PHYSICAL REVIEW D, VOLUME 70, 105003

Running coupling with minimal length

S. Hossenfelder
Department of Physics, University of Arizona, 1118 East 4th Street, Tucson, Arizona 85721, USA
(Received 21 June 2004; published 2 November 2004)

In models with large additional dimensions, the GUT scale can be lowered to values accessible by future colliders. Because of modification of the loop corrections from particles propagating into the extra dimensions, the logarithmic running of the couplings of the Standard Model is turned into a power law. These loop corrections are divergent and the standard way to achieve finiteness is the introduction of a cutoff. The question remains, whether the results are reliable as they depend on an unphysical parameter. In this paper, we show that this running of the coupling can be calculated within a model including the existence of a minimal length scale. The minimal length acts as a natural regulator and allows us to confirm cutoff computations.

DOI: 10.1103/PhysRevD.70.105003 PACS numbers: 11.10.Kk

I. INTRODUCTION

The Standard Model (SM) of particle physics yields an extremely precise theory for the electroweak and strong interaction. It is renormalizable and physical observables can be computed, its results proven by experimental data. The Standard Model allows us to improve our view of nature in many ways but leaves us with several unsolved problems.

Among them, the question of how to consistently describe quantum effects of gravity is without doubt one of the most challenging and exciting problems in physics of this century. When extrapolating the strength of the Standard Model interactions by using the renormalization group equations, the three couplings converge. Within the minimal supersymmetric extension of the Standard Model (MSSM), the couplings meet in one point (within the $\alpha_3(M_Z)$ uncertainty) close to $\approx 10^{16}$ GeV [1].

The study of models with large extra dimensions has recently received a great deal of attention. These models, which are motivated by string theory [2], provide us with an extension to the Standard Model in which observables can be computed and predictions for tests beyond the Standard Model can be addressed. This in turn might help us to extract knowledge about the underlying theory once we have data to analyze. The need to look beyond the Standard Model infected many experimental groups to search for such SM-violating processes (for a summary see, e.g., [3].

One of the most striking consequences of the large extra dimensions is that unification can occur at a lowered fundamental scale $M_{\rm f}$, caused by a power law-running of the gauge couplings. This modified running of the couplings was originally derived by Taylor and Veneziano [4] and has been analyzed in the context of the Standard Model by Dienes, Dudas and Gherghetta [5]. The lowered unification scale being one of the central issues of the models with large extra dimensions, the question of the running coupling has been addressed in a large number of further works [6–13], enlightening the subject in many regards. However, these loop corrections are divergent and the standard way to achieve finiteness is with the introduction of a cutoff Λ. In this case, the question remains whether these results are reliable as they depend on an unphysical parameter.

In this paper we want to demonstrate how the assumption of a minimal length scale $L_{\rm f}$ fits in this scenario naturally. Moreover, the minimal length removes ambiguities which come along with the cutoff renormalization.

Throughout the whole paper we use the conventions $c = \hbar = 1$, $M_{\rm f} = 1/L_{\rm f}$, and the notation $\epsilon = L_{\rm f}^2$. Latin indices run over all dimensions.

II. LARGE EXTRA DIMENSIONS

The recently proposed models of extra dimensions successfully fill the gap between theoretical conclusions and experimental possibilities as the extra hidden dimensions may have radii large enough to make them accessible to experiments. Thus, they are an approach towards a phenomenology of grand unified theories (GUTs) at TeV-scale.

There are different ways to build a model of extra dimensional space-time. Here, we want to mention only the most common ones:

(1) The ADD-model proposed by Arkani-Hamed, Dimopoulos and Dvali [14] adds d extra spacelike dimensions without curvature, in general, each of them compactified to the same radius R. All Standard Model particles are confined to our brane, while gravitons are allowed to propagate freely in the bulk.

(2) Within the model of universal extra dimensions (UXD) [2,5,15] all gauge fields (or in some extensions, also fermions) can propagate in the whole multidimensional space-time. The extra dimen-

*Electronic address: sabine@physics.arizona.edu

sions are compactified on an orbifold to reproduce Standard-Model gauge degrees of freedom.

(3) The setting of the model from Randall and Sundrum [16,17] is a five-dimensional space-time with an nonfactorizable—so called warped—geometry. The solution for the metric is found by analyzing the solution of Einsteins field equations with an energy density on our brane, where the SM particles live. In the Randall-Sundrum (RS) one model [16] the extra dimension is compactified, in the RS two model [17] it is infinite.

It might as well be that nature chose to realize a mixture of (1) and (2) or (2) and (3). For a more general review on the subject the reader is referred to [18]. In the following we will focus on the model (2) with d denoting the number of extra dimensions, keeping in mind that there might exist further dimensions.

In the model of UXDs the momentum into the extra dimensions is conserved for gauge boson interactions. Therefore, Kaluza-Klein (K-K) excitations can only be produced in pairs; modifications to Standard-Model processes do not occur at tree level but arise from loop contributions. Constraints from electroweak data and collider experiments thus allow radii to be as large as $1/R \sim$ TeV [19]. Throughout this paper, we fix $1/R = 1$ TeV as a representative value.

III. RUNNING COUPLING

In quantum field theory, the running of the gauge coupling constants is a consequence of the renormalization process, the energy dependence of the coupling constant arising from loop contributions to the propagator of the gauge field(s). In a four-dimensional space-time, these contributions are known to be logarithmically divergent, $\int d^4p/p^4 \sim \int dp/p \sim \ln p$. In a higher dimensional space-time, divergences get worse. As is well known, higher-dimensional field theories are generally nonrenormalizable. In this case one has to introduce a hard cutoff Λ in order to render the result finite. The existence of extra dimensions then yields a power law explicitly depending on the cutoff parameter Λ, which is expected to be in the range of the new fundamental scale.

There are a vast number of publications on this topic [6], examining the issue within various classes of unification models and special regard of one- and two-step models [5,8,9]. It has been investigated [8] how the chosen subset of particles allowed to propagate into the bulk can achieve a more precise unification point, and detailed analysis of two-loop corrections and threshold effects [7,12] have been given.

During the last years it has been pointed out that the relevant loop corrections suffer from increased UV-sensitivity and that, as a result, no precise statement can be made about the behavior of the gauge couplings without first removing the UV-problem (this has, e.g., been mentioned in [10,12]). A proposal to this has been made by Hebecker and Westphal [11] by using a soft breaking of the GUT-group symmetry. The fact that the theory is nonrenormalizable surely is due to the fact that it has to be viewed as an effective theory, designed to model a deeper yet not understood fundamental theory.

The power law-running of the gauge coupling in a higher-dimensional space-time can be explained by assuming that the β-function coefficient b_i at an energy Λ is proportional to the number of active flavors, meaning in this context the number of K-K-modes with excitation energies below Λ. In this case one finds

$$b_i \sim \Omega_d(\Lambda R)^d, \tag{1}$$

with Ω_d being the volume of the d-dimensional sphere

$$\Omega_d = \frac{\pi^{(d/2)}}{\Gamma(1 + d/2)}. \tag{2}$$

This dependence on the energy scale is also justified by hard cutoff computations. Introducing an infrared cutoff μ_0 as well as an ultraviolet cutoff Λ, the behavior of the one-loop corrections can be estimated as

$$\int_{\mu_0}^{\Lambda} \frac{d^{d+4}p}{p^4} \sim \Lambda^d - \mu_0^d. \tag{3}$$

Performing this calculation, one is faced with the problem that the result depends explicitly on the cutoff Λ. This forces one to interpret the cutoff as the renormalization scale μ, giving rise to one-loop-corrected values of the gauge coupling $\alpha_i(\Lambda)$ as functions of the value of this cutoff parameter. In many cases in quantum field theories this cutoff dependence is identical to the scale dependence, which can be computed using reliable renormalization schemes that do not depend on the regulator, e.g., dimensional regularization [20].

In particular, there remain several ambiguities using the cutoff formalism. The first problem at hand is whether the cutoff Λ agrees with the regularization scale μ. Further, the use of a cutoff on the K-K-tower immediately raises the question for the threshold of the modes and how they are correctly added to the tower. Especially regarding the first mode, when using the above arguments, below the energy $1/R$ there are no excitations of K-K-modes at all. The value $1/R$ thus acts essentially as an infrared cutoff. The higher dimensional theory is matched to the four-dimensional logarithmic running at this infrared cutoff. It is unclear within this procedure in which way the crossing of the thresholds is performed best and whether the matching point to the theory on the brane is chosen correctly. Since the value of the matching point is the onset of the power law-running, its value is crucial for the value of the unification scale.

Further, besides all educated arguments, the constant for the coefficient in (1) finally has to be fixed by hand. This modifies the slope of the running once the threshold is crossed. None of these problems alter the main point that the coupling constants get power law corrections and that they unify at a lowered scale. But they are unsatisfactory from a theoretical point of view and do not allow us to make predictions.

As the minimal length we introduce modifies the measure of the momentum space in the ultraviolet region, the troublesome loop contributions get finite. The minimal length acts as a natural regulator, but in contrast to computations using cutoff regularization techniques, we expect the result to depend on the new parameter, as it is an order parameter for physics beyond the Standard Model.

IV. MINIMAL LENGTH

A. General Motivation

Even if a full description of quantum gravity is not yet available, there are some general features that seem to go hand in hand with all promising candidates for such a theory. One of them is the need for a higher-dimensional space-time; another is the existence of a minimal length scale. As the success of string theory arises from the fact that interactions are spread out on the world sheet and no longer take place at one singular point, the finite extension of the string has to become important at small distances or high energies, respectively. Now that we are discussing the possibility of a lowered fundamental scale, we want to examine the modifications arising from this, as they might become observable soon. If we do so, we should clearly take into account the minimal length effects.

In perturbative string theory [21,22], the feature of a fundamental minimal length scale arises from the fact that strings cannot probe distances smaller than the string scale. If the energy of a string reaches this scale $M_s = \sqrt{\alpha'}$, excitations of the string can occur and increase its extension [23]. In particular, an examination of the space-time picture of high-energy string scattering shows that the extension of the string grows proportional to its energy [21] in every order of perturbation theory. Because of this, uncertainty in position measurement can never become arbitrarily small. For a review, see [24,25].

In this paper we will implement both of these phenomenologically motivated issues of string theory—the extra dimensions and the minimal length—into quantum field theory. We do not aim to derive them from a fully consistent theory of first principles. Instead, we will analyze the consequences for the running coupling and ask what conclusions might be drawn for the underlying theory.

B. Minimal Length in Quantum Mechanics

Naturally, the minimum length uncertainty is related to a modification of the standard commutation relations between position and momentum [26,27]. With the Planck scale as high as 10^{16} TeV, applications of this are of high interest, mainly for quantum fluctuations in the early universe and for inflation processes, and have been examined closely [28,29].

There are several approaches of how to deal with the generalization of the relation between momentum and wave vector (see, e.g., [30]). To incorporate the notion of a minimal length into ordinary quantum field theory we will apply a simple model, which has been worked out in detail in [31].

We assume, no matter how much we increase the momentum p of a particle, we can never decrease its wavelength below some minimal length $L_{\rm f}$ or, equivalently, we can never increase its wave vector k above $M_{\rm f}$. Thus, the relation between the momentum p and the wave vector k is no longer linear $p = k$ but a function [32] $k = k(p)$. This function $k(p)$ has to fulfill the following properties:

(1) For energies much smaller than the new scale, we reproduce the linear relation: for $p \ll M_{\rm f}$ we have $p \approx k$.

(2) It is an odd function (because of parity) and k is collinear to p (see also Fig. 5).

(3) The function asymptotically approaches the upper bound $M_{\rm f}$.

We will assume that $L_{\rm f} \ll R$ so that the spacing of the Kaluza-Klein excitations compared to energy scales $M_{\rm f}$ becomes almost continuous and we can use the integral form.

Lorentz-covariance is not added to the above list, as the proposed model can not provide conservation of this symmetry. This is easy to see if we imagine an observer who is boosted relative to the minimal length. He then would observe a contracted minimal length which would be even smaller than the minimal length. To resolve this problem it might be inevitable to modify the Lorentz-transformation. Several attempts to construct such transformations have been made [33] but no clear answers have been given yet. Therefore we will assume p is a Lorentz vector, aim to express all quantities in terms of p and otherwise have to cope with a lack of Lorentz-covariance in k-space. One might think of constructing a covariant relation, but since the only covariant quantity available is p^2 and thus a constant [34] which is fixed by (1), we had no upper bound (3).

A relation fulfilling the above properties might be put in the form

$$k_\mu = \hat{e}_\mu \xi(p_{\rm e}), \tag{4}$$

where the index e denotes the Euclidean norm and $\hat{e}_\mu$ is the unit vector in μ-direction. We will specify the exact form later (see end of this section).

The quantization of these relations is straightforward and follows the usual procedure. The commutators between the corresponding operators $\hat{k}$ and $\hat{x}$ remain in the standard form. Using the well-known commutation relations

$$[\hat{x}_i, \hat{k}_j] = \mathrm{i}\delta_{ij} \tag{5}$$

and inserting the functional relation between the wave vector and the momentum then yields the modified commutator for the momentum

$$[\hat{x}_i, \hat{p}_j] = +\mathrm{i}\frac{\partial p_i}{\partial k_j}. \tag{6}$$

This results in the generalized uncertainty relation

$$\Delta p_i \Delta x_j \geq \frac{1}{2}\left|\left\langle\frac{\partial p_i}{\partial k_j}\right\rangle\right|, \tag{7}$$

which reflects the fact that by construction it is not possible to resolve space-time distances arbitrarily well. Since k gets asymptotically constant, its derivative $\partial k/\partial p$ drops to zero and the uncertainty in (7) increases for high energies. The behavior of our particles thus agrees with those of the strings found by Gross [21] as mentioned above.

The form of the new operator $\hat{p}_i$ is most easily analyzed when we expand the inverted relation $p(k)$ in a power series with coefficients a_n. In general, in the one-dimensional case, suppose we have the series

$$p_x = k_x + \sum_{n\geq 1} a_n k_x^{2n+1}. \tag{8}$$

It can then be seen that in position representation the momentum operator takes the form

$$\hat{p}_x = -\mathrm{i}\partial_x + \sum_{n\geq 1} a_n(-\mathrm{i})^{2n+1}\partial_x^{2n+1}. \tag{9}$$

Since $k = k(p)$, we have for the eigenvectors $\hat{p}(\hat{k})|k\rangle = p(k)|k\rangle$ and so $|k\rangle \propto |p(k)\rangle$. We could now add that both sets of eigenvectors have to be a complete orthonormal system and therefore $\langle k'|k\rangle = \delta(k - k')$, $\langle p'|p\rangle = \delta(p - p')$. This seems to be a reasonable choice at first sight, since $|k\rangle$ is known from the low-energy regime. Unfortunately, now the normalization of the states is different because k is restricted to the Brillouin zone $-1/L_\mathrm{f}$ to $1/L_\mathrm{f}$.

To avoid the need to recalculate normalization factors, we choose the $|p(k)\rangle$ to be identical to the $|k\rangle$. Following the proposal of [26], this yields then a modification of the measure in momentum space.

To make this point more clearly, especially in the presence of compactified extra dimensions, let x be the uncompactified coordinates on our brane and y the coordinates in the direction of the compactified extra dimensions. Since each of the latter is compactified on the same radius R, we have for the d-dimensional volume $\mathrm{Vol}_d(y)$ of the extra dimensions

$$\mathrm{Vol}_d(y) = (2\pi R)^d. \tag{10}$$

In addition to this, the volume of momentum space $\mathrm{Vol}(p_y)$ in the extra dimensions is also finite,

$$\mathrm{Vol}(p_y) = \Omega_d \frac{L_\mathrm{f}^d}{(2\pi)^d}, \tag{11}$$

where we have assumed that in the limit of small R the K-K-modes have smooth spacing in the directions of the extra dimensions. Now consider the expansion of the wave function ϕ in terms of eigenfunctions $|k\rangle = |p(k)\rangle$,

$$|k\rangle = e^{\mathrm{i}(k_x x + k_y y)}, \tag{12}$$

where the wave vector in direction of the extra dimensions k_y is geometrically quantized in steps n/R. The expansion then reads

$$\phi(x, y) = \sum\!\!\!\!\!\!\int \frac{\mathrm{d}^3 k_x}{(2\pi)^{d+3}} \frac{e^{\mathrm{i}(k_x x + k_y y)}}{N}, \tag{13}$$

where N is the normalization factor which has to be correctly set in the presence of a minimal length. The eigenfunctions are normalized to

$$\begin{aligned}\langle p'(k')|p(k)\rangle &= (2\pi)^{3+d}\delta(k_x' - k_x)\delta_{k_y' k_y} R^d \\ &= (2\pi)^{3+d}\delta(p_x' - p_x)\left|\frac{\partial p_i}{\partial k_j}\right|\delta_{p_y' p_y} R^d,\end{aligned} \tag{14}$$

where the functional determinant of the relation is responsible for an extra factor accompanying the δ-functions. When taking the continuum limit of (14) we find with $\delta_{k_y' k_y} R^d \to \delta(k_y' - k_y)$ the usual normalization.

So the completeness relation of the modes takes the form

$$\sum\!\!\!\!\!\!\int \frac{\mathrm{d}^3 k_x}{(2\pi)^{d+3}} \frac{\langle k'|k\rangle}{N} = R^d \mathrm{Vol}_d(p_y). \tag{15}$$

To avoid a new normalization of the eigenfunctions, we take the factors into the integral by a redefinition of the measure in momentum space

$$\mathrm{d}^{d+3}k \to \mathrm{d}^{d+3}p\left|\frac{\partial k_i}{\partial p_j}\right|\frac{1}{\mathrm{Vol}_d(p_y)R^d}. \tag{16}$$

This redefinition has a physical interpretation because we expect the momentum space to be squeezed at high-momentum values and weighted less. In the standard scenario with a noncompact momentum space, we have $(2\pi)^d \mathrm{Vol}_d(p_y) = \mathrm{Vol}_d(y)$ and thus the factor cancels to one.

C. Minimal Length in Quantum Field Theory

To proceed towards quantum field theory we could now take the continuum limit of (6). The purpose of our computations is to express all quantities in terms of the momentum p, as we eventually wish to describe physical observables. Keeping the relations with the wave vector k gives back the familiar relations but does not allow us to connect to particle physics. However, in intermediate steps we can stick to the k-formalism and proceed with a minimum of modifications. Regarding the fact that we have to give up an easy transformation from coordinate space to momentum space, we go on with the wave vectors and can apply Fourier transformations.

When using the Feynman rules in k-space we first have to make sure that we use the right conservation law. As the relation between the wave vector and the momentum is no longer linear, k is not additive and it is not conserved in particle interactions although it is conserved for one propagating particle (since it is a function of a conserved quantity). So, the right conservation factor for the vertices with in- and outgoing momenta $p^{\rm n}$, where n labels the participating particles, and $p_\alpha^{\rm tot} = \sum_{\rm n} p_\alpha^{\rm n}$, the total sum of the momenta is

$$\delta^{4+d}[k(p_\alpha^{\rm tot})] = \delta^{4+d}[(p_\alpha^{\rm tot})] \left| \frac{\partial p_\nu}{\partial k_\mu} \right|. \tag{17}$$

Now what about the dynamics of the particle? In general, the Lagrangian $\mathcal{L}^\phi$ for a scalar field ϕ is derived by quantization of the energy momentum relation. So, we find in the continuous case

$$\mathcal{L}^\phi = \int {\rm d}^{d+4}x \phi[\hat{p}(k)^2 - m^2]\phi. \tag{18}$$

As before, the modification arises solely by the fact that $\hat{p}$ is now a function of k. The propagator can then be found in k-space by a Fourier transformation

$$\Delta^\phi(x) = \int d^{d+4}k \frac{e^{-ikx}}{p(k)^2 - m^2}, \tag{19}$$

and so

$$\Delta^\phi(k) = \frac{1}{p(k)^2 - m^2}. \tag{20}$$

As is well known, the Lagrangian in the given form leads to complications in the generating functional. Working in Minkowski-space, the path integral does not converge as the exponent, given by $\mathcal{L}$, is not positive definite. We adopt the usual procedure for this problem by performing a Wick rotation and changing to Euclidean space. In this case, the propagator takes the form

$$\Delta_{\rm e}^\phi(k) = \frac{1}{p_{\rm e}^2 + m^2}. \tag{21}$$

Similar derivations as for the scalar field apply for fermion fields and yield

$$\Delta^F(k) = \frac{1}{p\!\!\!/(k) - m}. \tag{22}$$

As expected, the propagator in k-space can, in general, be found by the replacement $k \rightarrow p(k)$. To derive the interaction terms, one has to couple gauge fields to the free Lagrangian. It has been shown in [31] that in an approximation in first order (first order as well in the couplings as in $M_{\rm f}$ or mixtures of both) the vertices are not modified.

To summarize, we have then the following procedure to compute diagrams:

(1) Make computations in k-space and apply usual Feynman rules.
(2) Take the propagator as a function of $p(k)$.
(3) Use conservation of momentum on the vertices $\delta[k(\sum p)]$.
(4) Finally, replace the k-integration via

$$\frac{{\rm d}^{d+4}k}{(2\pi)^{4+d}} \rightarrow \frac{{\rm d}^{d+4}p}{(2\pi)^{4+d}} \left| \frac{\partial k_\mu}{\partial p_\nu} \right| \frac{1}{{\rm Vol}_d(p_y)R^d}. \tag{23}$$

V. MINIMAL LENGTH AND RUNNING GAUGE-COUPLINGS

The aim of our calculations is an investigation of the running of the gauge couplings in an energy range $p \sim M_{\rm f}$. In the following, we will use the specific relation for $p(k)$ by choosing for the scalar function in (4)

$$\xi(p_{\rm e}) = \int_0^{p_{\rm e}} \exp\left(-\epsilon \frac{\pi}{4} p_{\rm e}'^2\right) {\rm d}p_{\rm e}', \tag{24}$$

where the factor $\pi/4$ is included to ensure that the limiting value is $L_{\rm f}$. A frequently used relation in the literature [28] is $\xi(p) = \tanh^{1/\gamma}(p^\gamma)$, with γ being some positive integer. Both of these choices for modeling the minimal length are compared in Fig. 5. As can be seen in the considered energy range, the differences are negligible. The model dependence at smaller energies will be addressed in the next section.

The Jacobian determinate of the function $k(p)$ is best computed by adopting spherical coordinates and can be approximated for $p \sim M_{\rm f}$ with

$$\left| \frac{\partial k_\mu}{\partial p_\nu} \right| \approx \exp\left(-\epsilon \frac{\pi}{4} p_{\rm e}^2\right). \tag{25}$$

Since this factor occurs as a modification to the measure in momentum space, we see clearly that the minimal length acts essentially as a cutoff regulator. However, in contrast to cutoff calculations in quantum field theory, here the cutoff has a physical interpretation and is cause for effects on its own. The regulator itself is a parameter of the model. It is the existence of a fundamental length which implies that processes involving high energies will be suppressed and the UV-behavior of the theory will be improved. So, we are able to perform an integration over the whole K-K-tower instead of truncating the high end.

As an example, we have computed the one-loop correction to the photon propagator, using the above derived steps. This may be found in Appendix A.

The effect of the minimal length on the integration over momentum space is essentially that the contributions at high momenta get suppressed and the loop results with high external momenta approach a constant value. We have two effects working against each other. On the one hand, we have the power law arising from the extra dimensions; on the other hand, we have the exponential suppression arising from the minimal length.

The relation between the higher-dimensional coupling constant $\tilde{g}_i$ and the four-dimensional coupling $g_i^2 = 4\pi\alpha_i$ is given by the volume of the extra dimensions

$$g_i = \tilde{g}_i \mathrm{Vol}_d(y). \tag{26}$$

To examine the running of the coupling constants α_i, we assume that above the supersymmetry breaking scale M_{SUSY} we are dealing with the MSSM, whereas below M_{SUSY} we have the symmetry groups of the Standard Model.

The summarized one-loop contributions arising from the structure constants groups of the SM (after inclusion of the factor 3/5 for α_1) read

$$b^{\mathrm{SM}} = [b_1^{\mathrm{SM}}, b_2^{\mathrm{SM}}, b_3^{\mathrm{SM}}] = [4, 10/3, -7]. \tag{27}$$

Within the MSSM, the number of fermion generations $n_{\mathrm{g}} = 3$, and the number of Higgs fields $n_{\mathrm{h}} = 2$, we have then above M_{SUSY} the coefficients

$$\begin{aligned} b &= [b_1, b_2, b_3] \\ &= [0, -6, -9] + n_{\mathrm{g}}[2, 2, 2] + n_{\mathrm{h}}[3/10, 1/2, 0] \\ &= [33/5, 1, -3]. \end{aligned} \tag{28}$$

As pointed out in [5], these supersymmetric b_i coefficients will change in a higher-dimensional space-time due to the different content of the superfields. This content of the K-K-excitations of the fields can be accommodated in hyper multiplets of $N = 2$ supersymmetry instead of the $N = 1$ supersymmetry in the four-dimensional space-time. Therefore, the modified one-loop contributions have factors different from the MSSM ones. In this paper, we will consider only the case in which all fermions are confined to the brane ($n_{\mathrm{g}} = 0$). Then the factors for the excitation modes are given by

$$\tilde{b} = [\tilde{b}_1, \tilde{b}_2, \tilde{b}_3] = [0, -4, -6] + n_{\mathrm{h}}[3/10, 1/2, 0]. \tag{29}$$

The running of the couplings above the scale of SUSY-breaking M_{SUSY} is given by the familiar expression

$$\frac{\tilde{\alpha}_i(q')}{\tilde{\alpha}_i(q)} = 1 - (b_i - \tilde{b}_i)[\pi(q,0) - \pi(q',0)] - \tilde{b}_i[\pi(q,d) - \pi(q',d)], \tag{30}$$

where $\pi(q,d)$ denotes the finite part of the scalar factor in the one-loop contribution, which leads to a renormalization of the gauge-field propagator. It should be noted that the inclusion of the minimal length does not remove infrared divergences. Thus, a proper regularization is still necessary, resulting in a difference between "bare" and "physical" couplings.

The higher-dimensional one-loop contributions to the propagator can now be calculated by using the formalism developed in Section IV. We find that the infrared regularized result can be given in the integral form (see Appendix A)

$$\begin{aligned} \pi(q,d) = 3b_i \frac{\alpha_i}{2\pi} \frac{(2\pi)^d}{\Omega_d} (\pi\tilde{\epsilon})^{d/2} \bigg[\int_0^1 \mathrm{d}x x(1-x)^{1+d/2} \\ \times \int_{\tilde{\epsilon}}^\infty \mathrm{d}z e^{-zxq^2} z^{-1-d/2} + \frac{1}{q^2} \frac{(d+4)}{2(d+3)} \tilde{\epsilon}^{-1-d/2} \\ \times \int_0^1 \mathrm{d}x (1-x)^{1+d/2} \Big(e^{-\tilde{\epsilon}xq^2} - 1 \Big) \bigg], \end{aligned} \tag{31}$$

with the abbreviation $\tilde{\epsilon} = \epsilon\pi/4$. The result does depend explicitly on the parameter ϵ since this is a physical quantity in our description. As expected, we find two effects: the first giving a power-law behavior (the power depending on d), which can be located in the power of z, the second an exponential drop due to the minimal length, which can be located in the nonzero lower bound of z-integration.

Let us briefly compare this with the result using the hard cutoff computation where the sliding scale q is identified with the cutoff Λ (see, e.g., [5]). It is obvious that in our scenario the role of the UV cutoff is given to L_{f}. We thus interpret the only free parameter as energy scale:

$$\begin{aligned} \frac{\tilde{\alpha}_i(q')}{\tilde{\alpha}_i(q)} = 1 - \alpha_i(q') \frac{b_i}{2\pi} \ln\frac{q}{q'} + \Theta(q-\mu_0)\alpha_i(q')\frac{\tilde{b}_i}{2\pi} \\ \times \bigg(\ln\frac{q}{\mu_0} - \frac{\Omega_d}{d} \chi_d^d [(qL_{\mathrm{f}})^d - (\mu_0 L_{\mathrm{f}})^d] \bigg). \end{aligned} \tag{32}$$

Here μ_0 is the matching point below which the four-dimensional logarithmic running is unmodified and χ_d is an unknown factor usually set to be equal to one. In the above expression, Θ denotes the Heaviside-function.

The comparison to our result is best done when making a power-series expansion of the integral form (31) for small ϵ. For $\Delta\pi(q,q',d) = \pi(q,d) - \pi(q',d)$ we find

$$\Delta\pi(q,q',0) = b_i \frac{\alpha_i}{2\pi} \bigg[\ln\Big(\frac{q}{q'}\Big) - \frac{1}{3}\tilde{\epsilon}(q^2 - q'^2) + \mathcal{O}(\tilde{\epsilon}^2) \bigg] \tag{33}$$

$$\Delta\pi(q,q',1) = b_i \frac{\alpha_i}{2\pi} \bigg[\frac{9}{32}\pi^3 \tilde{\epsilon}^{1/2}(q-q') - \frac{74}{105}\pi^{3/2}\tilde{\epsilon}(q^2 - q'^2) + \mathcal{O}(\tilde{\epsilon}^2) \bigg] \tag{34}$$

$$\Delta\pi(q, q', 2) = b_i \frac{\alpha_i}{2\pi}\Big[-\frac{2}{5}\pi^3\tilde{\epsilon}(q^2 \ln\tilde{\epsilon}q^2 - q'^2 \ln\tilde{\epsilon}q'^2) - \frac{\pi^3}{150}\tilde{\epsilon}(60\gamma - 89)(q^2 - q'^2) + \mathcal{O}(\tilde{\epsilon}^2)\Big] \quad (35)$$

$$\Delta\pi(q, q', 3) = b_i \frac{\alpha_i}{2\pi}\Big[\frac{656}{393}\pi^{9/2}\tilde{\epsilon}(q^2 - q'^2) - \frac{5}{32}\pi^6\tilde{\epsilon}^{3/2}(q^3 - q'^3) + \frac{2528}{9009}\pi^{9/2}\tilde{\epsilon}^2(q^4 - q'^4) + \mathcal{O}(\tilde{\epsilon}^3)\Big] \quad (36)$$

$$\Delta\pi(q, q', 4) = b_i \frac{\alpha_i}{2\pi}\Big[-\frac{4}{7}\pi^6\tilde{\epsilon}(q^2 - q'^2) - \frac{869 - 420\gamma}{2450}\pi^6\tilde{\epsilon}^2(q^4 - q'^4) + \frac{42}{245}\tilde{\epsilon}^2\pi^6(q^4 \ln\tilde{\epsilon}q^2 - q'^4 \ln\tilde{\epsilon}q'^2) + \mathcal{O}(\tilde{\epsilon}^3)\Big] \quad (37)$$

$$\Delta\pi(q, q', 5) = b_i \frac{\alpha_i}{2\pi}\Big[\frac{592}{1287}\pi^{15/2}\tilde{\epsilon}(q^2 - q'^2) - \frac{928}{2145}\pi^{15/2}\tilde{\epsilon}^2(q^4 - q'^4) + \frac{7}{128}\pi^9\tilde{\epsilon}^{5/2}(q^5 - q'^5) - \frac{8768}{109\,395}\pi^{15/2}\tilde{\epsilon}^3(q^6 - q'^6) + \mathcal{O}(\tilde{\epsilon}^4)\Big], \quad (38)$$

For $d = 0$ we find the familiar logarithmic divergence. For higher d we find that an odd number of extra dimensions leads to one-loop corrections with a power law, whereas for an even number of extra dimensions there is a mixture of the d-power term with a logarithmic contribution. It can be seen that in contradiction to the results from introduction of a cutoff in momentum space, the leading power is not d. This conclusion agrees with analyses from [12] using dimensional regularization. It is interesting to note that in the limit $R \gg L_f$ the result no longer depends on the value of the radius of the extra dimensions.

The scale μ_0 in (32) usually is chosen to be $1/R$. This yields a good agreement with our minimal length scenario for $1/R$ close to M_{SUSY} and particular values of χ_d. However, for even values of d the power law in (32) is not a good fit.

There are three main points which are new to our results:

(1) Using the minimal length we do not need to introduce an initial threshold (in addition to the symmetry breaking scale) as we can include *all* virtual K-K-excitations.

(2) There is no arbitrariness for the parameter χ_d or the identification of the energy scale.

(3) The couplings no longer run with a pure power law.

VI. NUMERICAL RESULTS

In the following we will compare the full result (31) to the cutoff result and give numerical values for χ_d in the parametrization (32). This numerical fit is optimized to best reproduce the unification point of the full result. We will set $\mu_0 = M_{SUSY} = 1/R$ and match the curve with the Standard Model result at this energy.

For the initial values, we use the data set [35]

$$M_Z = 91.197 \pm 0.007 \text{ GeV}$$

$$\alpha_1(M_Z)^{-1} = 58.98 \pm 0.04$$

$$\alpha_2(M_Z)^{-1} = 29.57 \pm 0.03$$

$$\alpha_3(M_Z)^{-1} = 8.5 \pm 0.5.$$

In Fig. 1 the result of our computation for fixed $d = 3$ and different values of L_f is shown. We see that the onset of the deviations from the four-dimensional result is roughly given by the inverse minimal length, and the unification point lies at an energy scale of the same order of magnitude. The value of the coupling at the unification point does not vary much and lies at $1/\alpha_i \approx 50$.

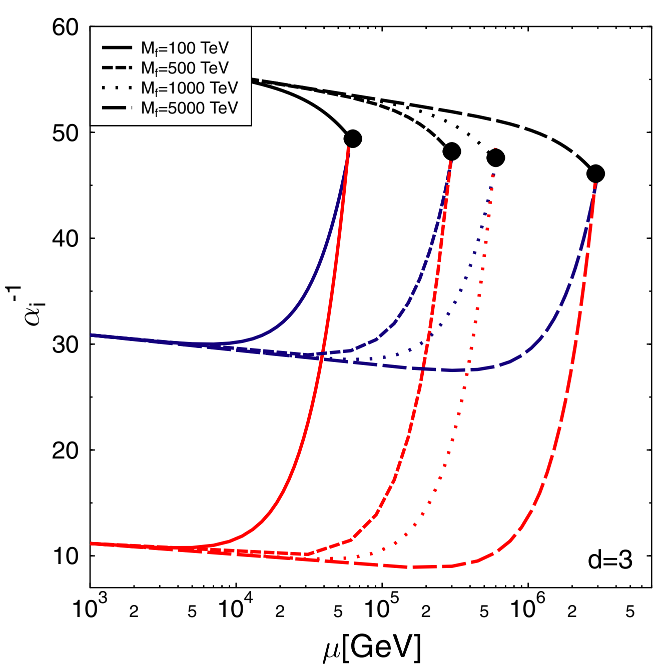

FIG. 1 (color online). The result for the running of the gauge-couplings for $M_f = 100, 500, 1000, 5000$ TeV and fixed $d = 3$.

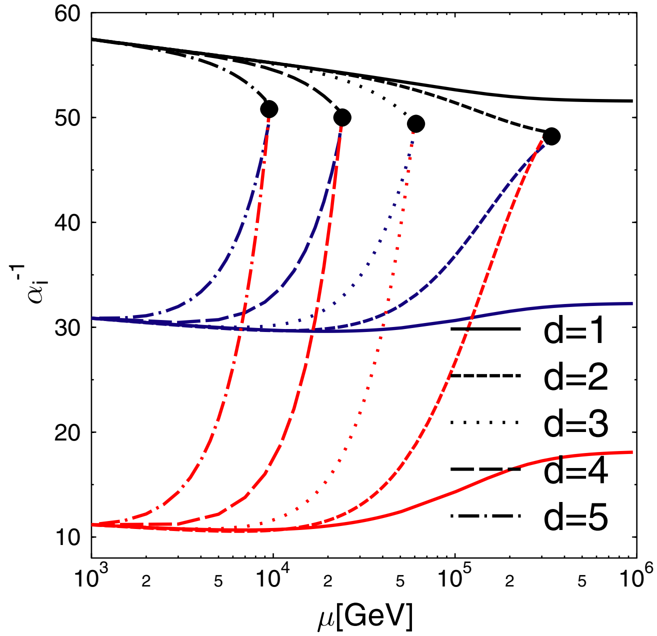

FIG. 2 (color online). The result for the running of the gauge-couplings for $d = 1, 2, 3$, four and fixed $M_{\rm f} = 100$ TeV.

Figure 2 shows the results of our computation for fixed $M_{\rm f} = 100$ TeV and different values of d. Here it shows clearly how the two factors — the power law-running and the dumping from the minimal length — act against each other. For $d = 1$ the minimal length avoids unification. For $d > 1$ it can be seen that a higher d leads to a faster running, and the unification point is reached before the exponential suppression becomes important.

Figure 3 shows a comparison of our result with the cutoff result, using the fitting parameter χ_d, whose values are depicted in Fig. 4. The errors are mainly due to the fact that in all cases the unification does not occur at one exact point.

Note that our specific choice of the functional relation, although not relevant for qualitative statements, introduces an additional model dependence at $p < M_{\rm f}$. To parametrize the lack of knowledge about the exact relation $k(p)$, consider the expansion

$$\xi(p_{\rm e}) = \sum_{i=0}^{n} c_i \left(\frac{p_{\rm e}}{M_{\rm f}}\right)^n e^{-\epsilon p_{\rm e}^2}, \tag{39}$$

with $c_0 = 1$. The parameters in this series can be trans-

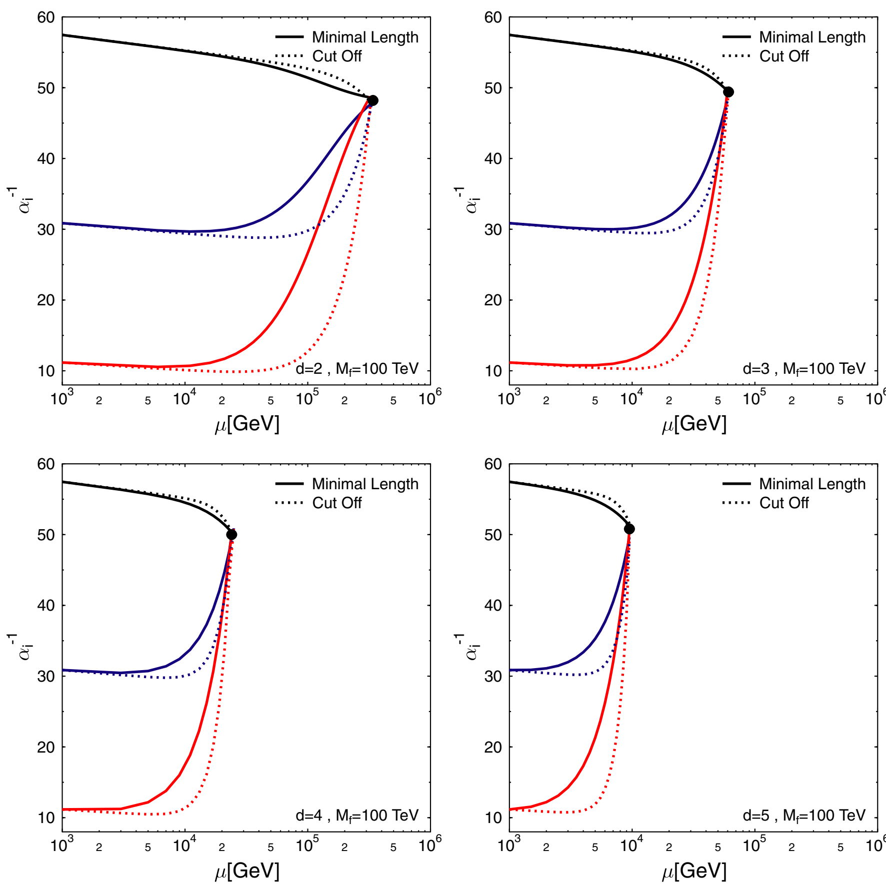

FIG. 3 (color online). $d = 2, 3, 4, 5$, $M_{\rm f} = 100$ TeV, $M_{\rm SUSY} = 1$ TeV. The dotted lines show the result with a hard cutoff, the solid lines the result from the minimal length.

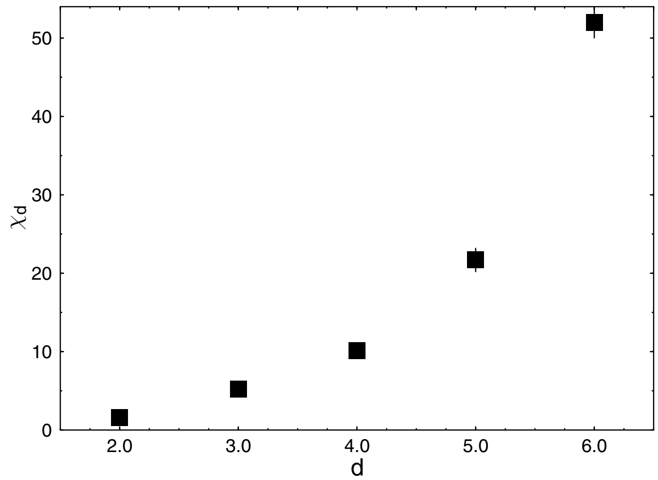

FIG. 4. The values of the fitting parameter χ_d. $\chi_2 = 2, 1 \pm 0.1$, $\chi_3 = 5.2 \pm 0.2$, $\chi_4 = 10.1 \pm 0.4$, $\chi_5 = 21.7 \pm 0.8$, and $\chi_6 = 52 \pm 1.9$.

formed into parameters in the functional determinant and further into parameters in the final expansion (33)–(38). The running of the coupling in this energy range therefore leads a direct connection to the behavior of the minimal length. The plot in Fig. 5 shows a comparison of different relations for $k(p)$. The dashed lines depict the function $\tanh^{1/\gamma}(p^\gamma)$ for different values of γ. The solid line between them shows our relation $\xi(p)$.

Further, we want to note that the above used assumption $R \gg L_f$, which justifies the replacement of the K-K-sum with an integral, leads numerically quiet good results even in the region where R and L_f differ only by 1 order of

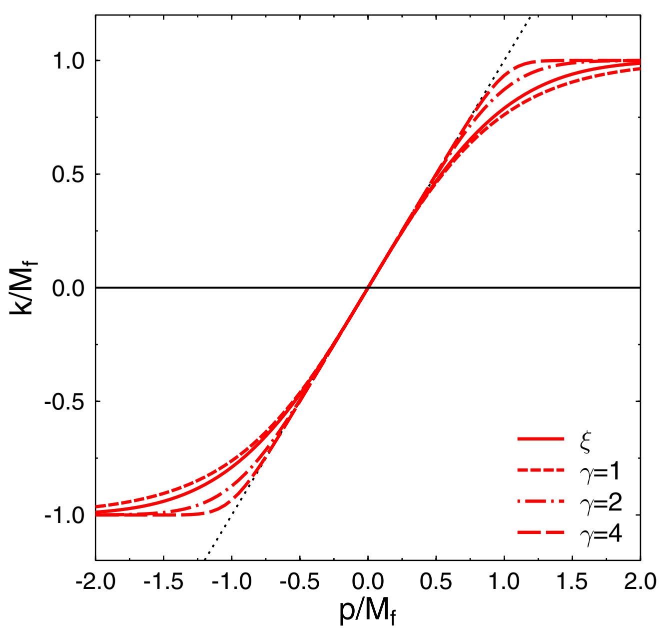

FIG. 5 (color online). The linear dotted line shows the case of no modification $k = p$.

magnitude. The approximation, however, breaks down for $R \to L_f$ as, in this case, the minimal length would avoid the existence of excitations at all.

VII. CONCLUSION

In this paper we computed the running of the gauge couplings in a higher- dimensional space-time at one-loop order. We proposed to remove the UV divergences with the introduction of a minimal length scale and examined the results on their dependence of the parameters. We found that the minimal length acts as a natural regulator. The scale dependence of the gauge couplings revealed a power law at energies below the inverse minimal length and stagnated at energies much higher than the inverse minimal length. In this high-energy region, the generalized uncertainty principle does not allow a further resolution of structures. The derived result for $d > 1$ confirms the cutoff regularized result and enriches the regularization scheme with a physical interpretation.

ACKNOWLEDGMENTS

I would like to thank Keith Dienes for valuable discussions and his contribution to this work. Further, I want to thank Stefan Hofmann and Jörg Ruppert for their answers as well as for their questions. This work was supported by the Postdoc-Programme of the German Academic Exchange Service (DAAD) and NSF Grant No. Phy-03-01998.

APPENDIX A

As an example, we compute the QED one-loop contribution to the photon propagator under inclusion of the modifications arising from the generalized uncertainty principle. The photon carries the external momentum q and therefore propagates on the brane. Here, we will treat the fermions circling in the loop as a higher-dimensional particle, even if we do not consider this case in the context of this paper. The result for loops of gauge bosons, which are allowed to leave the brane, is similar except for a constant factor arising from the structure constants of the gauge group. In the familiar way, all contributions can finally be summarized in the b_i coefficients.

Since the mass of the fermions is negligible at the energy scales $\approx M_f$ that we are interested in, we treat the particle as massless. Throughout this Appendix we perform the calculation in Euclidean space and suppress the index e.

With the abbreviation $\tilde{\epsilon} = \epsilon\pi/4$ the Feynman rules give, as explained in the text,

$$\Pi_{\mu\nu}(q, d) = e^2 \frac{(2\pi)^d}{\Omega_d} \tilde{\epsilon}^{d/2} \int \frac{d^{4+d}p}{(2\pi)^4} \mathrm{Tr}\left[\frac{\gamma_\mu}{\not{p}}\right] \times \left[\frac{\gamma_\nu}{\not{p} - \not{q}}\right] e^{-\tilde{\epsilon}p^2}, \tag{A1}$$

where the above expression is understood to result after the Wick-rotation, and where we have replaced the sum over K-K-modes by an approximate integral. We thus perform a higher-dimensional computation instead of using the effective theory on our brane. Since the external momentum q lies on our brane, it does not mix with the internal momenta p, and in an effective description the excitations therefore appear as a tower of massive particles. This effective theory on the brane is completely equivalent to the above one in the whole bulk.

As explained in the text, the zero mode needs further treatment because the b_i factors are different when lying on the brane only. This is taken into account with the second factor in (30) using the coefficients $\tilde{b}_i$. The zero mode is included in the above integral but with the wrong factor from the bulk modes. It therefore has to be subtracted and replaced with the brane-only term as in [5].

It should be noted that the above expression is gauge invariant as the formalism developed respects all symmetries in Euclidean space. To see this, contract the above expression with q. Gauge invariance then demands $q^\mu \Pi_{\mu\nu} = 0$. This can be written as

$$q^\mu \Pi_{\mu\nu}(q, d) \propto \int \frac{d^{4+d}p}{(2\pi)^{4+d}} \mathrm{Tr}\left[\frac{\not{q}}{\not{p}}\right]\left[\frac{\gamma_\nu}{\not{p} - \not{q}}\right] e^{-\tilde{\epsilon}p^2}. \quad \text{(A2)}$$

Now we rewrite the expression and return back to k-space to find

$$q^\mu \Pi_{\mu\nu}(q, d) \propto \int \frac{d^{4+d}k}{(2\pi)^{4+d}} \mathrm{Tr}\left[\frac{1}{\not{p}(k) - \not{q}} - \frac{1}{\not{p}(k)}\right]\gamma_\nu. \quad \text{(A3)}$$

Now we note that substituting $p' \to p - q$ in the first term does not modify the contours of integration, as the asymptotic value of $k(p')$ is still M_f. So the two terms are identical and cancel, keeping gauge invariance.

We then can assume

$$\Pi_{\mu\nu}(q, d) = \pi(q, d)(q_\mu q_\nu - g_{\mu\nu}q^2). \quad \text{(A4)}$$

By taking the trace of (A1) and using (A4), we find [36]

$$\pi(q, d) = \frac{e^2}{q^2}\frac{4(2+d)}{(3+d)}\frac{(2\pi)^d}{\Omega_d}\tilde{\epsilon}^{d/2}\int \frac{d^{4+d}p}{(2\pi)^4} \times \frac{p^2 - pq}{p^2(p-q)^2} e^{-\tilde{\epsilon}p^2}, \quad \text{(A5)}$$

Using a modified version of the Schwinger Proper time formalism

$$\frac{e^{-\tilde{\epsilon}p^2}}{p^2} = -\int_{\tilde{\epsilon}}^\infty \mathrm{d}z e^{-zp^2}, \quad \text{(A6)}$$

as well as the usual one with $\tilde{\epsilon} = 0$, we can further simplify the integral. At this stage it is apparent why the Euclidean norm is essential since the expression on the right side in (A9) otherwise would not converge.

We then arrive at

$$\pi(q, d) = \frac{e^2}{q^2}\frac{4(2+d)}{(3+d)}\frac{(2\pi)^d}{\Omega_d}\tilde{\epsilon}^{d/2} \times \int \frac{d^{4+d}p_e}{(2\pi)^4} \times \int_0^\infty \mathrm{d}z_1 \int_{\tilde{\epsilon}}^\infty \mathrm{d}z_2 (p^2 - pq) e^{-z_1(p-q)^2 - z_2 p^2}. \quad \text{(A7)}$$

After substituting $l := p - qz_1/(z_1 + z_2)$ and interchange of the z_i with the momentum integral, we can perform the momentum integration using the identities

$$\int \mathrm{d}^n x e^{-ax^2} = \left(\frac{\pi}{a}\right)^{n/2} \quad \text{(A8)}$$

$$\int \mathrm{d}^n x x^2 e^{-ax^2} = \frac{n}{2a}\left(\frac{\pi}{a}\right)^{n/2}. \quad \text{(A9)}$$

We use the further substitution $z_1 \to x := z_1/(z_1 + z_2)$ and relabel z_2 to z in order to allow an easy comparison to the standard result. Our expression for the one-loop correction then reads

$$\pi(q, d) = \frac{\alpha}{\pi q^2}\frac{(2+d)}{(3+d)}\frac{(2\pi)^d}{\Omega_d}(\pi\tilde{\epsilon})^{d/2} \times \int_0^1 \mathrm{d}x \int_{\tilde{\epsilon}}^\infty \mathrm{d}z e^{-zxq^2}\left(\frac{1-x}{z}\right)^{1+d/2}\left(\frac{d+4}{2z} - xq^2\right). \quad \text{(A10)}$$

Integrating the first term by parts yields

$$\pi(q, d) = 3b\frac{\alpha}{2\pi}\frac{(2\pi)^d}{\Omega_d}(\pi\tilde{\epsilon})^{d/2}\left[\int_0^1 \mathrm{d}x x(1-x)^{1+d/2} \times \int_{\tilde{\epsilon}}^\infty \mathrm{d}z e^{-zxq^2} z^{-1-d/2} + \frac{1}{q^2} \times \frac{(d+4)}{2(d+3)}\tilde{\epsilon}^{-1-d/2}\int_0^1 \mathrm{d}x(1-x)^{1+d/2} e^{-\tilde{\epsilon}xq^2}\right], \quad \text{(A11)}$$

where we have identified $b = 4/3$ as the beta-function coefficient of our single Dirac fermion. The second term in (A11) contains the infrared divergence. As we assume that the finite part of $\pi(q, d)$, which is of interest for our running coupling, fulfills the requirement $\pi(0, d) = 0$, which is necessary to preserve the pole-structure of the propagator, we subtract the divergent term and arrive at

$$\pi(q,d) = 3b\frac{\alpha}{2\pi}\frac{(2\pi)^d}{\Omega_d}(\pi\tilde{\epsilon})^{d/2}\left[\int_0^1 \mathrm{d}x x(1-x)^{1+d/2} \times \int_{\tilde{\epsilon}}^\infty \mathrm{d}z e^{-zxq^2} z^{-1-d/2} + \frac{1}{q^2} \times \frac{(d+4)}{2(d+3)}\tilde{\epsilon}^{-1-d/2}\int_0^1 \mathrm{d}x(1-x)^{1+d/2}\left(e^{-\tilde{\epsilon}xq^2} - 1\right)\right]. \quad \text{(A12)}$$

An analytic expansion in a power series in $\tilde{\epsilon}$ reveals the differences relative to the pure power law-running and is given in (33)–(38).

[1] U. Amaldi, W. de Boer, and H. Furstenau, Phys. Lett. B **260**, 447 (1991).
[2] I. Antoniadis, Phys. Lett. B **246**, 377 (1990); I. Antoniadis and M. Quiros, Phys. Lett. B **392**, 61 (1997); K. R. Dienes, E. Dudas, and T. Gherghetta, Nucl. Phys. **B537**, 47 (1999).
[3] G. Azuelos *et al.*, hep-ph/0204031.
[4] T. R. Taylor and G. Veneziano, Phys. Lett. B **212**, 147 (1988).
[5] K. R. Dienes, E. Dudas, and T. Gherghetta, Phys. Lett. B **436**, 55 (1998); K. R. Dienes, E. Dudas, and T. Gherghetta, hep-ph/9807522.
[6] C. P. Bachas, J. High Energy Phys. 11 (1998) 023; D. Ghilencea and G. G. Ross, Phys. Lett. B **442**, 165 (1998); E. G. Floratos and G. K. Leontaris, Phys. Lett. B **465**, 95 (1999); P. H. Frampton and A. Rasin, Phys. Lett. B **460**, 313 (1999); Z. Kakushadze, Nucl. Phys. **B548**, 205 (1999); Z. Berezhiani, I. Gogoladze, and A. Kobakhidze, Phys. Lett. B **522**, 107 (2001); J. Kubo, H. Terao, and G. Zoupanos, Nucl. Phys. **B574**, 495 (2000); D. M. Ghilencea and G. G. Ross, Nucl. Phys. **B606**, 101 (2001).
[7] M. Masip, Phys. Rev. D **62**, 065011 (2000); D. Dumitru and S. Nandi, Phys. Rev. D **62**, 046006 (2000).
[8] C. D. Carone, Phys. Lett. B **454**, 70 (1999).
[9] A. Perez-Lorenzana and R. N. Mohapatra, Nucl. Phys. **B559**, 255 (1999).
[10] R. Contino, L. Pilo, R. Rattazzi, and E. Trincherini, Nucl. Phys. **B622**, 227 (2002); K. w. Choi, H. D. Kim, and Y. W. Kim, J. High Energy Phys. 11, 033 (2002); N. Arkani-Hamed, A. G. Cohen, and H. Georgi, hep-th/0108089.
[11] A. Hebecker and A. Westphal, Ann. Phys. (N.Y.) **305**, 119 (2003).
[12] J. F. Oliver, J. Papavassiliou, and A. Santamaria, Phys. Rev. D **67**, 125004 (2003).
[13] A. Delgado and M. Quiros, Nucl. Phys. **B559**, 235 (1999).
[14] N. Arkani-Hamed, S. Dimopoulos, and G. R. Dvali, Phys. Lett. B **429**, 263 (1998); I. Antoniadis, N. Arkani-Hamed, S. Dimopoulos, and G. R. Dvali, Phys. Lett. B **436**, 257 (1998); N. Arkani-Hamed, S. Dimopoulos, and G. R. Dvali, Phys. Rev. D **59**, 086004 (1999).
[15] A. Delgado, A. Pomarol, and M. Quiros, Phys. Rev. D **60**, 095008 (1999); T. Appelquist, H. C. Cheng, and B. A. Dobrescu, Phys. Rev. D **64**, 035002 (2001); C. Macesanu, C. D. McMullen, and S. Nandi, Phys. Rev. D **66**, 015009 (2002); T. G. Rizzo, Phys. Rev. D **64**, 095010 (2001).
[16] L. Randall and R. Sundrum, Phys. Rev. Lett. **83**, 3370 (1999).
[17] L. Randall and R. Sundrum, Phys. Rev. Lett. **83**, 4690 (1999).
[18] V. A. Rubakov, Phys. Usp. **44**, 871 (2001); Y. A. Kubyshin, Lectures given at the XI. School Particles and Cosmology, Baksam, Russia, 2001 (unpublished), hep-ph/0111027; A. V. Kisselev, hep-ph/0303090.
[19] M. Masip and A. Pomarol, Phys. Rev. D **60**, 096005 (1999); T. G. Rizzo and J. D. Wells, Phys. Rev. D **61**, 016007 (2000).
[20] It should be mentioned that it is nonetheless possible to apply the dimensional regularization scheme. It is still possible to capture the infinities in a Γ-function since it has no poles when d is not an integer. However, to use this renormalization scheme, one needs to introduce a mass scale to assure the gauge couplings have the right power. For $d > 0$ the result depends explicitly on this mass scale, which might or might not agree with M_f and thus does not solve the problem.
[21] D. J. Gross and P. F. Mende, Nucl. Phys. **B303**, 407 (1988).
[22] D. Amati, M. Ciafaloni, and G. Veneziano, Phys. Lett. B **216**, 41 (1989).
[23] E. Witten, Phys. Today **50N5**, 28 (1997).
[24] L. J. Garay, Int. J. Mod. Phys. **A10**, 145 (1995).
[25] A. Kempf, hep-th/9810215.
[26] A. Kempf, G. Mangano, and R. B. Mann, Phys. Rev. D **52**, 1108 (1995).
[27] A. Kempf and G. Mangano, Phys. Rev. D **55**, 7909 (1997).
[28] W. G. Unruh, Phys. Rev. D **51**, 2827 (1995); S. F. Hassan and M. S. Sloth, Nucl. Phys. **B674**, 434 (2003); R. H. Brandenberger, S. E. Joras, and J. Martin, Phys. Rev. D **66**, 083514 (2002).
[29] U. H. Danielsson, Phys. Rev. D **66**, 023511 (2002); S. Shankaranarayanan, Classical Quantum Gravity **20**, 75 (2003); L. Mersini, M. Bastero-Gil, and P. Kanti, Phys. Rev. D **64**, 043508 (2001); A. Kempf, Phys. Rev. D **63**, 083514 (2001); A. Kempf and J. C. Niemeyer, Phys. Rev. D **64**, 103501 (2001); J. Martin and R. H. Brandenberger, Phys. Rev. D **63**, 123501 (2001); R. Easther, B. R. Greene, W. H. Kinney, and G. Shiu, Phys. Rev. D **67**, 063508 (2003); R. H. Brandenberger and J. Martin, Mod. Phys. Lett. A **16**, 999 (2001); M. Cavaglia and S. Das, hep-th/0404050.
[30] T. Padmanabhan, Phys. Rev. Lett. **78**, 1854 (1997); A. Smailagic, E. Spallucci, and T. Padmanabhan, hep-th/0308122; F. Scardigli, Phys. Lett. B **452**, 39 (1999); I. Dadic, L. Jonke, and S. Meljanac, Phys. Rev. D **67**, 087701 (2003); F. Brau, J. Phys. A **32**, 7691 (1999); R. Akhoury and Y. P. Yao, Phys. Lett. B **572**, 37 (2003).
[31] S. Hossenfelder, M. Bleicher, S. Hofmann, J. Ruppert, S. Scherer, and H. Stöcker, Phys. Lett. B **575**, 85 (2003).
[32] Note that this is similar to introducing an energy dependence of Planck's constant $\hbar$.
[33] G. Amelino-Camelia, Nature (London) **418**, 34 (2002); J. Magueijo and L. Smolin, Phys. Rev. Lett. **88**, 190403 (2002); M. Toller, Mod. Phys. Lett. A **18**, 2019 (2003); C. Rovelli and S. Speziale, Phys. Rev. D **67**, 064019 (2003).
[34] At least on shell.
[35] K. Hagiwara *et al.*, Phys. Rev. D**66**, 010001 (2002).
[36] As mentioned in [12] the trace over the higher dimensional γ-matrices yields an unwanted factor 2^d. This is due to the compactification scheme which is unsuitable for fermions as it does not properly reproduce the degrees of freedom on the brane. We, too, therefore drop this factor by hand.