mu\nu} + 4he^{-4\phi}(e^+ \wedge e^5)_{\mu\nu}. \tag{75}$$

Also we wish to comment on the choice of gauge. We always can, and will, choose the gauge $i_V A^A = 0$. We also know that $i_V F^A = 0$. Therefore if v is the coordinate along the integral curves of V, we have that A^A_μ is independent of v, in the gauge $i_V A^A = 0$. Therefore since we still have the freedom to perform v-independent gauge transformations, we may always choose a gauge such that

$$A^A_+ = A^A_- = 0, \tag{76}$$

and this is the gauge we work in henceforth. Now we know that each spacetime component of the Killing spinor equation can be reduced to the form (5) given in the introduction. All terms involving a single Γ^+ drop out. Furthermore, we have explicitly checked that all the terms of the form

$$[q_\mu + \Gamma^-(r_\mu + ir^A_\mu T^A + r_{\mu i}\Gamma^i)]\epsilon \tag{77}$$

vanish identically as a consequence of the constraints we have already derived. Thus, the Killing spinor equation reduces to

$$\delta\psi_\mu = (iq^A_\mu T^A + q_{\mu i}\Gamma^i)\epsilon. \tag{78}$$

Using the projections of Appendix B, it is easy to see that the spin connection components $\omega_{\mu i5}$ contribute only to the $q_{\mu i}\Gamma^i\epsilon$ terms, the $\omega^{(-)}_{\mu ij}$ components drop out, and the $\omega^{(+)}_{\mu ij}$ components contribute only to the $iq^A_\mu T^A\epsilon$ terms. The surviving components of the spin connection may be fixed in terms of the fluxes by demanding that the linearly independent terms (78) vanish. Thus, from the + component of $\delta\psi_\mu = 0$, we find

$$\omega^{(+)}_{+ij} = G^{(+)}_{+ij} - \tfrac{1}{2}F^A_{+5}J^A_{ij}, \tag{79}$$

$$\omega_{+i5} = G_{+i5} - F^A_{+j}J^{Aj}{}_i. \tag{80}$$

From the $-$ component we deduce

$$\omega^{(+)}_{-ij} = 0, \tag{81}$$

$$\omega_{-i5} = 0. \tag{82}$$

The 5 component gives

$$\omega^{(+)}_{5ij} = G^{(+)}_{5ij} + \frac{g}{2}A^A_5 J^A_{ij}, \tag{83}$$

$$\omega_{55i} = 4\partial_i\phi + 2G_{+-i} + \frac{1}{3}\epsilon_{ijkl}G^{jkl}. \tag{84}$$

Finally, from the i component we get

$$\omega_{ij5} = (G_{+-5} + \tfrac{3}{2}\partial_5\phi - 5m - 6he^{-4\phi})g_{ij}$$
$$+ G^{(-)}_{5ij} - G^{(+)}_{5ij} - F^{A(-)}_{ik}J^{Ak}{}_j, \tag{85}$$

$$\omega_{ijk}J^{Ajk} = 2gA^A_i - F^A_{i5} + \epsilon^{ABC}F^B_{j5}J^{Cj}{}_i - (3\partial_j\phi$$
$$+ G_{+-j})J^{Aj}{}_i. \tag{86}$$

This concludes the analysis of $\delta\psi_\mu = 0$. The practical advantages of calculating the necessary and sufficient conditions for supersymmetry in this fashion are obvious. From the integrability conditions of Appendix D, we may deduce that it is sufficient to impose the Bianchi identity for the Yang-Mills fields and $G_{(4)}$, the field equation for $G_{(4)}$, the $+$ component of the Yang-Mills field equation, and the $++$ component of the Einstein equation on this supersymmetric *Ansatz*. All other field equations are satisfied identically.

C. Introducing coordinates

In this subsection we will introduce coordinates for our problem. Our null Killing spinor induces a natural coordinate system in which the associated symmetries of the metric are manifest. Closely following [10], we introduce coordinates (v, u, z, x^M) such that the vectors dual to the basis one-forms are

$$e^+ = -\frac{\partial}{\partial v}, \tag{87}$$

$$e^- = -\frac{L\mathcal{F}}{2}\frac{\partial}{\partial v} + L\frac{\partial}{\partial u}, \tag{88}$$

$$e^5 = -\frac{B}{C}\frac{\partial}{\partial v} + \frac{1}{C}\frac{\partial}{\partial z}, \tag{89}$$

$$e^i = e^{i\mu}\partial_\mu. \tag{90}$$

Inverting we get

$$e^+ = -L^{-1}(du + \lambda), \tag{91}$$

$$e^- = dv + \tfrac{1}{2}\mathcal{F}du + Bdz + \nu, \tag{92}$$

$$e^5 = C(dz + \sigma), \tag{93}$$

$$e^i = e^i_M dx^M. \tag{94}$$

We have in fact fixed some of the gauge freedom in the metric. Now repeating an argument given in [10], we can deduce that L, $\mathcal{F}$, B, C, λ, ν, σ, and e^i_M may all be taken to be independent of v. The components of the spin connection for this metric were computed in [10]. We have performed the trivial modification to adjust to our slightly different conventions, and we give the result in Appendix C, where we use the following notation. Let Q be a q form on the base, satisfying $\mathcal{L}_V Q = 0$:

$$Q = \frac{1}{q!}Q(u, z, x^M)_{M_1...M_q}dx^{M_1} \wedge \cdots \wedge dx^{M_q}. \tag{95}$$

Define the exterior derivative restricted to the base

$$\tilde{d}Q \equiv \frac{1}{q!}\frac{\partial}{\partial x^{M_1}}Q_{M_2...M_{q+1}}dx^{M_1} \wedge \cdots \wedge dx^{M_{q+1}}, \tag{96}$$

and denote the Lie derivatives with respect to $\frac{\partial}{\partial u}$ and $\frac{\partial}{\partial z}$ acting on such forms by $\partial_u Q$, $\partial_z Q$, respectively. Then, defining

$$\mathcal{D}Q \equiv \tilde{d}Q - \sigma \wedge \partial_z Q + \lambda \wedge \partial_u Q, \tag{97}$$

we have

$$dQ = \mathcal{D}Q - Le^+ \wedge \partial_u Q + C^{-1}e^5 \wedge \partial_z Q. \tag{98}$$

Finally, we define the following quantities:

$$M_{ij} = \delta_{ik}(\partial_u e^k)_j, \tag{99}$$

$$\Lambda_{ij} = \delta_{ik}(\partial_z e^k)_j. \tag{100}$$

From the expression for the spin connection components in Appendix C, we may rewrite the constraints $\omega_{-i5} = \omega^{(+)}_{-ij} = 0$ as

$$\lambda = \lambda(u, x), \tag{101}$$

$$\mathcal{D}\lambda^{(+)} = 0. \tag{102}$$

D. Refining the classification

In this section, we have given the full set of necessary and sufficient conditions for the existence of a single arbitrary null spinor. In the next four sections, we will illustrate the computation of the constraints associated with the existence of additional spinors, for each of the four possibilities for the structure groups. As discussed in the introduction, this involves computing the expressions

$$[\Delta_\lambda, Q]\epsilon = 0, \tag{103}$$

$$[\mathcal{D}_\mu, Q]\epsilon = 0, \tag{104}$$

for a choice of Q compatible with the desired G-structure, and reducing them to canonical form, as a manifest sum of basis spinors. As has been stated, to do this for the most general choices of Q requires a lot of computation, and we will not perform the refined classification in full generality. For the structure groups SU(2), $\mathbb{R}^5$, and the identity, we will restrict attention to illustrative examples, for particular choices of Q. This means that we will make specific *Ansätze* for the additional Killing spinors. Also for computational convenience, unless otherwise stated we will restrict attention to bosonic configurations for which the Yang-Mills fields are of the form

$$A^A = A_i^A(x)e^i, \qquad F^A = \tfrac{1}{2}F_{ij}^A(x)e^i \wedge e^j, \tag{105}$$

but which are otherwise general. Note that this is the only assumption we make about the form of the bosonic fields; all the additional constraints we derive will follow as a consequence of the choice of Q.

IV. $[\mathbf{SU(2)}\ltimes\mathbb{R}^4] \times \mathbb{R}$ STRUCTURE

As we saw in the introduction, the simplest additional Killing spinors to incorporate are those which share the same isotropy group as the fiducial Killing spinor, and so imply no further reduction of the structure group. These spinors are parametrized by four real functions; given our assumption of the form of the Yang-Mills fields, we will completely analyze all $[SU(2)\ltimes\mathbb{R}^4] \times \mathbb{R}$ structures with more than one Killing spinor. In the first subsection we will derive the constraints, and in the second subsection we will construct a class of solutions for which the structure group is strictly $[SU(2)\ltimes\mathbb{R}^4] \times \mathbb{R}$ and not some subgroup.

A. Constraints

Consider an arbitrary $(+, +)$ spinor, which we denote by

$$(\delta + i\delta^A T^A)\epsilon. \tag{106}$$

As discussed in the introduction, assuming that this spinor is Killing does not imply any reduction of the structure group, but it does yield additional constraints on the intrinsic torsion. The commutators (103) for such spinors are easy to compute, since they just pick out the Yang-Mills terms in the supersymmetry variations. First consider

$$[\Delta_\lambda, \delta + i\delta^A T^A]\epsilon = -\tfrac{1}{2}F_{ij}^A\Gamma^{ij}\delta^B[T^A, T^B]\epsilon. \tag{107}$$

Reducing to canonical form, we get

$$\left[\frac{1}{4}\epsilon^{ABC}F_{ij}^A\delta^B J^{Cij} + \frac{i}{2}(F_{ij}^B J^{Bij}\delta^A - F_{ij}^A J^{Bij}\delta^B)T^A\right]\epsilon. \tag{108}$$

Next consider the Killing spinor equation. The commutator is

$$[\nabla_\mu - i(gA_\mu^A + \tfrac{1}{10}(\Gamma_{\mu\nu\sigma} - 8g_{\mu\nu}\Gamma_\sigma)F^{A\nu\sigma})T^A, \delta + i\delta^B T^B]\epsilon. \tag{109}$$

The $-$ component is particularly easy to evaluate. It reduces to

$$(\partial_-\delta + i\partial_-\delta^A T^A)\epsilon, \tag{110}$$

The $+$ component is

$$(\partial_+\delta + i\partial_+\delta^A T^A + \tfrac{1}{5}\Gamma^-[\Delta_\lambda, i\delta^A T^A])\epsilon. \tag{111}$$

The 5 component is

$$(\partial_5\delta - \tfrac{1}{5}[\Delta_\lambda, i\delta^A T^A] + i\partial_5\delta^A T^A)\epsilon, \tag{112}$$

and for the i component we have

$$[\partial_i\delta + i(\partial_i\delta^A + g\epsilon^{ABC}A_i^B\delta^C)T^A - \tfrac{1}{2}\epsilon^{ABC}(\tfrac{3}{5}F_{ij}^{A(+)} + F_{ij}^{A(-)})\delta^B J^{Cj}{}_k\Gamma^k]\epsilon.$$

Now, requiring the vanishing of the commutators acting on ϵ implies the following algebraic restrictions on the Yang-Mills fields in terms of the parameters δ^A:

$$\epsilon^{ABC}F_{ij}^A\delta^B J^{Cij} = 0, \qquad \delta^A F_{ij}^B J^{Bij} - \delta^B F_{ij}^A J^{Bij} = 0,$$
$$\epsilon^{ABC}F_{jk}^{A(-)}\delta^B J^{Cj}{}_i = 0. \tag{113}$$

We also get a set of differential constraints on the δ, δ^A which are

$$\partial_\mu\delta = 0, \tag{114}$$

$$\partial_-\delta^A = \partial_+\delta^A = \partial_5\delta^A = 0, \tag{115}$$

$$\partial_i\delta^A = -g\epsilon^{ABC}A_i^B\delta^C. \tag{116}$$

Since we have already assumed that ϵ is Killing, and we have derived that $\delta = \text{const}$, we take $\delta = 0$. We also have

$$\delta^A = \delta^A(x). \tag{117}$$

Given the Yang-Mills Bianchi identity, the integrability condition for (116) implies that $\epsilon^{ABC}F_{ij}^B\delta^C = 0$, and so implies the algebraic constraints. Therefore a bosonic configuration admitting the Killing spinor ϵ also admits the Killing spinor $i\delta^A T^A\epsilon$ if and only if Eq. (116) is satisfied. By performing an x dependent SU(2) gauge transformation (thus preserving $A_+^A = A_-^A = A_5^A = 0$), we may set

$$\delta^A = k\delta^{1A}, \tag{118}$$

for some constant k which can be eliminated by a constant rescaling, and so the existence of this Killing spinor implies and is implied by the condition that the Yang-Mills field is truncated to a U(1) subgroup,

$$A^2 = A^3 = 0. \tag{119}$$

Clearly, requiring the existence of a third linearly independent $(+, +)$ Killing spinor would imply that

$$A^1 = 0, \tag{120}$$

and then the configuration automatically admits a fourth linearly independent spinor. Thus, the $[SU(2)\ltimes\mathbb{R}^4] \times \mathbb{R}$ structure can admit one, two, or four linearly independent Killing spinors, depending on whether the Yang-Mills fields are SU(2), U(1), or zero. Incorporating additional $(+, +)$ spinors implies no further constraints on G, ϕ, or ω, beyond those derived in Sec. II. From these results, we also see that, when the Yang-Mills fields are truncated to a U(1) subgroup, Killing spinors always come in pairs; if ϵ_K is Killing, then so is the linearly independent spinor $iT^1\epsilon_K$, since T^1 commutes with the operators in the fermion supersymmetry variations. Similarly, when the Yang-Mills fields vanish, Killing spinors always come in groups of four.

B. Examples

Let us now present a family of solutions for which the structure group is strictly $[SU(2)\ltimes\mathbb{R}^4] \times \mathbb{R}$ and not some subgroup. It is easy to verify that the vacuum solution

$$ds^2 = (4hz)^{-2}[-dt^2 + dr^2 + dz^2 + ds^2(\mathcal{M}_4)], \tag{121}$$

where $\mathcal{M}_4$ is hyperkähler and $16h = g$, satisfies all the constraints of Sec. II, and the $++$ component of the Einstein equations is satisfied. Since the Yang-Mills fields vanish, these solutions admit Killing spinors in multiples of four. When $\mathcal{M}_4 = \mathbb{R}^4$, this is nothing but the maximally supersymmetric AdS_7 solution of the theory. Choosing any other hyperkähler base reduces the number of independent Killing spinors to four. One may show that demanding the existence of an arbitrary additional Killing spinor of the form (13) implies the constraints

$$\tilde{\nabla}_i \delta_j = 0, \tag{122}$$

$$\partial_\mu \theta = \partial_\mu \theta^A = 0, \tag{123}$$

$$\tilde{\nabla}_i \theta_j = 2m\theta\delta_{ij} + m\theta^A J_{ij}^A, \tag{124}$$

where $\tilde{\nabla}$ denotes the Levi-Civita connection on the base. Given that, for vanishing Yang-Mills fields Killing spinors come in linearly independent groups of four, the existence of a single one-form δ_i satisfying (122) implies the existence of three more linearly independent solutions, and thus (122) implies that either $\delta_i = 0$ or the base is flat. Similarly, since θ, θ^A are constant, and $\tilde{\nabla}J^A = 0$, (124) implies that $[\tilde{\nabla}_i, \tilde{\nabla}_j]\theta_k = 0$, and the existence of a one-form on the base satisfying this equation implies the existence of three more linearly independent solutions. Hence, either $\theta_j = 0$ or the base is flat. If the base is not flat, then we must have $\delta_i = \theta_i = 0$ and, given $m \neq$

0, that $\theta = \theta^A = 0$. Thus the solutions (121) preserve precisely four supersymmetries, with a strictly $[SU(2)\ltimes\mathbb{R}^4] \times \mathbb{R}$ structure, when the base is a nonflat hyperkähler manifold.

V. SU(2) STRUCTURE

In this section we will study configurations in the theory admitting an SU(2) structure. We will first derive the constraints for the existence of a particular choice of additional Killing spinor. We will then derive necessary and sufficient conditions for the existence of eight Killing spinors fixed by the same SU(2) . Finally, we will discuss some illustrative explicit solutions.

A. One additional Killing spinor

Consider an arbitrary linear combination of $(+, +)$ and $(-, +)$ spinors, which we denote by

$$[\delta + i\delta^A T^A + \Gamma^-(\theta + i\theta^A T^A)]\epsilon, \tag{125}$$

and where at least one of the θ, $\theta^A \neq 0$. To derive the constraints following from $\delta\lambda = 0$ for this spinor, we must impose

$$[\Delta_\lambda, \delta + i\delta^A T^A + \Gamma^-(\theta + i\theta^A T^A)]\epsilon = 0. \tag{126}$$

We have already calculated the $[\Delta_\lambda, \delta + i\delta^A T^A]\epsilon$ terms, in the previous section. Calculating the commutator

$$[\Delta_\lambda, i\theta^A \Gamma^- T^A]\epsilon, \tag{127}$$

while entirely straightforward, is a tedious exercise. We will therefore restrict attention to spinors with $\theta^A = 0$, but $\theta \neq 0$. The θ dependent terms in the commutator, reduced to canonical form, then become

$$[\Delta_\lambda, \theta\Gamma^-]\epsilon = \theta[-5\partial_+\phi + 2iJ^{Aij}G_{+ij}T^A + 2G_{+i5}\Gamma^i$$
$$+ 2G_{+-i}\Gamma^-\Gamma^i - 5\partial_5\phi\Gamma^-$$
$$+ 2iG_{5ij}J^{Aij}\Gamma^- T^A]\epsilon. \tag{128}$$

We may immediately deduce

$$G_{+i5} = G_{+-i} = \partial_5\phi = G_{5ij}^{(+)} = 0. \tag{129}$$

Then from the terms in $[\Delta_\lambda, \delta + i\delta^A T^A]\epsilon$, and using $G_{5ij}^{(+)} = 0$, we obtain

$$\partial_+\phi = 0, \tag{130}$$

$$\theta G_{+ij} J^{Aij} = -\tfrac{1}{4}(F_{ij}^B J^{Bij}\delta^A - F_{ij}^A J^{Bij}\delta^B). \tag{131}$$

Next, commuting the $-$ component of the Killing spinor equation, we deduce

$$\partial_-\theta = 0, \tag{132}$$

$$\partial_-\delta = -\theta[2(5m + 6he^{-4\phi}) + \tfrac{1}{2}F_{ij}^A J^{Aij}], \tag{133}$$

$$\partial_- \delta^A = 0. \tag{134}$$

From the $+$ component, we get

$$\partial_+ \theta = 0, \tag{135}$$

$$\partial_+ \delta = \theta \omega_{+5+}, \tag{136}$$

$$\partial_+ \delta^A = 0, \tag{137}$$

$$\omega_{++i} = 0. \tag{138}$$

The 5 component gives

$$\partial_5 \theta = 0, \tag{139}$$

$$\partial_5 \delta = \theta \omega_{55+}, \tag{140}$$

$$\partial_5 \delta^A = 0, \tag{141}$$

$$\omega_{5+i} = 0, \tag{142}$$

and from the i component we get

$$\partial_i \theta = -\theta \partial_i \phi, \tag{143}$$

$$\partial_i \delta = 0, \tag{144}$$

$$\partial_i \delta^A = -g\epsilon^{ABC} A_i^B \delta^C, \tag{145}$$

$$\tfrac{1}{2}\epsilon^{ABC}(\tfrac{3}{5}F_{ik}^{A(+)} + F_{ik}^{A(-)})\delta^B J^{Ck}{}_j$$
$$= \theta(\tfrac{1}{5}G_{+ij}^{(+)} + G_{+ij}^{(-)} - \omega_{ij+}), \tag{146}$$

$$(\tfrac{1}{4}F_{kl}^A J^{Akl} + 4he^{-4\phi})g_{ij} = F_{ik}^{A(-)} J^{Ak}{}_j. \tag{147}$$

As before, (145) implies that $\epsilon^{ABC} F_{ij}^A \delta^C = 0$. We may easily solve the differential equations for θ to find

$$\theta = ke^{-\phi}, \tag{148}$$

for some nonzero constant k; by a constant rescaling of the spinor, we may take $k = 1$. Furthermore, since $A_{ik}^{(+)} B^{(-)k}{}_j$ is in general symmetric and traceless on i, j, (147) implies that

$$F_{ij}^A J^{Aij} = -16he^{-4\phi}, \tag{149}$$

$$F_{ij}^{A(-)} = 0. \tag{150}$$

Summary.—Given the form (105) of the Yang-Mills fields, the following conditions, in addition to those of Sec. II, are necessary and sufficient for a bosonic configuration to admit the Killing spinors ϵ, $(\delta + i\delta^A T^A + \theta\Gamma^-)\epsilon$, $\theta \neq 0$. The functions θ, δ, and δ^A are required to satisfy

$$\theta = e^{-\phi}, \qquad \partial_- \delta = g, \qquad \partial_i \delta = 0,$$
$$\delta^A = \delta^A(x), \qquad \partial_i \delta^A = -g\epsilon^{ABC} A_i^B \delta^C. \tag{151}$$

The matter fields are constrained according to

$$\phi = \phi(x),$$

$$G_{(3)} = \left(4he^{-4\phi} - \frac{g}{2}e^\phi\right)e^+ \wedge e^- \wedge e^5 + e^+ \wedge P_1^{(-)}$$
$$+ e^- \wedge P_2^{(-)} + e^5 \wedge P_3^{(-)} + \frac{5}{2}\star_4 d\phi, \tag{152}$$

$$F^A = \tfrac{1}{2}F_{ij}^{A(+)} e^i \wedge e^j, \qquad F_{ij}^A J^{Aij} = -16he^{-4\phi},$$
$$\epsilon^{ABC} F_{ij}^B J^{Cij} = 0, \qquad \epsilon^{ABC} F_{ij}^B \delta^C = 0,$$

where $P_{1,2,3}^{(-)}$ are arbitrary anti-self-dual two-forms on the base, and $\star_4$ denotes the Hodge dual on the base. Finally, there are the following constraints on the independent components of the spin connection:

$$\omega_{(\mu\nu)-} = \omega_{-5i} = \omega_{+5i} = \omega_{++i} = 0, \qquad \omega_{-+5} = -\frac{g}{2}e^\phi, \qquad \omega_{-+i} = \omega_{5i5} = \partial_i\phi, \qquad \omega_{+5+} = \theta^{-1}\partial_+\delta,$$

$$\omega_{55+} = \theta^{-1}\partial_5\delta, \qquad \omega_{+ij} = -\omega_{ij+} = -P_1^{(-)}, \qquad \omega_{-ij} = -P_2^{(-)}, \qquad \omega_{5ij} = -\omega_{ij5} = -P_3^{(-)}, \tag{153}$$

$$\omega_{ijk} J^{Ajk} = 2gA_i^A - 3\partial_j\phi J^{Aj}{}_i,$$

and the remaining nonzero components may be read off from the equalities in Appendix C. It may be verified that it is sufficient to impose the Bianchi identities for the forms and the field equation for G, and all other equations of motion are implied by the existence of the pair of Killing spinors.

Clearly, demanding the existence of the second Killing spinor significantly reduces the complexity of the *Ansatz*, and much more of the field content is fixed in terms of the structure. Before considering some explicit examples, let

us now turn to the classification of all solutions admitting a strictly SU(2) structure with eight supersymmetries.

B. SU(2) structures with eight supersymmetries

In this subsection, we will classify all configurations admitting eight linearly independent Killing spinors sharing the common projection $\Gamma^{1234}\epsilon = \epsilon$ (note that this implies the common projection $\Gamma^{+-5}\epsilon = -\epsilon$). This is the maximal supersymmetry compatible with a strictly SU(2) structure. We will perform the classification in full

generality, and we thus relax the assumptions made about the form of the Yang-Mills fields for this subsection. First, consider the vanishing of $\delta\lambda$. Since this is linear in ϵ and contains no derivatives, it must vanish when acting on each of the eight basis spinors individually. It is a simple matter to verify that this implies that the matter fields are restricted to be of the following form:

$$\phi = \phi(x), \qquad F^A = \tfrac{1}{2}F_{ij}^{A(-)}e^i \wedge e^j,$$

$$G = 5(m + 2he^{-4\phi})e^+ \wedge e^- \wedge e^5 + \tfrac{1}{2}G_{-ij}^{(-)}e^- \wedge e^i \wedge e^j + \tfrac{1}{2}G_{+ij}^{(-)}e^+ \wedge e^i \wedge e^j + \tfrac{1}{2}G_{5ij}^{(-)}e^5 \wedge e^i \wedge e^j + \tfrac{5}{2}\star_4 d\phi. \tag{154}$$

The vanishing of $F_{+\mu}^A$, $F_{5\mu}^A$ implies that locally we may choose a gauge such that $A^A = A_i^A e^i$. Thus, acting on any of the eight Killing spinors, the supercovariant derivative reduces to the following form:

$$\mathcal{D}_- = \nabla_- + \tfrac{3}{2}(m + 2he^{-4\phi})\Gamma^{+5} - \tfrac{1}{2}\partial_i\phi\Gamma^{+i} - m\Gamma^+, \qquad \mathcal{D}_+ = \nabla_+ - \tfrac{3}{2}(m + 2he^{-4\phi})\Gamma^{-5} - \tfrac{1}{2}\partial_i\phi\Gamma^{-i} - m\Gamma^-,$$

$$\mathcal{D}_5 = \nabla_5 + \tfrac{3}{2}(m + 2he^{-4\phi})\Gamma^{+-} + \tfrac{1}{2}\partial_i\phi\Gamma^{5i} + m\Gamma^5, \tag{155}$$

$$\mathcal{D}_i = \nabla_i + iF_{ij}^A\Gamma^j T^A + \tfrac{1}{2}(G_{+ij}\Gamma^{+j} + G_{-ij}\Gamma^{-j} + G_{5ij}\Gamma^{5j}) + (m + 2he^{-4\phi})\Gamma_i^{+-5} + \tfrac{1}{2}\partial_i\phi - igA_i^A T^A + m\Gamma^i.$$

By assumption, there exist four Killing spinors of the form $[\delta + i\delta^A T^A + \Gamma^-(\theta + i\theta^A T^A)]\epsilon$, with four distinct choices of the functions θ, θ^A, at least one of which is nonzero in each case, and none of which is zero for all four. Let us evaluate $[\mathcal{D}_\mu, Q]\epsilon$ for one of these Killing spinors. From the $-$ component, we find the constraints

$$\partial_-\theta = \partial_-\theta^A = 0, \qquad \partial_-\delta = -2(5m + 6he^{-4\phi})\theta,$$
$$\partial_-\delta^A = -2(5m + 6he^{-4\phi})\theta^A. \tag{156}$$

Next, from the $+$ component, we find

$$\partial_+\theta = \partial_+\theta^A = \omega_{++i} = 0, \qquad \partial_+\delta = -\theta\omega_{++5},$$
$$\partial_+\delta^A = -\theta^A\omega_{++5}. \tag{157}$$

The 5 component gives

$$\partial_5\theta = \partial_5\theta^A = \omega_{5+i} = 0, \quad \partial_5\delta = -\theta\omega_{5+5}, \quad \partial_5\delta^A = -\theta^A\omega_{5+5}. \tag{158}$$

Finally, consider the i component and, in particular, the constraints derived from the vanishing of the $\Gamma^{-i}\epsilon$ term. These read

$$-\theta F_{ik}^A J^{Ak}{}_j + \tfrac{1}{2}F_{ij}^A\theta^A + 4h\theta e^{-4\phi}g_{ij} + 2he^{-4\phi}\theta^A J_{ij}^A = 0. \tag{159}$$

Recall that each F_{ij}^A is anti-self-dual. Then extracting the antisymmetric part of (159), and further decomposing in self-dual and anti-self-dual parts, we find

$$F_{ij}^A\theta^A = 0, \qquad hJ_{ij}^A\theta^A = 0. \tag{160}$$

Since by assumption there exist three linearly independent Killing spinors with different nonzero θ^A, these equations imply that $h = F^A = 0$. Since the Yang-Mills field strengths vanish, we may locally choose the gauge so that also $A^A = 0$. Then since each T^A commutes with the operators in the supersymmetry variations, we see that we may take four of the Killing spinors to be ϵ, $iT^A\epsilon$. It remains to determine the other four Killing spinors with nonzero θ, θ^A. Since iT^A commute with the operators in

the supersymmetry variations, once one of the Killing spinors, ϵ_K, with nonzero θ, θ^A is determined, the other three may be taken to be $iT^A\epsilon_K$. We thus only need to determine the single Killing spinor ϵ_K. The remaining constraints from the i component of the Killing spinor equation are

$$\partial_i\delta = \partial_i\delta^A = 0, \qquad \partial_i\theta = -\partial_i\phi\theta,$$
$$\partial_i\theta^A = -\partial_i\phi\theta^A, \qquad G_{ij+} = \omega_{ij+}. \tag{161}$$

We may easily solve for the differential equations for θ, θ^A to find

$$\theta = ke^{-\phi}, \qquad \theta^A = k^A e^{-\phi}, \tag{162}$$

for some constants k, k^A, at least one of which is nonzero. Now note from the differential equations for δ, δ^A that, if $k = 0$, then δ is constant and, if $k^A = 0$, then δ^A is constant. These constants may be taken to be zero, since we are free to add to ϵ_K a constant multiple of the four Killing spinors ϵ, $iT^A\epsilon$. Let us thus write

$$\delta = k\delta', \qquad \delta^A = k^A\delta^{A\prime}, \tag{163}$$

with no sum on A in the second of these equations. Now for each nonzero k, k^A, the associated δ', $\delta^{A\prime}$ obey the same set of differential equations, which schematically are

$$\partial_\mu f = B_\mu. \tag{164}$$

Locally the solution exists and is unique, up to a constant which as before may be taken to be zero. Hence,

$$\delta = kf(u, v, z, x), \qquad \delta^A = k^A f(u, v, z, x). \tag{165}$$

We may easily fix the v dependence of f from Eq. (156), finding

$$f(u, v, z, x) = gv + \hat{f}(u, z, x). \tag{166}$$

We must impose $\partial_i f = 0$, but the function $\hat{f}$ is otherwise arbitrary. Making a choice for it determines ω_{++5} and ω_{5+5} through Eqs. (157) and (158). Thus,

$$\epsilon_K = [kgv + k\hat{f} + iT^A k^A \hat{f} + e^{-\phi}\Gamma^-(k + ik^A T^A)]\epsilon, \tag{167}$$

and the remaining three Killing spinors are $iT^A \epsilon_K$. By taking linear combinations with constant coefficients of these four spinors, we may in fact choose the set of four Killing spinors to be

$$\epsilon'_K = [gv + \hat{f}(u, z, x) + e^{-\phi}\Gamma^-]\epsilon, \tag{168}$$

and $iT^A \epsilon'_K$. We have thus determined all eight Killing spinors, and the necessary and sufficient conditions for their existence. It is intriguing to note that all eight may be generated by repeated application of the three matrices iT^1, iT^2, and $[gv + f'(u, z, x) + e^{-\phi}\Gamma^-]$ to ϵ, and also that the third one of these matrices is precisely of the form we derived in the previous subsection, for a single additional Killing spinor. Perhaps for all configurations with an SU(2) structure, the additional Killing spinors are of this form; that is, perhaps these are the only three matrices defining an SU(2) structure which, together with their products, can commute with Δ_λ and $\mathcal{D}_\mu$, when acting on ϵ. We have not verified this conjecture. However, if it is true that there are only a few matrices Q that can generate additional Killing spinors $Q\epsilon$, then the problem of performing a fully refined classification will not be as formidable as it might appear. It will be interesting to explore this point in the future, and in other contexts. We will now turn to some explicit examples.

C. Examples

To illustrate the case of an SU(2) structure, let us first see how known $AdS_3 \times \mathcal{M}_4$ solutions of the theory arise in our formalism. This will also illustrate the effect of additional Killing spinors on the intrinsic torsion of the structure. We will then generalize the construction, to obtain new AdS_3 solutions.

1. $AdS_3 \times \mathcal{M}_4$ solutions

Let us make the following *Ansatz* for a second independent Killing spinor:

$$\epsilon_K = (gv + \Gamma^-)\epsilon. \tag{169}$$

This choice solves all the differential constraints on θ, δ; also, it implies that $\omega_{+5+} = \omega_{55+} = 0$, and $\phi = \text{const}$; without loss of generality, when $\phi = \text{const}$, by constant rescalings of the forms and the couplings we may take $\phi = 0$. We will take the metric to be of the direct product form

$$ds^2 = L^{-1}(z)dudv + C^2(z)dz^2 + h_{MN}(x)dx^M dx^N. \tag{170}$$

For this choice of metric, the constraint $\omega_{-+5} = -g/2$ is equivalent to

$$\partial_z L = -gLC, \tag{171}$$

from Appendix C, which is solved by

$$L = (gz)^{-2}, \qquad C = 2L^{1/2}, \tag{172}$$

and so by rescaling z the metric in the $+ - 5$ directions may be written as

$$\frac{4}{g^2}\left[z^2(-dt^2 + dy^2) + \frac{dz^2}{z^2}\right], \tag{173}$$

which is the metric on AdS_3 with AdS length $2g^{-1}$. Our choice of metric implies that $P^{(-)}_{1,2,3} = 0$, so the fluxes are

$$G_{(3)} = \left(4h - \frac{g}{2}\right)e^+ \wedge e^- \wedge e^5, \qquad F^A = f^{AB}J^B, \tag{174}$$

where

$$f^{[AB]} = 0, \qquad f^{AA} = -4h. \tag{175}$$

Since G is closed and coclosed, the Bianchi identity for $G_{(4)}$ is automatically satisfied, while the field equation for $G_{(4)}$ is

$$-8h\left(4h - \frac{g}{2}\right) - f^{AB}f^{AB} = 0. \tag{176}$$

We may rewrite the final constraint of (153) in covariant form as

$$\nabla_i J^A_{jk} + g\epsilon^{ABC}A^B_i J^C_{jk} = 0, \tag{177}$$

or equivalently,

$$A^A_i = -\frac{1}{8g}\epsilon^{ABC}J^{Bjk}\nabla_i J^C_{jk}. \tag{178}$$

Imposing the Yang-Mills Bianchi identity then implies that

$$f^{AB}J^B_{ij} = \frac{1}{2g}J^{Akl}R_{klij} \equiv \frac{1}{g}\mathfrak{R}^A_{ij}. \tag{179}$$

Taking the commutator of two SU(2) covariant derivatives acting on J^A, we may obtain

$$J^{Ak}_i R_{kj} = \mathfrak{R}^A_{ij} + \epsilon^{ABC}J^{Bk}_i \mathfrak{R}^C_{kj}, \tag{180}$$

where R_{ij} is the Ricci tensor and, hence, that

$$R = 4gf^{AA} = -16hg, \tag{181}$$

$$R_{ij} = \frac{R}{4}g_{ij}, \tag{182}$$

where R is the scalar curvature of the base. Hence, the Yang-Mills Bianchi identity implies that we must choose the base to be Einstein (with negative scalar curvature when $hg > 0$). The Yang-Mills fields are then determined by (179), and once f^{AB} is determined, (176) fixes h in terms of g. There are no further constraints, and all remaining field equations are identically satisfied.

Two Killing spinors.—Let us take the base to be H^4, equipped with the metric

$$ds_4^2 = \frac{4\mu^2}{(1-r^2)^2}\left(dr^2 + \frac{r^2}{4}\delta_{AB}\sigma^A\sigma^B\right), \tag{183}$$

where the σ^A are the right invariant one-forms on an S^3, $d\sigma^A = -\frac{1}{2}\epsilon^{ABC}\sigma^B\sigma^C$. Let us choose the vierbeins to be

$$e^1 = \frac{\mu r}{1-r^2}\sigma^1, \qquad e^2 = \frac{\mu r}{1-r^2}\sigma^3,$$

$$e^3 = \frac{\mu r}{1-r^2}\sigma^2, \qquad e^4 = -\frac{2\mu}{1-r^2}dr. \tag{184}$$

Then we may simply read off the full solution; we find that $g = \frac{28}{3}h$, $\mu^2 = \frac{7}{g^2}$, and the bosonic fields are given by

$$ds^2 = \frac{1}{g^2}ds^2(AdS_3) + \mu^2 ds^2(H^4), \qquad \phi = 0,$$

$$G_{(4)} = \frac{1}{2g\mu^2}e^{1234},$$

$$F^A = -\frac{1}{g\mu^2}\left(e^A \wedge e^4 + \frac{1}{2}\epsilon^{ABC}e^B \wedge e^C\right), \tag{185}$$

$$A^A = \frac{r^2}{g(r^2-1)}\sigma^A.$$

This solution admits precisely the two Killing spinors ϵ and $\epsilon_K = (gv + \Gamma^-)\epsilon$, and it may readily be verified that both of these spinors are null, as are their sum and difference. Uplifted to 11 dimensions, this solution is the AdS fixed point of the near-horizon limit of an $M5$ brane wrapped on a Cayley four-cycle in a Spin(7) manifold [30].

Four Killing spinors.—Now suppose that, in addition to the Killing spinors ϵ, ϵ_K, we demand the existence of the Killing spinor $iT^1\epsilon$. This imposes $F^2 = F^3 = 0$; also it implies the existence of the fourth Killing spinor $\epsilon_K' = iT^1(gv + \Gamma^-)\epsilon$. Since $A^2 = A^3 = 0$, we now have

$$\tilde{\nabla}J^1 = 0, \tag{186}$$

so the base must be chosen to be Einstein-Kähler, with $F^1 = g^{-1}\Re$, where $\Re$ is the Ricci form of the base. From the four-form field equation we find $g = 12h$. The four spinors ϵ, $iT^1\epsilon$, ϵ_K, ϵ_K' are null. However, we know from [19] that these solutions admit timelike spinors. It is easily checked that the four Killing spinors $\epsilon \pm \epsilon_K'$, $iT^1\epsilon \pm \epsilon_K$ are timelike. Uplifted to 11 dimensions, the metric is that of the AdS fixed point of the near-horizon limit of an $M5$ wrapped on a Kähler four-cycle in a Calabi-Yau fourfold [30].

Eight Killing spinors.—Finally, suppose that, in addition to the four Killing spinors of the previous paragraph, we demand the existence of the Killing spinor $iT^2\epsilon$. This imposes

$$F^A = h = 0, \tag{187}$$

$$G_{+-5} = \omega_{-+5} = -\frac{g}{2}, \tag{188}$$

$$\nabla J^A = 0, \tag{189}$$

and implies the existence of the additional Killing spinors $iT^3\epsilon$, $iT^2(gv + \Gamma^-)\epsilon$, $iT^3(gv + \Gamma^-)\epsilon$. The three J^A are required to be covariantly constant on the base, which must thus be taken to be Calabi-Yau. As in the previous example, these solutions admit both timelike and null Killing spinors. Lifted to ten dimensions, they are just the familiar $AdS_3 \times S^3 \times \mathcal{M}_4$ solutions of ten-dimensional supergravity.

The AdS examples we have given clearly illustrate the effect that the additional independent Killing spinors have on the torsion of the G-structure, which is SU(2) in each case. The base is progressively restricted from Einstein to Einstein-Kähler to Calabi-Yau, together with an associated truncation of the Yang-Mills fields from SU(2) to U(1) to zero, as additional Killing spinors are incorporated.

2. Generalizations: membranes with AdS_3 world volume

Let us now consider generalizing the known solutions given above, by allowing for a nonconstant dilaton. In fact, the only additional truncation of the matter fields we will make, beyond that implied by the required supersymmetry, is to set $P_{1,2,3}^{(-)} = 0$. Let us demand the existence of the second Killing spinor,

$$\epsilon_K = (gv + e^{-\phi}\Gamma^-)\epsilon. \tag{190}$$

This implies that $\omega_{+5+} = \omega_{55+} = 0$. Let us make the metric *Ansatz*

$$ds^2 = L^{-1}(x,z)dudv + C^2(x,z)dz^2 + h_{mn}(x)dx^M dx^N. \tag{191}$$

Now let us solve the constraints

$$\omega_{-+i} = \omega_{5i5} = \partial_i\phi. \tag{192}$$

These read

$$\frac{1}{2L}\partial_i L = -\frac{1}{C}\partial_i C = \partial_i\phi, \tag{193}$$

which are solved by

$$L = e^{2\phi}\tilde{L}(z), \qquad C = e^{-\phi}\tilde{C}(z). \tag{194}$$

Next, the constraint

$$\omega_{-+5} = -\frac{g}{2}e^\phi \tag{195}$$

becomes

$$\partial_z\tilde{L} = -g\tilde{C}\tilde{L}, \tag{196}$$

which as before is solved by

$$\tilde{L} = (gz)^{-2}, \qquad \tilde{C} = 2\tilde{L}^{1/2}. \tag{197}$$

Given our metric *Ansatz*, all the remaining constraints on the spin connection are satisfied, except for

$$\omega_{ijk}J^{Aij} = 2gA_i^A - 3\partial_j\phi J^{Aj}{}_i. \tag{198}$$

Conformally rescaling the base according to

$$h_{(4)} = e^{3\phi}\tilde{h}_{(4)}, \tag{199}$$

we find

$$\tilde{\omega}_{ijk}\tilde{J}^{Ajk} = 2gA_i^A, \tag{200}$$

where $J_{ij}^A = e^{3\phi}\tilde{J}_{ij}^A$, $\tilde{\omega}_{ijk}$ is the spin connection on the base with metric $\tilde{h}_{(4)}$, and for the remainder of this discussion all indices on "tilded" objects are raised with $\tilde{h}^{ij}$. The seven metric is thus

$$ds^2 = 4(ge^\phi)^{-2}\left[z^2(-dt^2 + dy^2) + \frac{dz^2}{z^2}\right] + e^{3\phi}d\tilde{s}^2, \tag{201}$$

the warped product of AdS_3 with some four manifold. Next, imposing the Bianchi identity for F^A [which we recall in our conventions for the field strength is of the modified form $D(e^\phi F^A) = 0$], we find that

$$F_{ij}^A = (ge^\phi)^{-1}\tilde{\Re}_{ij}^A, \tag{202}$$

and as before that

$$\tilde{R} = -16hg, \tag{203}$$

where the conformally rescaled metric $\tilde{h}$ is either Einstein, Kähler-Einstein, or Calabi-Yau, depending on whether the Yang-Mills fields are SU(2), U(1), or zero. It remains to impose the Bianchi identities and field equations for G. It is trivially verified that the four-form Bianchi identity, $d(e^{-2\phi}G_{(4)}) = 0$, is satisfied. Finally, the field equation for $G_{(4)}$ reduces to a single equation for ϕ:

$$\frac{5}{2}(\tilde{\nabla}^2\phi + 5\tilde{h}^{ij}\partial_i\phi\partial_j\phi) - 4hg + h^2e^{-5\phi}\left(32 + \frac{16}{n}\right) = 0, \tag{204}$$

where $n = 3$ for SU(2) Yang-Mills fields and $n = 1$ for U(1). When the Yang-Mills fields vanish, we must take $h = 0$. The examples in the previous subsection are clearly $\phi = 0$ solutions of this equation. When ϕ is not constant, by making the definition

$$\phi = \frac{1}{5}\log\left[f + \frac{4h}{g}\left(2 + \frac{1}{n}\right)\right], \tag{205}$$

we find that f obeys

$$\nabla^2 f - 8hgf = 0. \tag{206}$$

As a first example of a more general solution of this form, take the conformally rescaled base to be H^4, with squared radius $\frac{3}{4hg}$, and metric

$$d\tilde{s}^2 = \frac{3}{4hg}(dr^2 + \sinh^2 dd\Omega_3^2). \tag{207}$$

We take $f = f(r)$, and thus obtain

$$f'' + 3\coth rf' - 6f = 0. \tag{208}$$

Making the change of variable $u = \tanh^2 r$, and defining $f = (1 - u)^\alpha\psi$, $\alpha = (3 + \sqrt{33})/4$, converts this to hypergeometric form,

$$u(1 - u)\frac{d^2\psi}{du^2} + [c - (1 + a + b)u]\frac{d\psi}{du} - ab\psi = 0, \tag{209}$$

with $a = (3 + \sqrt{33})/4$, $b = (5 + \sqrt{33})/4$, $c = 2$. Depending on the solution we choose for ψ, we find that the metric is singular either at $r = 0$ or $r = \infty$, or both. The SU(2) Yang-Mills fields may be read off from (202). Imposing regularity at infinity, we choose $\psi = F(a, b; 1 + a + b - c; 1 - \tanh^2 r)$; we find that for large r, the solution asymptotes to our previous $AdS_3 \times H^4$ example. To investigate the behavior near $r = 0$, we define a new radial coordinate $\rho = r^{2/5}$; the metric approaches that of a cone over $AdS_3 \times S^3$,

$$ds^2 = d\rho^2 + \rho^2[R_1^2 ds^2(AdS_3) + R_2^2 ds^2(S^3)], \tag{210}$$

and the dilaton blows up. The full solution preserves two supersymmetries.

As a second example of a more general solution of this form, take the conformally rescaled base metric to be flat:

$$d\tilde{s}^2 = dr^2 + r^2 d\Omega_3^2. \tag{211}$$

This choice implies that $F^A = h = 0$. Let us set $f = f(r)$, so that up to an irrelevant multiplicative constant,

$$f = 1 + \frac{a}{r^2}. \tag{212}$$

The seven-dimensional metric is thus

$$ds^2 = 4g^{-2}\left(1 + \frac{a}{r^2}\right)^{-2/5}ds^2(AdS_3)$$
$$+ \left(1 + \frac{a}{r^2}\right)^{3/5}ds^2(\mathbb{R}^4). \tag{213}$$

Clearly, for large r the metric becomes that of $AdS_3 \times \mathbb{R}^4$. To investigate the behavior near $r = 0$, let us define $u = \frac{5}{2}a^{1/2}r^{2/5}$. Near $r = 0$, the metric becomes

$$ds^2 = du^2 + u^2\left[\frac{16}{25g^2}ds^2(AdS_3) + \frac{4}{25}ds^2(S^3)\right], \tag{214}$$

which is again that of a cone over $AdS_3 \times S^3$. Again, the dilaton blows up as we approach $r = 0$. The full solution preserves eight supersymmetries.

VI. $\mathbb{R}^5$ STRUCTURE

Next consider an arbitrary linear combination of $(+, +)$ and $(+, -)$ spinors, which we denote by

$$(\delta + i\delta^A T^A + \delta_i \Gamma^i)\epsilon, \tag{215}$$

and where at least one of the $\delta_i \neq 0$. It is convenient to introduce an orthonormal basis on the four-dimensional base. Defining $H^2 = \delta_i \delta^i$, we choose the basis

$$e_i^A = H^{-1} J_{ji}^A \delta^j, \qquad A = 1, \ldots, 3, \qquad e_i^4 = H^{-1} \delta_i. \tag{216}$$

In this basis, we have $\epsilon^{1234} = 1$. We will thus write $\epsilon^{ABC4} = \epsilon^{ABC}$. To avoid potential confusion with this mixing of spacetime and Yang-Mills indices, we will always place Yang-Mills indices on the J^A and the F^A "up" and spacetime indices "down." However, we will make no distinction between up and down indices on ϵ^{ABC}. The components of the J^A in this basis are

$$J_{B4}^A = -\delta^{AB}, \qquad J_{BC}^A = -\epsilon^{ABC}. \tag{217}$$

For computational convenience, we will restrict attention to Killing spinors which are not of the most general form compatible with an $\mathbb{R}^5$ structure, but which are rather of the form $\delta_i \Gamma^i \epsilon = H\Gamma^4 \epsilon$. Evaluating

$$[\Delta_\lambda, \delta_i \Gamma^i]\epsilon = 0, \tag{218}$$

and employing the constraints of Sec. III, we find the conditions

$$G_{-ij} = G_{ijk} = 0, \qquad \epsilon^{ABC} F_{BC}^A = 20(m + 2he^{-4\phi}),$$
$$\epsilon^{ABC} F_{4C}^B = \epsilon^{ABC} G_{5BC}. \tag{219}$$

Next, using $\omega_{\mu 4\nu} = -\nabla_\mu e_\nu^4$, we deduce from commuting the $-$ component of the Killing spinor equation that

$$\partial_- H = 0. \tag{220}$$

From the $+$ component we find

$$\partial_+ H = h = 0, \qquad \omega_{+4A} = G_{+4A}, \qquad G_{45A} = \tfrac{1}{2} F_{BA}^B. \tag{221}$$

The 5 component yields

$$\partial_5 H = 0, \qquad \omega_{54A} = G_{54A}, \tag{222}$$

while the B component gives

$$\partial_B H = F_{BC}^A = g = 0, \qquad \omega_{B4A} = \partial_4 \phi g_{AB},$$
$$\epsilon^{ACD} \omega_{BCD} = \tfrac{3}{2} \partial_4 \phi g_{AB} + \epsilon^{ABC} \partial_C \phi. \tag{223}$$

Since the gauge coupling g is required to be zero, an $\mathbb{R}^5$ structure of the form we have assumed is only admitted in the ungauged theory. The remaining constraints may be derived from the 4 component of the Killing spinor equation, and we find them to be of the form

$$H = \text{const}, \qquad \partial_5 \phi = 0, \qquad \epsilon^{ABC} F_{4C}^B = G_{5ij} = 0,$$
$$\omega_{4A4} = \partial_A \phi. \tag{224}$$

We have thus derived the complete set of additional constraints on a bosonic configuration for it to admit the second Killing spinor $\delta_i \Gamma^i \epsilon$. However, there are no solutions of this form in the gauged theory, since the coupling is zero. We have also verified that there are no vacuum solutions, or solutions with eight Killing spinors, with an $\mathbb{R}^5$ structure in the gauged theory. We will not consider this case any further.

VII. IDENTITY STRUCTURE

Making a more general *Ansatz* for a second Killing spinor than that consistent with an $[\mathrm{SU}(2)\ltimes\mathbb{R}^4] \times \mathbb{R}$, $\mathrm{SU}(2)$, or $\mathbb{R}^5$ structure implies that the structure group is reduced to the identity. The generic such Killing spinor is parametrized by 16 real functions. Computing the constraints associated with the existence of such a spinor is a lengthy computational exercise. To illustrate the case of an identity structure, we will instead restrict attention to Killing spinors of the simpler form

$$\theta_i \Gamma^{-i} \epsilon, \tag{225}$$

that is, pure $(-, -)$ spinors, in the language of the introduction.

A. Constraints

As in the previous section, we introduce an orthonormal basis on the base; as before, we define $H^2 = \theta_i \theta^i$, and choose

$$e_i^A = H^{-1} J_{ji}^A \theta^j, \qquad e_i^4 = H^{-1} \theta_i. \tag{226}$$

Commuting the dilatino variation, we may derive

$$\partial_+ \phi = \partial_i \phi = G_{+5i} = G_{+ij}^{(-)} = 0, \qquad G_{+-5} = \tfrac{1}{2} F_{4A}^A,$$
$$G_{45A} = \tfrac{1}{2} \epsilon^{ABC} F_{4C}^B. \tag{227}$$

Now, from the $+$ component of the Killing spinor equation, we get

$$\partial_+ H = \omega_{++\mu} = F_{4A}^A = \epsilon^{ABC} F_{4C}^B = \omega_{+ij}^{(-)} = 0. \tag{228}$$

The 5 component gives

$$\omega_{5+5} = \omega_{5+i} = \omega_{554} = \omega_{54A} = 0,$$
$$\partial_5 (\log H + \phi) = 4he^{-4\phi}. \tag{229}$$

Next, from the $-$ component we find

$$\partial_- H = \omega_{-+i} = G_{-+i} = G_{ijk} = G_{-ij} = 0,$$
$$\epsilon^{ABC} G_{5BC} = F_{BA}^B. \tag{230}$$

The A component yields the constraints

$$\partial_A H = \omega_{A54} = \omega_{AB4} = F^A_{4B} = 0, \qquad \omega_{ij+} = G_{ij+}. \tag{231}$$

Finally, the 4 component gives

$$\partial_4 H = \omega_{44A} = 0. \tag{232}$$

These, in addition to the constraints of Sec. II, are the full set of necessary and sufficient conditions on the bosonic fields of the theory for the existence of a second Killing spinor of the form (225). We see that demanding the existence of such a Killing spinor, with its associated identity structure, implies a radical simplification of the general problem. In fact, we will make one further simplifying assumption. By inspection of the constraints on the spin connection, we see that $de^5 = 0$ if and only if $\epsilon^{ABC} G_{5BC} = F^B_{BA} = 0$; we will assume that $G_{5AB} = 0$. Then since $de^5 = 0$, we may always choose our local coordinates such that $e^5 = dz$; that is, $C = 1$, $\sigma = 0$.

Summary.—Given the assumption that $G_{5AB} = 0$, demanding the existence of the second Killing spinor (225) implies that we may choose coordinates such that $e^5 = dz$, and that the matter fields and the function H are of the following form:

$$\phi = \phi(z), \qquad H = H(z), \qquad G_{(3)} = e^+ \wedge Q^{(+)},$$
$$F^A = \tfrac{1}{2} F^A_{BC} e^B \wedge e^C, \tag{233}$$

where $Q^{(+)}$ is a self-dual form on the base. There are the following constraints on ϕ, H and F^A:

$$\partial_z \log(He^\phi) = 4he^{-4\phi}, \qquad F^A_{AB} = 0,$$
$$\epsilon^{ABC} F^A_{BC} = 10(\partial_z \log H + 2m). \tag{234}$$

The only nonzero independent spin connection components are the following:

$$\omega_{-+5} = -\partial_z \log H, \qquad \omega_{+ij} = \omega^{(+)}_{+ij}, \qquad \omega_{ij+} = Q_{ij},$$
$$\omega_{454} = -\partial_z \log H,$$
$$\omega_{A5B} = (\tfrac{3}{2}\partial_z \log H + 5m)g_{AB} + \tfrac{1}{2}\epsilon^{CBD} F^C_{AD} \tag{235}$$
$$-\tfrac{1}{4}\epsilon^{ADE} F^B_{DE}, \qquad \omega_{iAB} = -g\epsilon^{ABC} A^C_i.$$

The degree of simplification in this case is quite remarkable.

B. Examples

Note that since $\omega_{(\mu\nu)+} = 0$, $\frac{\partial}{\partial u}$ is Killing. We will for simplicity seek solutions with $\omega_{ij+} = G_{ij+} = 0$. Then note that performing the rescalings

$$e^+ = H\tilde{e}^+, \qquad e^- = H\tilde{e}^-, \qquad e^5 = H\tilde{e}^5,$$
$$e^4 = H\tilde{e}^4, \tag{236}$$

the constraints on ω_{+-5} and ω_{454} imply that $d\tilde{e}^- = d\tilde{e}^+ = d\tilde{e}^4 = d\tilde{e}^5 = 0$; thus locally we may introduce coordinates such that

$$\tilde{e}^+ = du, \qquad \tilde{e}^- = dv, \qquad \tilde{e}^4 = dr, \qquad \tilde{e}^5 = dz. \tag{237}$$

Note that we have redefined the z coordinate. Furthermore, since u is Killing, the base is independent of u. We have thus solved all the constraints on the spin connection except

$$\omega_{A5B} = (\tfrac{3}{2}H^{-1}\partial_z \log H + 5m)g_{AB} + \tfrac{1}{2}\epsilon^{CBD} F^C_{AD}$$
$$-\tfrac{1}{4}\epsilon^{ADE} F^B_{DE}, \tag{238}$$

$$\omega_{iAB} = -g\epsilon^{ABC} A^C_i, \tag{239}$$

which we have written in terms of our new z coordinate.

The Yang-Mills Bianchi identities together with (239) then imply that, as in the case of the SU(2) structure,

$$e^\phi F^A_{ij} = \frac{1}{g}\Re^A_{ij}, \tag{240}$$

where the $\Re^A_{ij}$ are the curvatures of the right-hand spin bundle of the base with metric

$$ds^4 = H^2(z)dr^2 + \delta_{AB} e^A e^B. \tag{241}$$

Recall that $F^A_{4B} = 0$, and in addition to (238) and (239), we have the additional constraints, written in terms of our new z coordinate:

$$\partial_z \log(He^\phi) = 4hHe^{-4\phi}, \tag{242}$$

$$F^A_{AB} = 0, \tag{243}$$

$$\epsilon^{ABC} F^A_{BC} = 10(H^{-1}\partial_z \log H + 2m). \tag{244}$$

Let us now consider some explicit examples.

Two Killing spinors.—Recall that since the scalar curvature of the base is given by $R = J^{Aij}\Re^A_{ij} = gJ^{Aij}e^\phi F^A_{ij} = -ge^\phi\epsilon^{ABC} F^A_{BC}$. Equation (244) implies that R is a function only of z. Let us look for a solution with a base of the form

$$H^2[dr^2 + \rho^2(z)ds^2(S^3)], \tag{245}$$

so that

$$F^A_{BC} = -(H\rho)^{-2}e^{-\phi}\epsilon^{ABC} = \tfrac{5}{3}(H^{-1}\partial_z \log H + 2m)\epsilon^{ABC}. \tag{246}$$

We will seek a solution with $h = 0$, so that (242) implies that, up to an irrelevant multiplicative constant, $H = e^{-\phi}$. Then writing $H = f(z)e^{gz/5}$, the second equality of (246) becomes

$$-\rho^{-2} = \tfrac{5}{3}\partial_z \log f. \tag{247}$$

Next, using the expression for the spin connection given in Appendix C, (238) is

$$\partial_z(3\log\rho + 10\log f) = 0, \tag{248}$$

and hence

$$\rho^3 = \beta f^{-10}, \tag{249}$$

for some constant β. Hence, we find that

$$f^{-20/3} = \frac{4}{\beta} z + \alpha, \tag{250}$$

and so shifting z to eliminate the constant α, the metric and dilaton may be written as

$$ds^2 = \left(\frac{4z}{\beta}\right)^{-3/10} e^{2gz/5}[-dt^2 + d\mathbf{x}^2 + dz^2 + 4zds^2(S^3)],$$
$$e^\phi = \left(\frac{4z}{\beta}\right)^{3/20} e^{-gz/5}. \tag{251}$$

This describes the near-horizon limit of a IIB five-brane wrapped on an associative three-sphere [31].

Four Killing spinors.—Let now us seek U(1) solutions; this is equivalent to demanding the existence of the Killing spinor $iT^1\epsilon$. Then we find that the U(1) gauge field obeys $F^1_{1A} = 0$, and, hence, that $\omega_{A51} = -H^{-1}\partial_z \log H \delta_{1A}$, $\omega_{1A5} = \omega_{iA1} = 0$. Hence, the configuration also admits the Killing spinor $H\Gamma^-\Gamma^1\epsilon$. We rescale e^1, so that locally we can write the metric as

$$ds_7^2 = H^2 ds^2(\mathbb{R}^5) + ds^2(\mathcal{M}_2). \tag{252}$$

Consider solutions with $\phi = 0$. Then $H = -(4hz)^{-1}$, $F_{23} = 16h - g$, and, hence, the scalar curvature of $\mathcal{M}_2$ is

$$R = -\frac{g}{2}(16h - g), \tag{253}$$

and so $\mathcal{M}_2$ is either $\mathbb{R}^2$, S^2, or H^2, and is independent of the coordinates on the five orthogonal directions. Hence, we must impose $\omega_{A5B} = 0$, A, $B = 2, 3$, and therefore $12h = g$. Thus, the solution is the $AdS_5 \times H^2$ solution of Maldacena and Nunez [32], for which we have displayed the identity structure.

Eight Killing spinors.—Finally, let us look for solutions for which the Yang-Mills fields vanish. This is equivalent to demanding the existence of the eight Killing spinors ϵ, $iT^A\epsilon$, $H\Gamma^-\Gamma^i\epsilon$. The vanishing of the Yang-Mills fields implies that $\omega_{A5B} = -\partial_z \log H \delta_{AB}$. Then rescaling the e^A by H, we see that locally the metric can be put in the form

$$ds^2 = H^2 \eta_{\mu\nu} dx^\mu dx^\nu. \tag{254}$$

Furthermore we must require

$$\partial_z \log H = -2mH, \qquad \partial_z \log(He^\phi) = 4Hhe^{-4\phi}. \tag{255}$$

Consider first the case of vanishing topological mass, $h = 0$. Then up to irrelevant constants, $H = e^{-\phi} = \exp\frac{gz}{5}$. This solution is nothing but the reduction to seven dimensions of the linear dilaton solution in ten.

When $h \neq 0$, let us look for a solution with $\phi = 0$. We thus must have $16h = g$, $H^2 = (4hz)^{-2}$. This is just the maximally supersymmetric AdS_7 solution of the theory.

VIII. CONCLUSIONS AND OUTLOOK

In this work, a systematic formalism for performing complete G-structure classifications of supersymmetric bosonic configurations in supergravity theories has been presented, and illustrated in the context of $d = 7$, SU(2) gauged supergravity. The key notion for organizing the classification is that of the common isotropy group of the additional Killing spinors. The formalism has been used to derive a set of constraints associated with the existence of various additional Killing spinors, and these constraints have been exploited to derive numerous explicit solutions. The emphasis in this paper has been on illustrating the formalism, and no great effort has been made to derive new solutions; clearly, there is scope for a more careful analysis of the constraints in the future.

An important technical point that has been exploited throughout is that in performing G-structure classifications, it is unnecessary, and inefficient, to use spinor bilinears to derive the constraints; for more computational efficiency, it is much better to work directly with a specific spinor defined by a particular set of projections. The second key point that has been used is that, given a single Killing spinor, any other spinor may be constructed from it by acting with a matrix Q in the Clifford algebra. The problem of completely classifying all supersymmetric configurations in a given supergravity may then be restated as determining all possible sets of such matrices Q which commute with the operators in the fermion supersymmetry transformations, when acting on the fiducial Killing spinor.

The chief advantage of the G-structure formalism used here over other classification techniques is that the constraints one derives for the existence of the desired supersymmetries take the form of explicit algebraic constraints on the spin connection and the matter fields, supplemented by first order differential conditions on the functions parametrizing any additional Killing spinors. Presenting the necessary and sufficient conditions for the existence of N Killing spinors in this form naturally lends itself to exploiting the classification in the construction of explicit solutions, rendering G-structure classifications very useful for practical applications.

The only drawback of the formalism is, of course, the amount of computational effort required to compute the constraints in a theory of the complexity of the one studied here. It is to be expected that simpler theories, such as those with eight supercharges, would be much more tractable from the point of view of performing a refined classification in full generality, using the techniques of this paper; this will be interesting to investigate. It is of course possible that there is a yet more efficient

way of computing the constraints. The approach employed here is "bottom up," in the sense that one first assumes the existence of a single Killing spinor, and then incorporates additional Killing spinors iteratively. For the analysis of more complicated supergravities, it would be very useful to have a complementary "top down" approach, so that one could start with maximal supersymmetry and weaken the constraints progressively, in a controlled fashion. This would be particularly useful for classifying configurations admitting more than one half supersymmetry, since in this case the structure group is necessarily the identity, and this is the most complicated case to analyze using the iterative bottom up approach.

An obvious application of the formalism presented here is to the long-standing problem of completely classifying supersymmetric configurations in $d = 11$. The possible structure groups in $d = 11$ will coincide with the possible holonomy groups; these have been classified in [33], and there are 18 distinct possibilities. A G-structure analysis of the maximal structure groups [SU(5) and [Spin(7)⋉$\mathbb{R}^8$] × $\mathbb{R}$] has been given in [9,10]. The first step in performing the complete classification will be to construct, from a fiducial Killing spinor, the spaces of spinors fixed by each of the other 16 groups, as was done in the context of this paper in the introduction. Then it should certainly be practical to completely analyze the 13 additional structure groups fixing at most eight Killing spinors; analyzing the two groups [SU(2) and $\mathbb{R}^9$] fixing at most 16 will require considerably more effort. Finally, completely classifying the most generic case of an identity structure (which fixes at most 32 Killing spinors) will, without further insight, require an immense amount of calculation.

Nevertheless, a complete G-structure classification of all supersymmetric configurations in 11 dimensions (or in any other supergravity) will be of significant value. An in-depth analysis of the constraints derived from a standard classification of simple five-dimensional supergravities has already yielded some very interesting solutions, such as supersymmetric AdS_5 black holes [34,35], and supersymmetric black rings [36–38]. One would hope that a complete G-structure analysis in 11 dimensions would reveal a wealth of surprising new phenomena.

ACKNOWLEDGMENTS

APPENDIX A: CONVENTIONS

We work in almost plus signature, $\eta_{\mu\nu} =$ diag$(-, +, \ldots, +)$. Seven-dimensional spacetime indices are denoted by Greek letters $\mu, \nu, \ldots$. Orthonormal in-

dices on the four-dimensional base are denoted by lower case Roman letters, $i, j, \ldots$; spacetime indices on the base are denoted by upper case Roman letters $M, N, \ldots$. In an orthonormal frame the Dirac algebra is

$$\{\Gamma_\mu, \Gamma_\nu\} = 2g_{\mu\nu}. \tag{256}$$

This tells us that Γ_0 is anti-Hermitian and the Γ_i ($i = 1, \ldots, 6$) are Hermitian. Following the appendix to Chapter 1 in [39], we have that the charge conjugation matrix C satisfies

$$C^T = C, \qquad C^\dagger C = I, \qquad \Gamma_\mu^T = -C\Gamma_\mu C^{-1}. \tag{257}$$

We can therefore choose

$$C = I. \tag{258}$$

This implies that Γ_0 is real and the Γ_i are imaginary. We will choose a representation (there are two inequivalent ones) such that

$$\Gamma_0\Gamma_1\Gamma_2\Gamma_3\Gamma_4\Gamma_5\Gamma_6 = -I. \tag{259}$$

We also have the identity

$$\Gamma_{\alpha_1 \ldots \alpha_n} = \frac{(-)^{[n/2]+1}}{(7-n)!} \epsilon_{\alpha_1 \ldots \alpha_n \beta_1 \ldots \beta_{7-n}} \Gamma^{\beta_1 \ldots \beta_{7-n}}. \tag{260}$$

We choose the orientation to be given by $\epsilon^{0123456} = +1$. Our null basis is defined by

$$e^\pm = \frac{1}{\sqrt{2}}(e^0 \pm e^6), \tag{261}$$

so that $\epsilon^{+-12345} = 1$. The orientation on the base is chosen to be given by $\epsilon^{+-ijkl5} = \epsilon^{ijkl}$.

The Dirac conjugate $\bar{\epsilon}_a$ of an anticommuting spinor ϵ^a is defined as

$$\bar{\epsilon}_a = (\epsilon^a)^\dagger \Gamma_0, \tag{262}$$

and we also define

$$\bar{\epsilon}^a = \epsilon^{ab}\bar{\epsilon}_b, \tag{263}$$

where ϵ^{ab} is a constant antisymmetric matrix satisfying $\epsilon_{ab}\epsilon^{bc} = -\delta_a^c$ that is used to raise and lower spinor indices according to $\epsilon^a \equiv \epsilon^{ab}\epsilon_b$, and $\epsilon^{12} = 1$. On the other hand, the symplectic-Majorana conjugate ϵ^C of ϵ is defined to be

$$(\epsilon^C)^a = (\epsilon^T)_b. \tag{264}$$

Symplectic-Majorana spinors are those for which (263) is equal to (264); namely,

$$(\epsilon^T)^a = \bar{\epsilon}^a. \tag{265}$$

Given four spinors $\epsilon_1, \ldots, \epsilon_4$, the Fierz identity is

$$\bar{\epsilon}_1 \epsilon_2 \bar{\epsilon}_3 \epsilon_4 = \frac{1}{8}\Big[\bar{\epsilon}_1 \epsilon_4 \bar{\epsilon}_3 \epsilon_2 + \bar{\epsilon}_1 \Gamma_\mu \epsilon_4 \bar{\epsilon}_3 \Gamma^\mu \epsilon_2$$
$$- \frac{1}{2} \bar{\epsilon}_1 \Gamma_{\mu\nu} \epsilon_4 \bar{\epsilon}_3 \Gamma^{\mu\nu} \epsilon_2$$
$$- \frac{1}{3!} \bar{\epsilon}_1 \Gamma_{\mu\nu\rho} \epsilon_4 \Gamma^{\mu\nu\rho} \epsilon_2 \Big]. \tag{266}$$

APPENDIX B: USEFUL PROJECTIONS

We give here the full set of projections satisfied by the fiducial basis spinor that we have employed in deriving the constraints:

$$\Gamma^+ \epsilon = 0, \tag{267}$$

$$\Gamma^{+-} \epsilon = -\epsilon, \tag{268}$$

$$\Gamma^{ijkl} \epsilon = -\epsilon^{ijkl} \epsilon, \tag{269}$$

$$A_{ij}^{(-)} \Gamma^{ij} \epsilon = 0, \tag{270}$$

$$\Gamma^{ijk} \epsilon = \epsilon^{ijkl} \Gamma_l \epsilon, \tag{271}$$

$$J_{ij}^A \Gamma^{ij} \epsilon^a = 8i T_b^{Aa} \epsilon^b, \tag{272}$$

$$T^{Aa}{}_b \Gamma_i \epsilon^b = -\frac{i}{2} J_{ij}^A \Gamma^j \epsilon^a. \tag{273}$$

APPENDIX C: SPIN CONNECTION COMPONENTS

The exterior derivatives of the basis one forms are given by

$$de^+ = e^+ \wedge (\mathcal{D}\log L + \partial_u \lambda) + e^5 \wedge [-(LC)^{-1}\partial_z \lambda]$$
$$+ e^+ \wedge e^5 C^{-1} \partial_z \log L - L^{-1}\mathcal{D}\lambda, \tag{274}$$

$$de^- = e^+ \wedge L(\tfrac{1}{2}\mathcal{D}\mathcal{F} + \tfrac{1}{2}\lambda\partial_u\mathcal{F} - \partial_u\nu + \sigma\partial_u\beta)$$
$$+ e^5 \wedge C^{-1}(-\tfrac{1}{2}\lambda\partial_z\mathcal{F} + \partial_z\nu - \sigma\partial_z B - \mathcal{D}B)$$
$$+ e^+ \wedge e^5 LC^{-1}(\tfrac{1}{2}\partial_z\mathcal{F} - \partial_u B) + \mathcal{D}\nu$$
$$- \tfrac{1}{2}\mathcal{D}\mathcal{F} \wedge \lambda - \mathcal{D}B \wedge \sigma, \tag{275}$$

$$de^5 = e^+ \wedge (-CL\partial_u\sigma) + e^5 \wedge (\partial_z\sigma - \mathcal{D}\log C)$$
$$+ e^+ \wedge e^5(-L\partial_u \log C) + C\mathcal{D}\sigma, \tag{276}$$

$$de^i = \mathcal{D}e^i + e^+ \wedge (-L\partial_u e^i) + e^5 \wedge (C^{-1}\partial_z e^i). \tag{277}$$

Then the full set of nonzero spin connection components is

$$\omega_{-+5} = \frac{1}{2C}\partial_z\log L, \qquad \omega_{-+i} = \frac{1}{2}(\mathcal{D}\log L + \partial_u\lambda)_i,$$
$$\omega_{-5i} = \frac{1}{2LC}(\partial_z\lambda)_i, \qquad \omega_{-ij} = \frac{1}{2L}\mathcal{D}\lambda_{ij}, \tag{278}$$

$$\omega_{++5} = LC^{-1}\Big(\frac{1}{2}\partial_z\mathcal{F} - \partial_u B\Big),$$
$$\omega_{++i} = L\Big(\frac{1}{2}\mathcal{D}\mathcal{F} + \frac{1}{2}\lambda\partial_u\mathcal{F} - \partial_u\nu + \sigma\partial_u B\Big)_i,$$
$$\omega_{+-5} = \omega_{-+5}, \qquad \omega_{+-i} = \omega_{-+i},$$
$$\omega_{+5i} = \frac{1}{2C}\Big(-\frac{1}{2}\lambda\partial_z\mathcal{F} + \partial_z\nu - \sigma\partial_z B - \mathcal{D}B + LC^2\partial_u\sigma\Big)_i,$$
$$\omega_{+ij} = LM_{[ij]} + \frac{1}{2}\Big(\mathcal{D}\nu - \frac{1}{2}\mathcal{D}\mathcal{F} \wedge \lambda - \mathcal{D}B \wedge \sigma\Big)_{ij}, \tag{279}$$

$$\omega_{5+-} = \omega_{-+5}, \qquad \omega_{5+5} = L\partial_u\log C,$$
$$\omega_{5+i} = \omega_{+5i}, \qquad \omega_{5-i} = \omega_{-5i},$$
$$\omega_{55i} = (\mathcal{D}\log C - \partial_z\sigma)_i, \tag{280}$$
$$\omega_{5ij} = -C^{-1}\Lambda_{[ij]} - \tfrac{1}{2}C\mathcal{D}\sigma_{ij},$$

$$\omega_{i+-} = \omega_{-+i}, \qquad \omega_{i+5} = -\omega_{+5i}, \qquad \omega_{i+j} = LM_{(ij)} + \tfrac{1}{2}(\mathcal{D}\nu - \tfrac{1}{2}\mathcal{D}\mathcal{F} \wedge \lambda - \mathcal{D}B \wedge \sigma)_{ij}, \qquad \omega_{i-5} = -\omega_{-5i},$$
$$\omega_{i-j} = \omega_{-ij}, \qquad \omega_{i5j} = -C^{-1}\Lambda_{(ij)} - \tfrac{1}{2}C\mathcal{D}\sigma_{ij},$$
$$\omega_{ijk} = \hat{\omega}_{ijk} + \sigma_i\Lambda_{[jk]} + \sigma_{[j}\Lambda_{k]i} + \sigma_{[j}\Lambda_{|i|k]} - \lambda_i M_{[jk]} - \lambda_{[j}M_{k]i} - \lambda_{[j}M_{|i|k]}, \tag{281}$$

where $\hat{\omega}_{ijk}$ is the spin connection on the base.

APPENDIX D: INTEGRABILITY CONDITIONS

In [19] it was shown that we may obtain the following integrability condition from commuting the Killing spinor equation with $\delta\lambda$:

$$\sqrt{5}\Gamma^\mu[\mathcal{D}_\mu, \Delta_\lambda]\epsilon^a = \left[\frac{1}{2}P + \frac{1}{6}Q_{\mu\nu\sigma}\Gamma^{\mu\nu\sigma} + \frac{e^{2\phi}}{96}d(e^{-2\phi}G)_{\mu\nu\sigma\tau\rho}\Gamma^{\mu\nu\sigma\tau\rho}\right]\epsilon^a + \left[iR^A_\mu\Gamma^\mu + \frac{ie^{-\phi}}{6}D(e^\phi F^A)_{\mu\nu\sigma}\Gamma^{\mu\nu\sigma}\right]T^{Aa}{}_b\epsilon^b$$

$$+ \sqrt{5}\left[\frac{1}{60}G_{\mu\nu\sigma\tau}\Gamma^{\mu\nu\sigma\tau}\delta^a_b + \frac{3i}{5}F^A_{\mu\nu}\Gamma^{\mu\nu}T^{Aa}{}_b + (8h - g)\delta^a_b\right]\delta\lambda^b, \tag{282}$$

where D denotes the gauge-covariant exterior derivative, P, Q, R are defined by Eqs. (36)–(38), and the dilaton, four-form, and two-form field equations are, respectively, $P = 0$, $Q = 0$, and $R^A = 0$. It was also shown that the integrability condition for the Killing spinor equation is

$$\Gamma^\nu[\mathcal{D}_\mu, \mathcal{D}_\nu]\epsilon^a = \left[-\frac{1}{2}E_{\mu\nu}\Gamma^\nu + e^{2\phi}d(e^{-2\phi}G)^{\nu\sigma\tau\rho\xi}\left(-\frac{1}{120}g_{\mu\nu}\Gamma_{\sigma\tau\rho\xi} + \frac{1}{200}\Gamma_{\mu\nu\sigma\tau\rho\xi}\right) + \frac{1}{10}Q^{\nu\sigma\tau}\left(\frac{1}{2}\Gamma_{\mu\nu\sigma\tau} - g_{\mu\nu}\Gamma_{\sigma\tau}\right)\right]\epsilon^a$$

$$+ \left[\frac{ie^{-\phi}}{5}D(e^\phi F^A)^{\nu\sigma\tau}\left(2g_{\mu\nu}\Gamma_{\sigma\tau} + \frac{1}{6}\Gamma_{\mu\nu\sigma\tau}\right) - \frac{i}{5}R^{A\nu}(-4g_{\mu\nu} + \Gamma_{\mu\nu})\right]T^{Aa}{}_b\epsilon^b$$

$$+ \left[\partial_\mu\phi\delta^a_b - \frac{i}{25}F^{A\nu\sigma}(8g_{\mu\nu}\Gamma_\sigma - \Gamma_{\mu\nu\sigma})T^{Aa}{}_b + \frac{2}{5}m'\Gamma_\mu\delta^a_b + \frac{1}{25}G^{\nu\sigma\tau\rho}\left(-\frac{2}{3}g_{\mu\nu}\Gamma_{\sigma\tau\rho}\right.\right.$$

$$\left.\left.+ \frac{1}{4}\Gamma_{\mu\nu\sigma\tau\rho}\right)\delta^a_b\right]\sqrt{5}\delta\lambda^b = 0, \tag{283}$$

where the Einstein equations are $E_{\mu\nu} = 0$.

When a single null Killing spinor exists, one may readily verify that it is sufficient to impose the Bianchi identities for the forms, the field equations for $G_{(4)}$, the + component of the Yang-Mills field equations, and the + + component of the Einstein equations. When additional Killing spinors are incorporated, more of the field equations and Bianchi identities are identically satisfied; working out which ones may be done straightforwardly case by case.

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