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Gravity from spinors

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We investigate a possible unified theory of all interactions which is based only on fundamental spinor fields. The vielbein and metric arise as composite objects. The effective quantum gravitational theory can lead to a modification of Einstein's equations due to the lack of local Lorentz symmetry. We explore the generalized gravity with global instead of local Lorentz symmetry in first order of a systematic derivative expansion. At this level diffeomorphisms and global Lorentz symmetry allow for two new invariants in the gravitational effective action. The one which arises in the one loop approximation to spinor gravity is consistent with all present tests of general relativity and cosmology. This shows that local Lorentz symmetry is tested only very partially by present observations. In contrast, the second possible new coupling is severely restricted by present solar system observations.

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I. INTRODUCTION

Can a fundamental theory of all interactions be based only on spinors? The fermions as crucial constituents of matter are indeed described by spinor fields. In contrast, the interactions are mediated by bosons which do not have the transformation properties of spinors. Any realistic spinor theory has therefore to account for the bosons as bound states. In principle, this poses no problem since bosons may be composed of an even number of fermions. In a fundamental theory, however, we need bosons with very particular properties: The graviton is connected to the symmetry of general coordinate transformations (diffeomorphisms) and the gauge interactions are mediated by gauge bosons with spin one. Furthermore, scalar fields are needed in order to achieve the spontaneous breaking of the electroweak symmetry and possibly also extended symmetries like grand unified gauge symmetries. This raises[1] a first challenge: Can gravity arise from a spinor field theory?

Several proposals in this direction have discussed "pregeometry" [1] or "metric from matter" [2], inspired by the observation that the matter fluctuations in a gravitational background field can generate a kinetic term for the graviton [3]. While the introduction of a diffeomorphism invariant action for the spinors is rather straightforward [1], the arguments presented in favor of local Lorentz symmetry are less convincing. The main obstacle is the absence of an object transforming as a spin connection that could be constructed as a polynomial of the spinor fields.[2] Concentrating on a well-defined spinor action as a polynomial in the fermionic Grassmann variables, the models that have been proposed thus far exhibit only global instead of local Lorentz invariance [6]. Only very recently[3] a locally Lorentz invariant polynomial spinor action has been found [7].

In this paper, we explore the alternative of a spinor action that respects only the global and not the local Lorentz transformations. Then also the gravitational theory for the vielbein which emerges in this setting will exhibit only a global Lorentz symmetry. The quantum fluctuations will lead to a theory with a massless graviton bound state as well as further massless bosonic excitations which are responsible for a particular form of torsion. We will see that local Lorentz symmetry is actually not required by observation. A new invariant, generated by one loop spinor gravity and violating local Lorentz symmetry, is compatible with all present tests of general relativity.

[1]The old question of a fundamental spinor theory has been discussed in different contexts by De Broglie, Heisenberg, and many others. Here we address more specifically the problem if gravity can arise from spinors.

[2]Elements of the Grassmann algebra are polynomials in the spinor fields and can be classified according to their rank. Local Lorentz transformations do not change the rank of an object. A covariant derivative needs a spin connection which must have rank 0. However, no object of rank 0 with the requested inhomogeneous transformation properties can be constructed from spinor polynomials. Neither the limiting process proposed in [1] nor the covariant derivative suggested in [2] have a well-defined meaning as polynomials in the spinor fields. In particular, we note that the inverse of the "Grassmann matrix" $M_{\alpha\beta} = \psi_\alpha \bar{\psi}_\beta$ does not exist for Grassmann variables with "spinor index" α, β. We note, however, the construction of a supersymmetric action with local Lorentz symmetry in a setting with nonlinear fields, including additional bosonic fields [4]. Other approaches integrate out a bosonic connection [5].

[3]The present work was performed before this finding.

Once the graviton can be associated with a bound state of fermions, the explanation of the other bosonic degrees of freedom could follow a well established road. A higher dimensional gravity theory can induce four-dimensional gauge interactions by "spontaneous compactification" [8,9]. The gauge symmetries are then related to the isometries of "internal space." The number of generations of massless or light fermions is connected to the "chirality index" [10] which depends on the topology and symmetries of internal space. A nonvanishing index requires a vanishing higher dimensional mass term for the fermions [11] and we therefore need an effective higher dimensional theory with massless fermions and gravitons. Both the "constituents" and "bound states" need to be massless. Finally, the higher dimensional metric also contains four-dimensional scalar fields with the properties required for spontaneous symmetry breaking. Rather realistic models with the gauge interactions of the standard model, three chiral generations of quarks and leptons, spontaneous symmetry breaking, and an interesting hierarchical pattern of fermion masses and mixings have been proposed [12] based on 18 dimensional gravity coupled to a Majorana-Weyl spinor.

The requirement of a local polynomial spinor action which is invariant under general coordinate transformations leads to the proposal [13] of "spinor gravity" as a possible fundamental theory of all interactions. In the present accompanying paper, we elaborate on this proposal and put it into a somewhat more general context. In particular, we discuss here the role of the additional "gravitational" degrees of freedom which are due to the lack of local Lorentz symmetry. These massless excitations, which have not been discussed previously, lead to possible modifications of Einstein's gravity on macroscopic scales. Comparison with the present status of observations will reveal that the usual assumption of local Lorentz symmetry is actually very poorly tested. It is possibly to add to Einstein's action a new invariant which preserves global but not local Lorentz symmetry and which is nevertheless consistent with all present observations. In this case, the new massless gravitational degrees of freedom couple only to macroscopic spin. A second paper[4] [14] will discuss the nonlinear geometrical structures of our setting.

Fermion bilinears transforming as vector fields under general coordinate transformations can be obtained from derivatives, $\tilde{E}_\mu^m = i\bar{\psi}\gamma^m\partial_\mu\psi/2$ + H.c. Here $\psi(x)$ denotes Grassmann variables in the spinor representation of the d-dimensional Lorentz group and we have introduced the associated Dirac matrices γ^m such that $\tilde{E}_\mu^m$ is a vector with respect to global Lorentz rotations.[5] From $\tilde{E}_\mu^m$ we can construct a composite operator with the transformation properties of the metric, $\tilde{E}_\mu^m\tilde{E}_{\nu m}$. However, the action has to be a polynomial in the spinors and no object transforming as the inverse metric can be used in order to contract the "lower world indices" connected to the derivatives. The only possible choice for a diffeomorphism invariant action therefore contracts d derivatives with the totally antisymmetric tensor $\epsilon^{\mu_1\cdots\mu_d}$. Invariance under global Lorentz rotations can be achieved similarly by contraction with $\epsilon_{m_1\ldots m_d}$. In consequence, it is indeed possible to construct an invariant action as a local polynomial in the spinor fields and their derivatives

$$S_E = \alpha\int d^dx \det[\tilde{E}_\mu^m(x)], \quad \tilde{E}_\mu^m = \frac{i}{2}\bar{\psi}\gamma^m\partial_\mu\psi + \text{H.c.} \quad (1)$$

We emphasize that we have no spin connection at our disposal. Therefore the bilinear $\tilde{E}_\mu^m$ does *not* transform as a tensor under *local* Lorentz transformations. Instead, its transformation property is characterized by an additional inhomogeneous piece involving the derivative of the Lorentz-transformation parameter.

In consequence, the action S_E is invariant under *global* but not *local* Lorentz transformations. This is an important difference as compared to the standard formulation of gravity ("Einstein gravity"). We will explore both the phenomenological and conceptual aspects of this difference. Actually, the action (1) is not the only invariant with diffeomorphism and global Lorentz symmetry—other invariants are discussed in [14]. We will see that the lack of local Lorentz symmetry leads to a generalized version of gravity.

Within "spinor gravity" the "global vielbein" $E_\mu^m(x)$ can be associated with the expectation value of the fermion bilinear $\tilde{E}_\mu^m(x)$. As usual, the metric obtains then by contraction with the invariant tensor η_{mn} which lowers the Lorentz indices

$$E_\mu^m(x) = \langle\tilde{E}_\mu^m(x)\rangle, \qquad g_{\mu\nu}(x) = E_\mu^m(x)E_{\nu m}(x). \qquad (2)$$

On the level of the composite bosonic fields E_μ^m and $g_{\mu\nu}$, the inverse vielbein and metric $E_m^\mu(x)$, $g^{\mu\nu}(x)$ are well defined provided $E = \det(E_\mu^m) \neq 0$. The field equations for the vielbein and metric can, at least in principle, be computed from Eq. (1) plus an appropriate regularization of the functional measure. This approach realizes the general idea that both geometry and topology can be associated with the properties of appropriate correlation functions [16]—in the present case the two point functions for spinors.

Because of the lack of local Lorentz symmetry, the global vielbein contains additional degrees of freedom that are not described by the metric. Correspondingly, the effective theory of gravity will also exhibit new invariants not present in Einstein gravity. These invariants are consistent with global but not local Lorentz symmetry. Indeed, we may use a nonlinear field decomposition

[4]The content of this second paper was contained in the first version of the present paper.

[5]In three dimensions, a similar object can be used to characterize the order parameter of liquid He^3 [15].

$E_\mu^m(x) = e_\mu^m(x) H_m{}^n(x)$, where e_μ^m describes the usual "local vielbein" and $H_m{}^n$ the additional degrees of freedom. These additional degrees of freedom are massless Goldstone-boson-like excitations due to the spontaneous breaking of a global symmetry. In Einstein gravity, $H_m{}^n$ would be the gauge degrees of freedom of the local Lorentz transformations and therefore drop out of any invariant action. In contrast, the generalized gravity discussed here will lead to new propagating massless gravitational degrees of freedom.

Indeed, the kinetic terms for $H_m{}^n$ can be inferred from the most general effective action for the vielbein which contains two derivatives and is invariant under diffeomorphisms and global Lorentz transformations:

$$\Gamma_{(2)} = \frac{\mu}{2} \int d^d x E\{-R + \tau_A [D^\mu E_m^\nu D_\mu E_\nu^m - 2 D^\mu E_m^\nu D_\nu E_\mu^m] + \beta_A D_\mu E_m^\mu D^\nu E_\nu^m\}. \quad (3)$$

Here the curvature scalar R is constructed from the metric $g_{\mu\nu}$ which is also used to lower and raise world indices in the usual way. The covariant derivative D_μ contains the connection $\Gamma_{\mu\nu}{}^\lambda$ constructed from $g_{\mu\nu}$ but no spin connection. Because of the missing spin connection, the last two terms $\sim \tau_A, \beta_A$ are invariant under global but not local Lorentz transformations. They induce the kinetic term for $H_m{}^n$. The effective action (3), together with a "cosmological constant" term $\sim \int d^d x E$, constitutes the first order in a systematic derivative expansion. In the one loop approximation to spinor gravity, one finds $\beta_A = 0$.

In this paper we discuss the viability of generalized gravity [17] in a setting with only global Lorentz symmetry. For this purpose, we analyze the consequences of the effective action (3) in four dimensions. In complete analogy to Einstein gravity, we discuss the solutions of the field equations derived from the effective action (3) in the presence of suitable sources associated with an energy-momentum tensor. In principle, the energy-momentum tensor contains an antisymmetric part $T_A^{\mu\nu}$ which reflects the presence of anomalous spin interactions for the fermions. These effects are, however, much too small to be observable. Then $T_A^{\mu\nu}$ can be neglected and test particles couple to the metric in the usual way. We find that for $\beta_A = 0$ neither Newtonian gravity nor the isotropic Schwarzschild or the cosmological Friedmann solutions are modified. This also holds for the emission, propagation, and detection of gravitational waves and for all tests of general relativity in post-Newtonian gravity. For vanishing β_A, our generalized gravity is therefore consistent with all present observations of general relativity. We conclude that a violation of local Lorentz symmetry by the invariant $\sim \tau_A$ in Eq. (3) remains unconstrained experimentally. On the other hand, for $\beta_A \neq 0$ we find a modification of the Schwarzschild solution similar to a Jordan-Brans-Dicke theory [18]: Whereas $g_{00} = -B(r)$ behaves as usual as $B(r) = 1 - r_s/r$ (with r_s the Schwarzschild radius), one obtains $g_{rr} = A(r) = (1 - \gamma r_s/r)^{-1}$, where $\gamma \approx 1 + \beta_A$. This imposes a severe bound $|\beta_A| \lesssim 5 \times 10^{-5}$ [19]. In view of this bound, the modifications of cosmology are too small to be presently observable.[6]

This paper is organized as follows. In Sec. II, we recapitulate the transformation properties of spinor fields and bilinears and the construction of the polynomial action (1). The effective bosonic action for fermion bilinears is formulated in Sec. III. This setting describes our version of quantum gravity. In Sec. IV we start a general discussion of gravity theories with only global instead of local Lorentz symmetry. There we classify the possible invariants with up to two derivatives and formulate the effective action in first order in a systematic derivative expansion. In addition to the terms present in Einstein gravity, it contains the two invariants (3) with dimensionless coefficients τ_A and β_A. The corresponding generalized gravitational field equations are derived in Sec. V. In Secs. VI and VII, we discuss the linear approximation to the field equations. Beyond the graviton of Einstein gravity, the spectrum of excitations contains a new set of massless fields described by an antisymmetric tensor field $c_{\mu\nu}$. However, this field does not couple to the symmetric part of the energy-momentum tensor but rather to the antisymmetric part which reflects the internal degrees of freedom of the spinors. We show in Sec. VIII that the new interactions mediated by the exchange of $c_{\mu\nu}$ play no macroscopic role and do not affect the observational effects of linear gravity. We also establish that the invariant $\sim \tau_A$ is compatible with all tests of general relativity in first nonleading order in post-Newtonian gravity.

In the linear approximation, one finds for $\beta_A \neq 0$ also an additional massless vector field w_μ. Again, it couples only to the antisymmetric part of the energy-momentum tensor. More important, a nonvanishing coupling β_A modifies also the linearized equation for the degrees of freedom contained in the metric—more precisely the coupling of the "conformal factor" σ. In the Newtonian approximation this effect only renormalizes Newton's constant. Beyond Newtonian gravity $\beta_A \neq 0$ affects the tests of general relativity.

Going beyond the linear and post-Newtonian approximations, we discuss the modifications of the general isotropic static solution and the homogeneous isotropic cosmological solution for the full field equations. In Sec. IX, we present the generalization of the isotropic static metric for a gravity theory with only global Lorentz symmetry. The corresponding modification of the

[6]Spinor gravity may lead to other long range degrees of freedom not contained in E_μ^m. These could lead to interesting modifications of gravity like quintessence [20,21].

Schwarzschild solution for $\beta_A \neq 0$ is discussed in Sec. X. In Sec. XI we turn to the most general homogeneous and isotropic cosmological solution within our setting of generalized gravity. We find that the solutions of Einstein gravity remain also solutions of generalized gravity as long as $\beta_A = 0$. For $\beta_A \neq 0$, one finds a difference in the value of the Planck mass appearing in the cosmological equations as compared to the one inferred from the Newtonian approximation. In view of the solar system bounds on $|\beta_A|$, this effect is too small in order to be presently observable. For small enough nonvanishing $|\beta_A|$, the generalized gravity with only global Lorentz symmetry obeys all present tests of general relativity. Similar to the Brans-Dicke theory, the model with small nonzero β_A can be used to quantify the experimental precision of general relativity. It is therefore interesting in its own right and merits further quantitative studies in the future—even though spinor gravity may finally result in $\beta_A = 0$.

In Secs. XII and XIII, we make a first attempt to compute the bosonic effective action for spinor gravity. For this purpose we express the fermionic functional integral in terms of a "partially bosonized" functional integral. We briefly explore the classical approximation to the field equations. In Sec. XIII, we discuss the generalized Dirac operator in an arbitrary "background geometry" E^m_μ. This defines the one loop approximation to spinor gravity. One loop spinor gravity involves only the new invariant $\sim\tau_A$ that is not restricted by observation. In our conclusions in Sec. XIV we finally discuss the prospects for spinor gravity as a candidate for a unified theory of all interactions.

II. INVARIANT SPINOR ACTION

Our basic entities are spinor fields $\psi(x)$ which are represented by anticommuting Grassmann variables and transform as irreducible spinor representations under the d-dimensional Lorentz group $SO(1, d-1)$:

$$\delta_{\mathcal{L}}\psi = -\tfrac{1}{2}\epsilon_{mn}\Sigma^{mn}\psi, \qquad \Sigma^{mn} = -\tfrac{1}{4}[\gamma^m, \gamma^n]. \quad (4)$$

Here the Dirac matrices obey $\{\gamma^m, \gamma^n\} = 2\eta^{mn}$, and Lorentz indices are raised and lowered by $\eta^{mn} = \eta_{mn} = \mathrm{diag}(-1, +1, \ldots, +1)$. Under d-dimensional general coordinate transformations, the spinor fields transform as scalars,

$$\delta_\xi \psi = -\xi^\nu \partial_\nu \psi, \quad (5)$$

such that $\partial_\mu\psi$ is a vector. Similarly, the spinor fields $\bar{\psi}(x)$ transform as

$$\delta_{\mathcal{L}}\bar{\psi} = \tfrac{1}{2}\bar{\psi}\epsilon_{mn}\Sigma^{mn}, \qquad \delta_\xi\bar{\psi} = -\xi^\nu\partial_\nu\bar{\psi}. \quad (6)$$

For Majorana spinors in $d = 0, 1, 2, 3, 4\mathrm{mod}8$, one has $\bar{\psi} = \psi^T C$ where C obeys[7] $(\Sigma^T)^{mn} = -C\Sigma^{mn}C^{-1}$. Otherwise $\bar{\psi}$ may be considered as an independent spinor, with an involutive mapping between ψ and $\bar{\psi}$ associated with complex conjugation in spinor space. In even dimensions, the irreducible spinors are Weyl spinors obeying $\bar{\gamma}\psi = \psi$ with $\bar{\gamma} = \eta\gamma^0\ldots\gamma^{d-1}$, $\eta^2 = (-1)^{d/2-1}$, $\bar{\gamma}^2 = 1$, $\bar{\gamma}^\dagger = \bar{\gamma}$. Majorana-Weyl spinors exist for $d = 2\mathrm{mod}8$.

We want to construct an action that is a polynomial in $\psi, \bar{\psi}$ and invariant under global Lorentz transformations and general coordinate transformations. Our basic building block is a spinor bilinear[8]

$$\tilde{E}^m_\mu = \frac{i}{2}(\bar{\psi}\gamma^m\partial_\mu\psi - \partial_\mu\bar{\psi}\gamma^m\psi). \quad (7)$$

It transforms as a vector under general coordinate transformations,

$$\delta_\xi\tilde{E}^m_\mu = -\partial_\mu\xi^\nu\tilde{E}^m_\nu - \xi^\nu\partial_\nu\tilde{E}^m_\mu, \quad (8)$$

and as a vector under global Lorentz rotations

$$\delta_{\mathcal{L}}\tilde{E}^m_\mu = \epsilon^m{}_n\tilde{E}^n_\mu. \quad (9)$$

For irreducible spinors in $d = 2, 3, 9\mathrm{mod}8$, one has $\bar{\psi}\gamma^m\psi = 0$ such that $\tilde{E}^m_\mu = i\bar{\psi}\gamma^m\partial_\mu\psi$. From $\tilde{E}^m_\mu$ we can easily construct a composite field transforming like the metric

$$\tilde{g}_{\mu\nu} = \tilde{E}^m_\mu\tilde{E}^n_\nu\eta_{mn}. \quad (10)$$

However, no object transforming as the inverse metric can be constructed as a polynomial in the spinor fields. The spinor polynomials contain only "lower world indices" μ, ν which are induced by derivatives. The only possible coordinate invariant polynomial must therefore involve precisely d derivatives, contracted with the totally antisymmetric ϵ tensor. In particular, the scalar density $\tilde{E} = \det(\tilde{E}^m_\mu)$ can be written as a spinor polynomial

$$\tilde{E} = \frac{1}{d!}\epsilon^{\mu_1\ldots\mu_d}\epsilon_{m_1\ldots m_d}\tilde{E}^{m_1}_{\mu_1}\ldots\tilde{E}^{m_d}_{\mu_d} = \det(\tilde{E}^m_\mu). \quad (11)$$

Therefore a possible invariant action reads

$$S_E = \alpha\int d^dx\tilde{E}. \quad (12)$$

It involves d derivatives and $2d$ powers of ψ. We note that the ways to construct invariants are restricted by the absence of objects transforming as the inverse metric or the inverse vielbein. All invariants contain $\epsilon^{\mu_1\ldots\mu_d}$, where the indices $\mu_1\ldots\mu_d$ have to be contracted with derivatives. On the other hand, the construction of invariants with respect to the global Lorentz symmetry is not unique [14] since we have the invariant tensor η_{mn} and spinor bilinears not involving derivatives at our disposal.

[7] For details see [11].

[8] In [14] we generalize this construction to an bilinear $\tilde{E}^m_\mu = i\bar{\psi}\gamma^m\partial_\mu\psi$ that is not necessarily Hermitian.

With respect to local Lorentz transformations, $\delta_{\mathcal{L}}\tilde{E}^m_\mu$ acquires additional inhomogeneous pieces. In fact, if $\epsilon_{mn}(x)$ depends on the space-time coordinate one has

$$\delta_{\mathcal{L}}\tilde{E}^m_\mu = \epsilon^m{}_n\tilde{E}^n_\mu + \Big(\frac{i}{8}\bar{\psi}\gamma^{[mnp]}\psi\partial_\mu\epsilon_{np} + \frac{i}{4}\bar{\psi}\gamma^n\psi\partial^\mu\epsilon^m{}_n + \text{H.c.}\Big), \tag{13}$$

with $\gamma^{[mnp]}$ the totally antisymmetrized product of three γ^m matrices, $\gamma^{[mnp]} = \frac{1}{6}(\gamma^m\gamma^n\gamma^p - \gamma^m\gamma^p\gamma^n + \cdots)$. The piece $\sim\bar{\psi}\gamma^n\psi$ drops out—recall that for Majorana spinors in $d = 2, 3, 9$mod8 the antisymmetry under the exchange of Grassmann variables implies $\bar{\psi}\gamma^n\psi = 0$ [11]. The piece $\sim\bar{\psi}\gamma^{[mnp]}\psi$ remains, however. In consequence, the action S_E is only invariant under *global* Lorentz rotations, but not *local* Lorentz rotations. Two spinor configurations related to each other by a *local* Lorentz transformation are not equivalent to each other.[9]

III. BOSONIC EFFECTIVE ACTION

In order to construct the quantum effective action for our model with classical action (12), we introduce fermionic sources $\bar{\eta}$ and bosonic sources J^μ_m. The fermionic sources are Grassmann variables transforming as

$$\delta_{\mathcal{L}}\bar{\eta} = \tfrac{1}{2}\bar{\eta}\epsilon_{mn}\Sigma^{mn}, \qquad \delta_\xi\bar{\eta} = -\xi^\nu\partial_\nu\bar{\eta} - (\partial_\nu\xi^\nu)\bar{\eta}, \tag{14}$$

such that $S_\eta = -\int d^dx\bar{\eta}\psi$ is invariant. The bosonic sources multiply the fermion bilinear $\tilde{E}^m_\mu$. With the vector density J^μ_m transforming as

$$\delta_{\mathcal{L}}J^\mu_n = -J^\mu_m\epsilon^m{}_n, \qquad \delta_\xi J^\mu_m = -\partial_\nu(\xi^\nu J^\mu_m) + \partial_\nu\xi^\mu J^\nu_m, \tag{15}$$

the source term

$$S_J = -\int d^dxJ^\mu_m\tilde{E}^m_\mu \tag{16}$$

is again invariant. The generating functional[10]

$$W[\bar{\eta}, J] = \ln Z[\bar{\eta}, J] = \ln\int D\psi\exp\{-(S + S_\eta + S_J)\} \tag{17}$$

is therefore an invariant functional of $\bar{\eta}$ and J provided that the functional measure $\int D\psi$ is free of anomalies. The vielbein is now defined as the expectation value of $\tilde{E}^m_\mu$:

$$\frac{\delta W}{\delta J^\mu_m} = E^m_\mu = \langle\tilde{E}^m_\mu\rangle = \frac{i}{2}\langle\bar{\psi}\gamma^m\partial_\mu\psi - \partial_\mu\bar{\psi}\gamma^m\psi\rangle. \tag{18}$$

We use the symbol E^m_μ instead of the usual e^m_μ in order to recall that E^m_μ does not transform as a vector under *local* Lorentz transformations. We omit here[11] the fermionic sources $\bar{\eta}$ such that W is only a functional of J. Also E^m_μ depends on J. The effective action[12] for the vielbein is constructed by the usual Legendre transform

$$\Gamma[E^m_\mu] = -W[J^\mu_m] + \int d^dxJ^\mu_mE^m_\mu, \tag{19}$$

where $J^\mu_m[E^n_\nu]$ obtains by inverting Eq. (18). It obeys the identity

$$\frac{\delta\Gamma}{\delta E^m_\mu} = J^\mu_m. \tag{20}$$

If W is an invariant functional of J, we conclude that Γ is an invariant functional of the vielbein E^m_μ.

Equation (20) is the exact gravitational field equation for the quantum field theory defined by S_E and an appropriate functional measure $\mathcal{D}\psi$. In the presence of nongravitational degrees of freedom, the "physical" source J^μ_m should be associated with the energy-momentum tensor defined by

$$T^{\mu\nu} = E^{-1}E^{m\mu}J^\nu_m. \tag{21}$$

For example, the energy-momentum tensor receives contributions from the spinor fields as well as other possible bosonic composite fields beyond the vielbein. If the four-dimensional effective action is obtained by dimensional reduction from a higher dimensional theory, the source J^μ_m also accounts for the gauge bosons and scalars which arise from the higher dimensional bosonic fields. If we collect all contributions to the effective action involving fields other than the vielbein in Γ', we can formally write $J^\mu_m = -\langle\delta\Gamma'/\delta E^m_\mu\rangle$, where the bracket indicates that J^μ_m has to be evaluated for the given physical state.[13] For

[9]On the level of $\tilde{E}^m_\mu$, one may formulate a "new" local Lorentz transformation by using the transformation rule (9) instead of (13) such that S_E is invariant. However, this transformation cannot be formulated on the level of ψ. Since the transformation of the functional measure is not defined the effective gravitational action will not obey such a symmetry.

[10]For $d = 5, 6, 7$mod8 we have to use a spinor $\bar{\psi}$ not related to ψ by $\bar{\psi} = \psi^TC$. One should therefore use sources $\bar{\eta}$ and η multiplying ψ and $\bar{\psi}$ and a functional measure involving ψ and $\bar{\psi}$.

[11]The fermionic part of the effective action is discussed in [14].

[12]The sources J can be generalized to multiply arbitrary fermion bilinears. The "bosonic effective action" Γ [22] contains then all information about the correlation functions of the system. It can formally be obtained as the sum over two particle irreducible graphs [23].

[13]For the formal construction, one introduces the sources J^μ_m only as technical devices and puts them to zero at the end of the computation. However, one has to compute $\Gamma + \Gamma'$ with field equation $\delta\Gamma/\delta E^\mu_m + \delta\Gamma'/\delta E^\mu_m = 0$. The piece from the variation of Γ' can then be reinterpreted as a nonvanishing "physical source."

example, J_m^μ may account for the presence of a macroscopic massive object or for a relativistic plasma in cosmology. This setting is completely analogous to the treatment of standard gravity. As usual in gravity, we define[14] the metric by

$$g_{\mu\nu} = E_\mu^m E_{\nu m}. \tag{22}$$

We next show that for most practical purposes $T^{\mu\nu}$ can be identified with the usual energy-momentum tensor. Consider first the effective action Γ_0' for fields with trivial Lorentz-transformation properties as scalars or gauge bosons (i.e., fields carrying only "world indices" μ, ν and no spinor or Lorentz index). Then the dependence of Γ' on the gravitational degrees of freedom arises only via the metric

$$\Gamma_0'[E_\mu^m] = \Gamma_0'[g_{\nu\rho}[E_\mu^m]]. \tag{23}$$

In particular, this holds for structureless point particles. In standard gravity the energy-momentum tensor $T_{(g)}^{\mu\nu}$ is defined as

$$T_{(g)}^{\mu\nu} = -\frac{2}{\sqrt{g}}\frac{\delta\Gamma_0'}{\delta g_{\mu\nu}}. \tag{24}$$

Using the definition (21) we find

$$T^{\mu\nu} = -E^{-1}E^{m\mu}\frac{\delta\Gamma_0'}{\delta g_{\rho\sigma}}\frac{\delta g_{\rho\sigma}}{\delta E_\nu^m} = T_{(g)}^{\mu\nu}, \tag{25}$$

and conclude that $T^{\mu\nu}$ is symmetric and coincides indeed with the standard energy-momentum tensor.

One may object that the situation is different for fields with nontrivial Lorentz-transformation properties as, for example, spinors. Then the gravitational couplings can typically not be written only in terms of the metric but involve explicitly the vielbein. We will discuss in Sec. VIII that the fermion contribution to the energy-momentum tensor involves an antisymmetric part [13,14] proportional to the spin. Nevertheless, for standard macroscopic gravitational sources, the spin averages out and only the symmetric part of $T^{\mu\nu}$ needs to be retained.[15] For stars, dust, and radiation we can write the gravitational field equation in terms of the usual energy-momentum tensor

$$\frac{\delta\Gamma}{\delta E_\mu^m} = EE_{\rho m}T_{(g)}^{\rho\mu}. \tag{26}$$

Similar considerations hold for test particles used to probe the gravitational fields generated by other bodies. The action for photons depends only on the metric—their trajectory can therefore be computed as usual once $g_{\mu\nu}$ is known. Similarly, macroscopic test particles follow the geodesics defined by $g_{\mu\nu}$. Throughout this paper, we will assume that gravity is tested by point particles or light. Testable differences between our setting and Einstein's gravity can therefore only result from possible differences of the solutions of the field Eq. (26) as compared to the Einstein equations.

At this point we would like to stress that a successful computation of $\Gamma[E_\mu^m]$ (together with $T^{\mu\nu}$ or Γ') is equivalent to a well-defined theory of quantum gravity. The gravitational field Eq. (20) includes all quantum fluctuations. Also, the motion of test particles can directly be inferred from Γ'. The difficult part is, of course, the computation of Γ. In particular, this requires a well-defined functional measure $\mathcal{D}\psi$ which preserves diffeomorphisms and global Lorentz symmetry.

IV. SYMMETRIES AND INVARIANTS

We will make a first attempt to a very approximate computation of $\Gamma[E_\mu^m]$ in Secs. VII and VIII. First, we want to exploit the general structure of the effective action, in particular, the symmetries. This will allow us a first judgment if a theory with only global Lorentz symmetry is viable at all. Perhaps surprisingly, we find a new diffeomorphism invariant involving second derivatives of the vielbein which seems compatible with all present tests of gravity. This invariant respects global but not local Lorentz symmetry. We conclude that the local character of the Lorentz symmetry is only very partially tested—an invariant violating the local symmetry seems to be allowed and remains essentially unconstrained. On the other hand, we also discuss a second global invariant which modifies post-Newtonian gravity. Its coefficient is severely constrained.

In Secs. IV, V, VI, VII, VIII, IX, X, and XI, we discuss the properties of a generalized version of gravity which features only global instead of local Lorentz invariance. We discuss the most general setting consistent with these symmetries. Within spinor gravity, this generalizes the action (12) to an arbitrary polynomial action for spinors with invariance under general coordinate and global Lorentz transformations. (See [14] for a discussion of possible invariants.) Our discussion will be based purely on symmetry and a derivative expansion of the effective action. It is therefore more general than the specific one loop approximation discussed in [13]. The only assumption entering implicitly the following discussion is that the functional measure preserves diffeomorphism and global Lorentz symmetry, being free of anomalies [24].

In the following, we will mainly concentrate on four dimensions, $d = 4$. This permits a direct comparison of the solution of our generalized gravitational field Eq. (26) with observation. Embedding the four-dimensional effective theory in a more fundamental higher dimensional theory, we assume only that the "ground state" properties of "internal space" are consistent with the four-

[14] In principle, $g_{\mu\nu}$ differs from the fermionic four point function $\langle\tilde{g}_{\mu\nu}\rangle$ [cf. Eq. (10)].

[15] Note that orbital angular momentum does not contribute to the antisymmetric part of $T^{\mu\nu}$.

dimensional diffeomorphisms and global Lorentz rotations. Other details are not important for our discussion of the purely gravitational part. Of course, there could be additional light degrees of freedom influencing cosmology or the macroscopic laws, like the cosmon of quintessence [20].

Let us therefore discuss the most general structure of a gravitational effective action which involves the vielbein E_μ^m and is invariant under diffeomorphisms and the global Lorentz symmetry. The characteristic mass scale will be the Planck mass. For macroscopic phenomena on length and time scales much larger than the Planck length, we can expand $\Gamma[E_\mu^m]$ in the number of derivatives. For a given number of derivatives Γ can be composed of terms that are each invariant under general coordinate transformations and global Lorentz rotations. As compared to Einstein's gravity, we will find new invariants which involve the new physical degrees of freedom in E_μ^m not described by the metric.

In lowest order in the derivative expansion the unique invariant is ($E = \det E_\mu^m$, $g = |\det g_{\mu\nu}| = E^2$)

$$\Gamma_1 = \int d^d x E = \pm \int d^d x \sqrt{g}. \tag{27}$$

In four dimensions this is a cosmological constant. In the following, we will assume that some mechanism makes the effective cosmological constant very small—for example, the dynamical mechanism proposed for quintessence [20,21]. We mainly will discard this term for the following phenomenological discussion. This is, of course, a highly nontrivial assumption, meaning that spinor gravity admits an (almost) static solution with a large three-dimensional characteristic length scale (at least the size of the horizon).

For the construction of invariants involving derivatives of E_μ^m, we can employ the antisymmetric tensor

$$\Omega_{\mu\nu}{}^m = -\tfrac{1}{2}(\partial_\mu E_\nu^m - \partial_\nu E_\mu^m). \tag{28}$$

Let us first look for possible polynomials in E_μ^m and $\Omega_{\mu\nu}{}^m$. We will concentrate on even dimensions where we need an even power of $\Omega_{\mu\nu}{}^m$ because of global Lorentz invariance. Any polynomial invariant with two derivatives must be of the form

$$\Gamma_{2,p} = \int d^d x \Omega_{\mu_1\mu_2}{}^{n_1} \Omega_{\mu_3\mu_4}{}^{n_2} E_{\mu_5}^{m_5} \ldots E_{\mu_d}^{m_d} \epsilon^{\mu_1\ldots\mu_d} A_{n_1 n_2 m_5 \ldots m_d}, \tag{29}$$

where A should be constructed from η and ϵ tensors and has to be symmetric in (n_1, n_2) and totally antisymmetric in $(m_5 \ldots m_d)$. Only for $d = 4$ we can take $A_{n_1 n_2} = \eta_{n_1 n_2}$ whereas no polynomial two-derivative invariant exists for $d > 4$. (The polynomial invariant generalizing (29) in d dimensions involves $d/2$ factors of Ω and therefore $d/2$ derivatives of E_μ^m.) We observe that $\Gamma_{2,p}$ contains only one ϵ tensor and therefore violates parity. We will assume that the gravitational effective action preserves parity, at least to a very good approximation, and discard the polynomial invariant (29).

There is, however, no strong reason why the effective action should be a polynomial in E_μ^m. Whenever $E \neq 0$ we can construct the inverse vielbein

$$E_{m_1}^{\mu_1} = \frac{1}{(d-1)! E} \epsilon^{\mu_1\ldots\mu_d} \epsilon_{m_1\ldots m_d} E_{\mu_2}^{m_2} \ldots E_{\mu_d}^{m_d} = \frac{1}{E} \frac{\partial E}{\partial E_{\mu_1}^{m_1}}, \tag{30}$$

which obeys $E_\mu^m E_m^\nu = \delta_\mu^\nu$, $E_m^\mu E_\mu^n = \delta_m^n$. This allows us to define the inverse metric

$$g^{\mu\nu} = E_m^\mu E^{m\nu}, \qquad g^{\mu\nu} g_{\nu\rho} = \delta_\rho^\mu \tag{31}$$

which can be used to raise world indices, e.g.,

$$\Omega^{\mu\nu}{}_m = \eta_{mn} g^{\mu\rho} g^{\nu\sigma} \Omega_{\rho\sigma}{}^n. \tag{32}$$

We conclude that nonpolynomial invariants exist in arbitrary dimensions. They are well defined as long as $E \neq 0$ and may become singular in the limit $E \to 0$. Within spinor gravity, the nonpolynomial invariants are induced by the fluctuation (or loop) effects [13].

On the level of two derivatives, three linearly independent nonpolynomial invariants are given by

$$\begin{aligned}
\Gamma_{2,1} &= \int d^d x E \Omega^{\mu\nu}{}_m \Omega_{\mu\nu}{}^m, \\
\Gamma_{2,2} &= \frac{1}{2} \int d^d x E (D_\mu E_m^\mu)(D^\nu E_\nu^m), \\
\Gamma_{2,3} &= \frac{1}{4} \int d^d x E (D^\mu E_m^\nu + D^\nu E_m^\mu)(D_\mu E_\nu^m + D_\nu E_\mu^m).
\end{aligned} \tag{33}$$

For the latter two invariants, we introduce the covariant derivative

$$\begin{aligned}
D_\mu E_\nu^m &= \partial_\mu E_\nu^m - \Gamma_{\mu\nu}{}^\lambda E_\lambda^m, \\
D_\mu E_m^\nu &= \partial_\mu E_m^\nu + \Gamma_{\mu\lambda}{}^\nu E_m^\lambda,
\end{aligned} \tag{34}$$

where the affine connection involves the inverse metric

$$\Gamma_{\mu\nu}{}^\lambda = \tfrac{1}{2} g^{\lambda\rho}(\partial_\mu g_{\nu\rho} + \partial_\nu g_{\mu\rho} - \partial_\rho g_{\mu\nu}). \tag{35}$$

We emphasize that the covariant derivative acting on E_μ^m does not contain a spin connection since m is only a global Lorentz index. Using the relations (22) and (30), the connection can be expressed in terms of E_μ^m in a nonpolynomial way. For $d \neq 4$, or requesting parity invariance, there are no more independent invariants in this order of the derivative expansion.

The most general invariant bosonic effective action involving up to two derivatives of the vielbein can therefore be written as a linear combination of Γ_0 and $\Gamma_{2,1}$, $\Gamma_{2,2}$, $\Gamma_{2,3}$. As a convenient parametrization, we use

$$\Gamma = \epsilon \Gamma_0 + \mu(I_1 + \tau_A I_2 + \beta_A I_3), \tag{36}$$

with

$$I_1 = \frac{1}{2}\int d^d x E\{D^\mu E^\nu_m D_\nu E^m_\mu - D_\mu E^\mu_m D^\nu E^m_\nu\}, \quad (37)$$

$$I_2 = \frac{1}{2}\int d^d x E\{D^\mu E^\nu_m D_\mu E^m_\nu - 2D^\mu E^\nu_m D_\nu E^m_\mu\}, \quad (38)$$

$$I_3 = \frac{1}{2}\int d^d x E D_\mu E^\mu_m D^\nu E^m_\nu. \quad (39)$$

We will see that μ determines the effective Planck mass. This is most apparent if we rescale E^m_μ by an arbitrary unit of mass m in order to make the vielbein and the metric dimensionless, $E^m_\mu = m\bar{E}^m_\mu$. This replaces $\epsilon \to \bar{\epsilon} = \epsilon m^d$, $\mu \to \bar{\mu} = \mu m^{d-2}$. The precise relation between μ and Newton's constant will be given in Sec. VII. In the following, we will assume that this rescaling has been done and omit the bars on E^m_μ, μ, and ϵ such that μ and ϵ have dimension mass^{d-2} and mass^d, respectively. The remaining two dimensionless parameters τ_A and β_A account for possible deviations from Einstein's gravity. We will find that tight observational bounds exist only for the parameter β_A. It is therefore very interesting that the one loop contribution to β_A vanishes [13].

We close this section by noting that the three invariants can also be interpreted in terms of torsion. Indeed, we may define a different connection $\tilde{\Gamma}_{\mu\nu}{}^\lambda$ and a new covariant derivative $\tilde{D}_\mu$ such that the vielbein is covariantly conserved

$$\tilde{D}_\mu E^m_\nu = \partial_\mu E^m_\nu - \tilde{\Gamma}_{\mu\nu}{}^\lambda E^m_\lambda = 0. \quad (40)$$

This fixes the connection as[16]

$$\tilde{\Gamma}_{\mu\nu}{}^\lambda = (\partial_\mu E^m_\nu) E^\lambda_m, \quad (41)$$

and comparison with Eq. (34) identifies[17] the contorsion

$$E^\lambda_m D_\mu E^m_\nu = \tilde{\Gamma}_{\mu\nu}{}^\lambda - \Gamma_{\mu\nu}{}^\lambda = L_{\mu\nu}{}^\lambda. \quad (42)$$

We note that the antisymmetric part of $\tilde{\Gamma}_{\mu\nu}{}^\lambda$ is the torsion tensor[18]

$$\tilde{\Gamma}_{\mu\nu}{}^\lambda - \tilde{\Gamma}_{\nu\mu}{}^\lambda = -2\Omega_{\mu\nu}{}^m E^\lambda_m = T_{\mu\nu}{}^\lambda. \quad (43)$$

For the invariant I_3 we observe the identity

$$D^\mu E^m_\mu = (\tilde{\Gamma}_\mu{}^{\mu\lambda} - \Gamma_\mu{}^{\mu\lambda}) E^m_\lambda. \quad (44)$$

Since Eq. (40) implies the existence of d covariantly conserved vector fields the connection $\tilde{\Gamma}$ (41) is curvature free.

V. CURVATURE SCALAR AND FIELD EQUATIONS

Of course, the usual curvature scalar R can be constructed from the metric $g_{\mu\nu}$ and the connection Γ such that $R[g_{\mu\nu}]$ is invariant. With $g_{\mu\nu}[E^m_\rho]$ given by Eq. (22), this yields another invariant involving two derivatives of the vielbein, namely,

$$\Gamma_{2,R} = \int d^d x E R[g_{\mu\nu}[E^m_\rho]]. \quad (45)$$

The invariants (33) and (45) are not linearly independent; however,

$$\Gamma_{2,1} - \Gamma_{2,3} = -2\Gamma_{2,2} + \Gamma_{2,R}, \Gamma_{2,R} = -2I_1. \quad (46)$$

This follows by partial integration and use of the commutator identity for two covariant derivatives

$$[D_\rho, D_\sigma] E^m_\mu = R_{\rho\sigma\mu}{}^\nu E^m_\nu. \quad (47)$$

For practical computational purposes, it is sometimes convenient to use an alternative parametrization of Γ with $\Gamma_{2,R}, \Gamma_{2,1}$, and $\Gamma_{2,2}$ as independent invariants. We may rewrite the effective action in the form

$$\Gamma = \epsilon\Gamma_0 - \delta\Gamma_{2,R} + \zeta\Gamma_{2,1} + \xi\Gamma_{2,2}, \quad (48)$$

where

$$\zeta = \tau_A \mu, \qquad \xi = (\beta_A - \tau_A)\mu, \qquad \delta = \tfrac{1}{2}(1 - \tau_A)\mu. \quad (49)$$

In analogy to the Einstein equation, we can then write the field equation in absence of sources in the form

$$2\delta(R_{\mu\nu} - \tfrac{1}{2}R g_{\mu\nu}) = \hat{T}_{\mu\nu} - \epsilon g_{\mu\nu}. \quad (50)$$

Here the contribution from the invariants $\Gamma_{2,1}$ and $\Gamma_{2,2}$ is formally written as a part of the energy-momentum tensor

$$\begin{aligned}\hat{T}_{\mu\nu} = {}& \zeta\{4\Omega_{\mu\rho m}\Omega_\nu{}^{\rho m} - 2(D_\rho \Omega^\rho{}_{\nu m})E^m_\mu \\ & - \Omega_{\sigma\rho}{}^m \Omega^{\sigma\rho}{}_m g_{\mu\nu}\} + \xi\{E^\sigma_m \partial_\sigma (D^\rho E^m_\rho) g_{\mu\nu} \\ & - \partial_\mu (D^\rho E^m_\rho) E_{\nu m} + \tfrac{1}{2} D^\sigma E^m_\sigma D^\rho E_{\rho m} g_{\mu\nu}\}. \end{aligned} \quad (51)$$

Details of the derivation of $\hat{T}_{\mu\nu}$ can be found in Appendix A. The tensor $\hat{T}_{\mu\nu}$ can be decomposed into a symmetric and an antisymmetric part:

$$\hat{T}_{\mu\nu} = \hat{T}^{(s)}_{\mu\nu} + \hat{T}^{(a)}_{\mu\nu}, \quad (52)$$

and the field Eq. (50) implies that the antisymmetric part must vanish in the absence of sources

$$\hat{T}^{(a)}_{\mu\nu} = \tfrac{1}{2}(\hat{T}_{\mu\nu} - \hat{T}_{\nu\mu}) = 0. \quad (53)$$

[16]This connection is often called Weizenböck connection and discussed in the context of teleparallel theories [25]. We stress, however, that the usual teleparallel theories are equivalent reformulations of Einstein's gravity, in contrast to the present work.

[17]Since the left-hand side of Eq. (42) is a tensor, this shows that $\tilde{\Gamma}_{\mu\nu}{}^\lambda$ indeed transforms as a connection under general coordinate transformations. Of course, this can also be checked by direct computation from the analogue of Eq. (8).

[18]In Ref. [17] the torsion tensor $T_{\mu\nu\rho}$ is denoted by $R_{\mu\nu\rho}$.

Because of the Bianchi identity, the symmetric part is convariantly conserved

$$D_\nu \hat{T}^{(s)\mu\nu} = 0. \tag{54}$$

We note that $\hat{T}_{\mu\nu}$ can contain a piece proportional to the Einstein tensor $R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}$. The definition of the "gravitational energy-momentum tensor" $\hat{T}_{\mu\nu}$ is therefore not unique and Eq. (51) should be considered as a formal tool. In particular, for $\tau_A \neq 0$ the coefficient 2δ should not be associated with the Planck mass which is rather related to μ. If one chooses to collect the gravitational effects beyond Einstein gravity in a gravitational energy-momentum tensor, a better definition would collect the pieces from I_2, I_3 instead of $\Gamma_{2,1}, \Gamma_{2,2}$. This subtracts from $\hat{T}_{\mu\nu}$ (51) a piece $(2\delta - \mu)(R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu})$. In the presence of matter fluctuations (i.e., from the spinor fields) or expectation values of composite fields beyond the vielbein, the energy-momentum tensor will receive additional contributions, $\hat{T}_{\mu\nu} \rightarrow \hat{T}_{\mu\nu} + T_{\mu\nu}$.

VI. LINEARIZED GRAVITY

In the next sections, we will study the possible phenomenological consequences of the generalized gravity (36). We start the investigation of possible observable effects with a discussion of weak gravity. This will also reveal the "particle content" of this theory. The spectrum of small fluctuations around flat space can be investigated by linearization (for vanishing cosmological constant $\epsilon = 0$). In the linear approximation we write

$$\begin{gathered} E_\mu^m = \delta_\mu^m + \delta E_\mu{}^m, \qquad E_m^\mu = \delta_m^\mu + \delta E_m{}^\mu, \\ \delta E_\mu{}^m = \tfrac{1}{2}k_{\mu\nu}\eta^{\nu m} = \tfrac{1}{2}(h_{\mu\nu} + a_{\mu\nu})\eta^{\nu m}, \\ \delta E_m{}^\mu = -\tfrac{1}{2}(h_{\rho\nu} + a_{\rho\nu})\delta_m^\rho \eta^{\nu\mu}, \end{gathered} \tag{55}$$

with symmetric and antisymmetric parts

$$h_{\mu\nu} = h_{\nu\mu}, \qquad a_{\mu\nu} = -a_{\nu\mu}, \tag{56}$$

and

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}. \tag{57}$$

In this and in the next two sections, we raise and lower the indices of $h_{\mu\nu}$ and $a_{\mu\nu}$ with $\eta^{\mu\nu}$, $\eta_{\mu\nu}$ such that

$$\delta E_m{}^\mu = -\tfrac{1}{2}\eta_{m\rho}(h^{\rho\mu} + a^{\rho\mu}), \qquad g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu}. \tag{58}$$

We observe that the antisymmetric fluctuation $a_{\mu\nu}$ does not contribute to the fluctuation of the metric. In linear order one finds

$$\Omega_{\mu\nu}{}^m = -\tfrac{1}{4}\eta^{\rho m}(\partial_\mu h_{\nu\rho} - \partial_\nu h_{\mu\rho} + \partial_\mu a_{\nu\rho} - \partial_\nu a_{\mu\rho}), \tag{59}$$

$$D^\mu E_\mu^m = \tfrac{1}{2}\eta^{\rho m}(\partial^\mu a_{\mu\rho} - \partial^\mu h_{\mu\rho} + \partial_\rho h_\mu^\mu), \tag{60}$$

and the invariants in quadratic order therefore read

$$\begin{aligned} \Gamma_{2,1} = \frac{1}{8}\int d^dx\{&\partial^\mu a^{\nu\rho}\partial_\mu a_{\nu\rho} - \partial^\mu a^{\nu\rho}\partial_\nu a_{\mu\rho} \\ &- 2\partial^\mu h^{\nu\rho}\partial_\nu a_{\mu\rho} + \partial^\mu h^{\nu\rho}\partial_\mu h_{\nu\rho} - \partial^\mu h^{\nu\rho}\partial_\nu h_{\mu\rho}\}, \end{aligned} \tag{61}$$

and

$$\begin{aligned} \Gamma_{2,2} = \frac{1}{8}\int d^dx(&\partial_\mu a^{\mu\rho} - \partial_\mu h^{\mu\rho} + \partial^\rho h_\mu^\mu)(\partial^\nu a_{\nu\rho} \\ &- \partial^\nu h_{\nu\rho} + \partial_\rho h_\nu^\nu). \end{aligned} \tag{62}$$

We next decompose $h_{\mu\nu}$ and $a_{\mu\nu}$ in orthogonal irreducible representations of the Poincaré group:

$$\begin{gathered} h_{\mu\nu} = \sum_{k=1}^4 h_{\mu\nu}^{(k)} = \sum_{k=1}^4 (P^k)_{\mu\nu}^{\rho\sigma} h_{\rho\sigma}, \\ a_{\mu\nu} = \sum_{l=1}^2 a_{\mu\nu}^{(l)} = \sum_{l=1}^2 (\bar{P}^l)_{\mu\nu}^{\rho\sigma} a_{\rho\sigma}. \end{gathered} \tag{63}$$

Here ($\partial^2 = \eta^{\rho\sigma}\partial_\rho\partial_\sigma$)

$$\begin{gathered} h_{\mu\nu}^{(1)} = b_{\mu\nu}, \qquad h_{\mu\nu}^{(2)} = \frac{1}{d-1}\left(\eta_{mu\nu} - \frac{\partial_\mu\partial_\nu}{\partial^2}\right)\sigma, \\ h_{\mu\nu}^{(3)} = \frac{\partial_\mu\partial_\nu}{\partial^2} f, \qquad h_{\mu\nu}^{(4)} = \partial_\mu v_\nu + \partial_\nu v_\mu, \\ a_{\mu\nu}^{(1)} = c_{\mu\nu}, \qquad a_{\mu\nu}^{(2)} = \partial_\mu(v_\nu + w_\nu) - \partial_\nu(v_\mu + w_\mu) \end{gathered} \tag{64}$$

obey the constraints

$$\begin{gathered} \partial_\mu b^{\mu\nu} = 0, \qquad \eta^{\mu\nu} b_{\mu\nu} = 0, \qquad \partial_\mu v^\mu = 0, \\ \partial_\mu c^{\mu\nu} = 0, \qquad \partial_\mu w^\mu = 0, \end{gathered} \tag{65}$$

such that

$$\begin{gathered} h_\mu^\mu = \sigma + f, \qquad \partial_\mu h^{\mu\nu} = \partial^2 v^\nu + \partial^\nu f, \\ \partial_\mu\partial_\nu h^{\mu\nu} = \partial^2 f, \qquad \partial_\mu a^{\mu\nu} = \partial^2(v^\nu + w^\nu). \end{gathered} \tag{66}$$

This yields the effective action (36) in quadratic order ($\epsilon = 0$)

$$\begin{aligned} \Gamma = \frac{\mu}{8}\int d^dx\Big\{&\partial^\mu b^{\nu\rho}\partial_\mu b_{\nu\rho} - \left(\frac{d-2}{d-1} - \beta_A\right)\partial^\mu\sigma\partial_\mu\sigma \\ &+ \tau_A\partial^\mu c^{\nu\rho}\partial_\mu c_{\nu\rho} + \beta_A\partial^2 w^\mu\partial^2 w_\mu\Big\}. \end{aligned} \tag{67}$$

We observe that f and v_μ are pure gauge degrees of freedom and do not appear in Γ. In addition to the usual metric degrees of freedom $b_{\mu\nu}$ and σ spinor gravity contains the new massless fields $c_{\mu\nu}$ and w_μ. In the presence of a local Lorentz symmetry (as in the usual setting) $c_{\mu\nu}$ and w_μ would be the gauge degrees of freedom of the local Lorentz group. Here they rather have the character of Goldstone degrees of freedom associated with the spontaneous breaking of the global Lorentz symmetry. In fact, the vielbein of a flat ground state, $E_\mu^m = \delta_\mu^m$, spontaneously breaks the global rotations acting on the index m. It remains invariant, however, under a combined global Lorentz transformation and coordinate rotation, the latter acting on the index μ. We note that for $\beta_A = 0$ the only modification of standard gravity would be the additional antisymmetry field $c_{\mu\nu}$. This leads to the speculation that within spinor gravity the field w^μ could correspond to the gauge degree of freedom of a yet unidentified, perhaps nonlinear symmetry. We will see that the coupling τ_A is not restricted by present observation.

VII. LINEARIZED FIELD EQUATIONS

For the derivation of the field equations, it is useful to write the effective action (67) in a form which uses explicitly the projectors. Defining $k_{\mu\nu} = h_{\mu\nu} + a_{\mu\nu}$, $k^{(k)}_{\mu\nu} = h^{(k)}_{\mu\nu}$ for $k = 1\ldots4$, $k^{(5)}_{\mu\nu} = a^{(1)}_{\mu\nu}$, $k^{(6)}_{\mu\nu} = a^{(2)}_{\mu\nu}$, and correspondingly $P^5 = \bar{P}^1$, $P^6 = \bar{P}^2$, one finds

$$\Gamma = -\frac{\mu}{8} k^{\mu\nu} \partial^2 \Big(\sum_{k=1}^{6} A_k P^k\Big)_{\mu\nu}^{\rho\sigma} k_{\rho\sigma}, \tag{68}$$

with

$$A_1 = 1, \qquad A_2 = -(d-2) + (d-1)\beta_A, \qquad A_4 = 0, \qquad A_5 = \tau_A. \tag{69}$$

The projectors

$$\begin{aligned} P^1 &= \frac{1}{2}(\delta^\rho_\mu \delta^\sigma_\nu + \delta^\sigma_\mu \delta^\rho_\nu) - \frac{1}{d-1}\eta_{\mu\nu}\eta^{\rho\sigma} - \frac{1}{2}\Big(\frac{\partial_\mu\partial^\rho}{\partial^2}\delta^\sigma_\nu + \frac{\partial_\nu\partial^\rho}{\partial^2}\delta^\sigma_\mu + \frac{\partial_\mu\partial^\sigma}{\partial^2}\delta^\rho_\nu + \frac{\partial_\nu\partial^\sigma}{\partial^2}\delta^\rho_\mu\Big) \\ &\quad + \frac{1}{d-1}\Big(\frac{\partial_\mu\partial_\nu}{\partial^2}\eta^{\rho\sigma} + \frac{\partial^\rho\partial^\sigma}{\partial^2}\eta_{\mu\nu}\Big) + \frac{d-2}{d-1}\frac{\partial_\mu\partial_\nu\partial^\rho\partial^\sigma}{\partial^4}, \qquad P^2 = \frac{1}{d-1}\Big(\eta_{\mu\nu} - \frac{\partial_\mu\partial_\nu}{\partial^2}\Big)\Big(\eta^{\rho\sigma} - \frac{\partial^\rho\partial^\sigma}{\partial^2}\Big), \\ P^3 &= \frac{\partial_\mu\partial_\nu\partial^\rho\partial^\sigma}{\partial^4}, \qquad P^4 = \frac{1}{2}\Big(\frac{\partial_\mu\partial^\rho}{\partial^2}\delta^\sigma_\nu + \frac{\partial_\nu\partial^\rho}{\partial^2}\delta^\sigma_\mu + \frac{\partial_\mu\partial^\sigma}{\partial^2}\delta^\rho_\nu + \frac{\partial_\nu\partial^\sigma}{\partial^2}\delta^\rho_\mu\Big) - 2\frac{\partial_\mu\partial_\nu\partial^\rho\partial^\sigma}{\partial^4}, \\ P^5 &= \frac{1}{2}\Big(\frac{\partial_\mu\partial^\rho}{\partial^2}\delta^\sigma_\nu - \frac{\partial_\nu\partial^\rho}{\partial^2}\delta^\sigma_\mu - \frac{\partial_\mu\partial^\sigma}{\partial^2}\delta^\rho_\nu + \frac{\partial_\nu\partial^\sigma}{\partial^2}\delta^\rho_\mu\Big), \qquad P^6 = \frac{1}{2}(\delta^\rho_\mu\delta^\sigma_\nu - \delta^\sigma_\mu\delta^\rho_\nu) - P_5, \end{aligned} \tag{70}$$

obey $(P^k)^2 = P^k$ and are orthogonal $P^j P^k = 0$ for $j \neq k$.

In order to derive the linear field equations in the presence of sources—e.g., matter concentrations—we add to Γ a term involving the symmetric energy-momentum tensor $T_{\mu\nu} = T_{\nu\mu}$ of matter and radiation

$$\begin{aligned} \Gamma_M &= -\frac{1}{2}\int d^dx h^{\mu\nu}T_{\mu\nu} = -\frac{1}{2}\int d^dx h_{\mu\nu}T^{\mu\nu} \\ &= -\frac{1}{2}\int d^dx\Big\{b_{\mu\nu}\Big(T^{\mu\nu} - \frac{1}{d-1}T^\rho_\rho\eta^{\mu\nu} \\ &\quad + \frac{1}{d-1}\frac{\partial_\mu\partial_\nu}{\partial^2}T^\rho_\rho\Big) + \frac{1}{d-1}\sigma T^\rho_\rho\Big\}. \end{aligned} \tag{71}$$

Here we have used the linear energy-momentum conservation $\partial_\mu T^{\mu\nu} = 0$ for the last line. We will motivate in the next section the omission of the antisymmetric part of $T_{\mu\nu}$ in more detail.

The field equations follow from the variation of $\Gamma + \Gamma_M$, Eqs. (69) and (71)

$$-\partial^2 \sum_k A_k (P^k)^{\rho\sigma}_{\mu\nu} k_{\rho\sigma} = \frac{2}{\mu} T_{\mu\nu}. \tag{72}$$

It can be projected on the irreducible representations

$$-\partial^2 A_k k^{(k)}_{\mu\nu} = \frac{2}{\mu}(P_k)^{\rho\sigma}_{\mu\nu} T_{rho\sigma}, \tag{73}$$

where we note $(P_k)^{\rho\sigma}_{\mu\nu}T_{\rho\sigma} = 0$ for $k = 3, 4, 5, 6$ due to the symmetry of $T_{\mu\nu}$ and $\partial_\mu T^{\mu\nu} = 0$. This yields

$$\begin{aligned} -\partial^2 h^{(1)}_{\mu\nu} &= \frac{2}{A_1\mu}\Big(T_{\mu\nu} - \frac{1}{d-1}T^\rho_\rho\eta_{\mu\nu} + \frac{1}{d-1}\frac{\partial_\mu\partial_\nu}{\partial^2}T^\rho_\rho\Big) \\ -\partial^2 h^{(2)}_{\mu\nu} &= \frac{2}{A_2(d-1)\mu}\Big(\eta_{\mu\nu} - \frac{\partial_\mu\partial_\nu}{\partial^2}\Big)T^\rho_\rho, \end{aligned} \tag{74}$$

in accordance[19] with a variation of Eqs. (67) and (71) with respect to $b_{\mu\nu}$ and σ. The remaining field equations can be written as

$$\tau_A\partial^2 c_{\mu\nu} = 0, \qquad \beta_A\partial^2 w_\mu = 0, \tag{75}$$

since nonvanishing sources for $c_{\mu\nu}$ and w_μ can arise only from the antisymmetric part of a generalized energy-momentum tensor.

We can now compute the Newtonian limit by inserting $T_{\mu\nu} = \rho\delta_{\mu0}\delta_{\nu0}$ and considering a time independent metric with Newtonian potential

$$\phi = -\tfrac{1}{2}h_{00} = -\tfrac{1}{2}(h^{(1)}_{00} + h^{(2)}_{00}). \tag{76}$$

For $d = 4$ one has ($\Delta = \partial^i\partial_i$)

$$-\Delta h^{(1)}_{00} = \frac{4\rho}{3\mu}, \qquad -\Delta h^{(2)}_{00} = -\frac{\rho}{3\mu}\Big(1 - \frac{3}{2}\beta_A\Big)^{-1}, \tag{77}$$

or

[19] In this respect it is crucial that $b_{\mu\nu}$ multiplies the correctly projected $(P^1T)^{\mu\nu}$ in Eq. (71).

$$\Delta\phi = \frac{\rho}{2\mu}\frac{1-2\beta_A}{1-\frac{3}{2}\beta_A} = 4\pi G_N\rho = \frac{\rho}{2\bar{M}^2}. \tag{78}$$

This fixes Newton's constant G_N or the reduced Planck mass $\bar{M}^2 = M_p^2/8\pi$ as

$$\bar{M}^2 = \frac{1-\frac{3}{2}\beta_A}{1-2\beta_A}\mu. \tag{79}$$

We will see below that β_A has to be small such that $\bar{M}^2$ is essentially given by μ. However, within Newtonian gravity the couplings τ_A and β_A are not constrained.

The linear approximation governs the emission, propagation, and detection of gravitational waves. Those are described by $h^{(1)}_{\mu\nu} = b_{\mu\nu}$. For example, the emission of gravitational waves from pulsars is the same as in Einstein gravity. However, the effective reduced Planck mass extracted from the gravitational radiation of pulsars is given by $\bar{M}^2_{\text{pulsar}} = \mu$ (since $A_1 = 1$). In view of the tight limit for $|\beta_A|$ derived in Sec. X the difference between the gravitational constant measured in Newtonian gravity (79) and the one relevant for pulsars seems to be too small in order to be observable.

Actually, if Γ is given by the one loop approximation, one finds that β_A vanishes. Indeed, we may compute $\Omega_{\mu\nu\rho}$ in the linearized approximation

$$\begin{aligned}\Omega_{\mu\nu\rho} &= -\tfrac{1}{4}\{\partial_\mu h_{\nu\rho} + \partial_\mu a_{\nu\rho} - (\mu \leftrightarrow \nu)\}\\ &= -\tfrac{1}{4}\Big\{\partial_\mu(b_{\nu\rho} + c_{\nu\rho}) - \partial_\nu(b_{\mu\rho} + c_{\mu\rho}) + \frac{1}{d-1}\\ &\quad\times(\eta_{\nu\rho}\partial_\mu\sigma - \eta_{\mu\rho}\partial_\nu\sigma) - \partial_\rho(\partial_\mu w_\nu - \partial_\nu w_\mu)\Big\}.\end{aligned} \tag{80}$$

The totally antisymmetric part therefore only involves $c_{\nu\rho}$:

$$\Omega_{[\mu\nu\rho]} = -\tfrac{1}{2}\partial_{[\mu}c_{\nu\rho]}. \tag{81}$$

We will see in Sec. XIII that the modified Dirac operator in our generalized gravity can be written in the form $\mathcal{D} = \mathcal{D}_E + \frac{1}{4}\Omega_{[\mu\nu\rho]}\gamma^{\mu\nu\rho}_{(3)}$ (151). Here $\mathcal{D}_E$ can depend only on $b_{\nu\rho}$ and σ as a consequence of local Lorentz invariance. We conclude that $\mathcal{D}$ does not depend on w_μ. In consequence, the one loop expression $\text{Tr}\ln\mathcal{D}$ cannot lead to a term $\sim\partial^2 w^\mu \partial^2 w_\mu$ in quadratic order. Comparison with Eq. (67) implies $\beta_A = 0$.

VIII. ANOMALOUS SPIN INTERACTIONS AND POST-NEWTONIAN GRAVITY

In this section we discuss the anomalous couplings of gravitational degrees of freedom to the spin of fermions. This is related to possible anomalous spin interactions and the issue of post-Newtonian gravity beyond the linear approximation. For our purpose, we need information about the coupling of the vielbein E^m_μ to spinor fields ψ. From symmetry arguments [14], one expects that the missing spin connection reflects itself in a modification of the covariant spinor kinetic term

$$\mathcal{L}_\psi = i\bar{\psi}\gamma^\mu D^{(E)}_\mu\psi + i\sigma_A\Omega_{[\mu\nu\rho]}\bar{\psi}\gamma^{\mu\nu\rho}_{(3)}\psi. \tag{82}$$

Here the covariant derivative $D^{(E)}_\mu$ is constructed as usual and involves the standard spin connection constructed from E^m_μ and its derivatives (see Sec. XIII for details). The anomalous term proportional to the coupling σ_A reflects the violation of local Lorentz symmetry in the spinor coupling. In the classical approximation to spinor gravity one has $\sigma_A = 1/4$.

As mentioned already, the first term in Eq. (82) couples only to $h_{\mu\nu}$. It gives the standard contribution of fermionic particles to the symmetric energy-momentum tensor $T^{\mu\nu}_{(g)}$ (24). In contrast, the anomalous second part yields in the linear approximation

$$\mathcal{L}_A = -\frac{i}{2}\sigma_A\partial_{[\mu}c_{\nu\rho]}\bar{\psi}\gamma^{\mu\nu\rho}_{(3)}\psi. \tag{83}$$

Using partial integration one finds [13] that the coupling of $c_{\nu\rho}$ to the spinors,

$$\mathcal{L}_A \sim c_{\nu\rho}\partial_\mu(\bar{\psi}\gamma^{[\mu}\gamma^\nu\gamma^{\rho]}\psi) \sim c_{\nu\rho}\epsilon^{\nu\rho\mu\sigma}\partial_\mu S_\sigma, \tag{84}$$

involves the density of the spin vector S_σ. (We recall that there is no coupling to $a^{(2)}_{\nu\rho}$.) In consequence, the exchange of $c_{\nu\rho}$ induces a dipole-dipole interaction between fermions with infinite range. However, its strength is only gravitational $\sim\mu^{-1}\sim\bar{M}^{-2}$ and therefore suppressed as compared to the magnetic dipole interaction by a factor $\sim(m_e/\bar{M})^2 \sim 10^{-44}$—many orders of magnitude too small to be observable [26].

We conclude that we can safely neglect the anomalous spinor coupling and concentrate on the symmetric energy-momentum tensor $T^{\mu\nu}_{(g)}$. Indeed, as compared to Newtonian gravity the macroscopic spin forces are doubly suppressed. First, the lack of spin coherence of macroscopic bodies leads to suppression factors $S_{\text{tot}}\cdot m/M_{\text{tot}}$ for each body involved. Here S_{tot} is the total spin (in units of $\hbar$) and M_{tot} the total mass of the body composed of particles with (average) mass m. Second, a dipole-dipole force decays very fast $\sim r^{-3}$.

We will next see that the particular form of the coupling of $c_{\nu\rho}$ to the spinor field also has important consequences for post-Newtonian gravity. All effects from a violation of the local Lorentz symmetry by the invariant $\sim\tau_A$ are severely suppressed.

The expansion in weak gravitational fields can be extended beyond linear order. A general framework for the effects beyond Newtonian gravity is given by the "parametrized Post-Newtonian formalism" (PPN) [27]. In principle, one should perform a systematic computation of all PPN parameters for the field equations following from the effective gravitational action (3) without local

Lorentz invariance. For weak gravitational fields the higher order corrections are computed iteratively: One uses the results of the linear approximation (cf. the preceding two sections) in order to derive the field equations for the derivations of E^m_μ from the linear approximation. The PPN formalism needs at most the next-to-linear terms for some of the vielbein components. Deviations from Einstein's gravity therefore involve the cubic couplings arising from the invariants $\sim\tau_A$ and β_A.

In the remainder of this section, we show that for $\beta_A = 0$ the invariant τ_A gives precisely the same PPN results as Einstein's gravity. The coupling τ_A is therefore not constrained by any one of the tests of general relativity for weak fields (i.e., up to the order used for the PPN formalism). This demonstrates that local Lorentz symmetry is actually very poorly tested—an additional invariant violating local Lorentz symmetry can be added and remains essentially undetectable by present means. We will not pursue a systematic PPN discussion of the influence of the coupling β_A since we believe that the strongest constraints arise for isotropic gravitational fields which will be discussed in complete nonlinear order in the next sections.

For $\beta_A = 0$ there is no distinction between Einstein gravity and our generalized gravity in the linear approximation provided $T^{\mu\nu} = T^{\mu\nu}_{(g)}$. (We have shown above that the neglection of the antisymmetric part of $T^{\mu\nu}$ is indeed a very good approximation.) In particular, $h_{\mu\nu}$ takes in the linear approximation the same values as for Einstein's gravity and $a_{\mu\nu} = 0$. In principle, deviations on the PPN level could arise if the invariant $\sim\tau_A$ produces cubic terms $\sim h^3$ or $h^2 a$. Then the field equations for the corrections in next-to-linear order would have additional source terms $\sim h_l^2$, where h_l is the metric in linear order. (Recall $a_l = 0$.) We will show that such cubic terms are not present at the PPN level.

By straightforward algebraic manipulations, one establishes that the invariant $\sim\tau_A$ only involves the totally antisymmetric part of $\Omega_{\mu\nu\rho}$

$$I_2 = \frac{3}{2}\int d^d x E \Omega^{[\mu\nu\rho]}\Omega_{[\mu\nu\rho]}. \tag{85}$$

Expanding $\Omega_{[\mu\nu\rho]}$ up to terms quadratic in h and a, one finds ($\partial_{[\mu}a_{\nu\rho]} = \partial_{[\mu}c_{\nu\rho]}$)

$$\Omega_{[\mu\nu\rho]} = -\tfrac{1}{2}\partial_{[\mu}a_{\nu\rho]} - G_{[\mu\nu\rho]} + \cdots, \tag{86}$$

with

$$G_{\mu\nu\rho} = \tfrac{1}{8}(\partial_\mu h_{\nu\sigma})h_{\rho\tau}\eta^{\sigma\tau}. \tag{87}$$

The dots denote terms $\sim ah$ and a^2 which are not relevant for our purpose. The crucial point is that h appears only in quadratic order in $\Omega_{[\mu\nu\rho]}$ and I_2 therefore contains no term $\sim h^3$. On the PPN level possible modification can therefore arise only from

$$I_2^{(3)} = \int d^d x G^{[\mu\nu\rho]}\partial_{[\mu}c_{\nu\rho]}. \tag{88}$$

This term results in a gravitational source term for $c_{\nu\rho}$ which is $\sim h_l^2$.

On the PPN level, the source $\sim\partial_\mu G^{[\mu\nu\rho]}$ vanishes. Since the source is already $\sim h^2$, we only need to take into account the Newtonian part $(h_l)_{00}$. In this case, one infers $G_{[\mu\nu\rho]} = 0$ since an antisymmetrization over two equal indices $\nu = \rho = 0$ is involved. This concludes our argument. Can one conceive future experiments that could detect the field $c_{\nu\rho}$ as a manifestation of the violation of local Lorentz symmetry for the case $\beta_A = 0$? The answer tends to be negative: It is simply very hard to produce a macroscopic $c_{\nu\rho}$ field with observable strength. Even if one would succeed, the measurement would require a probe with coherent spin.

IX. GENERAL ISOTROPIC STATIC SOLUTION

The comparison with Einstein gravity should, of course, not be restricted to the linear approximation. The two prominent examples where nonlinear effects play a role are the Schwarzschild solution and cosmology. They will be discussed in Secs. IX, X, and XI. In this section, we discuss the Schwarzschild solution for the generalized gravity corresponding to the effective action (36) or (48). For this purpose we describe the most general static solution of the nonlinear gravitational field equations under the assumption of isotropy. Obviously, this goes beyond Newtonian gravity and linearized gravity. In Sec. X we concentrate on $d = 4$ and compare our general solution with the Schwarzschild solution in Einstein gravity. For $\beta_A = 0$ we find that the standard Schwarzschild solution is also a solution to the nonlinear field equation of our generalized gravity. The parameter τ_A therefore remains unconstrained. On the other hand, for $\beta_A \neq 0$ we find a difference already in the lowest order of the post-Newtonian expansion. Recent precision observations put a severe bound on the parameter β_A.

A rotation acts on the vielbein E^m_μ as a coordinate rotation leaving $r^2 = \sum_{i=1}^{d-1} x_i^2$ invariant, accompanied by a simultaneous suitable global Lorentz rotation acting on the index m. The most general rotation invariant vielbein takes the form[20]

$$E^0_0 = f(r), \qquad E^j_0 = h(r)x_j, \qquad E^0_i = g(r)x_i, \qquad E^j_i = c(r)\delta_{ij} + k(r)x_i x_j. \tag{89}$$

We can rescale $r = D(r')r'$ in order to fix $c(r) = 1$. Similarly, a radius dependent rescaling of the clocks $dt = dt' + rF(r)dr$ leaves $d\xi^m = E^m_\mu dx^\mu = E'^m_\mu dx'^\mu$ invariant

[20] There should be no confusion between $f(r)$ and the gauge degree of freedom f in Sec. VI.

$$d\xi^0 = f(r)dt + g(r)rdr = f(r)dt' + [f(r)F(r) + g(r)]rdr = f(r)dt' + g'(r)rdr, \quad d\xi^i = h(r)x_i dt + dx_i + k(r)rx_i dr = h(r)x_i dt' + dx_i + [k(r) + h(r)F(r)]rx_i dr = h(r)x_i dt' + dx_i + k'(r)rx_i dr. \tag{90}$$

With $g' = g + fF$, $k' = k + hF$ we can use this freedom in order to fix k as a function of f, g, and h such that[21]

$$gf = h(1 + r^2 k). \tag{91}$$

With this coordinate choice we remain with three free functions $f(r)$, $g(r)$, and $h(r)$.

From Eqs. (89) with (91) and $c = 1$, we can compute the metric

$$g_{00} = -B(r), \quad g_{0i} = 0, \quad g_{ij} = \delta_{ij} + \frac{A(r) - 1}{r^2} x_i x_j, \tag{92}$$

which corresponds for $d = 4$ to the line element of the Schwarzschild metric in standard (polar) coordinates,

$$ds^2 = -B(r)dt^2 + A(r)dr^2 + r^2(d\vartheta^2 + sin^2\vartheta d\varphi^2). \tag{93}$$

Here we have used Eq. (91) in order to obtain $g_{0i} = 0$. The functions $A(r)$, $B(r)$ are related to $f(r)$, $g(r)$, $h(r)$ by

$$B = f^2 - r^2h^2, \quad A = 1 - r^2g^2 + 2r^2k + r^4k^2 = \frac{g^2}{h^2}B. \tag{94}$$

We choose as the three independent functions $A(r)$, $B(r)$, and $h(r)$ with $f = \sqrt{B + r^2h^2}$, $g = \sqrt{A/B} \cdot h$, and $k(r) = r^{-2}(\sqrt{A + r^2h^2A/B} - 1)$. We emphasize that we have here one more free function $h(r)$ in addition to the two functions $A(r)$, $B(r)$ characterizing the metric. In case of *local* Lorentz symmetry $h(r)$ would correspond to a gauge degree of freedom—here it is not.

Inserting the vielbein

$$E_0^0 = \sqrt{B + r^2h^2}, \quad E_0^j = hx_j, \quad E_i^0 = h\sqrt{A/B}x_i, \quad E_{ij} = \delta_{ij} + \frac{x_i x_j}{r^2}\left(\sqrt{A + r^2h^2A/B} - 1\right), \tag{95}$$

we compute in Appendix A the invariants

$$Y_{2,1} = \Omega_{\mu\nu}{}^m \Omega^{\mu\nu}{}_m = \frac{d-2}{2r^2}\left(1 + \frac{1}{A} - 2\sqrt{\frac{B + r^2h^2}{AB}}\right) + \frac{1}{8AB(B + r^2h^2)} \times [B'^2 + 4rB'h(h + rh') - 4B(h + rh')^2], \tag{96}$$

and

$$Y_{2,2} = \frac{1}{2}(D^\mu E_\mu^m)(D^\nu E_{\nu m}) = \frac{(d-2)^2}{2r^2}\left(\sqrt{\frac{B + r^2h^2}{AB}} - 1\right)^2 + \frac{d-2}{2rAB}\left(1 - \sqrt{\frac{AB}{B + r^2h^2}}\right)[B' + 2rh(h + rh')] + \frac{1}{8AB(B + r^2h^2)}[B' + 2rh(h + rh')]^2 - \frac{1}{2AB}[(d-1)h + rh']^2. \tag{97}$$

With $E = \sqrt{AB}$, the effective action (48) becomes

$$\Gamma = \int d^dx\sqrt{AB}\{\epsilon - \delta R[A, B] + \zeta Y_{2,1} + \xi Y_{2,2}\}, \tag{98}$$

where (cf. Appendix A)

$$R[A, B] = -\frac{B''}{AB} + \frac{B'}{2AB}\left(\frac{A'}{A} + \frac{B'}{B}\right) + \frac{d-2}{rA}\left(\frac{A'}{A} - \frac{B'}{B}\right) + \frac{(d-3)(d-2)}{r^2}\left(1 - \frac{1}{A}\right). \tag{99}$$

Since our ansatz covers the most general rotation invariant static vielbein (up to coordinate transformations), the field equations for this symmetric situation can be obtained by varying Γ with respect to A, B, and h. For small h we observe that $Y_{2,1}$ and $Y_{2,2}$ are quadratic in h (or derivatives of h). The field equation $\delta\Gamma/\delta h = 0$ admits therefore always solutions with $h = 0$. In this situation the remaining field equations for A and B can be obtained by inserting $h = 0$ into $Y_{2,1}$ and $Y_{2,2}$:

$$Y_{2,1}(h = 0) = \frac{d-2}{2r^2}\left(1 - \sqrt{\frac{1}{A}}\right)^2 + \frac{1}{8A}\left(\frac{B'}{B}\right)^2, \quad Y_{2,2}(h = 0) = \frac{1}{2}\left[\frac{d-2}{r}\left(1 - \sqrt{\frac{1}{A}}\right) - \frac{1}{2\sqrt{A}}\frac{B'}{B}\right]^2. \tag{100}$$

X. MODIFICATION OF THE SCHWARZSCHILD SOLUTION

In the following we concentrate on $d = 4$ where the resulting effective action reads after partial integration ($\epsilon = 0$)

$$\Gamma[A, B] = 8\pi\delta\int dtdr\sqrt{AB}\left\{\frac{1}{A} - 1 - \frac{rA'}{A^2} + \tilde{\zeta}\left[\left(1 - \frac{1}{\sqrt{A}}\right)^2 + \frac{r^2}{8A}\left(\frac{B'}{B}\right)^2\right] + 2\tilde{\xi}\left(1 - \frac{1}{\sqrt{A}} - \frac{r}{4\sqrt{A}}\frac{B'}{B}\right)^2\right\}. \tag{101}$$

The contributions $\sim\tilde{\zeta} = \zeta/(2\delta)$, $\tilde{\xi} = \xi/(2\delta)$ do not vanish. For $\tilde{\zeta} > 0$, $\tilde{\xi} \geq 0$ they give a strictly positive contribution to Γ whenever $A \neq 1$, $B' \neq 0$, i.e., for all

[21]A suitable function F exists provided $r^2h^2 \neq f^2$.

geometries which differ from a flat space-time. Standard gravity is recovered for[22] $\tilde{\zeta} = \tilde{\xi} = 0$ and leads to the well-known Schwarzschild solution $B = A^{-1} = 1 - r_s/r$ with Schwarzschild radius $r_s = 2mG_N$ related to the total mass m of the object.

The spherically symmetric static field equations obtain by variation of $\Gamma[A, B]$ with respect to A and B. Equivalently, we may express[23] Γ in terms of two new functions $\alpha(\rho)$, $\beta(\rho)$ and a rescaled radial coordinate ρ

$$\alpha = A^{-1/2}, \qquad \beta = \ln B, \qquad \rho = \ln(r/r_s), \tag{102}$$

as

$$\Gamma[\alpha, \beta] = 8\pi\delta r_s \int dt d\rho e^{\rho} e^{\beta/2}\left\{\alpha - \frac{1}{\alpha} + 2\dot{\alpha} + \tilde{\zeta}\left[\frac{(1-\alpha)^2}{\alpha} + \frac{\alpha}{8}\dot{\beta}^2\right] + \frac{2\tilde{\xi}}{\alpha}\left(1 - \alpha - \frac{\alpha}{4}\dot{\beta}\right)^2\right\}. \tag{103}$$

Here a dot denotes a derivative with respect to ρ. The field equation from the variation with respect to β reads

$$\alpha^2 - 1 + 2\dot{\alpha}\alpha = -\tilde{\zeta}\left\{(1-\alpha)^2 - \frac{1}{8}\alpha^2\dot{\beta}^2 - \frac{1}{2}\alpha^2(\dot{\beta} + \ddot{\beta}) - \frac{1}{2}\alpha\dot{\alpha}\dot{\beta}\right\} + \tilde{\xi}\left\{2\alpha\dot{\alpha} + \frac{1}{2}\alpha(\dot{\alpha}\dot{\beta} + \alpha\ddot{\beta}) - \left(2 + \frac{1}{2}\alpha\dot{\beta}\right)\left(1 - \alpha - \frac{\alpha\dot{\beta}}{4}\right)\right\}, \tag{104}$$

and from the variation with respect to α we obtain

$$1 - \frac{1}{\alpha^2} + \dot{\beta} = \tilde{\zeta}\left(1 - \frac{1}{\alpha^2} + \frac{\dot{\beta}^2}{8}\right) + 2\tilde{\xi}\left[\left(1 + \frac{\dot{\beta}}{4}\right)^2 - \frac{1}{\alpha^2}\right]. \tag{105}$$

We may first recover the usual Schwarzschild solution for $\tilde{\zeta} = \tilde{\xi} = 0$ where

$$\frac{\partial\alpha^2}{\partial\rho} = 1 - \alpha^2, \qquad \frac{\partial\beta}{\partial\rho} = \frac{1}{\alpha^2} - 1 \tag{106}$$

implies the solution[24]

$$\alpha^2 = e^{\beta} = 1 - e^{-\rho}, \qquad \frac{1}{A} = B = 1 - \frac{r_s}{r}. \tag{107}$$

For general $\tilde{\zeta}$, $\tilde{\xi}$ we make for large ρ the ansatz

$$\alpha^2 = 1 - \gamma e^{-\sigma\rho}, \qquad \ln\beta = 1 - e^{-\sigma\rho}. \tag{108}$$

Dropping terms $\sim e^{-2\sigma\rho}$ or smaller Eqs. (104) and (105) results in

$$\gamma - \gamma\sigma + \frac{1}{2}\tilde{\zeta}(\sigma - \sigma^2) + \tilde{\xi}\left(\gamma\sigma - \gamma + \frac{\sigma}{2} - \frac{\sigma^2}{2}\right) = 0,$$
$$\gamma - \sigma - \tilde{\zeta}\gamma - \tilde{\xi}(2\gamma - \sigma) = 0, \tag{109}$$

with solution

$$\sigma = 1, \quad \gamma = \frac{1 - \tilde{\xi}}{1 - \tilde{\zeta} - 2\tilde{\xi}} = \left(1 - \frac{\tilde{\zeta} + \tilde{\xi}}{1 - \tilde{\xi}}\right)^{-1} = \frac{1 - \beta_A}{1 - 2\beta_A}. \tag{110}$$

One finds for $r \gg r_s$ a behavior similar as for a Jordan-Brans-Dicke [18] theory:

$$B = 1 - \frac{r_s}{r}, \qquad A^{-1} = 1 - \gamma\frac{r_s}{r}. \tag{111}$$

As expected, Newtonian gravity [encoded in $B(r)$] remains unaffected. On the other hand, post-Newtonian gravity is modified. Strong observational bounds from the solar system imply that γ must be very close to 1 [19]

$$\gamma - 1 \approx \beta_A \approx \tilde{\zeta} + \tilde{\xi} = (2.1 \pm 2.3)10^{-5}. \tag{112}$$

This seems to be the most stringent bound on the parameter β_A.

For $\beta_A = 0$, as suggested by the one loop approximation, one has $\gamma = 1$ and there is no correction to lowest order post-Newtonian gravity. This extends to the full Schwarzschild solution. Indeed, for $\tilde{\xi} = -\tilde{\zeta}$ the field equations (104) and (105) reduce to the standard field equations (106) for arbitrary values of τ_A. For $\beta_A = 0$ the spherically symmetric static solution does not distinguish between our version of generalized gravity and Einstein gravity.

Finally, we observe that for $\beta_A \neq 0$ the full solution can be found numerically whereby the initial value problem has one more free integration constant since Eq. (104) also involves $\ddot{\beta}_A(\rho)$. It would be interesting to investigate if this has an effect on the singularity. Furthermore, the solutions with $h = 0$ are not the only candidates for the description of the gravitational effects of massive bodies. One may explore a behavior for $r \gg r_s$ where

$$h(r) = \eta r_s^{1/2} r^{-3/2}. \tag{113}$$

We postpone the analysis of such modified solutions to a future investigation.

IX. HOMOGENEOUS ISOTROPIC COSMOLOGY

In this section we investigate cosmologies with a homogeneous and isotropic metric and a flat spatial hypersurface, i.e., a situation where the Killing vectors of the (three) spatial translations commute. Again, this tests our generalized gravity beyond the linear approximation. The most general vielbein consistent with these symmetries can be brought to the form

[22] This condition is sufficient but not necessary.
[23] There should be no confusion of $\beta(\rho)$ and the coupling constant β_A in the effective action (36).
[24] The two integration constants of the general solution of Eq. (106) are an additive constant in ρ which is absorbed in r_s and a multiplicative constant in B which can be set to one by appropriate time rescaling.

$$E_0^0 = 1, \qquad E_0^i = 0, \qquad E_i^0 = 0, \qquad E_i^j = a(t)\delta_i^j. \tag{114}$$

Here $a(t)$ is the usual scale factor which is related to the Hubble parameter by $H(t) = \dot{a}(t)/a(t)$. We concentrate on three space and one time dimensions, $d = 4$. The nonvanishing components of the metric and affine connection read

$$g_{00} = -1, \qquad g_{ij} = a^2(t)\delta_{ij}, \tag{115}$$

and

$$\Gamma_{ij}^0 = Hg_{ij}, \qquad \Gamma_{0i}^j = H\delta_i^j. \tag{116}$$

They result in a curvature scalar

$$R = 12H^2 + 6\dot{H}. \tag{117}$$

(In this section dots denote time derivatives.) The only independent nonvanishing components of $\Omega_{\mu\nu}{}^m$ are

$$\Omega_{i0}{}^j = -\Omega_{0i}{}^j = \tfrac{1}{2}\dot{a}\delta_i^j, \tag{118}$$

and therefore

$$Y_{2,1} = \Omega_{\mu\nu}{}^m\Omega^{\mu\nu}{}_m = -\tfrac{3}{2}H^2. \tag{119}$$

With

$$D^\mu E_\mu^0 = -3H, \qquad D^\mu E_\mu^i = 0, \tag{120}$$

the second invariant becomes

$$Y_{2,2} = \tfrac{1}{2}D^\mu E_\mu^m D^\nu E_{\nu m} = -\tfrac{9}{2}H^2. \tag{121}$$

Since we consider the most general vielbein consistent with the symmetries we can again derive the relevant field equation by variation of an effective action with respect to a

$$\Gamma[a] = 2\delta \int d^4x a^3\left\{-\frac{R}{2} + \tilde{\zeta}Y_{2,1} + \tilde{\xi}Y_{2,2}\right\} + \Delta\Gamma. \tag{122}$$

Here $\Delta\Gamma$ accounts for an energy-momentum tensor of matter $T_{\mu\nu}$ which has the usual coupling to the metric. From $\delta\Delta\Gamma/\delta a = (\delta\Delta\Gamma/\delta g^{ij})(\delta g^{ij}/\delta a) = (\sqrt{g}g_{ij}p/2)\times(-2a^{-3}\delta_{ij}) = -3pa^2$, one finds formally

$$\Delta\Gamma = -\int d^4x p a^3. \tag{123}$$

By partial integration one has

$$\Gamma[a] = 2\delta \int d^4x\left\{a\dot{a}^2\left(3 - \tfrac{3}{2}\tilde{\zeta} - \tfrac{9}{2}\tilde{\xi}\right) - \frac{p}{2\delta}a^3\right\}, \tag{124}$$

and we infer the field equation from $\delta(\Gamma + \Delta\Gamma)/\delta a = 0$, i.e.,

$$-(3\dot{a}^2 + 6a\ddot{a})\left(1 - \frac{1}{2}\tilde{\zeta} - \frac{3}{2}\tilde{\xi}\right) = \frac{3}{2\delta}pa^2, \tag{125}$$

or

$$2\dot{H} + 3H^2 = -\left(1 - \tfrac{1}{2}\tilde{\zeta} - \tfrac{3}{2}\tilde{\xi}\right)^{-1}\frac{p}{2\delta}. \tag{126}$$

Combining Eq. (126) with energy-momentum conservation, $\dot{\rho} + 3H(\rho + p)$, this yields the standard Friedmann cosmology. Of course, these equations can also be derived by inserting the ansatz (114) into the field Eq. (50) cf. Appendix A.

The only difference from Einstein gravity turns out to be the different value of the Planck mass which can be extracted from cosmological observations as compared to the one inferred from local gravity measurements. Denoting the reduced Planck mass for cosmological observations by

$$\bar{M}_c^2 = 2\delta - \tfrac{1}{2}\zeta - \tfrac{3}{2}\xi = (1 - \tfrac{3}{2}\beta_A)\mu, \tag{127}$$

and comparing with Newtonian gravity (79), we infer the ratio

$$\frac{\bar{M}_c^2}{\bar{M}^2} = 1 - 2\beta_A. \tag{128}$$

This affects quantitative cosmology like nucleosynthesis or the cosmic microwave background. In view of the severe bound on β_A from post-Newtonian gravity derived in the preceding section these effects are very small, however. For $\beta_A = 0$ we recover precisely the standard Friedmann cosmology. Of course, this could be modified by a cosmological constant or other degrees of freedom not contained in E_μ^m. In particular, we note that the classical action (12) is dilatation invariant whereas the quantum effects induce a dilatation anomaly. For a suitable form of the anomaly this could lead to quintessence [20,21].

XII. PARTIAL BOSONIZATION

Our aim is a computation of $\Gamma[E_\mu^m]$ for the action (12). An explicit solution of the functional integral (17) seems out of reach and we have to proceed to approximations. There is no obvious small parameter in the problem since the parameter α can be rescaled to an arbitrary value by a rescaling of ψ. Nonperturbative approximations will be hard to control but they should give at least an insight into the qualitative structure.

A convenient tool in our context is partial bosonization. This method reformulates the fermionic functional integral (17) into an equivalent functional integral involving both bosonic and fermionic degrees of freedom. In this formulation the dynamical role of the fluctuations in the "gravitational degrees of freedom" will become apparent. The reformulation is achieved [28–31] by use of a functional integral over fermions ψ and bosons $\hat{\chi}^m_\mu$, i.e.,

$$W'[J] = \ln \int D\psi D\hat{\chi}^n_\nu \exp\left\{\int d^dx[\alpha \det(\tilde{E}^m_\mu - \hat{\chi}^m_\mu) - \alpha \det(\tilde{E}^m_\mu) + J^\mu_m \hat{\chi}^m_\mu]\right\}. \tag{129}$$

We show in Appendix B that the free energy W is equivalent to the one of the original theory (17) up to a local polynomial in $\det(J^\mu_m)$

$$W'[J] = W[J] + \int d^dx F[\det J^\mu_m(x)]. \tag{130}$$

Performing derivatives of W with respect to J at $J = 0$, one obtains the connected correlation functions for composite fermion bilinears $\tilde{E}^m_\mu$. We see that all connected correlation functions involving less than d powers of $\tilde{E}^m_\mu$ are equal for the new "partially bosonized" functional integral and the original theory. (The fermionic correlation functions are equal anyhow.) In particular, the expectation value $\langle \tilde{E}^m_\mu \rangle$ and the two point function can equally well be computed in the partially bosonized setting. The first difference appears in the connected correlation function for d powers of $\tilde{E}^m_\mu$. These differences in the high order correlation functions are not relevant for our discussion and we will omit the prime on W from now on, treating the partially bosonized theory as an equivalent version of the original fermionic theory.

It is apparent from Eqs. (18) and (129) that the expectation value of $\hat{\chi}^m_\mu$ is given by the vielbein or fermion bilinear

$$\langle \hat{\chi}^m_\mu \rangle = E^m_\mu = \frac{i}{2}\langle \bar{\psi}\gamma^m \partial_\mu \psi - \partial_\mu \bar{\psi}\gamma^m \psi \rangle. \tag{131}$$

Using the definition of the effective action (19) and performing a variable shift $\hat{\chi}^m_\mu = E^m_\mu + \chi^m_\mu$, one obtains a convenient implicit functional integral expression

$$\Gamma[E^m_\mu] = -\ln \int D\psi D\chi^n_\nu \exp\left\{\int d^dx[\alpha \det(\tilde{E}^m_\mu - E^m_\mu - \chi^m_\mu) - \alpha \det(\tilde{E}^m_\mu) + J^\mu_m \chi^m_\mu]\right\}, \tag{132}$$

where J^μ_m is given by Eq. (20).

The classical approximation to Γ neglects all fluctuation effects and simply reads

$$\Gamma_{cl} = \tilde{\alpha} \int d^dx E, \qquad E = \det(E^m_\mu). \tag{133}$$

One easily infers the classical field Eq. (20)

$$\frac{\tilde{\alpha}}{(d-1)!}\epsilon^{\mu_1\dots\mu_d}\epsilon_{m_1\dots m_d} E^{m_2}_{\mu_2}\cdots E^{m_d}_{\mu_d} = J^{\mu_1}_{m_1}. \tag{134}$$

Whenever $E \neq 0$ we may introduce the inverse vielbein E^μ_m obeying the relations (30). For nonzero E the classical field equation therefore becomes

$$\tilde{\alpha} E E^\mu_m = J^\mu_m. \tag{135}$$

We can use this form in order to show that the field Eq. (135) has for $J^\mu_m \to 0$ only solutions with

$$E = 0. \tag{136}$$

Indeed, the classical field Eq. (135) implies that a nonzero finite value of E is in contradiction with $J^\mu_m = 0$. Of course, $E = 0$ does not require $E^m_\mu = 0$. For example, a possible solution is ($\bar{D} < d - 1$)

$$E^m_\mu = \begin{cases} \delta^m_\mu & \text{for } \mu = 0\dots\bar{D}, m = 0\dots\bar{D} \\ 0 & \text{otherwise.} \end{cases} \tag{137}$$

For $\bar{D} = 3$ this would describe a flat four-dimensional space-time geometry which we may associate with Minkowski space later. The remaining $d - 4$ dimensions would not admit a metric description, however. We also observe a large degeneracy of possible classical solutions. Finally, in the presence of a nonvanishing energy-momentum tensor (21), the classical solution reads (for $E \neq 0$)

$$\tilde{\alpha} g_{\mu\nu} = T_{\mu\nu}. \tag{138}$$

XIII. GENERALIZED DIRAC OPERATOR AND LOOP EXPANSION

This situation is expected to change drastically once the fluctuation effects are included. A simple approximation includes only the fermionic fluctuations in one loop order. For this purpose, we put $\chi^m_\mu = 0$ in Eq. (132) and expand

$$\det(\tilde{E}^m_\mu - E^m_\mu) = (-1)^d E[1 - E^\mu_m \tilde{E}^m_\mu + 0(\tilde{E}^2)]. \tag{139}$$

This yields the quadratic term in ψ,

$$S_{(2)} = -\frac{i\tilde{\alpha}}{2}\int d^dx E(\bar{\psi}\gamma^m E^\mu_m \partial_\mu \psi - \partial_\mu \bar{\psi} E^\mu_m \gamma^m \psi), \tag{140}$$

and the one loop expression

$$\Gamma = \tilde{\alpha}\int d^dx E + \Gamma_{(1l)},$$
$$\Gamma_{(1l)} = -\ln \int D\psi \exp\{-S_{(2)}[\psi, E^m_\mu]\}. \tag{141}$$

The Gaussian Grassmann integration for the fermionic one loop contribution can be evaluated explicitly. We concentrate here on Majorana spinors,[25] where $\bar{\psi}$ and ψ are identified (up to the matrix C)

[25] For Dirac spinors the relevant operator $S^{(2)}_F$ turns out to be the same [17], but there is no factor 1/2 in (142). Also the matrix C is absent. This plays no role since $\det C = 1$.

$$\Gamma_{(1l)} = -\tfrac{1}{2}\ln\det S_F^{(2)}, \qquad (142)$$

with

$$S_F^{(2)} = -2i\tilde{\alpha}C[E\gamma^\mu\partial_\mu + \tfrac{1}{2}\partial_\mu(E\gamma^\mu)], \qquad (143)$$

where

$$\gamma^\mu = E_m^\mu\gamma^m. \qquad (144)$$

Up to irrelevant constants we can also write

$$\Gamma_{(1l)} = -\tfrac{1}{2}\mathrm{Tr}\ln(E\mathcal{D}), \qquad (145)$$

$$\mathcal{D} = \gamma^\mu\partial_\mu + \frac{1}{2E}\gamma^m\partial_\mu(EE_m^\mu) = \gamma^\mu\hat{D}_\mu. \qquad (146)$$

We call $\mathcal{D}$ the generalized Dirac operator and observe the appearance of a "covariant derivative"

$$\hat{D}_\mu = \partial_\mu + \frac{1}{2E}E_\mu^m\partial_\nu(EE_m^\nu). \qquad (147)$$

[For Weyl spinors one should either multiply $\mathcal{D}$ by an appropriate projection operator $(1+\bar{\gamma})/2$ or work within a reduced space of spinor indices, using $C\gamma^m$ instead of γ^m since only $C\gamma^m$ acts in the reduced space.] The contribution from the derivative acting on the vielbein can also be written in the form

$$\mathcal{D} = \gamma^m(E_m^\mu\partial_\mu - \Omega_m), \qquad \Omega_m = -\frac{1}{2E}\partial_\mu(EE_m^\mu). \qquad (148)$$

It is instructive to compare the generalized Dirac operator $\mathcal{D}$ with the corresponding operator $\mathcal{D}_E$ in Einstein gravity. The latter is constructed from the Lorentz covariant derivative D_μ which appears in the spinor kinetic term (Majorana spinors)

$$i\bar{\psi}\gamma^\mu D_\mu\psi = i\bar{\psi}\gamma^m e_m^\mu\Big(\partial_\mu - \frac{1}{2}\omega_{\mu np}\Sigma^{np}\Big)\psi = i\bar{\psi}\mathcal{D}_E\psi$$
$$= i\bar{\psi}\gamma^\mu\partial_\mu\psi - \frac{i}{4}\Omega_{[mnp]}\bar{\psi}\gamma_{(3)}^{mnp}\psi. \qquad (149)$$

Here $\gamma_{(3)}^{mnp}$ is the totally antisymmetrized product of three γ matrices $\gamma_{(3)}^{mnp} = \gamma^{[m}\gamma^n\gamma^{p]}$, and $\Omega_{[mnp]}$ corresponds to the total antisymmetrization of

$$\Omega_{mnp} = -\tfrac{1}{2}e_m^\mu e_n^\nu(\partial_\mu e_{\nu p} - \partial_\nu e_{\mu p}). \qquad (150)$$

Replacing e_μ^m by E_μ^m, one finds

$$\mathcal{D} = \mathcal{D}_E[E] + \tfrac{1}{4}\Omega_{[mnp]}[E]\gamma_{(3)}^{mnp}. \qquad (151)$$

For the fermionic loop contribution the only difference between spinor gravity and standard gravity concerns the totally antisymmetric piece $\sim\Omega_{[mnp]}$.

Neglecting the piece $\sim\Omega_{[mnp]}$, the first contribution $\mathcal{D}_E[E_\mu^m]$ is covariant with respect to both general coordinate and *local* Lorentz transformations. Replacing $\mathcal{D}\to\mathcal{D}_E$ in the integral (145) will therefore lead to a one loop effective action Γ_{1l} with these symmetries. This is a gravitational effective action of the standard type. Expanded in the number of derivatives, one will find the curvature scalar plus higher derivative invariants like $R^2, R_{\mu\nu}R^{\mu\nu}$ etc. However, the additional piece $\sim\Omega_{[mnp]}[E_\mu^m]$ violates the local Lorentz symmetry and only preserves a global Lorentz symmetry. We therefore expect the appearance of new terms in the effective action which are invariant under global but not local Lorentz rotations. According to Eq. (151), all additional terms must involve $\Omega_{[mnp]}$ or derivatives thereof. They vanish for $\Omega_{[mnp]} = 0$. As discussed at the end of Sec. VII, the linear approximation to $\Omega_{[mnp]}$ only involves $c_{\nu\rho}$ and we can conclude that $\beta_A = 0$. This concerns precisely the "dangerous term" restricted by observation (112). We conclude that one loop spinor gravity is consistent with all tests of general relativity.

This simple argument can be confirmed by an explicit computation [13]. The overall coefficient μ in the effective action (36), as well as ϵ, depends on the precise choice of the regularization. In contrast, the relative coefficients τ_A and β_A are regularization independent and characterized by the de Witt coefficients [9,32] of the generalized Dirac operator. One obtains [6,13]

$$\tau_A = 3, \qquad \beta_A = 0. \qquad (152)$$

We finally observe that the trace in Eq. (145) involves a trace over spinor indices as well as an integration over space coordinates or, equivalently, a momentum integral in Fourier space. As it stands, these integrations are highly divergent in the ultraviolet and the integral (145) needs a suitable regularization. This regularization should preserve the invariance under general coordinate transformations. If possible, it should also preserve the global Lorentz symmetry. However, there may be obstructions in the form of "gravitational anomalies" [24] for $d = 6\mathrm{mod}4$. At present it is not known if such anomalies occur in spinor gravity. For the time being, we neglect this possible complication and assume global Lorentz symmetry of the effective action.

We do not claim quantitative accuracy for our one loop evaluation of the bosonic effective action. In particular, the value of the coefficient τ_A may be affected by higher loop orders. Also, dimensional reduction from a higher dimensional spinor gravity theory will affect the effective four-dimensional value of τ_A. In contrast, our finding $\beta_A = 0$ may be more robust. First of all, one may perform a similar computation [14] by the solution of the Schwinger-Dyson equation (without using partial bosonization). In lowest order, the nontrivial contribution to Γ will again be characterized by Eq. (145) while the coefficient of the "classical contribution" (133) will be modified [22]. More generally, all higher orders in the evaluation of Γ using the Schwinger-Dyson approach will involve powers of the exact fermionic propagator in the "background" of a vielbein E_μ^m [22]. One may expand the exact inverse fermionic propagator $\Gamma_F^{(2)}$ [the generalization of $S_F^{(2)}$ (143)] up to terms linear and quadratic in $\partial_\mu E_\nu^m$. If the corresponding operator $\mathcal{D}$ [the generalization of Eq. (151)] does not contain the linear field w^μ (64) we can conclude that β_A vanishes to all orders. First investigations [14] suggest that such a property could be related to a hidden nonlinear symmetry. As an interesting alternative, $\beta_A = 0$ could be associated with an infrared stable partial fixed point in the flow of generalized couplings.

XIV. CONCLUSIONS

In [13] and this paper, we have formulated a proposal for a unified theory based only on spinor fields. We insist on a well-defined action which is a polynomial in the spinor Grassmann field. The action is invariant under general coordinate and global Lorentz transformations, whereas local Lorentz symmetry may be violated. Within spinor gravity, the vielbein is not a fundamental field but rather arises as a composite object or bound state. It is described by the expectation value of a fermion bilinear. The metric can be formed as usual from the product of two vielbeins. As a consequence of the missing local Lorentz symmetry the vielbein contains, however, new physical degrees of freedom not described by the metric. This leads to a version of generalized gravity with global instead of local Lorentz symmetry. We discuss in detail the observational consequences of such a generalization. In particular, we find that the form suggested by the one loop approximation to spinor gravity is compatible with all present tests of general relativity. We conclude that the local character of the Lorentz symmetry is tested only very partially by observations.

Can spinor gravity serve as a candidate for a fundamental theory of all interactions? Several important steps have to be taken before this question can be answered. First of all, only a well-defined and diffeomorphism invariant regularization procedure for the functional measure would make the expectation values of fermion bilinears explicitly calculable. Second, the most general form of the classical action admits many independent polynomials which are invariant with respect to diffeomorphisms and global Lorentz rotations. At this stage, the corresponding dimensionless couplings remain undetermined. A predictive unified theory would have to select a particular point in the high dimensional space of possible couplings. One possibility is that this particular point is associated with an enhanced symmetry [7]. As an interesting alternative, the renormalization flow of the couplings could reveal a fixed point which would render spinor gravity renormalizable. If such a fixed point exists, the number of relevant (or marginal) parameters at the fixed point would determine the number of free parameters entering the predictions for physical quantities. If there is only one relevant direction, it could be associated with the overall mass scale of the theory by dimensional transmutation. In such a case no free dimensionless coupling would remain and spinor gravity would become completely predictive. Only the number of dimensions would influence the outcome of a calculation of fermion bilinears like the vielbein. Third, it remains to be shown that spinor gravity formulated in a suitable dimension $d > 4$ admits an interesting ground state with a small characteristic length scale for the $d - 4$ internal dimensions and a large scale for the observed dimensions. The isometries of this ground state should be the gauge group $SU(3) \times SU(2) \times U(1)$ of the standard model (up to tiny effects of electroweak symmetry breaking) and the chirality index should account for three generations of quarks and leptons. Obviously, the way towards such a goal is still long, but, we believe, worthwhile pursuing.

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APPENDIX A: FIELD EQUATIONS

In this appendix, we give details for the field equations of generalized gravity which are used in the main text. We use the form (48) for the effective action. In order to derive the field equations for the effective action (48), we expand the inverse vielbein in linear order:

$$E_m^\mu = \bar{E}_m^\mu + \delta E_m{}^\mu. \tag{A1}$$

For the vielbein, metric, and determinant this implies

$$\begin{aligned} E_\nu^n &= \bar{E}_\nu^n - \bar{E}_\nu^m \bar{E}_\mu^n \delta E_m{}^\mu, \\ g^{\mu\nu} &= \bar{g}^{\mu\nu} + \bar{E}^{m\nu}\delta E_m{}^\mu + \bar{E}^{m\mu}\delta E_m{}^\nu, \\ E &= \bar{E}(1 - \bar{E}_\mu^m \delta E_m{}^\mu). \end{aligned} \tag{A2}$$

Omitting the bars, the first variation of the invariants reads

$$\delta(\epsilon\Gamma_0 - \bar{\delta}\Gamma_{2,R}) = -\int d^d x E\{\bar{\delta}(2R_\mu^m - RE_\mu^m) + \epsilon E_\mu^m\}\delta E_m{}^\mu, \tag{A3}$$

and

$$\delta\Gamma_{2,1} = \int d^d x E\{4\Omega_{\mu\nu}{}^n\Omega^{m\nu}{}_n + 2(D_\nu\Omega^{\rho\nu}{}_n)E_\rho^m E_\mu^n - \Omega_{\nu\rho}{}^n\Omega^{\nu\rho}{}_n E_\mu^m\}\delta E_m{}^\mu, \tag{A4}$$

$$\delta\Gamma_{2,2} = \int d^d x E\{E_n^\rho \partial_\rho (D^\nu E_\nu^n) E_\mu^m - \partial_\mu (D^\nu E_\nu^m) + \frac{1}{2} D^\nu E_\nu^n D_\rho E_n^\rho E_\mu^m\} \delta E_m{}^\mu. \tag{A5}$$

We therefore obtain the field equation from $\delta\Gamma/\delta E_m{}^\mu = 0$ as

$$\begin{aligned}\bar{\delta}(2R_\mu^m - RE_\mu^m) + \epsilon E_\mu^m =& \zeta\{4\Omega_{\mu\nu}{}^n \Omega^{m\nu}{}_n + 2(D_\nu \Omega^{\rho\nu}{}_n) E_\rho^m E_\mu^n - \Omega_{\nu\rho}{}^n \Omega^{\nu\rho}{}_n E_\mu^m\} \\ &+ \xi\{E_n^\rho \partial_\rho (D^\nu E_\nu^n) E_\mu^m - \partial_\mu (D^\nu E_\nu^m) + \tfrac{1}{2} D^\nu E_\nu^n D_\rho E_n^\rho E_\mu^m\}.\end{aligned} \tag{A6}$$

By multiplication with $E_{\nu m}$ we can bring this into the form of a modified Einstein equation

$$\begin{aligned}2\bar{\delta}(R_{\mu\nu} - \tfrac{1}{2} R g_{\mu\nu}) = & -\epsilon g_{\mu\nu} + \zeta\{4\Omega_{\mu\rho m}\Omega_\nu{}^{\rho m} - 2(D_\rho \Omega^\rho{}_{\nu m}) E_\mu^m - \Omega_{\sigma\rho}{}^m \Omega^{\sigma\rho}{}_m g_{\mu\nu}\} + \xi\{E_m^\sigma \partial_\sigma (D^\rho E_\rho^m) g_{\mu\nu} \\ & - \partial_\mu (D^\rho E_\rho^m) E_{\nu m} + \tfrac{1}{2} D^\sigma E_\sigma^m D^\rho E_{\rho m} g_{\mu\nu}\}.\end{aligned} \tag{A7}$$

This yields the field Eq. (50) for generalized gravity.

We next want to specialize to the general static and isotropic ansatz (95) of Secs. IX and X. We first compute the tensor $\Omega_{\mu\nu}{}^m = -\frac{1}{2}(\partial_\mu E_\nu^m - \partial_\nu E_\mu^m)$. With $\Omega_{0i}{}^m = \frac{1}{2}\partial_i E_0^m$, one finds

$$\Omega_{0i}{}^0 = \frac{x_i}{2r} f', \qquad \Omega_{0i}{}^j = \frac{h}{2}\delta_{ij} + h' \frac{x_i x_j}{2r}. \tag{A8}$$

Similarly, we obtain

$$\Omega_{ki}{}^0 = 0, \qquad \Omega_{ki}{}^j = -\frac{k}{2}(\delta_{jk} x_i - \delta_{ji} x_k). \tag{A9}$$

For the invariant $\Gamma_{2,1}$ we calculate $Y_{2,1}$ which yields Eq. (96):

$$\begin{aligned}Y_{2,1} &= \Omega_{\mu\nu}{}^m \Omega^{\mu\nu}{}_m = g^{ij}\{\Omega_{ik}{}^n \Omega_{jl}{}^n g^{kl} - 2\Omega_{0i}{}^0 \Omega_{0j}{}^0 g^{00} + 2\Omega_{0i}{}^k \Omega_{0j}{}^k g^{00}\} \\ &= \frac{k^2}{2}\{(g^{ij}\delta_{ij})(g^{kl} x_k x_l) - g^{ij} g^{jk} x_i x_k\} + \frac{h^2}{2} g^{00}(g^{ij}\delta_{ij}) + \frac{1}{2}\left(h'^2 + \frac{2hh'}{r} - \frac{f'^2}{r^2}\right) g^{00}(g^{kl} x_k x_l) \\ &= \frac{d-2}{2}\left(\frac{k^2 r^2}{A} - \frac{h^2}{B}\right) - \frac{1}{2AB}[(h + rh')^2 - f'^2] \\ &= \frac{d-2}{2r^2}\left(1 + \frac{1}{A} - 2\sqrt{\frac{B + r^2 h^2}{AB}}\right) + \frac{1}{8AB(B + r^2 h^2)}[B'^2 + 4rB'h(h + rh') - 4B(h + rh')^2].\end{aligned} \tag{A10}$$

Here we use

$$g^{00} = -\frac{1}{B(r)}, \qquad g^{ij} = \delta_{ij} - \frac{A(r) - 1}{A(r) r^2} x_i x_j, \tag{A11}$$

and

$$g^{ij}\delta_{ij} = d - 2 + \frac{1}{A}, \qquad g^{kl} x_k x_l = \frac{r^2}{A}, \qquad g^{ij} g^{jk} = \delta_{ik} - \frac{A^2 - 1}{A^2 r^2} x_i x_k, \qquad g^{ij} g^{jk} x_i x_k = \frac{r^2}{A^2}. \tag{A12}$$

In order to evaluate the invariant $\Gamma_{2,2}$ we need

$$\tilde{D}^\mu E_\mu^m = g^{\mu\nu}(\partial_\mu E_\nu^m - \Gamma_{\mu\nu}^\lambda E_\lambda^m) = \left(\delta_{ij} - \frac{A-1}{Ar^2} x_i x_j\right)\partial_i E_j^m - \Gamma_\mu{}^{\mu i} E_i^m. \tag{A13}$$

Here we have employed the explicit form of the connection in our Cartesian coordinates

$$\begin{aligned}&\Gamma_{00}{}^0 = 0, \quad \Gamma_{00}{}^i = \frac{B'}{2rA} x_i, \quad \Gamma_{0i}{}^0 = \frac{B'}{2rB} x_i, \quad \Gamma_{ij}{}^0 = 0, \quad \Gamma_{i0}{}^j = 0, \\ &\Gamma_{ij}{}^k = \frac{A-1}{Ar^2}\left(\delta_{ij} x_k - \frac{x_i x_j x_k}{r^2}\right) + \frac{A'}{2r^3 A} x_i x_j x_k, \quad \Gamma_\mu{}^{\mu 0} = 0, \quad \Gamma_\mu{}^{\mu i} = \left\{(d-2)\frac{A-1}{Ar^2} + \left(\frac{A'}{A} - \frac{B'}{B}\right)\frac{1}{2rA}\right\} x_i.\end{aligned} \tag{A14}$$

This yields

$$D^\mu E^0_\mu = \frac{1}{\sqrt{AB}}[(d-1)h + rh'], \qquad D^\mu E^k_\mu = x_k\left\{\frac{d-2}{r^2}\left(\sqrt{\frac{B+r^2h^2}{AB}} - 1\right) + \frac{1}{2r\sqrt{AB}\sqrt{B+r^2h^2}}[B' + 2rh(h+rh')]\right\}, \tag{A15}$$

and we infer the invariant $Y_{2,2}$ in Eq. (97).

We can also compute the components of the tensor $\hat{T}_{\mu\nu}$ in the field Eq. (50) and (51),

$$\hat{T}_{\mu\nu} = \zeta\hat{T}^{(1)}_{\mu\nu} + \xi\hat{T}^{(2)}_{\mu\nu}. \tag{A16}$$

One finds for $h = 0$ and $d = 4$

$$\hat{T}^{(1)}_{00} = -\frac{B'}{Ar} + \frac{A'B'}{4A^2} + \frac{B'^2}{8AB} - \frac{B''}{2A} + \frac{B}{r^2}\left(1 - \frac{1}{\sqrt{A}}\right)^2. \tag{A17}$$

Combining this with the corresponding expressions for $\hat{T}^{(1)}_{ij}$, $\hat{T}^{(2)}_{00}$, and $\hat{T}^{(2)}_{ij}$, one may compute the isotropic field equations in a formally more direct but computationally more cumbersome way as compared to Sec. X. For this purpose, we also need

$$\begin{aligned} R_{00} &= \frac{d-2}{2}\frac{B'}{rA} + \frac{B''}{2A} - \frac{A'B'}{4A^2} - \frac{B'^2}{4AB}, \\ R_{ij} &= \delta_{ij}\left\{\frac{A'}{2rA^2} - \frac{B'}{2rAB} + (d-3)\frac{A-1}{Ar^2}\right\} + \frac{x_ix_j}{r^2}\left\{\frac{B'^2}{4B^2} + \frac{B'A'}{4AB} - \frac{B''}{2B} + \frac{B'}{2rAB} + \left(d-2-\frac{1}{A}\right)\frac{A'}{2rA} - (d-3)\frac{A-1}{Ar^2}\right\}, \\ R_{0i} &= 0, \end{aligned} \tag{A18}$$

and

$$R = -\frac{B''}{AB} + \frac{B'}{2AB}\left(\frac{A'}{A} + \frac{B'}{B}\right) + \frac{d-2}{rA}\left(\frac{A'}{A} - \frac{B'}{B}\right) + (d-3)(d-2)\frac{A-1}{Ar^2}. \tag{A19}$$

Let us finally turn to the computation of the field equation relevant for cosmology, with the ansatz (114) for the vielbein. For a computation of $\hat{T}_{\mu\nu}$, we need the components of $E^m_\mu D_\rho \Omega^\rho_{\nu m}$, i.e.,

$$\begin{aligned} E^m_0 D_\rho \Omega^\rho{}_{0m} = 0, \qquad D_\rho\Omega^\rho{}_{0i} = D_\rho\Omega^\rho{}_{i0} = 0, \\ E^m_j D_\rho \Omega^\rho{}_{im} = \tfrac{1}{2}a^2(\dot{H} + 2H^2)\delta_{ij}, \end{aligned} \tag{A20}$$

and $\Omega_{\mu\rho m}\Omega_\nu{}^{\rho m}$, i.e.,

$$\begin{aligned} \Omega_{0\rho m}\Omega_0{}^{\rho m} = \tfrac{3}{4}H^2, \qquad \Omega_{0\rho m}\Omega_i{}^{\rho m} = 0 \\ , \Omega_{i\rho m}\Omega_j{}^{\rho m} = -\tfrac{1}{4}H^2a^2\delta_{ij}. \end{aligned} \tag{A21}$$

We may also use

$$E^\sigma_m \partial_\sigma(D^\rho E^m_\rho) = -3\dot{H}. \tag{A22}$$

This yields

$$\begin{aligned} \hat{T}_{00} &= \tfrac{3}{2}(\zeta + 3\xi)H^2, \\ \hat{T}_{ij} &= -\tfrac{1}{2}(\zeta + 3\xi)(2\dot{H} + 3H^2)g_{ij}. \end{aligned} \tag{A23}$$

With

$$\begin{aligned} R_{00} - \tfrac{1}{2}Rg_{00} &= 3H^2, \\ R_{ij} - \tfrac{1}{2}Rg_{ij} &= -(3H^2 + 2\dot{H})g_{ij}, \end{aligned} \tag{A24}$$

one finally finds

$$\begin{aligned} (3H^2 + 2\dot{H})\left(1 - \frac{\tilde{\zeta}}{2} - \frac{3\tilde{\xi}}{2}\right) &= -\frac{p}{2\delta}, \\ 3H^2\left(1 - \frac{\tilde{\zeta}}{2} - \frac{3\tilde{\xi}}{2}\right) &= \frac{\rho}{2\delta}. \end{aligned} \tag{A25}$$

The first equation coincides with Eq. (126), whereas the combination of both equations ensures the energy-momentum conservation of matter:

$$\dot{\rho} = -3H(\rho + p). \tag{A26}$$

APPENDIX B: FUNCTIONAL IDENTITY FOR PARTIAL BOSONIZATION

In this appendix we derive the partially bosonized functional integral (129). For this purpose we use the identity [recall $\tilde{E}^m_\mu = i(\bar{\psi}\gamma^m\partial_\mu\psi - \partial_\mu\bar{\psi}\gamma^m\psi)/2$]

$$\int D\hat{\chi}^n_\nu \exp\int d^dx\{\alpha\det(\tilde{E}^m_\mu - \hat{\chi}^m_\mu) - J^\mu_m(\tilde{E}^m_\mu - \hat{\chi}^m_\mu)\} = \exp\int d^dx\tilde{V}[J^\mu_m(x)], \tag{B1}$$

with

$$\tilde{V}(J^\mu_m) = \ln\int d\tilde{\chi}^n_\nu \exp[-\tilde{\alpha}\det(\tilde{\chi}^m_\mu) + J^\mu_m\tilde{\chi}^m_\mu] \tag{B2}$$

an even function of J^μ_m for d even and $\tilde{\alpha} = (-1)^{d+1}\alpha$. (We omit the irrelevant additive constant for $J = 0$.) Furthermore, $\tilde{V}$ is invariant under global Lorentz transformations of J^μ_m. More generally, $\tilde{V}$ remains invariant under all special linear transformations acting on the $d \times$

d matrix J from the left or right. [This follows from the invariance of the integral (B2) under accompanying (inverse) special linear transformations acting on the integration variable $\tilde{\chi}$.] Therefore $\tilde{V}$ can only be a function of the determinant $\det(J_m^\mu)$. The definition of the integral (B2) may be somewhat formal (even after subtraction of the value for $J = 0$) since $\det(\tilde{\chi}_\mu^m)$ has positive and negative eigenvalues. (Note that $\tilde{\alpha}$ is imaginary for a Minkowski signature.) We assume that it can be suitably regularized such that $\tilde{V}$ is analytic in J. We then conclude that $\tilde{V}$ is an analytic function of $\det(J_m^\mu)$:

$$\tilde{V}(J_m^\mu) = F[\det(J_m^\mu)] = \beta_1 \det(J_m^\mu) + \beta_2[\det(J_m^\mu)]^2 + \cdots. \tag{B3}$$

The relation (130) is now easily obtained by performing in Eq. (129) the functional integral over $\hat{\chi}_\mu^m$ using Eq. (B1).

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