# Riemannian light cone from vanishing birefringence in premetric vacuum electrodynamics

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We consider premetric electrodynamics with a local and linear constitutive law for the vacuum. Within this framework, we find quartic Fresnel wave surfaces for the propagation of light. If we require (i) the Fresnel equation to have only real solutions and (ii) the vanishing of birefringence in vacuum, then a Riemannian light cone is implied. No proper Finslerian structure can occur. This is generalized to dynamical equations of any order.

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## I. INTRODUCTION AND MOTIVATION

Recently, the physics of the electromagnetic field, without assumptions about the metrical structure of the underlying spacetime, has gained renewed interest. On the one hand, this general ansatz is needed for a proper interpretation of experiments testing Lorentz invariance. In such approaches it is not allowed to make assumptions about the underlying geometric structure, in particular, about a metric of spacetime. On the contrary, by using properties of the evolution of the electromagnetic field, one likes to establish the metrical structure of spacetime (here ''metrical'' may be more general than the ordinary Riemannian or Minkowskian metric). The general structure of Maxwell equations can serve as a test theory for searches for Lorentz violation in the photon sector [1]. On the other hand, it is a general task to explore the structure of the electromagnetic field and the geometry it defines, see, e.g., [2,3].

There are two main effects in the realm of ray optics based on the Maxwell equations: One effect is *birefringence* and the other one *anisotropy* of the propagation of light[1]. Both effects are well known from the physics of light propagation in general media, such as in crystals, for example. The basics of the general formalism have been laid down in [2]. The explicit calculations of these effects have been carried through to first order in these anomalous effects by Kostelecky and coworkers and by others [1,3–6] (for a possible birefringence caused by a torsion of spacetime, see [7–10] and also [11]). In these approaches the first step is to confront the result with the possible observations of birefringence. From astrophysical observations [1], the parameters responsible for birefringence must be smaller than $10^{-32}$ and, thus, can safely be neglected. The remaining anisotropy in the photon propagation is given by a symmetric second-rank tensor. By an appropriate coordinate transformation, this tensor becomes proportional to the unit tensor. Accordingly, there is an adapted coordinate system such that light propagation is isotropic and defines a Riemannian metric. This is a remarkable result that may be due to the approximation used. In this work we show that this result holds exactly. That is, we show, provided we assume a local and linear constitutive law for the vacuum, that

$$\left.\begin{array}{c}\text{Maxwell equations}\\ +\text{only real sols. of Fresnel eq.}\\ +\text{vanish. birefringence in vac.}\end{array}\right\} \Rightarrow \begin{array}{c}\text{pseudo-Riemannian}\\ \text{metric.}\end{array}$$

## II. OBSERVATIONAL AND EXPERIMENTAL FACTS

As discussed, the best estimate on birefringence effects of the vacuum have been given by an analysis of Kostelecky and Mewes [1]. Their results show that the birefringence parameter is smaller than $10^{-32}$. This estimate is independent of the coordinate system chosen since it is an effect which cannot be transformed to zero.

Since the resulting anisotropy can be transformed away, it cannot be understood as an effect solely within the photon sector. Thus, the coordinates used for the description of the anisotropy experiments have to be fixed by some other physical process. In these experiments, this is realized by some solid like, for example, the interferometer arm or the optical resonator. The length of the resonator or of the interferometer arm is determined by the laws of quantum physics and of electrodynamics, see, e.g., [12,13]. Consequently, the search for an anisotropy of the propagation of light has to be interpreted as a comparison between the laws of quantum physics, like the Dirac equation, the Pauli exclusion principle, etc., and the

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[1]A further propagation effect of electromagnetic radiation is the precession of its polarization. This will not be discussed in this paper.

 

Maxwell equations. The most recent experiments searching for an anisotropy of the velocity of light yield no effect to the order of $\Delta c/c \leq 10^{-15}$ [14].

## III. SOME PREMETRIC ELECTRODYNAMICS

The Maxwell equations, expressed in terms of the excitations $\mathcal{D}, \mathcal{H}$ and the field strengths $E, B$, read

$$\underline{d}\mathcal{D} = \rho, \qquad \underline{d}\mathcal{H} - \dot{\mathcal{D}} = j, \tag{1}$$

$$\underline{d}B = 0, \qquad \underline{d}E + \dot{B} = 0. \tag{2}$$

We mark the exterior derivative in three dimensions with an underline: $\underline{d}$. The dot denotes a Lie derivative with respect to the vector field $\partial_t$. The electric charge density is $\rho$, the current density $j$. For the formulation of the Maxwell equations, we use the calculus of exterior differential forms. We take the notation from [2], compare also, e.g., Frankel [15], Lindell [16], or Russer [17].

The four dimensional form of the Maxwell equations

$$dH = J, \quad H = \mathcal{D} - \mathcal{H} \wedge dt, \quad J = \rho - j \wedge dt, \tag{3}$$

$$dF = 0, \quad F = B + E \wedge dt, \tag{4}$$

shows that they are generally covariant under diffeomorphisms and there is no need of a metric of spacetime [2].

The set of Eqs. (3) and (4) is incomplete. What is missing is the constitutive law of the vacuum (the spacetime relation). If we assume *locality* and *linearity,* then $H = \kappa(F)$, with the local and linear operator $\kappa$. If we decompose the 2-forms $H$ and $F$ in their components (here $i, j = 0, 1, 2, 3$), then $H = H_{ij}dx^i \wedge dx^j/2$ and $F = F_{ij}dx^i \wedge dx^j/2$. Accordingly,

$$H_{ij} = \tfrac{1}{2}\kappa_{ij}{}^{kl}F_{kl} \quad \text{with} \quad \kappa_{ij}{}^{kl} = -\kappa_{ji}{}^{kl} = -\kappa_{ij}{}^{lk}. \tag{5}$$

Here $\kappa_{ij}{}^{kl}$ is the constitutive tensor of spacetime with 36 independent components. With the help of the contravariant Levi-Civita symbol $\epsilon^{ijmn} = \pm 1, 0$, we can introduce the equivalent constitutive tensor *density* of spacetime

$$\chi^{ijkl} := \tfrac{1}{2}\epsilon^{ijmn}\kappa_{mn}{}^{kl}. \tag{6}$$

Incidentally, the covariant Levi-Civita symbol, which we will use below, is denoted by a circumflex: $\hat{\epsilon}_{ijmn} = \pm 1, 0$. Since no metric is available at this stage, we have to distinguish these two symbols.

Alternatively, we can express (5) in a six component version, which is sometimes more convenient. In terms of blocks with 3-dimensional indices $a, b, \ldots = 1, 2, 3$, we find

$$\begin{pmatrix} \mathcal{H}_a \\ \mathcal{D}^a \end{pmatrix} = \begin{pmatrix} \mathfrak{C}^b{}_a & \mathfrak{B}_{ba} \\ \mathfrak{A}^{ba} & \mathfrak{D}_b{}^a \end{pmatrix} \begin{pmatrix} -E_b \\ B^b \end{pmatrix}. \tag{7}$$

Obviously, $\mathfrak{A}$ is the 3–dimensional *permittivity* matrix and $\mathfrak{B}$ the reciprocal of the *permeability* matrix. The matrices $\mathfrak{C}$ and $\mathfrak{D}$ describe electric-magnetic cross terms (which vanish in Maxwell-Lorentz vacuum electrodynamics in Cartesian coordinates). In (7), for the components of the electromagnetic field, we took a vectorlike notation

$$\mathcal{H} = \mathcal{H}_a \vartheta^a, \quad E = E_a \vartheta^a, \tag{8}$$

$$\mathcal{D} = \mathcal{D}^b \hat{\epsilon}_b, \quad B = B^b \hat{\epsilon}_b, \tag{9}$$

with the 3-dimensional coframe $\vartheta^a$ and the 2-form basis $\hat{\epsilon}_a = \hat{\epsilon}_{bcd}\vartheta^c \wedge \vartheta^d/2$. By straightforward algebra, the constitutive $3 \times 3$ matrices $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}$, and $\mathfrak{D}$ can be related to the 4-dimensional constitutive tensor density (6) by

$$\mathfrak{A}^{ba} := \chi^{0a0b}, \tag{10}$$

$$\mathfrak{B}_{ba} := \tfrac{1}{4}\hat{\epsilon}_{acd}\hat{\epsilon}_{bef}\chi^{cdef}, \tag{11}$$

$$\mathfrak{C}^a{}_b := \tfrac{1}{2}\hat{\epsilon}_{bcd}\chi^{cd0a}, \tag{12}$$

$$\mathfrak{D}_a{}^b := \tfrac{1}{2}\hat{\epsilon}_{acd}\chi^{0bcd}. \tag{13}$$

## IV. QUARTIC WAVE SURFACE FOR THE PROPAGATION OF LIGHT

The propagation of light in *local* and *linear* premetric vacuum electrodynamics is characterized by the generalized Fresnel equation [2]

$$M_0 k_0^4 + M_1 k_0^3 + M_2 k_0^2 + M_3 k_0 + M_4 = 0, \tag{14}$$

where $k_0$ is the zeroth component of the 4–wave covector $k$. The coefficients $M_i$ are homogeneous functions of degree $i$ in the spatial components $k_a$ of the wave covector:

$$M_i := M^{a_1 \ldots a_i} k_{a_1} \cdots k_{a_i}. \tag{15}$$

The Fresnel equation results from a solvability condition for a 3-vector equation $W^{ab}k_b = 0$ on the jump surfaces [2,18]; here

$$W^{ab} := (k_0^2 \mathfrak{A}^{ab} + k_0 k_d [\mathfrak{C}^a{}_c \epsilon^{cdb} + \mathfrak{C}^b{}_c \epsilon^{cda}] + k_e k_f \epsilon^{aec}\epsilon^{bfd}\mathfrak{B}_{cd}) \tag{16}$$

is a $3 \times 3$ matrix, the determinant of which has to vanish, see [18]. Equation (15) is valid in a special anholonomic frame with $\vartheta^0 = k$.

The equation for the jump surfaces can also be obtained in an analogous way as effective partial differential equation for the components of the radiating electromagnetic potential after removing all gauge freedom. This equation, for all initial conditions or all sources of sufficient regularity, should possess a unique solution in some future causality cone (this corresponds to a finite propagation velocity of the solutions). The necessary and sufficient condition for that is the hyperbolicity of the differential operator [19]. Furthermore, the differential operator is hyperbolic if the corresponding polynomial is hyperbolic [19]. This means that (14) is required to possess four *real* solutions for $k_0$ which need not to be different. The condition of the hyperbolicity or, equivalently, the condition for the existence and the uniqueness of the solutions, is the fundamental fact behind the particular signature for the metric which we are going to derive (see also [20] for another example).

The $M^{a_1 \dots a_i}$'s in (15) are cubic in the $3 \times 3$ matrices $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, and $\mathfrak{D}$, see [2]:

$$M = \det\mathfrak{A}, \tag{17}$$

$$M^a = -\hat{\epsilon}_{bcd}(\mathfrak{A}^{ba}A^{ce}\mathfrak{C}^d{}_e + \mathfrak{A}^{ab}\mathfrak{A}^{ec}\mathfrak{D}_e{}^d), \tag{18}$$

$$\begin{aligned} M^{ab} = {} & \tfrac{1}{2}A^{(ab)}[(\mathfrak{C}^d{}_d)^2 + (\mathfrak{D}_c{}^c)^2 - (\mathfrak{C}^c{}_d + \mathfrak{D}_d{}^c) \\ & \times (\mathfrak{C}^d{}_c + \mathfrak{D}_c{}^d)] + (\mathfrak{C}^d{}_c + \mathfrak{D}_c{}^d)(\mathfrak{A}^{c(a}\mathfrak{C}^{b)}{}_d \\ & + \mathfrak{D}_d{}^{(a}\mathfrak{A}^{b)c}) - \mathfrak{C}^d{}_d\mathfrak{A}^{c(a}\mathfrak{C}^{b)}{}_c - \mathfrak{D}_c{}^{(a}\mathfrak{A}^{b)c}\mathfrak{D}_d{}^d \\ & - \mathfrak{A}^{dc}\mathfrak{C}^{(a}{}_c\mathfrak{D}_d{}^{b)} + (\mathfrak{A}^{(ab)}\mathfrak{A}^{dc} - \mathfrak{A}^{d(a}\mathfrak{A}^{b)c})\mathfrak{B}_{dc}, \end{aligned} \tag{19}$$

$$\begin{aligned} M^{abc} = {} & \epsilon^{de(c}[\mathfrak{B}_{df}(\mathfrak{A}^{ab)}\mathfrak{D}_e{}^f - \mathfrak{D}_e{}^a\mathfrak{A}^{b)f}) + \mathfrak{B}_{fd}(\mathfrak{A}^{ab)}\mathfrak{C}^f{}_e \\ & - \mathfrak{A}^{f|a}\mathfrak{C}^{b)}{}_e) + \mathfrak{C}^a{}_f\mathfrak{D}_e{}^{b)}\mathfrak{D}_d{}^f + \mathfrak{D}_f{}^a\mathfrak{C}^{b)}{}_e\mathfrak{C}^f{}_d], \end{aligned} \tag{20}$$

$$M^{abcd} = \epsilon^{ef(c}\epsilon^{|gh|d}\mathfrak{B}_{hf}[\tfrac{1}{2}\mathfrak{A}^{ab)}\mathfrak{B}_{ge} - \mathfrak{C}^a{}_e\mathfrak{D}_g{}^{b)}]. \tag{21}$$

Computer plots of the 4th-order surface of the generalized Fresnel Eq. (14) have been prepared by Tertychniy [21].

We solve (14) with respect to the frequency $k_0$, keeping the 3-covectors $k_a$ fixed. We find the four solutions

$$k^{\uparrow}_{0(1)} = \sqrt{\alpha} + \sqrt{\beta + \frac{\gamma}{\sqrt{\alpha}}} - \delta, \tag{22}$$

$$k^{\uparrow}_{0(2)} = \sqrt{\alpha} - \sqrt{\beta + \frac{\gamma}{\sqrt{\alpha}}} - \delta, \tag{23}$$

$$k^{\downarrow}_{0(1)} = -\sqrt{\alpha} + \sqrt{\beta - \frac{\gamma}{\sqrt{\alpha}}} - \delta, \tag{24}$$

$$k^{\downarrow}_{0(2)} = -\sqrt{\alpha} - \sqrt{\beta - \frac{\gamma}{\sqrt{\alpha}}} - \delta. \tag{25}$$

We introduced the abbreviations

$$\alpha := \frac{1}{12M_0}\left[\frac{a}{(b + \sqrt{c})^{1/3}} + (b + \sqrt{c})^{1/3} - 2M_2\right] + \delta^2, \tag{26}$$

$$\beta := \frac{1}{12M_0}\left[-\frac{a}{(b + \sqrt{c})^{1/3}} - (b + \sqrt{c})^{1/3} - 4M_2\right] + 2\delta^2, \tag{27}$$

$$\gamma := \frac{1}{4M_0}(2\delta M_2 - M_3) - 2\delta^3, \tag{28}$$

$$\delta := \frac{M_1}{4M_0}, \tag{29}$$

with

$$a := 12M_0M_4 - 3M_1M_3 + M_2^2, \tag{30}$$

$$b := \tfrac{27}{2}M_0M_3^2 - 36M_0M_2M_4 - \tfrac{9}{2}M_1M_2M_3 + \tfrac{27}{2}M_1^2M_4 + M_2^3, \tag{31}$$

$$c := 4(b^2 - a^3). \tag{32}$$

Earlier investigations on light propagation in general linear media and on Fresnel-Kummer surfaces includes the important work of Schultz *et al.*[22] and Kiehn *et al.* [23].

## V. VANISHING BIREFRINGENCE

Vanishing birefringence means that there is only one future and only one past directing light cone. In order to achieve this, one has to identify two pairs of solutions. There are these two possibilities[2]:

$$k_{0(1)}{}^{\uparrow} = k_{0(2)}{}^{\uparrow}, \quad k_{0(1)}{}^{\downarrow} = k_{0(2)}{}^{\downarrow}, \text{ i.e., } \quad \beta = \gamma = 0, \tag{33}$$

$$k_{0(1)}{}^{\uparrow} = k_{0(1)}{}^{\downarrow}, \quad k_{0(2)}{}^{\uparrow} = k_{0(2)}{}^{\downarrow}, \text{ i.e., } \quad \alpha = \gamma = 0. \tag{34}$$

For the case (33), the solution degenerates to

$$k^{\uparrow}_0 = \sqrt{\alpha} - \delta, \qquad k^{\downarrow}_0 = -\sqrt{\alpha} - \delta, \tag{35}$$

and for the case (34) to

$$k^{\uparrow}_0 = \sqrt{\beta} - \delta, \qquad k^{\downarrow}_0 = -\sqrt{\beta} - \delta. \tag{36}$$

The equation $\gamma = 0$, which is valid for both cases, has the simple solution

$$M_3 = \frac{M_1M_2}{2M_0} - \frac{1}{8}\frac{M_1^3}{M_0^2} = \frac{M_1}{8M_0^2}(4M_0M_2 - M_1^2). \tag{37}$$

This can be inserted into $a$ and $b$, but presently we do not need the explicit expressions. The functions $\alpha$ and $\beta$ can be written as

$$\alpha = \frac{3M_1^2 - 8M_0M_2}{48M_0^2} + \xi, \tag{38}$$

[2]For $\gamma = \delta = 0$, we have $k_{0(1)}{}^{\uparrow} = -k_{0(2)}{}^{\downarrow}$ and $k_{0(2)}{}^{\uparrow} = -k_{0(1)}{}^{\downarrow}$. However, this is irrelevant for birefringence.

$$\beta = \frac{6M_1^2 - 16M_0M_2}{48M_0^2} - \xi, \tag{39}$$

with

$$\xi := \frac{1}{12M_0}\left[\frac{a}{(b+\sqrt{c})^{1/3}} + (b+\sqrt{c})^{1/3}\right]. \tag{40}$$

Since either $\beta = 0$ or $\alpha = 0$, we can add (38) and (39) and find

$$\alpha \quad \text{or} \quad \beta = \frac{3M_1^2 - 8M_0M_2}{16M_0^2}, \tag{41}$$

corresponding to (33) or to (34), respectively.

Hence in all cases the light cones turn out to be

$$k_0^{\uparrow\downarrow} = \pm\sqrt{\frac{3M_1^2 - 8M_0M_2}{16M_0^2}} - \frac{M_1}{4M_0}. \tag{42}$$

Accordingly, the quartic wave surface in this case reads

$$[(k_0 - k_0^{\uparrow})(k_0 - k_0^{\downarrow})]^2 = 0. \tag{43}$$

We drop the square and find

$$\left(k_0 + \frac{M_1}{4M_0} - \sqrt{\frac{3M_1^2 - 8M_0M_2}{16M_0^2}}\right)\times \left(k_0 + \frac{M_1}{4M_0} + \sqrt{\frac{3M_1^2 - 8M_0M_2}{16M_0^2}}\right) = 0. \tag{44}$$

Multiplication yields

$$\left(k_0 + \frac{M_1}{4M_0}\right)^2 - \frac{3M_1^2 - 8M_0M_2}{16M_0^2} = 0 \tag{45}$$

or

$$k_0^2 + \frac{1}{2}\frac{M_1}{M_0}k_0 + \frac{1}{2}\frac{M_2}{M_0} - \frac{1}{8}\left(\frac{M_1}{M_0}\right)^2 = 0. \tag{46}$$

If we substitute the $M_i$'s according to (15), we have

$$g^{ij}k_ik_j := k_0^2 + \frac{1}{2}\frac{M^a}{M}k_0k_a + \frac{1}{8}\left(4\frac{M^{ab}}{M} - \frac{M^aM^b}{M^2}\right)k_ak_b = 0. \tag{47}$$

This form is quadratic in the wave 4-covector $k_i$ and thus constitutes, up to a scalar factor, a Riemannian metric. Equation (47) represents our main result. It is clear that there is a coordinate system so that the metric $g^{ij}$ acquires the ordinary Minkowski form: $g^{ij} \stackrel{*}{=} \mathrm{diag}(+1, -1, -1, -1)$. Therefore, intrinsically it is not possible to have an anisotropic speed of light.

From the condition of the existence of a unique solution (or from hyperbolicity), Eq. (47) has to possess two real solutions for any given spatial $k_a$. As a consequence, the signature of the metric $g^{ij}$ is $(+1, -1, -1, -1)$. Accordingly, the signature of the metrical structure is a consequence of the existence of a unique solution of the Maxwell equations in a future causal cone for arbitrary sources with compact support.

Let us look at a specific example. If we exclude, besides birefringence, also *electric-magnetic cross terms* in the spacetime relation (7), then $\mathfrak{C} = \mathfrak{D} = 0$ and, according to (18), $M^a = 0$. If we substitute this into (47), we find

$${k_0}^2 + \frac{M^{ab}k_ak_b}{2M} = 0. \tag{48}$$

It can be shown [24] that one arrives also at this result by only forbidding the existence of electric-magnetic cross terms, that is, this condition is stronger than the requirement of vanishing birefringence. Clearly then, for the Minkowskian signature we have

$$\frac{M^{ab}k_ak_b}{2M} < 0, \tag{49}$$

see also (47). The flat Minkowski spacetime of special relativity is a subcase of (48). Then, in Cartesian coordinates, $M^{ab}$ is a constant. This is a consequence of the constancy of the constitutive matrices $\mathfrak{A}^{ba}$ and $\mathfrak{B}_{ba}$. Because of (7), we find $\mathcal{D}^a = -\mathfrak{A}^{ba}E_b$ and $\mathcal{H}_a = \mathfrak{B}_{ba}B^b$. Thus,

$$\mathfrak{A} = -\varepsilon_0\mathbf{1}_3, \quad \mathfrak{B} = \frac{1}{\mu_0}\mathbf{1}_3, \tag{50}$$

where $\mathbf{1}_3$ denotes the 3-dimensional unit matrix. If we substitute this into (17) to (19), we find $M = -\varepsilon_0^3$, $M^a = 0$, and $M^{ab} = (2\varepsilon_0^2/\mu_0)\mathbf{1}^{ab}$, that is,

$$\frac{M^{ab}}{2M} = -\frac{1}{\varepsilon_0\mu_0}\mathbf{1}^{ab} = -c^2\mathbf{1}^{ab} \tag{51}$$

is negative, with $c$ as the speed of light in vacuum.

Note that the vanishing of birefringence is not equivalent to the validity of the reciprocity relation as discussed in [2].

## VI. A UNIQUE LIGHT CONE IS INCOMPATIBLE WITH A FINSLERIAN GEOMETRY

Now we would like to generalize the result obtained above: *For all hyperbolic partial differential equations a vanishing birefringence of the characteristic cones defines merely a Riemannian structure.* There is no way to have two characteristic cones with a Finslerian structure. In fact, the restriction to hyperbolic partial differential equations is necessary for physical reasons: only for hyperbolic partial differential equations one has a unique solution in the future half space for prescribed initial values or prescribed source, see [19].

Let us now prove the above statement: Any characteristic surface is given by a polynomial of order $p$ in the covector $k_i$, which is "normal" to the characteristic surface[3],

$$H(k) = \mathfrak{g}^{i_1 i_2 \ldots i_p} k_{i_1} k_{i_2} \cdots k_{i_p}. \tag{52}$$

In order to be based on a hyperbolic differential operator, this polynomial also has to be hyperbolic, that is, there should exist $p$ real solutions $k_0 = k_0(k_a)$ (see, e.g., [19])

$$H(k) = \prod_{m=0}^{p} (k_0 - k_{0(m)}). \tag{53}$$

This specifies a splitting of the characteristic cone into $p$ sheets.

Now we want to restrict the number of cones to 2. In order to be able to identify an equal number of cones, we choose $p = 2q$. After the identification of the first $q$ solutions and the last $q$ solutions, respectively, we have as characteristic polynomial

$$\begin{aligned} H(k) &= \mathfrak{g}^{i_1 i_2 \ldots i_{2q}} k_{i_1} k_{i_2} \cdots k_{i_{2q}} \\ &= (k_0 - k_{0(1)})^q (k_0 - k_{0(2)})^q \\ &= [k_0^2 - (k_{0(1)} + k_{0(2)})k_0 + k_{0(1)}k_{0(2)}]^q, \end{aligned} \tag{54}$$

where the two solutions $k_{0(1)}$ and $k_{0(2)}$ are homogeneous functions of the spatial components $k_a$.

We differentiate this relation with respect to $k_a$ and set subsequently $k_a = 0$. This results in $k_{0(1,2)}(k_a = 0) = 0$. For the zeroth derivative we get

$$\mathfrak{g}^{\,00\ldots0} = 1. \tag{55}$$

The first derivative reads

$$\begin{aligned} &2n\mathfrak{g}^{i_1 \cdots i_{2q-1} a} k_{i_1} \cdots k_{i_{2q-1}} \\ &\quad = q[k_0^2 - (k_{0(1)} + k_{0(2)})k_0 + k_{0(1)}k_{0(2)}]^{q-1} \\ &\quad\quad \times \left[ -\frac{\partial}{\partial k_a}(k_{0(1)} + k_{0(2)})k_0 + \frac{\partial}{\partial k_a}(k_{0(1)}k_{0(2)}) \right], \end{aligned} \tag{56}$$

which, for $k_a \to 0$, yields

$$2\mathfrak{g}^{(0\ldots0a)} = -\frac{\partial}{\partial k_a}(k_{0(1)} + k_{0(2)}). \tag{57}$$

This can be integrated to

$$k_{0(1)} + k_{0(2)} = -2\mathfrak{g}^{(0\ldots0a)} k_a =: -2g^{0a} k_a \tag{58}$$

(no constant must be added because the $k_{0(m)}$'s are homogenous in $k_a$).

Analogously, we calculate the second derivative and perform the limit $k_a \to 0$,

$$2(2q-1)\mathfrak{g}^{(0\ldots0ab)} = 4\mathfrak{g}^{(0\ldots0a)}\mathfrak{g}^{(0\ldots0b)} + \frac{\partial^2}{\partial k_a \partial k_b}(k_{0(1)}k_{0(2)}), \tag{59}$$

where we used (58). Therefore,

$$\begin{aligned} k_{0(1)}k_{0(2)} &= [(2q-1)\mathfrak{g}^{(0\ldots0ab)} - 2\mathfrak{g}^{(0\ldots0a)}\mathfrak{g}^{(0\ldots0b)}]k_a k_b \\ &=: g^{ab} k_a k_b. \end{aligned} \tag{60}$$

If we substitute (58) and (60) into (54), then merely a Riemannian metric shows up,

$$\begin{aligned} k_0^2 - (k_{0(1)} + k_{0(2)})k_0 + k_{0(1)}k_{0(2)} &= \\ k_0^2 + 2g^{0a}k_0 k_a + g^{ab}k_a k_b &= g^{ij}k_i k_j, \end{aligned} \tag{61}$$

with $g^{00} = 1$. No Finslerian metric does occur. The underlying metric $g^{ij}$ has to be of signature $\pm 2$. Otherwise it would not lead, for prescribed $k_a$, to two real solutions $k_0$. Again, the metric $g^{ij}$ has to possess the signature $(+1, -1, -1, -1)$. ■

## VII. DISCUSSION

As our main result, we have shown that radiative vacuum solutions of the general Maxwell equations that do not show birefringence define—up to a scale transformation—a Riemannian metric. Thus, the requirement of vanishing birefringence automatically yields a Riemannian structure. No Finslerian metric can be introduced in this way. As a consequence, no intrinsic anisotropy in the propagation of light can be found (intrinsic in the sense of using merely the Maxwell equations). It is always possible to make a coordinate transformation to a locally Minkowskian frame. This applies also to a hypothetical higher order version of the generalized inhomogeneous Maxwell equation like $\partial_j(\chi^{ijkl}F_{kl}/2) + \partial_j\partial_m(\chi^{ijmkl}F_{kl}) = J^i$. Only if non-Minkowskian coordinates are related to or fixed by other physical phenomena, then one may speak about an anisotropy of the speed of light. Such phenomena may be related to quantum matter described by some Dirac-like equation. Accordingly, this anisotropy is defined *only* with respect to another physical phenomenon.

This situation is, of course, present in current tests searching for an *anisotropy of the propagation of light*, like the modern tests using optical cavities [14]. In these tests the isotropy of the velocity of light is tested with respect to the length of the cavity. This length is determined by the Dirac equation but, in part, also by the Maxwell equations. However, it turns out that for the used materials the latter influence the length of the cavity only marginally so that the length is mainly determined by the Dirac equation. Therefore, *Michelson-Morley tests are tantamount to a comparison of the Maxwell with the Dirac equation.*

This result also shows that the generalized Maxwell equations alone cannot cover the anisotropy effects of

[3]Strictly, a covector or 1-form is visualized by two parallel planes. If $\phi = 0$ describes the jump surface, then $k_i = \partial_i\phi$. Thus the two planes visualizing the 1-form are parallel to the tangent plane of the jump surface.

light described in the kinematical framework of Robertson-Mansouri-Sexl [25–27]. In the same way as in this kinematical framework, one has to make a comparison between the propagation of light and a length standard. This length standard is given as such within the Robertson-Mansouri-Sexl framework. In the present framework of dynamical test theories, this is replaced by a comparison of the Maxwell and the Dirac equation. In this sense, one may take the framework including a generalized Maxwell *and* a generalized Dirac equation as the dynamical replacement for the old Robertson-Mansouri-Sexl framework. However, one may want to go further to the appreciably more general *standard model extension* (SME) of Kostelecký and collaborators [28], which contains more than a single generalized Dirac equation. In any case, the birefringence of light and also of Dirac matter waves in vacuum is truly beyond the Robertson-Mansouri-Sexl scheme, but is included in the SME of Kostelecký.

Our main result only relies on the fourth order Fresnel Eq. (14). All propagation phenomena which lead to characteristic equations of fourth order lead to a Riemannian metric if one does not allow birefringence. Therefore, this also applies to the characteristics of a generalized Dirac equation where the $\gamma$-matrices are not assumed to fulfill a Clifford algebra. If the Dirac characteristics do not show birefringence, then we can conclude that the $\gamma$-matrices will fulfill a Clifford algebra. This also follows from our general result in Sec. VI.

Furthermore, our result can also be applied to the WKB approximation of generalized particle field equation as, e.g., the generalized Dirac equation [20,29,30]. As a result, one arrives at a scalar-vector-tensor theory where the dispersion relation induces a splitting of the mass shells according to $0 = k_0^2 - g^{ab}(p_a + \alpha_a)(p_b + \alpha_b) + \alpha^2$. The equation of motion for the corresponding point particle is that of a charged particle in Riemannian spacetime with a position and time dependent mass.

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