Gauge invariant effective action for the Polyakov line in the SU(N) Yang-Mills theory at high temperatures

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We integrate out fast varying quantum fluctuations around static A_4 and A_i fields for the SU(N) gauge group. By assuming that the gluon fields are slowly varying but allowing for an arbitrary amplitude of A_4 we obtain two variants of the effective high-temperature theory for the Polyakov line. One is the effective action for the gauge-invariant eigenvalues of the Polyakov line, and it is explicitly Z(N) symmetric. The other is the effective action for the Polyakov line itself as an element of the SU(N). In this case the theory necessarily includes the spatial components A_i to ensure its gauge invariance under spatial gauge transformations. We derive the 1-loop effective action in the electric and magnetic sectors, summing up all powers of A_4.

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I. INTRODUCTION

Finite-temperature quantum chromodynamics (QCD) is an intensely studied field. At finite temperature gluons obey periodic boundary conditions in imaginary time, leading to the quantized Matsubara frequencies $\omega_k = 2\pi kT$ for gluons. At very high temperatures nonzero Matsubara frequencies decouple from a theory as they become infinitely heavy. This is called dimensional reduction [1] since the decoupled heavy modes are time-dependent ones. Neglecting all modes except the static zero Matsubara frequencies leaves a 3D theory

$$-\frac{1}{4g^2}\int d^4x F_{\mu\nu}^2 \to -\frac{1}{4g^2T}\int d^3x[F_{ij}^2 + 2(D_i^{ab}A_4^b)^2], \tag{1}$$

which only contains the static gluonic modes with the coupling constant $g_3^2 = g^2T$. The long-range forces mediated by the static gluons lead to the IR divergencies, because in strict perturbation theory the gluons are massless. Owing to the chromomagnetic sector, the high-temperature perturbation theory explodes already at a few-loop level [2–4] and is hence only applicable at academically high temperatures [5]. The region of intermediate temperatures is of much bigger interest. For example, the deconfinement phase transition in the pure Yang-Mills theory is believed to take place in this region.

In the pure Yang-Mills theory the center symmetry plays a crucial part in the description of the confinement-deconfinement phase transition [2,6–8]. The order parameter for the latter is the average of the trace of the Polyakov line:

$$L(x) = \mathcal{P}\exp\left(i\int_0^{1/T} dx_4 A_4\right). \tag{2}$$

The order parameter $\langle \mathrm{Tr}L\rangle$ is zero in the confined phase below the critical temperature and assumes, after a proper regularization, a nonzero value in the deconfined phase above the critical temperature. For a recent confirmation in lattice simulations see [9,10]. The Polyakov line is not invariant under gauge transformations belonging to the gauge group center. One hence concludes that if $\langle \mathrm{Tr}L\rangle = 0$ then the Z(N) symmetry is manifest. This situation describes confinement. If for any reason $\langle \mathrm{Tr}L\rangle \neq 0$ then the symmetry must have been broken. This corresponds to the deconfined phase.

At high but not infinitely high temperatures the tree-level action (1) is not sufficient to study gluon fluctuations. When one includes quantum corrections all the nonzero Matsubara modes of the gluons show up in the loops. The 1-loop [4,11] and 2-loop [12] potential energies as functions of A_4 are known. They are periodic functions of the eigenvalues of A_4 in the adjoint representation with period $2\pi T$. This reflects the symmetry of the Z(N) vacua. The curvature of the potential at its minima gives the leading order Debye mass for "electric" gluons. The zero energy minima of the potential are at quantized values of A_4 corresponding to the center-of-group values of the Polyakov line, where $L \in \mathrm{Z}(N)$. At high temperatures the system oscillates around one of these minima. At lower temperatures, however, the fluctuations around the minimum increase and eventually the system undergoes a phase transition to $\langle \mathrm{Tr}L\rangle = 0$. At the same time, one expects that near the phase transition point the fluctuations are long-range. To study those fluctuations, one needs an effective low-momenta theory which, however, does not assume that the A_4 component is small.

The nonzero temperatures explicitly break the 4D Euclidean symmetry of the theory down to the 3D Euclidean symmetry, so that the spatial A_i and time A_4 components of the Yang-Mills field play different roles and should be treated differently. One can always choose a

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gauge where A_4 is time-independent. Taking $A_4(x)$ to be static is not a restriction of any kind on the fields but merely a convenient gauge choice, and we shall imply this gauge throughout the paper. It is also a possible gauge choice at $T = 0$ but in that limiting case it is unnatural as one usually wishes to preserve the 4D symmetry. In this gauge, the Polyakov line (2) is not a path-ordered but a simple exponent of $A_4(x)$. It rotates under x-dependent gauge transformations. Therefore, an effective action for the Polyakov line, which should be invariant under spatial gauge transformations, cannot depend on the gradients of $L(x)$ alone, but rather on the *covariant* derivatives of $L(x)$, involving the spatial components of the background-field A_i. The aim of this work is to compute the effective action expanding it in covariant derivatives but *keeping all powers of* A_4. It is an extension of the previous work [13–15] to higher gauge groups. We use the background-field method for the gluons and evaluate the 1-loop action through the functional determinant formalism. We integrate out fast varying quantum fluctuations about the static background (A_i, A_4) by making an expansion in spatial covariant derivatives. This method was originally developed in [16] for zero temperature QCD. We compute terms quadratic in covariant derivatives, that is the electric E^2 terms where $E_i^a = D_i^{ab}A_4^b$, and the quartic "magnetic" $B_\parallel^2$ terms where $B_i^a = \frac{1}{2}\epsilon_{ijk}(\partial_i A_j^a - \partial_j A_i^a + f^{abc}A_i^b A_j^c)$. While we consider a general electric field, we restrict ourselves to a magnetic field parallel to A_4, $B_i = B_{i\parallel}$. The structure of the SU(3) effective action turns out to be much richer than in the SU(2) case, as there are far more invariants which one can build out of A_4 and the electric and the magnetic fields.

As a byproduct of our study, we obtain the effective action for the *eigenvalues* of the Polyakov line. Contrary to the Polyakov line itself which is an element of the SU(N) group and rotates under spatial gauge transformations, its eigenvalues are gauge invariant. Therefore, the derivative expansion of the effective action for the eigenvalues goes in ordinary rather than covariant derivatives. Such an action was first obtained in Ref. [17] for the SU(2) gauge group and reaffirmed in Ref. [13]; here we extend it first to the more intricate case of the SU(3) gauge group and then to the general SU(N). The resulting action is explicitly Z(N) symmetric.

We expect that our results are suitable to study the correlation functions of the Polyakov line not too far from the transition point where it experiences fluctuations that are large in amplitude but presumably mainly long ranged.

II. COVARIANT DERIVATIVE EXPANSION OF 1-LOOP QUANTUM ACTION

In [13] we obtained the 1-loop effective action for the pure Yang-Mills SU(2) theory. We started with the Yang-Mills partition function and decomposed the gluon fields into background fields and quantum fluctuations around them. For a 1-loop approximation we expanded the action to quadratic order in the gluonic quantum fluctuations. The resulting bilinear form for the quantum fluctuations is degenerate, so we chose a background Lorenz gauge for them. This in turn gives rise to the Faddeev-Popov ghost determinant in the background field. The 1-loop effective action was obtained by integrating over the ghost fields as well as over the gluon quantum fluctuations. It is expressed in terms of functional determinants which is an economic and aesthetic method of getting gauge invariant results[1]:

$$S_{1-\mathrm{loop}} = \log(\det W)^{-1/2} + \log\det(-D^2). \tag{3}$$

The first contribution here comes from integrating out the gluon fields, while the second comes from the ghost fields. Since the gluons transform under the adjoint representation of the color group the operators D^2, $W_{\mu\nu}$ consist of matrices in the adjoint representation. The latter is given by

$$W_{\mu\nu}^{ab} = -[D^2(A)]^{ab}\delta_{\mu\nu} - 2f^{acb}F_{\mu\nu}^c(A), \tag{4}$$

where

$$D_\mu^{ab} = \partial_\mu \delta^{ab} + f^{acb}A_\mu^c \tag{5}$$

is the covariant derivative in the adjoint representation. All gluon fields A_μ are background fields. Equation (3) is independent of the gauge group and hence valid for any SU(N). In addition, it is invariant under general gauge transformations of the A_μ fields. We can use this property to make the A_4 component static. Then, there still remains a freedom to make time-independent gauge transformations,

$$\begin{aligned} A_4(x) &\to U(x)A_4(x)U^\dagger(x), \\ A_i(x) &\to U(x)A_i(x)U^\dagger(x) + iU(x)\partial_i U^\dagger(x), \end{aligned} \tag{6}$$

which can be used to bring A_4 to a diagonal form in the fundamental representation. The general SU(N) case is considered in section XI B. For the SU(3) gauge group we can write (λ^a are eight Gell-Mann matrices, see the Appendix)

$$A_4 = A_4^3(x)\frac{\lambda^3}{2} + A_4^8(x)\frac{\lambda^8}{2}. \tag{7}$$

This gauge fixing leaves A_i an arbitrary SU(3) matrix, generally speaking both space and time dependent.

After fixing the gauge such that A_4 is diagonal and static, one is still left with a freedom of gauge transformations of a special type. For the SU(2) group they were pointed out in Ref. [13], section 10; in the SU(3) case the remaining gauge transformations are of the form

[1]Our convention for the sign of the action is such that the partition function $\mathcal{Z} = \exp(+S)$.

$$A_\mu \to S A_\mu S^\dagger + iS\partial_\mu S^\dagger,$$
$$S(x,t) = \exp\left\{ i\frac{\lambda^3}{2}[\alpha_3(x) + 2\pi t T n_3] + i\frac{\lambda^8}{2}[\alpha_8(x) + 2\pi t T n_8]\right\}, \quad (8)$$

where n_3 and $n_8\sqrt{3}$ are both even or both odd integers. One cannot take rotations about axes other than the 3rd and 8th ones because it will make A_4 nondiagonal, and one cannot take the time dependence other than linear because that would make A_4 time dependent. The space-dependent functions $\alpha_{3,8}$ may be arbitrary. The fact that n_3 and $n_8\sqrt{3}$ must be both even or both odd integers follows from the requirement that the gauge transformation in the adjoint representation, given by the 8×8 matrix $O^{ab} = 2\mathrm{Tr}(St^aS^\dagger t^b)$, must be periodic in time. In components, the transformation (8) reads:

$$A_4^{3,8} \to A_4^{3,8} + 2\pi T n_{3,8}, \quad (9)$$

$$A_i^{3,8} \to A_i^{3,8} + \partial_i \alpha_{3,8}, \quad (10)$$

$$A_i^1 \to \cos\beta_3 A_i^1 + \sin\beta_3 A_i^2, \quad (11)$$

$$A_i^2 \to -\sin\beta_3 A_i^1 + \cos\beta_3 A_i^2, \quad (12)$$

$$A_i^4 \to \cos\frac{\beta_8+\beta_3}{2} A_i^4 + \sin\frac{\beta_8+\beta_3}{2} A_i^5, \quad (13)$$

$$A_i^5 \to -\sin\frac{\beta_8+\beta_3}{2} A_i^4 + \cos\frac{\beta_8+\beta_3}{2} A_i^5, \quad (14)$$

$$A_i^6 \to \cos\frac{\beta_8-\beta_3}{2} A_i^6 + \sin\frac{\beta_8-\beta_3}{2} A_i^7, \quad (15)$$

$$A_i^7 \to -\sin\frac{\beta_8-\beta_3}{2} A_i^6 + \cos\frac{\beta_8-\beta_3}{2} A_i^7, \quad (16)$$

$$\beta_{3,8} = 2\pi t T n_{3,8} + \alpha_{3,8}. \quad (17)$$

The Polyakov line is transformed by a diagonal matrix

$$\mathrm{diag}(e^{\pi i[n_3+(n_8/\sqrt{3})]}, e^{-\pi i[n_3-(n_8/\sqrt{3})]}, e^{-(2\pi i/\sqrt{3})n_8}), \quad (18)$$

which for n_3 and $n_8\sqrt{3}$ both even or both odd becomes an element of the group center, $\mathrm{diag}(e^{2\pi ik/3}, e^{2\pi ik/3}, e^{2\pi ik/3})$, $k = 0, 1, 2$.

Thus we see that the Z(N) symmetry of the effective action is a consequence of the symmetry under gauge transformations of a special type, Eq. (8). It should be stressed, however, that the same gauge transformations make the spatial components of the background field A_i time dependent, even if one starts from purely static A_i's, with their Matsubara frequencies directly related to $n_{3,8}$ according to Eqs. (11)–(16). Consequently, if one wishes to see the Z(N) symmetry of the effective action (3) explicitly, one has to sum up all powers in time derivatives of A_i. This seems to be a formidable problem which we do not attempt to solve here. We expand the effective action (3) to the second order in the electric field and hence to the second order in $\dot{A}_i$ only. Therefore, our effective action will not be explicitly Z(N) symmetric. To be more precise, part of the invariants we calculate will be explicitly Z(N) symmetric, namely, those which remain invariant under the transformations (11)–(16), but some of the invariants will be not. The effective action we are computing is for the static A_i fields and is invariant only under static gauge transformations (6).

With our background field technique, we are also able to solve simultaneously another physical problem. Namely, one may be interested in the effective quantum action for the *eigenvalues* of the Polyakov line. Contrary to A_4 and to the Polyakov line as a unitary matrix, its eigenvalues *are invariant* under spatial gauge transformations. Therefore, the effective action for the eigenvalues can be expanded in ordinary "short" rather than covariant derivatives, and the background A_i field may then be set to zero. In this setting, the A_i fields (as well as the rapidly changing components of the A_4 fields) are understood and treated as quantum fluctuations over which one integrates. The resulting effective action for the eigenvalues of the Polyakov line, including its spatial derivatives to the second order, explicitly obeys the Z(N) symmetry. It has been first found for the SU(2) gauge group in Ref. [17] and reaffirmed in Ref. [13]; here we extend this result to the SU(3) case in section XI A and to the general SU(N) gauge group in section XI B.

The functional determinants in Eq. (3) are UV divergent which reflects the divergence of the coupling constant. Since QCD is a renormalizable theory the divergence can be absorbed in the definition of the tree-level coupling constant. To do so, we have to properly normalize and regularize the functional determinants. The former is obtained by normalizing to the free zero gluon contribution, for the latter we introduce a Pauli-Villars cutoff M in momentum space. For the gluon determinant this results in the "quadrupole formula"

$$\det(-D^2)_{\mathrm{r,n}} \equiv \frac{\det(-D_\mu^2)}{\det(-\partial_\mu^2)}\frac{\det(-\partial_\mu^2+M^2)}{\det(-D_\mu^2+M^2)} = \exp\left\{-\int_0^\infty \frac{ds}{s}\mathrm{Sp}[(1-e^{-sM^2}) \times (e^{sD_\mu^2} - e^{s\partial_\mu^2})]\right\}, \quad (19)$$

where the rewriting in the last line is due to a trick originally introduced by Schwinger [18]. The functional trace Sp can be taken by inserting any complete basis. We choose the plane wave basis:

$$\mathrm{S\,p}e^{-sK} = \mathrm{Tr}\int d^4x \lim_{y\to x}\int \frac{d^4p}{(2\pi)^4} e^{-ip\cdot y} e^{-sK} e^{ip\cdot x} = \mathrm{Tr}\int d^4x \int \frac{d^4p}{(2\pi)^4} e^{-sK(\partial_\alpha \to \partial_\alpha + ip_\alpha)} 1. \quad (20)$$

Here Tr is the remaining trace over color and Lorentz indices. The **1** at the end is meant to emphasize that the shifted operator acts on unity, so that, for example, any term that has a ∂_α in the exponent and is brought all the way to the right, will vanish. According to (20) we now have

$$\begin{aligned}\log\det(-D^2)_{\mathrm{r,n}} = &-\int d^3x \sum_{k=-\infty}^{\infty}\int \frac{d^3p}{(2\pi)^3} \\ &\times \int_0^\infty \frac{ds}{s}(1-e^{-sM^2})e^{-sp^2} \\ &\times \mathrm{Tr}\{\exp[s\mathcal{A}^2 + sD_i^2 + 2isp_iD_i] \\ &- \exp[-s\omega_k^2]\},\end{aligned} \quad (21)$$

where we introduced the adjoint matrix

$$\mathcal{A} = f^{acb}A_4^c + i\omega_k\delta^{ab}. \quad (22)$$

Similarly to Eq. (19) we have to normalize and regularize the gluon functional determinant. Again after an insertion of a plane wave basis we obtain

$$\begin{aligned}\log(\det W)^{-1/2}_{\mathrm{r,n}} = &\frac{1}{2}\int d^3x \sum_{k=-\infty}^{\infty}\int \frac{d^3p}{(2\pi)^3} \\ &\times \int_0^\infty \frac{ds}{s}(1-e^{-sM^2})e^{-sp^2} \\ &\times \mathrm{Tr}\{\exp[(s\mathcal{A}^2 + sD_i^2 + 2isp_iD_i)^{ab} \\ &\times \delta_{\mu\nu} + 2sf^{acb}F^c_{\mu\nu}] - \exp[-s\omega_k^2]\}.\end{aligned} \quad (23)$$

The regularized and normalized effective 1-loop action is hence given by the sum of Eq. (21) and Eq. (23). Expressed in this way one can now expand in powers of spatial covariant derivatives D_i. This introduces a potential energy and a kinetic energy in terms of color-electric and color-magnetic fields, since the latter are identified as

$$\begin{aligned}&i[D_i, \mathcal{A}] = i[D_i, D_4] = F_{i4} = E_i \quad \text{and} \\ &\frac{1}{2}\epsilon_{ijk}F_{jk} = \frac{i}{2}[D_j, D_k] = B_i.\end{aligned} \quad (24)$$

In [13] we studied the effective action for the gauge group SU(2). We expanded to quadratic order in the electric and magnetic fields, but did not retain mixing terms between electric and magnetic fields or derivative terms of the electric field, which all exist at that order. The gauge-invariant structures that we obtained were $A_4^aA_4^a$, $E_i^aE_i^a$, $B_i^aB_i^a$, $(E_i^aA_4^a)^2$, and $(B_i^aA_4^a)^2$. For SU(3), which is the case that we are considering here, one expects and finds more invariants.

III. THE POLYAKOV LINE

The order parameter for the transition from a confined to a deconfined phase in a pure Yang-Mills theory is the average of the trace of the Polyakov line,

$$L(x) = \mathcal{P}\exp\left\{i\int_0^{\beta=\frac{1}{T}} dtA_4(t,x)\right\}. \quad (25)$$

As usual $\mathcal{P}$ denotes path ordering. For our explicit calculations we shall use the freedom to rotate the A_4 field to a diagonal form and consider it to be static. In this gauge, $L(x)$ is a diagonal matrix $\exp(iA_4/T)$. For an SU(N) there are $(N-1)$ diagonal generators. In SU(2) this is the third Pauli matrix τ_3, in SU(3) one has the λ^3 and λ^8 Gell-Mann matrices (the general SU(N) case will be considered in section XI B). Writing A_4 in the basis of these diagonal generators automatically ensures that it is traceless.

In particular we choose the following parametrization ($t^a = \lambda^a/2$, see Appendix)

$$A_4 = A_4^3t^3 + A_4^8t^8. \quad (26)$$

From this we can then construct A_4 in the adjoint representation as $A_4^{\mathrm{adj}} = A_4^{ab} = if^{acb}A_4^c$. We find that it has the following nonzero eigenvalues

$$\begin{aligned}&\pm\phi_1 = \pm A_4^3, \qquad \pm\phi_2 = \pm\frac{A_4^3 + A_4^8\sqrt{3}}{2}, \\ &\pm\phi_3 = \pm\frac{A_4^3 - A_4^8\sqrt{3}}{2}.\end{aligned} \quad (27)$$

Introducing the rescaled variables $a_3 \equiv A_4^3/(2\pi T)$, $a_8 \equiv A_4^8/(2\pi T)$, and $\nu_i = \phi_i/(2\pi T)$, Eq. (27) becomes

$$\begin{aligned}&\pm\nu_1 = \pm a_3, \qquad \pm\nu_2 = \pm\frac{a_3 + a_8\sqrt{3}}{2}, \\ &\pm\nu_3 = \pm\frac{a_3 - a_8\sqrt{3}}{2}.\end{aligned} \quad (28)$$

The Polyakov line $L = \exp(iA_4/T)$ is a diagonal unitary matrix:

$$\begin{aligned}L &= \begin{pmatrix} e^{\pi i[a_3 + a_8/\sqrt{3})]} & 0 & 0 \\ 0 & e^{-\pi i[a_3 - (a_8/\sqrt{3})]} & 0 \\ 0 & 0 & e^{-(2\pi i/\sqrt{3})a_8}\end{pmatrix} \\ &= \begin{pmatrix} e^{(2\pi i/3)(\nu_1+\nu_2)} & 0 & 0 \\ 0 & e^{(-2\pi i/3)(\nu_1+\nu_3)} & 0 \\ 0 & 0 & e^{(-2\pi i/3)(\nu_2-\nu_3)}\end{pmatrix}.\end{aligned} \quad (29)$$

Here we expressed it once in terms of a_3 and a_8 and once through the eigenvalues of A_4^{adj}. It is easy to see that the Polyakov line (29) assumes values of the group center,

$e^{2\pi ik/3}\mathbf{1}_3$, $k = 0, 1, 2$, for integer values of ν_i. This knowledge will be of use to understand the potential energy in the next section.

IV. THE POTENTIAL ENERGY

The potential energy is the contribution to the 1-loop effective action that corresponds to constant background fields, hence all spatial variations are set to zero. Since in Yang-Mills theories gauge invariance requires that derivatives always show up in their covariant form, we obtain the potential by setting D_i equal to zero. From Eq. (3) and Eqs. (21) and (23) we see that $F_{\mu\nu}$ does not contribute since it is either linear (F_{4i}) or quadratic (F_{ij}) in the covariant derivatives. Therefore, we find that at zeroth order in D_i

$$\begin{aligned} S^{(0)}_{1-\text{loop}} &= -[\text{logdet}(-D^2)]^{(0)} \\ &= \int d^3x \sum\!\!\!\!\!\!\int e^{-sp^2} \text{Tr}(e^{-s\mathcal{A}^2} - e^{-s\omega_k^2}) \\ &= -\int d^3x P(A_4). \end{aligned} \tag{30}$$

Here we introduce compact notations:

$$\sum\!\!\!\!\!\!\int = \sum_{k=-\infty}^{\infty} \int \frac{d^3p}{(2\pi)^3} \int_0^\infty \frac{ds}{s},$$

$$\sum\!\!\!\!\!\!\int^M = \sum_{k=-\infty}^{\infty} \int \frac{d^3p}{(2\pi)^3} \int_0^\infty \frac{ds}{s}(1 - e^{-sM^2}).$$

For Eq. (30) we need the matrix $\mathcal{A}^{ab}$. With the Ansatz (26) we can construct and express it in terms of the eigenvalues of A_4^{adj}, Eq. (27):

$$\mathcal{A} = \begin{pmatrix} i\omega_k & -\phi_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \phi_1 & i\omega_k & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & i\omega_k & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i\omega_k & -\phi_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & \phi_2 & \omega_k & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & i\omega_k & \phi_3 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\phi_3 & \omega_k & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & i\omega_k \end{pmatrix}. \tag{31}$$

We then find that

$$\begin{aligned} \text{Tr}[e^{-s\mathcal{A}^2} - e^{-s\omega_k^2}] = \sum_{n=1,2,3} [&(e^{-s(\phi_n-\omega_k)^2} + e^{-s(\phi_n+\omega_k)^2}) \\ &- 2e^{-s\omega_k^2}] \end{aligned} \tag{32}$$

where the sum is over the adjoint eigenvalues. In the SU(2) case there is just one term yielding [13]

$$\begin{aligned} 2\sum\!\!\!\!\!\!\int e^{-sp^2}[e^{-s(\phi-\omega_k)^2} - e^{-s\omega_k^2}] \\ = -\frac{1}{12\pi^2 T}\phi^2(2\pi T - \phi)^2|_{\text{mod}2\pi T}. \end{aligned} \tag{33}$$

In the SU(3) case one obtains [4]

$$P(A_4) = \frac{(2\pi)^2 T^3}{3} \sum_{n=1,2,3} \nu_n^2(1-\nu_n)^2|_{\text{mod}1}, \tag{34}$$

where again

$$\begin{aligned} \nu_1 &= \frac{A_4^3}{2\pi T}, \qquad \nu_2 = \frac{1}{2}\frac{A_4^3 + A_4^8\sqrt{3}}{2\pi T}, \\ \nu_3 &= \frac{1}{2}\frac{A_4^3 - A_4^8\sqrt{3}}{2\pi T}. \end{aligned} \tag{35}$$

We hence see that the potential energy is periodic in the eigenvalues of A_4^{adj} with period $2\pi T$. We plot it as a function of A_4^3 and A_4^8 in Fig. 1. In these axes, the minima and maxima of the potential energy form a regular triangle lattice, with lattice spacing $2\pi T$. The zero-energy minima are at integer values of ν_n, $n = 1, 2, 3$, where the Polyakov line assumes values from the group center. The maxima of the potential energy correspond to the Polyakov line $L_{\text{max}} = \text{diag}(1, e^{2\pi i/3}, e^{-2\pi i/3})$ (and those obtained from this one by multiplying it by the elements of the group center) whose $\text{Tr}L_{\text{max}} = 0$. At high temperatures $L(x)$ oscillates somewhat around the trivial center-of-group minima of the potential. However, as one lowers the temperature the fluctuations of the Polyakov line around these perturbative minima become stronger and eventually a transition to a phase with $\langle \text{Tr}L \rangle = 0$ occurs, despite the fact that the potential energy is maximal at $\text{Tr}L = 0$. The aim of the rest of the paper is to derive the effective action for the fluctuations of $L(x)$ in a broad range of its variation beyond the trivial minima.

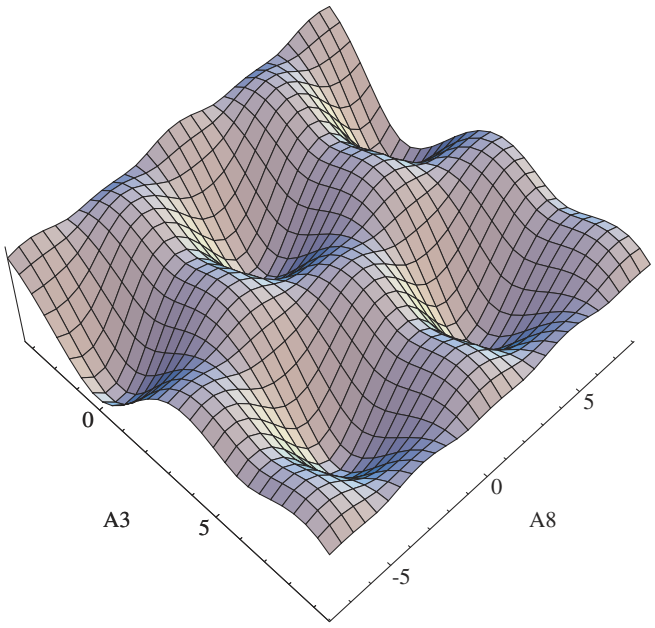

FIG. 1 (color online). Potential energy as function of A_4^3 and A_4^8 forms a double-periodic triangle lattice. The Z(3) symmetry is the symmetry under lattice translations.

V. THE ELECTRIC SECTOR

As in [13] we are interested in the leading terms for the kinetic energy in the color-electric sector, meaning terms quadratic in the electric field. To obtain these, we have to expand Eqs. (21) and (23) to quadratic order in D_i, because the electric field is identified as Eq. (24). Since D_i shows up in the exponents in Eqs. (21) and (23) we use the following two "master formulae" ([13]) for an expansion of exponentials of noncommuting operators:

$$\begin{aligned} e^{A+B} &= e^A + \int_0^1 d\alpha e^{\alpha A} B e^{(1-\alpha)A} \\ &\quad + \int_0^1 d\alpha \int_0^{1-\alpha} d\beta e^{\alpha A} B e^{\beta A} B e^{(1-\alpha-\beta)} A \\ &\quad + \ldots, \\ [B, e^A] &= \int_0^1 d\gamma e^{\gamma A} [B, A] e^{(1-\gamma)A}. \end{aligned} \tag{36}$$

Here B contains combinations of D_i and A denotes the rest. Equations (21) and (23) as well as Eq. (36) are independent of the color group. We found in [13] that for any SU(N) the individual contributions from the ghosts and gluons to the order that we are interested in are

$$\begin{aligned} [\mathrm{logdet}(-D^2)_{\mathrm{r,n}}]^{(2)}_{\mathrm{E}} &= -\int d^3x \sum\!\!\!\!\!\!\int^M e^{-sp^2} [I_1 + I_2], \\ [\log(\det W)^{-1/2}_{\mathrm{r,n}}]^{(2)}_{\mathrm{E}} &= \int d^3x \sum\!\!\!\!\!\!\int^M e^{-sp^2} \left[2I_1 + 2I_2 + \frac{I_3}{2} \right], \end{aligned} \tag{37}$$

so that the 1-loop contribution to the kinetic energy in the electric sector is

$$[S_{1-\mathrm{loop}}]_{\mathrm{E}} = \int d^3x \sum\!\!\!\!\!\!\int^M e^{-sp^2} \left[I_1 + I_2 + \frac{I_3}{2} \right]. \tag{38}$$

The structures I_i in Eqs. (37) and (38) are given by [13]

$$\begin{aligned} I_1 &= s^3 \int_0^1 d\alpha \left\{ -\frac{1}{2} + \alpha(1-\alpha) \right. \\ &\quad \left. + \frac{2}{9} sp^2 \left[1 - \frac{3}{2}\alpha(1-\alpha) \right] \right\} \\ &\quad \times \mathrm{Tr} e^{(1-\alpha)s\mathcal{A}^2} \{\mathcal{A}, E_i\} e^{\alpha s \mathcal{A}^2} \{\mathcal{A}, E_i\}, \end{aligned} \tag{39}$$

$$I_2 = -s^2 \left(\frac{1}{2} - \frac{2}{9} sp^2 \right) \mathrm{Tr} e^{s\mathcal{A}^2} (2E_i^2 + i\{\mathcal{A}, [D_i, E_i]\}), \tag{40}$$

$$I_3 = 8s^2 \int_0^1 d\alpha \frac{1}{2} \mathrm{Tr} e^{(1-\alpha)s\mathcal{A}^2} E_i e^{\alpha s \mathcal{A}^2} E_i, \tag{41}$$

where all matrices are in the adjoint representation. So far the gauge group has not been specified. For SU(2) the result was presented in [13]. For SU(3) we chose A_4 to be diagonal in the fundamental representation, Eq. (26), and obtain the adjoint matrix $\mathcal{A}$ in Eq. (31). The electric field is $E_i^{ab} = i f^{acb} E_i^c$. The second term in the invariant I_2 is zero if the gluon fields obey the equation of motion. We leave it away for the time being but shall return to it later. After all integrations and the summation over all Matsubara frequencies, we find the following structure for the electric kinetic energy:

$$\begin{aligned} [S_{1-\mathrm{loop}}]_{\mathrm{E}} &= \int d^3x \{ [(E_i^1)^2 + (E_i^2)^2] \lambda_1(\nu) + [(E_i^4)^2 \\ &\quad + (E_i^5)^2] \lambda_4(\nu) + [(E_i^6)^2 + (E_i^7)^2] \lambda_6(\nu) \\ &\quad + (E_i^3)^2 \lambda_3(\nu) + (E_i^8)^2 \lambda_8(\nu) + E_i^3 E_i^8 \lambda_{38}(\nu) \}. \end{aligned} \tag{42}$$

All functions $\lambda_j(\nu)$ are functions of the rescaled eigenvalues of A_4^{adj}, $\nu = \nu_1, \nu_2, \nu_3$, Eq. (35). For the explicit sums over the Matsubara frequencies, one has to define the range of definition for the ϕ_n, respectively, the ν_n. This defines the functional form of the λ_j. In this section we shall consider the range $0 \le |\phi_i| \le 2\pi T$, which corresponds to $0 \le |A_4^3|, \sqrt{3}|A_4^8| \le 2\pi T$. Outside this interval the functions have to be computed separately. We would like to stress once more that we do not expect Z(3) symmetric results here, as we obtained them for the potential (34), which would be equivalent to invariance under large time-dependent gauge transformations. We broke this symmetry by making the background A_i components static. As discussed before, to recover this symmetry one has to find the action to all orders in time derivatives.

We start by looking at the coefficients in front of the electric field orthogonal to A_4 in color space, $E_i^{1-2,4-7}$, namely $\lambda_{1,4,6}$. We shall split the results into a contribution from the nonzero Matsubara frequencies (denoted by a prime) and into the $\omega_{k=0}$ contribution as $\lambda_i = \lambda_i' + \lambda_{i0}$, where $i = 1, 4, 6$. We find that as long as the individual eigenvalues ν_i are either between 0 and 1 or between -1 and 0 then the functional form of the λ_i' is given by

$$\begin{aligned} \lambda_1'(\nu) &= -\frac{11}{96\pi^2 T} \left[6(\gamma_{\mathrm{E}} - \log\mu) + 2\psi\left(\frac{\nu_1}{2}\right) \right. \\ &\quad \left. + 2\psi\left(-\frac{\nu_1}{2}\right) + \psi\left(\frac{\nu_2 - \nu_3}{2}\right) + \psi\left(-\frac{\nu_2 - \nu_3}{2}\right) \right], \end{aligned} \tag{43}$$

$$\begin{aligned} \lambda_4'(\nu) &= -\frac{11}{96\pi^2 T} [6(\gamma_{\mathrm{E}} - \log\mu) + 2\psi\left(\frac{\nu_2}{2}\right) + 2\psi\left(-\frac{\nu_2}{2}\right) \\ &\quad + \psi\left(\frac{\nu_1 + \nu_3}{2}\right) + \psi\left(-\frac{\nu_1 + \nu_3}{2}\right)], \end{aligned} \tag{44}$$

$$\lambda_6'(\nu) = -\frac{11}{96\pi^2 T}\Big[6(\gamma_{\rm E} - \log\mu) + 2\psi\Big(\frac{\nu_3}{2}\Big) + 2\psi\Big(-\frac{\nu_3}{2}\Big) + \psi\Big(\frac{\nu_1+\nu_2}{2}\Big) + \psi\Big(-\frac{\nu_1+\nu_2}{2}\Big)\Big]. \tag{45}$$

Here ψ is the digamma function

$$\psi(\nu) = \frac{\partial}{\partial\nu}\log\Gamma(\nu). \tag{46}$$

For the λ_{i0} their functional form depends on the region of definition of each of the three ν_i separately. It is, however, possible to express them in terms of the ϕ_i before specifying the intervals for the latter:

$$\lambda_{10} = \frac{1}{2\pi^2}\Big[\frac{5\pi}{3|\phi_1|} + \frac{|\phi_2|\pi(10\phi_2^3 + 23\phi_2^2\phi_3 + 10\phi_2\phi_3^2 + \phi_3^3)}{12\phi_2(\phi_2-\phi_3)(\phi_2+\phi_3)^3} - \frac{|\phi_3|\pi(\phi_2^3 + 10\phi_2^2\phi_3 + 23\phi_2\phi_3^2 + 10\phi_3^3)}{12\phi_3(\phi_2-\phi_3)(\phi_2+\phi_3)^3}\Big], \tag{47}$$

$$\lambda_{40} = \frac{1}{2\pi^2}\Big[\frac{5\pi}{3|\phi_2|} + \frac{|\phi_3|\pi(-10\phi_3^3 + 23\phi_3^2\phi_1 - 10\phi_3\phi_1^2 + \phi_1^3)}{12\phi_3(\phi_1+\phi_3)(\phi_1-\phi_3)^3} + \frac{|\phi_1|\pi(-\phi_3^3 + 10\phi_3^2\phi_1 - 23\phi_3\phi_1^2 + 10\phi_1^3)}{12\phi_1(\phi_1+\phi_3)(\phi_1-\phi_3)^3}\Big], \tag{48}$$

$$\lambda_{60} = \frac{1}{2\pi^2}\Big[\frac{5\pi}{3|\phi_3|} + \frac{|\phi_1|\pi(10\phi_1^3 - 23\phi_1^2\phi_2 + 10\phi_1\phi_2^2 - \phi_2^3)}{12\phi_1(\phi_1+\phi_2)(\phi_1-\phi_2)^3} + \frac{|\phi_2|\pi(\phi_1^3 - 10\phi_1^2\phi_2 + 23\phi_1\phi_2^2 - 10\phi_2^3)}{12\phi_2(\phi_1+\phi_2)(\phi_1-\phi_2)^3}\Big]. \tag{49}$$

Note that there is no explicit factor $1/T$, since the variables are not rescaled yet. One can easily see that these functions can be obtained from each other as $\lambda_{40} = \lambda_{10}(\phi_1 \to \phi_2, \phi_2 \to -\phi_1)$ and $\lambda_{60} = \lambda_{40}(\phi_2 \leftrightarrow \phi_3)$. To show how the functions (47)–(49) depend on the region of definition of ϕ_i we show some examples. If all three $\nu_i = \phi_i/(2\pi T)$ are defined to be in the interval from 0 to +1, then we have

$$\begin{aligned}
\lambda_{10} &= \frac{1}{12\pi^2 T}\Big[\frac{5}{\nu_1} - \frac{\nu_2^2}{(\nu_2+\nu_3)^3} + \frac{\nu_2}{(\nu_2+\nu_3)^2} + \frac{9}{4(\nu_2+\nu_3)}\Big],\\
\lambda_{40} &= \frac{1}{12\pi^2 T}\Big[\frac{5}{\nu_2} + \frac{11}{4(\nu_1+\nu_3)}\Big],\\
\lambda_{60} &= \frac{1}{12\pi^2 T}\Big[\frac{5}{\nu_3} + \frac{11}{4(\nu_1+\nu_2)}\Big].
\end{aligned} \tag{50}$$

If all three ν_i lie between -1 and 0, then the above functions just change their global sign. If we put ν_1 and ν_3 in the positive region from 0 to 1 and ν_2 into the negative one from -1 to 0, then the functions change to

$$\begin{aligned}
\lambda_{10} &= \frac{1}{12\pi^2 T}\Big[\frac{5}{\nu_1} + \frac{11}{4(\nu_3-\nu_2)}\Big],\\
\lambda_{40} &= \frac{1}{12\pi^2 T}\Big[-\frac{5}{\nu_2} + \frac{11}{4(\nu_1+\nu_3)}\Big],\\
\lambda_{60} &= \frac{1}{12\pi^2 T}\Big[\frac{5}{\nu_3} - \frac{\nu_1^2}{(\nu_1-\nu_2)^3} + \frac{\nu_1}{(\nu_1-\nu_2)^2} + \frac{9}{4(\nu_1-\nu_2)}\Big].
\end{aligned} \tag{51}$$

Adding up the contributions from the nonzero Matsubara frequencies, Eqs. (43)–(45), and the λ_{i0} terms from Eqs. (47)–(49), one finds that the functions are not symmetric in the ν_i between 0 and 1 (or between -1 and 0), which would have been the requirement for the Z(3) equivalence.

The remaining coefficient functions $\lambda_{3,8,38}(\nu)$ in Eq. (42), which are connected to the electric field parallel to A_4 in color space, viz. E_i^3 and E_i^8, can be expressed in terms of the function

$$h(\nu) = \frac{1}{|\nu|} + 2(\log\mu - \gamma_{\rm E}) - \psi(\nu) - \psi(-\nu). \tag{52}$$

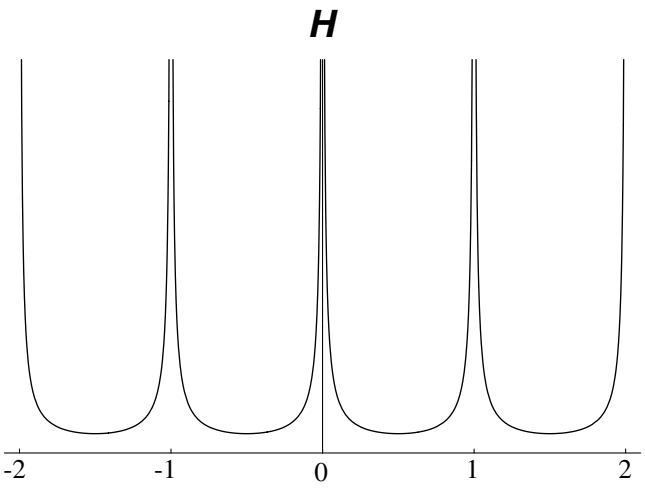

FIG. 2. The periodic function $H(\nu)$ plotted versus ν.

In the positive region $0 \le \nu \le 1$ we can use that $\psi(\nu) + 1/\nu = \psi(1+\nu)$ and Eq. (52) becomes

$$h(\nu) = 2(\log\mu - \gamma_{\rm E}) - \psi(\nu) - \psi(1-\nu) \equiv 2\log\mu + H(\nu), \tag{53}$$

$$H(\nu) = -\psi(\nu) - \psi(1-\nu) - 2\gamma_{\rm E}. \tag{54}$$

Inside the interval $0 \le \nu \le 1$ the function is symmetric with respect to the replacement $\nu \to 1-\nu$. Outside this interval the function is continued by periodicity with period 1. We plot the cutoff-independent function $H(\nu)$ in Fig. 2. We would like to stress that the term $1/\nu$ is always the (sole) contribution of the zero Matsubara frequency ω_0.

If we keep all the ν_n either between -1 and 0 or between 0 and 1 we find the following functional form for the coefficients in the parallel sector:

$$\lambda_3(\nu) = \frac{11}{192\pi^2 T}[4h(\nu_1) + h(\nu_2) + h(\nu_3)], \tag{55}$$

$$\lambda_8(\nu) = \frac{11}{64\pi^2 T}[h(\nu_2) + h(\nu_3)], \tag{56}$$

$$\lambda_{38}(\nu) = \frac{11}{32\sqrt{3}\pi^2 T}[h(\nu_2) - h(\nu_3)]. \tag{57}$$

They are periodic functions in the eigenvalues of $A_4^{\rm adj}$.

Similar to the case of SU(2) discussed in [13], the periodicity in the parallel sector comes from the fact that the electric field components $E_i^{3,8} = \partial_i A_4^{3,8} + f^{(3,8)c(3,8)} A_i^c A_4^{3,8} = \partial_i A_4^{3,8}$ do not depend on A_i and are by themselves invariant under special time-dependent gauge transformation (8) leading to the shift of $A_4^{3,8}$ by integers. Since the action has to be invariant under (8), it can only happen if the coefficient functions $\lambda_{3,8,38}(\nu)$ are periodic functions of ν_m which indeed they are owing to the periodicity of the functions $h(\nu)$. The rest electric field components necessarily contain the A_i fields which become fast time dependent under the transformation (8), see Eqs. (11)–(16). The full electric field is $E_i^a = D_i^{ab} A_4^b - \dot{A}_i^a$. Therefore, the transformation (8) changes the components $E_i^{1,2,4,5,6,7}$, and the invariance under (8) does not request that the corresponding coefficient functions $\lambda_{1,4,6}(\nu)$ are periodic. The Z(3) symmetry in the "transverse" electric sector has to take place only when one collects all powers of $\dot{A}_i$.

All functions $\lambda_1, \ldots, \lambda_8$ (but not the cross term λ_{38}) contain the same UV divergent contribution

$$\frac{11}{16\pi^2 T}\log\mu. \tag{58}$$

The parameter μ is the UV-cutoff in divergent series:

$$\sum_{k=1}^{\infty}\frac{1}{k} \to \sum_{k=1}^{\mu}\frac{1}{k} \equiv \log\mu, \tag{59}$$

and is related to the Pauli-Villars mass as

$$\mu = \frac{M}{4\pi T}e^{\gamma_{\rm E}}. \tag{60}$$

This subtraction scale for the running coupling constant has been known previously [19] and was also obtained in [13].

The divergence (58) is necessary to cancel the UV divergence from the running coupling constant in the tree-level action:

$$-\frac{F_{\mu\nu}^2}{4g^2(M)} = -\frac{F_{\mu\nu}^2}{8\pi^2}\log\frac{M}{\Lambda}\frac{11}{12}N_c = -(E_i^2 + B_i^2)\frac{11}{16\pi^2}\log\frac{M}{\Lambda}. \tag{61}$$

If we add the tree-level and the 1-loop action then the result should be UV finite. This is obtained by choosing the scale M in Eq. (61) to be equal to the Pauli-Villars mass, which corresponds to the evaluation of the running coupling constant at the scale $4\pi T/\exp(\gamma_{\rm E})$. In the effective action we then have to replace the Pauli-Villars cutoff M by Λ in all our functions λ_i from Eq. (42).

VI. GAUGE-INVARIANT STRUCTURES

Written in the form above the 1-loop action Eq. (42) does not look covariant. This is, however, a consequence of the fact that we chose A_4 to be diagonal. It is easy to show that Eq. (42) can be rewritten as

$$[S_{1-\rm loop}]_{\rm E} = \int d^3x \sum_{i=1}^{5}\Phi_i f_i(\nu_1, \nu_2, \nu_3) \tag{62}$$

with the following five invariant structures:

$$\Phi_1 = {\rm Tr}[E_i A_4]{\rm Tr}[E_i A_4], \quad \Phi_2 = {\rm Tr}[E_i E_i], \quad \Phi_3 = {\rm Tr}[E_i A_4 E_i A_4], \quad \Phi_4 = {\rm Tr}[E_i A_4^2 E_i A_4^2], \quad \Phi_5 = {\rm Tr}[E_i A_4^3]{\rm Tr}[E_i A_4^3]. \tag{63}$$

All the traces are over matrices in the adjoint representation. For our particular choice (26) for A_4 these structures are composed as follows:

$$\Phi_1 = 4\pi^2T^2[\sqrt{3}E_i^8(\nu_2-\nu_3)+E_i^3(2\nu_1+\nu_2+\nu_3)]^2,$$
$$\Phi_2 = 3[(E_i^aE_i^a)],$$
$$\Phi_3 = 2\pi^2T^2[-2\nu_2\nu_3(E_i^{1^2}+E_i^{2^2})+2\nu_1\nu_3(E_i^{4^2}+E_i^{5^2})+2\nu_1\nu_2(E_i^{6^2}+E_i^{7^2})+E_i^{3^2}(4\nu_1^2+\nu_2^2+\nu_3^2)+3E_i^{8^2}(\nu_2^2+\nu_3^2)$$
$$+2\sqrt{3}E_i^3E_i^8(\nu_2-\nu_3)], \quad (64)$$
$$\Phi_4 = 8\pi^4T^4[2\nu_2^2\nu_3^2(E_i^{1^2}+E_i^{2^2})+2\nu_1^2\nu_3^2(E_i^{4^2}+E_i^{5^2})+2\nu_1^2\nu_2^2(E_i^{6^2}+E_i^{7^2})+E_i^{3^2}(4\nu_1^4+\nu_2^4+\nu_3^4)+3E_i^{8^2}(\nu_2^4+\nu_3^4)$$
$$+2\sqrt{3}E_i^3E_i^8(\nu_2^4-\nu_3^4)],$$
$$\Phi_5 = 64\pi^6T^6[\sqrt{3}E_i^8(\nu_2^3-\nu_3^3)+E_i^3(2\nu_1^3+\nu_2^3+\nu_3^3)]^2.$$

Since the coefficients f_i in front of these structures in Eq. (62), which reproduce Eq. (42), are rather lengthy we do not write them down explicitly here. But it is easy to reconstruct them with Eq. (64). We would like to stress, however, that the UV logarithm is now only contained in the coefficient f_2 in front of Φ_2. This is expected, since only that structure shows up in the classical action Eq. (61).

VII. THE MAGNETIC SECTOR

To get the magnetic fields in the effective action we have to expand Eqs. (21) and (23) to the quartic order in D_i. The basic idea of the calculation is, again, to use master Eqs. (36) to drag covariant derivatives D_i to the right. What enters in the expansion is

$$D_ie^{s\mathcal{A}^2} = e^{s\mathcal{A}^2}D_i - is\int_0^1 d\delta e^{\delta s\mathcal{A}^2}\{\mathcal{A}, E_i\}e^{(1-\delta)s\mathcal{A}^2},$$
$$D_iD_je^{s\mathcal{A}^2} = e^{s\mathcal{A}^2}D_iD_j - is\int_0^1 d\delta e^{\delta s\mathcal{A}^2}[D_iD_j, \mathcal{A}^2]e^{(1-\delta)s\mathcal{A}^2}, \quad (65)$$

where

$$[D_iD_j, \mathcal{A}^2] = -iD_i\{\mathcal{A}, E_j\} - i\{\mathcal{A}, E_i\}D_j.$$

In this way one ultimately obtains gauge-invariant combinations of the electric field in the fourth power, mixed terms containing both electric and magnetic fields, derivatives of the electric field and, finally, magnetic field squared. In this paper we restrict ourselves to the latter terms quadratic in the magnetic field B_i. This means that we treat D_i and A_4 as commuting operators and neglect the commutators $[D_i, A_4]$. Moreover, unlike for the electric field, we would like to restrict ourselves to a magnetic field parallel to A_4, i.e., $B_i = B_{i\|}$. In that case we see from the Bianchi identity $[F_{ij}, A_4] = i([D_i, E_j] - [D_j, E_i])$ that we can neglect derivatives of the electric field as well. Hence we can also treat D_i and F_{ij} as commuting operators.

As shown in [13] there is only one structure that contributes to quadratic order in the magnetic field:

$$[S^{(2)}_{1-\mathrm{loop}}]_{\mathrm{M}} = [\log(\det W)^{-1/2}_{\mathrm{r,n}} + \log\det(-D^2)_{\mathrm{r,n}}]^{(2)}_{\mathrm{M}}$$
$$= \frac{11}{48\pi^{3/2}}\int d^3x\sum_{k=-\infty}^{\infty}\int_0^\infty\frac{ds}{\sqrt{s}}\mathrm{Tr}(1-e^{-sM^2})$$
$$\times(e^{s\mathcal{A}^2}B_kB_k). \quad (66)$$

Equation (66) is so far independent of the gauge group. For SU(3) we have the matrix $\mathcal{A}$ given by (31), and the magnetic field is in the adjoint representation, $B_i^{ab} = if^{acb}B_i^c$ where B_i^c is as in Eq. (24). After the integrations over s and p and the summation over the Matsubara frequencies ω_k we find the following result:

$$[S_{1-\mathrm{loop}}]_{\mathrm{M}} = \int d^3x\{(B_i^3)^2\kappa_3(\nu)+(B_i^8)^2\kappa_8(\nu) + B_i^3B_i^8\kappa_{38}(\nu)\}. \quad (67)$$

Again the coefficients κ_i are functions of the rescaled eigenvalues of A_4^{adj}, $\nu = \nu_1, \nu_2, \nu_3$, and their functional form will depend on the region of definition of them. As before we chose $0 \le |\phi_i| \le 2\pi T$. The fact that the result (67) looks noninvariant is again a consequence of the diagonal form of A_4. However, Eq. (67) can be rewritten in terms of the invariants (63) with an obvious replacement $E_i^{3,8} \to B_i^{3,8}$ and $E_i^{1,4,6} \to 0$. All the functions κ_j are linear combinations of the function h, defined in Eq. (52), with arguments $\nu_{1,2,3}$:

$$\kappa_3(\nu) = \frac{11}{192\pi^2T}[4h(\nu_1)+h(\nu_2)+h(\nu_3)] = \lambda_3(\nu),$$
$$\kappa_8(\nu) = \frac{33}{192\pi^2T}[h(\nu_2)+h(\nu_3)] = \lambda_8(\nu), \quad (68)$$
$$\kappa_{38}(\nu) = \frac{11}{32\sqrt{3}\pi^2T}[h(\nu_2)-h(\nu_3)] = \lambda_{38}(\nu).$$

All of these functions are symmetric in the rescaled eigenvalues ν of A_4^{adj}, which corresponds to the equivalence of the Z(3)-symmetric points. The reason for this is similar to the one we discussed for the parallel electric sector. Since the magnetic field

$$B_i^a = \frac{1}{2}\epsilon_{ijk}(\partial_jA_k^a - \partial_kA_j^a + \epsilon^{abc}A_j^bA_k^c) \quad (69)$$

does not have any explicit time dependence, a fast time-dependent gauge transformation does not affect B_i^2, nei-

ther for static nor for time-dependent A_i fields. The invariance of the action with respect to the special time-dependent gauge transformation (8) requires then that Eq. (67) is Z(3)-symmetric, which amounts to the periodicity of all functions $\kappa(\nu)$. Indeed, they are all periodic.

The functions κ_3, κ_8 have the same UV divergent contribution

$$\frac{11}{16\pi^2 T}\log\mu, \tag{70}$$

while the cross term κ_{38} is UV finite. This divergence removes the one from the tree-level action, Eq. (61). Adding Eq. (61) and Eq. (67) corresponds to the evaluation of the running coupling constant at the scale $4\pi T/\exp(\gamma_{\rm E})$. In the effective action we then have to replace the Pauli-Villars cutoff M by Λ in all our functions κ_i from Eq. (67).

VIII. INVARIANCE UNDER PERMUTATIONS OF COLOR AXES

An important check of the results (42) and (67) is their invariance under color rotations corresponding to the permutation of color axes. We can, for example, rotate from the color-1 axes to the color-4 axes, or from the color-4 axes to the color-6 axes. This corresponds to finding the matrices S and U such that:

$$S^\dagger \lambda^1 S = \lambda^4, \tag{71}$$

$$U^\dagger \lambda^4 U = \lambda^6. \tag{72}$$

They are given by

$$S = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \qquad U = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{73}$$

We shall show explicitly how this works for the first of these two rotations, namely, the one with S around λ^1. We chose A_4 to be diagonal in the fundamental representation, Eq. (26),

$$A_4 = A_4^3 t^3 + A_4^8 t^8. \tag{74}$$

Under the rotation $S^\dagger A_4 S$ we find the transformations

$$A_4^3 \to \frac{A_4^3 + \sqrt{3}A_4^8}{2} \equiv \tilde{A}_4^3, \qquad A_4^8 \to \frac{3A_4^3 - \sqrt{3}A_4^8}{2\sqrt{3}} \equiv \tilde{A}_4^8. \tag{75}$$

The electric and magnetic fields transform with $S^{ab} = \frac{1}{2}\mathrm{Tr}(S^\dagger \lambda^a S \lambda^b)$ to

$$S^{ab}E_i^b \equiv \tilde{E}_i^a = \left(E_i^4, E_i^5, \frac{E_i^3 + \sqrt{3}E_i^8}{2}, -E_i^1, -E_i^2, -E_i^6, E_i^7, \frac{\sqrt{3}E_i^3 - E_i^8}{2}\right), \tag{76}$$

$$S^{ab}B_i^b \equiv \tilde{B}_i^a = \left(0, 0, \frac{B_i^3 + \sqrt{3}B_i^8}{2}, 0, 0, 0, 0, \frac{\sqrt{3}B_i^3 - B_i^8}{2}\right). \tag{77}$$

We hence see that the structures orthogonal and parallel to A_4 transform separately. We shall start by checking the results in the electric sector. For the orthogonal part we have to show that

$$\begin{aligned} \lambda_1(A_4^3, A_4^8) &= \lambda_4(\tilde{A}_4^3, \tilde{A}_4^8), \\ \lambda_4(A_4^3, A_4^8) &= \lambda_1(\tilde{A}_4^3, \tilde{A}_4^8), \\ \lambda_6(A_4^3, A_4^8) &= \lambda_6(\tilde{A}_4^3, \tilde{A}_4^8), \end{aligned} \tag{78}$$

and for the parallel part the requirement is

$$\begin{aligned} &E_i^{3^2}\lambda_3(A_4^3, A_4^8) + E_i^{8^2}\lambda_8(A_4^3, A_4^8) + E_i^3 E_i^8 \lambda_{38}(A_4^3, A_4^8) \\ &\quad = \tilde{E}_i^{3^2}\lambda_3(\tilde{A}_4^3, \tilde{A}_4^8) + \tilde{E}_i^{8^2}\lambda_8(\tilde{A}_4^3, \tilde{A}_4^8) + \tilde{E}_i^3 \tilde{E}_i^8 \lambda_{38}(\tilde{A}_4^3, \tilde{A}_4^8). \end{aligned} \tag{79}$$

There is one additional subtlety, however. Here we expressed the functions through A_4^3, A_4^8, instead of through the eigenvalues ν_i. We mentioned before that the functional form of the λ_i depends on the interval of definition for the individual ν_i. So if we compare functions of A_4^3, A_4^8 to functions of $\tilde{A}_4^3$, $\tilde{A}_4^8$, we have to classify these functions according to their support in ν_i, i.e., the ν_i's that correspond to A_4^3, A_4^8 can lie in different intervals than the ν_i's corresponding to $\tilde{A}_4^3$, $\tilde{A}_4^8$. For a comparison we then have to choose different functional forms of the λ_i.

It turns out, however, that as long as $A_4^3/(2\pi T)$ and $\sqrt{3}A_4^8/(2\pi T)$ are between -1 and 1 (then also the ν_i are in that region) we can check the contribution of the zero Matsubara mode and of the nonzero modes separately.

We start by checking the contribution of the zero Matsubara mode in the orthogonal electric sector, Eqs. (47)–(49). It is independent of the region of definition of $A_4^3/(2\pi T)$ and $A_4^8/(2\pi T)$, since it is not rescaled. First we express it in terms of A_4^3 and A_4^8, then we make the rotation to $\tilde{A}_4^3$, $\tilde{A}_4^8$ according to Eq. (75). And we find indeed that $\lambda_{10}(A_4^3, A_4^8) = \lambda_{40}(\tilde{A}_4^3, \tilde{A}_4^8)$, $\lambda_{40}(A_4^3, A_4^8) = \lambda_{10}(\tilde{A}_4^3, \tilde{A}_4^8)$, and $\lambda_{60}(A_4^3, A_4^8) = \lambda_{60}(\tilde{A}_4^3, \tilde{A}_4^8)$. The λ_i' in Eqs. (43)–(45) have the same functional form for individual ν_i in the positive or negative region. Therefore, we first express them in terms of $A_4^3/(2\pi T)$ and $A_4^8/(2\pi T)$ and then make the rotation Eq. (75). As expected we find that $\lambda_1'(A_4^3, A_4^8) = \lambda_4'(\tilde{A}_4^3, \tilde{A}_4^8)$, $\lambda_4'(A_4^3, A_4^8) = \lambda_1'(\tilde{A}_4^3, \tilde{A}_4^8)$, and $\lambda_6'(A_4^3, A_4^8) = \lambda_6'(\tilde{A}_4^3, \tilde{A}_4^8)$. Therefore, the electric sec-

tor orthogonal to A_4 is indeed invariant under color rotations.

The functional form of Eqs. (55)–(57) for the electric field parallel to A_4 depends on the region of definition of each of the three ν_n. It is again the contribution of the zero Matsubara frequency which is responsible for that. It comes in as the $1/|\nu|$ term in the definition of the function $h(\nu)$ in Eq. (52). If $0 \leq \nu \leq 1$ then one has $+1/\nu$, while it becomes $-1/\nu$ if $-1 \leq \nu \leq 0$. As discussed above, however, the color rotation Eq. (75) can rotate individual ν_n from a positive region of definition into the negative one, and vice versa. This problem can be circumvented, however, by replacing the argument of the function $h(\nu_n)$ in the coefficients $\lambda_{3,8,38}(\nu)$ of Eqs. (55)–(57) by the fractional part of ν_n. Then it is easy to check that the invariance requirement Eq. (79) is fulfilled for each value of A_4^3 and $\sqrt{3}A_4^8$ between $-2\pi T$ and $2\pi T$, where the functions $h(\nu_n)$ are defined.

What remains is to check the results for the magnetic sector of the theory, Eqs. (67) and (68). Since the invariance requirements Eqs. (78) and (79) remain the same in the electric sector if we replace $\lambda_i \to \kappa_i$ and $E_i \to B_i$ we have already shown the invariance of the magnetic sector parallel to A_4, since according to Eq. (68) $\kappa_{3,8,38} = \lambda_{3,8,38}$.

Similarly one checks the invariance under other color rotations, for example, the one with U in (73).

IX. THE "EQUATION OF MOTION" TERM

We finally return to the second term in the invariant I_2, Eq. (40). In the derivation of our results in the previous sections we ignored this term since it vanishes if the gluon fields obey the equation of motion. Its contribution to the effective action is

$$S_{\mathrm{EoM}} = \int d^3x \sum_{k=-\infty}^{\infty} \int \frac{d^3p}{(2\pi)^3} \int_0^\infty \frac{ds}{s} e^{-sp^2} \times \left[-s^2 \left(\frac{1}{2} - \frac{2}{9} sp^2 \right) \mathrm{Tr} e^{s\mathcal{A}^2} (i\{\mathcal{A}, [D_i, E_i]\}) \right], \quad (80)$$

which after integrations and summation yields

$$S_{\mathrm{EoM}} = \int d^3x \frac{1}{8\pi} \left[\frac{(D_i E_i)^a A_4^a}{\pi T} - \frac{(D_i E_i)^b A_4^b}{\sqrt{A_4^c A_4^c}} \right]. \quad (81)$$

Here the first term comes exclusively from the nonzero Matsubara frequencies, while the second term is the contribution of the zero Matsubara frequency alone. The result is similar to the one that we obtained for the case of SU(2). The only difference is the prefactor: while it is $1/8$ here, it was $1/12$ in [13]. Equation (81) is zero if the classical equation of motion is satisfied. If the background field does not satisfy the equation of motion one can integrate Eq. (81) by parts which yields:

$$S_{\mathrm{EoM}} = \frac{1}{8\pi^2} \int d^3x \left\{ E_i^a E_i^a \left(\frac{\pi}{|A_4|} - \frac{1}{T} \right) + \frac{\pi}{|A_4|^3} (E_i^a A_4^a) \times (E_i^b A_4^b) - \partial_i \left[E_i^a A_4^a \left(\frac{\pi}{|A_4|} - \frac{1}{T} \right) \right] \right\}. \quad (82)$$

The last term here is a full derivative, and the remaining two have to be added to our results found before in the electric sector.

X. COMPARISON WITH PREVIOUS WORK

In a related work by Chapman [20] an effective action for the static modes in a pure Yang-Mills theory was obtained by means of a covariant derivative expansion. While we kept all orders of A_4 in the present work the author of [20] expands to quadratic order in the A_4 fields. For a comparison we hence have to expand our functions to quadratic order in A_4. The result for the electric sector is

$$\frac{11\zeta(3)}{768\pi^2 T} \Big\{ 2 \frac{A_4^3 A_4^8}{(2\pi T)^2} \Big[-E_i^{42} - E_i^{52} + E_i^{62} + E_i^{72} + 8\sqrt{3} E_i^3 E_i^8 \Big] + \frac{(A_4^3)^2}{(2\pi T)^2} [8(E_i^{1^2} + E_i^{2^2}) + 36 E_i^{3^2} + 12 E_i^{8^2} + 11(E_i^{4^2} + E_i^{5^2} + E_i^{6^2} + E_i^{7^2})] + \frac{(A_4^8)^2}{(2\pi T)^2} [4(E_i^{1^2} + E_i^{2^2} + E_i^{3^2}) + 12 E_i^{8^2} + 3(E_i^{4^2} + E_i^{5^2} + E_i^{6^2} + E_i^{7^2})] \Big\}, \quad (83)$$

while in the magnetic sector we find

$$\frac{11\zeta(3)}{384\pi^2 T} \Big\{ 8\sqrt{3} \frac{A_4^3 A_4^8}{(2\pi T)^2} B_i^3 B_i^8 + \frac{(A_4^3)^2}{(2\pi T)^2} [18 B_i^{3^2} + 6 B_i^{8^2}] + \frac{(A_4^8)^2}{(2\pi T)^2} [2 B_i^{3^2} + 6 B_i^{8^2}] \Big\}. \quad (84)$$

This agrees with the results found in [20] if the gauge group is chosen to be SU(3) and the magnetic field is restricted to be parallel to A_4.

While this paper was in preparation, Ref. [21] appeared on the web, where a similar expansion in covariant derivatives of the effective action at high temperatures was performed, see also [22]. Many of our results coincide with theirs: in the electric sector the terms in the effective action where the electric field is parallel to the A_4 field, i.e., our functions $\lambda_{3,8,38}$, and our results in the (parallel) magnetic sector, $\kappa_{3,8,38}$. While we consider a parallel magnetic field, the authors of [21] have a general result. The corresponding functions in Ref. [21] are periodic in $A_4^{3,8}$, but get an infinite contribution from the zero Matsubara frequencies.

XI. EFFECTIVE ACTION FOR THE EIGENVALUES OF THE POLYAKOV LINE

A. SU(3) case

In the previous sections, we have computed the 1-loop effective action for the Polyakov line $L(x) = \exp[iA_4(x)/T]$ as a slowly varying element of SU(3). Since $L(x)$ rotates under spatial gauge transformations (6) one has to introduce a nonzero A_i background field in order to maintain gauge invariance of the effective action under spatial gauge transformations, and expand it in covariant derivatives D_i. If, however, only time-independent A_i's are introduced, as it is required by static gauge invariance, the Z(3) symmetry of the effective action is generally lost, as anticipated in section II and checked in section V. In order to restore the Z(3) symmetry of the effective action, one needs to sum up all time derivatives of A_i, since the Z(3) symmetry is a consequence of the invariance under fast time-dependent gauge transformations (8). This seems to be a formidable task which lies beyond the scope of this study; therefore we have to restrict ourselves to purely static A_4, A_i fields at the cost that the Z(3) symmetry of our effective action is not manifest.

One can, however, be interested in another problem, namely, in the effective quantum action for the *eigenvalues* of the Polyakov line. Contrary to A_4 and to the Polyakov line as a unitary matrix, its eigenvalues *are invariant* under spatial gauge transformations. Therefore, the effective action for the eigenvalues can be expanded in ordinary short rather than covariant derivatives, and the background A_i field may then be set to zero. In this setting, the A_i fields, as well as the rapidly changing components of the A_4 fields, are understood and treated as quantum fluctuations over which one integrates.

Fortunately, one does not need to solve this problem anew: in fact the result can be obtained from the more general case considered above. Assuming the gauge where A_4 is static and diagonal (such that the eigenvalues of the Polyakov line are given by Eq. (29) expressed through $\nu_{1,2,3}$) all one practically needs to do is to set A_i to zero, which means putting to zero all components of the magnetic field B_i and the nondiagonal components of the electric field $E_i^{1,2,4,5,6,7}$. What is left are the $E_i^{3,8}$ components of the electric field whose definition is $E_i^{3,8} = \partial_i A_4^{3,8} + f^{(3,8)c(3,8)} A_i^c A_4^{3,8} = \partial_i A_4^{3,8}$, i.e., they are independent of A_i. One can easily check that these components correspond to combinations of the invariants (63) and (64) in which A_i is canceled, even if one does not assume A_4 to be diagonal. We remark that our coefficient functions of A_4 in front of $E_i^{3,8}$ coincide with those computed recently in Ref. [21].

In terms of the quantities $\nu_{1,2,3}(x)$ the diagonal components of the electric field are, according to Eq. (35),

$$E_i^3 = \partial_i \nu_1 2\pi T, \qquad E_i^8 = \frac{\partial_i(\nu_2 - \nu_3)}{\sqrt{3}} 2\pi T. \tag{85}$$

The last line in Eq. (42) gives the needed two-derivative terms in the gauge-invariant effective action for the eigenvalues of the Polyakov line. Assembling terms proportional to $h(\nu_1), h(\nu_2), h(\nu_3)$ and using $\nu_1 = \nu_2 + \nu_3$ we get a remarkably symmetric expression:

$$S^{\rm kin} = \int d^3x T \frac{11}{12} \Big\{ -2 \log\Big(\frac{4\pi T}{\Lambda e^{\gamma_E}}\Big) \times [(\partial_i \nu_1)^2 + (\partial_i \nu_2)^2 + (\partial_i \nu_3)^2] + [(\partial_i \nu_1)^2 H(\nu_1) + (\partial_i \nu_2)^2 H(\nu_2) + (\partial_i \nu_3)^2 H(\nu_3)] \Big\}, \tag{86}$$

where

$$H(\nu) = [-\psi(\nu) - \psi(1-\nu) - 2\gamma_E]_{\rm mod 1}, \tag{87}$$

and the eigenvalues of the Polyakov loop in terms of $\nu_{1,2,3}(x)$ are given by Eq. (29). Since the Z(3) symmetry consists in shifting $\nu_{1,2,3}$ (i.e., the eigenvalues of A_4 in the adjoint representation) by integers, and the function $H(\nu)$ is periodic with period 1, this action is explicitly Z(3) symmetric. It should be reminded that $\nu_{1,2,3}$ are not independent but satisfy $\nu_1 = \nu_2 + \nu_3$, see Eq. (35). The renormalization has been performed in the Pauli-Villars scheme. If another, e.g., the Modified Minimal Subtraction ($\overline{\rm MS}$) scheme is used, one has to replace Λ by $e^{1/22}\Lambda_{\overline{\rm MS}}$ [23].

The function $H(\nu)$ (which appears also in the SU(2) case [13,17]) is plotted in Fig. 2. It goes as $\frac{1}{\nu}$ or as $\frac{1}{1-\nu}$ as ν approaches 0 or 1, respectively. The singularity is due to the contribution of the zero Matsubara frequency, i.e., of the static quantum fluctuations. One may wish to disregard this contribution but then the periodicity of $H(\nu)$ is lost, together with the Z(3) symmetry of the action.

Equation (86) together with Eq. (34) for the potential energy is the SU(3) effective action for the eigenvalues of the Polyakov line (29), up to two spatial derivatives in the 1-loop order. It is interesting that both the "potential" and "kinetic" energy parts are sums over eigenvalues of A_4 in the adjoint representation.

B. General SU(N) case

As before, it is possible and convenient to choose the gauge such that A_4 is static and diagonal in the fundamental representation:

$$A_4(x) = A_4^a t^a = 2\pi T \,{\rm diag}(a_1, a_2, ..., a_N), \quad a_1 + a_2 + ... + a_N = 0, \tag{88}$$

where the common factor $2\pi T$ is taken for future convenience. The Polyakov line is then a diagonal SU(N) unitary matrix,

$$L(x) = {\rm diag}(e^{2\pi i a_1}, e^{2\pi i a_2}, ..., e^{2\pi i a_N}). \tag{89}$$

The effective action must be invariant under time-dependent gauge transformations, generalizing Eq. (8):

$$A_4 \to SA_4S^\dagger + iS\partial_t S^\dagger, \tag{90}$$

$$S = \mathrm{diag}(e^{i[2\pi tT\mu_1+\alpha_1(x)]}, \ldots, e^{i[2\pi tT\mu_N+\alpha_N(x)]}), \tag{91}$$

$$\sum_{m=1}^{N}\mu_m = 0, \qquad \sum_{m=1}^{N}\alpha_m(x) = 0. \tag{92}$$

This gauge transformation amounts to shifting $a_m \to a_m + \mu_m$. The time frequencies μ_m are quantized because this gauge transformation written down for the adjoint representation must be periodic in time, meaning that $O^{ac}(t=0)O^{bc}(t=1/T) = \delta^{ab}$ where $O^{ab} = 2\mathrm{Tr}(St^aS^\dagger t^b)$, $\mathrm{Tr}t^at^b = \frac{1}{2}\delta^{ab}$. This periodicity in time leads to the requirement that all differences

$$\mu_m - \mu_n = \text{integers}. \tag{93}$$

Given the condition (92), the general solution to Eq. (93) is

$$\mu_m = \frac{k_0}{N} + k_m, \qquad \sum_{m=0}^{N} k_m = 0, \tag{94}$$

where $k_0, k_1, ..., k_N$ are all integer numbers. Under the gauge transformation (91) the Polyakov line is multiplied by a diagonal matrix

$$S^\dagger(0)S(1/T) = \mathrm{diag}(e^{2\pi i\mu_1}, \ldots, e^{2\pi i\mu_N}) \in \mathrm{Z}(N), \tag{95}$$

which belongs to the group center when μ's are given by Eq. (94).

It follows from the general formulae of section II that the effective action is a functional of the eigenvalues of A_4 in the *adjoint* representation. If A_4 in the fundamental representation is diagonal and given by Eq. (88), the eigenvalues of the adjoint $(N^2-1)\times(N^2-1)$ matrix $A^{ab}_{4\mathrm{adj}} = if^{acb}A^c_4$ are

$$\text{adjoint eigenvalues} = \pm 2\pi T(a_m - a_n) \equiv \pm 2\pi T\nu_{mn}. \tag{96}$$

There are $(N-1)$ zero eigenvalues whose number is the rank of the group, and $N(N-1)$ pairwise ($\pm$) nonzero eigenvalues. Under gauge transformation (91) the adjoint eigenvalues apparently shift by integers,

$$\nu_{mn} = (a_m - a_n) \to \nu_{mn} + (\mu_m - \mu_n) = \nu_{mn} + \text{integers} \tag{97}$$

owing to Eq. (93). Therefore, the Z(N) symmetry of the effective action is manifest if it is periodic in the adjoint eigenvalues ν_{mn} with period 1.

With A_i set to zero, the calculation of the invariants (39)–(41) simplifies. Basically, they are sums of functionals of the adjoint eigenvalues ν_{mn}. The tree-level kinetic energy $E^a_iE^a_i/2 = \mathrm{Tr}(\partial_iA_4)^2 = (2\pi T)^2\times\sum_{m=1}^{N}(\partial_ia_m)^2$ can be also written as a sum over adjoint eigenvalues, using the identity

$$\sum_{m=1}^{N}(\partial_ia_m)^2 = \frac{1}{N}\sum_{m<n}^{N}[\partial_i(a_m - a_n)]^2, \tag{98}$$

which is satisfied when $\sum_1^N a_n = 0$ is taken into account. We thus obtain the following 1-loop effective action for the eigenvalues of the Polyakov loop in a general SU(N) theory:

$$S = -\sum_{m>n}^{N}\int d^3x \times \left\{(\partial_i\nu_{mn})^2\frac{11}{12}T\left[2\log\left(\frac{4\pi T}{\Lambda e^{\gamma_E}}\right) - H(\nu_{mn})\right] + \frac{(2\pi)^2T^3}{3}\nu^2_{mn}(1-\nu_{mn})^2\right\}, \tag{99}$$

$$\nu_{mn} = a_m - a_n,$$

where the function $H(\nu)$ is given by Eq. (87) and plotted in Fig. 2; it is periodic with period 1. There are $N(N-1)/2$ similar terms in Eq. (99), however it should be kept in mind that there are only $N-1$ independent variables $a_m(x)$ through which the eigenvalues of the Polyakov line are expressed. At $N=3$ there are three terms and the general result (99) comes to Eq. (86). At $N=2$ there is only one term, and the result coincides with that of Refs. [13,17]. The minima of the potential energy lie at integer values of ν_{mn} corresponding to the Polyakov line belonging to the group center. The maxima correspond to the Polyakov line $L_{\max} = \mathrm{diag}(e^{i\pi(N-1/N)}, e^{i\pi(N-3/N)}, \ldots, e^{-i\pi(N-1/N)})$ (and those which one gets from this one by multiplying it by elements of the center), and it has the property that $\mathrm{Tr}L_{\max} = 0$. It is interesting that $H(\nu)$ is positive definite, so that the kinetic energy changes sign at certain values of ν depending on the temperature. It may signal an instability of the trivial (perturbative) vacua, as one lowers the temperature.

XII. CONCLUSIONS

We derived an effective 1-loop action for the Polyakov line in a pure Yang-Mills theory in two settings: (1) for the Polyakov line as a unitary matrix rotating under spatial gauge transformations, (2) for the gauge-invariant eigenvalues of the Polyakov line. We do not assume that the Polyakov line is close to the elements of the center, meaning that we are collecting all powers of the A_4 field in the effective action while expanding in its (covariant) derivatives.

In case 2 the effective action is an expansion in ordinary spatial derivatives of the Polyakov line eigenvalues, and is explicitly symmetric under discrete Z(N) transformations of the eigenvalues. In case 1 the effective action is an expansion in covariant derivatives of A_4, which necessarily include the spatial components A_i of the background field. We expand the action to include all invariants of the type E_i^2 where E_i is the electric field and of the type $B^2_{i\|}$ where B_i is the magnetic field parallel, in the SU(3) sense, to A_4. We have checked our results by

reducing them to the previously studied SU(2) case, and by comparing them with the previously known expansion up to quadratic terms in A_4. We stress, however, that we collect all powers of A_4 in our effective action.

In case 1 our results are symmetric with respect to the group center only in part of the gauge invariants we compute but not in all of them. The reason is that the Z(N) symmetry is actually a consequence of the invariance under fast time-dependent gauge transformations. Once A_i fields are introduced to ensure gauge invariance of the effective action under static transformations of the Polyakov line, the time-dependent transformations generate large time derivatives of A_i. Unless all powers in $\dot{A}_i$ are collected in the effective action, the Z(N) symmetry is not manifest. Since we expand to the order of E^2 only (hence to the second order in $\dot{A}_i$) we cannot observe the Z(N) symmetry. Our results in case 1 apply to static (A_i, A_4) background fields and to the SU(3) group only.

We hope that the results may be of some help to study correlation functions of the Polyakov line not too far from the transition point where it experiences fluctuations that are large in amplitude but presumably mainly long ranged.

APPENDIX: BASICS ABOUT SU(3)

The generators of the gauge group SU(3) are

$$t^a = \frac{\lambda^a}{2}, \tag{A1}$$

where the matrices λ^a are given by

$$\lambda^1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda^2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$
$$\lambda^3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda^4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$
$$\lambda^5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \quad \lambda^6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \tag{A2}$$
$$\lambda^7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad \lambda^8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}.$$

The generators satisfy

$$[t^a, t^b] = i f^{abc} t^c. \tag{A3}$$

In addition to the totally antisymmetric f^{abc} there are totally symmetric d^{abc} structure constants which are defined according to

TABLE I.

(a, b, c)	f^{abc}	(a, b, c)	d^{abc}
123	1	118	$1/\sqrt{3}$
147	1/2	146	1/2
156	−1/2	157	1/2
246	1/2	228	$1/\sqrt{3}$
257	1/2	247	−1/2
345	1/2	256	1/2
367	−1/2	338	$1/\sqrt{3}$
458	$\sqrt{3}/2$	344	1/2
678	$\sqrt{3}/2$	355	1/2
		366	−1/2
		377	−1/2
		448	$-1/(2\sqrt{3})$
		558	$-1/(2\sqrt{3})$
		668	$-1/(2\sqrt{3})$
		778	$-1/(2\sqrt{3})$
		888	$-1/(\sqrt{3})$

$$\{t^a, t^b\} = \frac{1}{3}\delta^{ab} + d^{abc} t^c. \tag{A4}$$

The nonvanishing values of f^{abc} and d^{abc} are summarized in Table I. The t^a matrices fulfill the following relations

$$t^a_{ij} t^a_{kl} = \frac{1}{2}\left[\delta_{il}\delta_{jk} - \frac{1}{3}\delta_{ij}\delta_{kl}\right], \tag{A5}$$

$$\mathrm{Tr} t^a = 0, \tag{A6}$$

$$\mathrm{Tr}(t^a t^b) = \frac{\delta^{ab}}{2}, \tag{A7}$$

$$\mathrm{Tr}(t^a t^b t^c) = \frac{1}{4}(d^{abc} + i f^{abc}), \tag{A8}$$

$$\mathrm{Tr}(t^a t^b t^a t^c) = -\frac{1}{12}\delta^{bc}. \tag{A9}$$

The structure constants satisfy the Jacobi identities:

$$f^{abe} f^{ecd} + f^{cbe} f^{aed} + f^{dbe} f^{ace} = 0, \tag{A10}$$

$$f^{abe} d^{ecd} + f^{cbe} d^{aed} + f^{dbe} d^{ace} = 0. \tag{A11}$$

Additionally

$$f^{abe} f^{cde} = \frac{2}{3}(\delta^{ac}\delta^{bd} - \delta^{ad}\delta^{bc}) + (d^{ace} d^{bde} - d^{bce} d^{ade}). \tag{A12}$$

By defining the 8×8 matrices F^a and D^a such that

$$(F^a)^{bc} = -i f^{abc}, \tag{A13}$$

$$(D^a)^{bc} = d^{abc}, \tag{A14}$$

the Jacobi identities (A10) and (A11) take the form

$$[F^a, F^b] = if^{abc}F^c, \tag{A15}$$

$$[F^a, D^b] = if^{abc}D^c. \tag{A16}$$

And the fact that $f^{abb} = 0$ and $d^{abb} = 0$ implies

$$\mathrm{Tr}F^a = 0, \qquad \mathrm{Tr}D^a = 0. \tag{A17}$$

Some more useful relationships are:

$$f^{acd}f^{bcd} = 3\delta^{ab}, \tag{A18}$$

$$F^aF^a = 3, \tag{A19}$$

$$\mathrm{Tr}(F^aF^b) = 3\delta^{ab}, \tag{A20}$$

$$f^{acd}d^{bcd} = 0, \tag{A21}$$

$$F^aD^a = 0, \tag{A22}$$

$$\mathrm{Tr}(F^aD^b) = 0, \tag{A23}$$

$$d^{acd}d^{bcd} = \frac{5}{3}\delta^{ab}, \tag{A24}$$

$$D^aD^a = \frac{5}{3}, \tag{A25}$$

$$\mathrm{Tr}(D^aD^b) = \frac{5}{3}\delta^{ab}, \tag{A26}$$

$$\mathrm{Tr}(F^aF^bF^c) = i\frac{3}{2}f^{abc}, \tag{A27}$$

$$\mathrm{Tr}(D^aF^bF^c) = \frac{3}{2}d^{abc}, \tag{A28}$$

$$\mathrm{Tr}(D^aD^bF^c) = i\frac{5}{6}f^{abc}, \tag{A29}$$

$$\mathrm{Tr}(D^aD^bD^c) = -\frac{1}{2}d^{abc}, \tag{A30}$$

$$\mathrm{Tr}(F^aF^bF^aF^c) = \frac{9}{2}\delta^{bc}. \tag{A31}$$

In particular for any matrix in the fundamental representation

$$A = A^at^a \tag{A32}$$

one can construct the adjoint representation according to

$$A^{ab} = if^{acb}A^c. \tag{A33}$$

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