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Entanglement in relativistic quantum field theory

Yu Shi

Center for Advanced Study, Tsinghua University, Beijing 100084, China
Department of Physics, University of Illinois at Urbana-Champaign, Urbana, Illinois 61891, USA
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I present some general ideas about quantum entanglement in relativistic quantum field theory, especially entanglement in the physical vacuum. Here, entanglement is defined between different single particle states (or modes), parametrized either by energy-momentum together with internal degrees of freedom, or by spacetime coordinate together with the component index in the case of a vector or spinor field. In this approach, the notion of entanglement between different spacetime points can be established. Some entanglement properties are obtained as constraints from symmetries, e.g., under Lorentz transformation, space inversion, time reverse, and charge conjugation.

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Quantum entanglement is a notion about the structure of a quantum state of a composite system, referring to its nonfactorization in terms of states of subsystems. It is regarded as an essential quantum characteristic [1,2]. Entanglement with environment is also crucial in decoherence, i.e., the emergence of classical phenomena in a quantum foundation, and may even be a possible explanation of superselection rules [3,4]. There have been a lot of activities on various aspects of entanglement, including some recent works which take into account relativity [5,6]. Investigations on entanglement in quantum field theories may provide useful perspectives on field theory issues. On the other hand, as the framework of fundamental physics, incorporating relativity, quantum field theory may be useful in deepening our understanding of entanglement. Besides, entanglement due to environmental perturbation may also be helpful in understanding spontaneous symmetry breaking. Most of the methods in field theory adopt Heisenberg or interaction picture, and do not need the explicit form of the underlying quantum state living in an infinite-dimensional Hilbert space. Nevertheless, in many circumstances, it is still important to know the nature of the quantum state, most notably the vacuum. In this paper, as an extension of some previous discussions on nonrelativistic quantum field theories [7], I present some general ideas about the nature of entanglement in relativistic quantum field theory, and constraints from symmetries. Such investigations may offer useful insights on the structures of the vacua in quantum field theories on one hand, and on quantum information in relativistic regime on the other hand.

First, I describe the basic method here of characterizing entanglement in quantum field theory. In quantum field theory, the dynamical variables are field operators (in real spacetime) or annihilation and creation operators (in energy-momentum space), in terms of which any observable can be expressed. Spacetime coordinate plus the component index in the case of a vector or spinor field, or energy-momentum plus internal degrees of freedom (such as being particle or antiparticle, spin, polarization, etc.) are merely parameters. These parameters define the modes in either the real spacetime or the momentum space, and exactly provide the labels for the (distinguishable) subsystems, between which entanglement can be well defined, in the same manner as that for distinguishable nonrelativistic quantum mechanical systems. In other words, consider the Hilbert space as composed of the Hilbert spaces for all the modes, parametrized either by the spacetime or by the momentum, together with whatever other degrees of freedom. Therefore, in momentum space, a mode, parametrized by the energy-momentum together with internal degrees of freedom, is entangled with other modes if the quantum state cannot be factorized as a direct product of the state of this mode and the rest of the system. Similarly, in real spacetime, a mode parametrized by the spacetime coordinate is entangled with other modes if the quantum state cannot be factorized as a direct product of the state of this mode and the rest of the system. The basis of the Hilbert space at each specified mode can be arbitrarily chosen to be a orthonormal set of eigenstates at this mode. A convenient, but not necessary, basis of the modes in the momentum space is the occupation-number states, as previously used in some related investigations [7–10]. In the real spacetime, one can use the eigenstates of the local density $\phi^\dagger(x)\phi(x)$, where $\phi(x)$ is the field operator. The concept of "local operation," as used in theories of entanglement, is generalized to an operation only acting on a subsystem. In real spacetime, this generalization is consistent with the usual meaning, but I have naturally incorporated relativity: one can consider *entanglement between different spacetime points.*

When different fields, i.e., particle species, coexist, these different fields are clearly distinguishable subsystems, between which entanglement can be defined. In some effective or approximate theories, different fields may be related by an additional symmetry, e.g., the isospin in nuclear physics, and thus can be treated as a single field with an additional degree of freedom. In a semiclassical setting, entanglement between fields and charges was discussed previously [11].

In a composite system, there is a complex pattern of entanglement, which is still only partially understood in theories of entanglement. For simplicity, here we focus on the bipartite entanglement between a subsystem and its complementary subsystem, i.e., the rest of the system.

In the following, we first stay in momentum space until we shift to real spacetime later on.

It is instructive to start with the simple case of free field theories. Under canonical quantization, the Hamiltonians can be written as $\mathcal{H} = \int d^3k k_0 N_k$ for a real scalar field, where N_k is particle number operator, $\mathcal{H} = \int d^3k k_0 (N_k + N_k^c)$ for a complex scalar field, $\mathcal{H} = \int d^3k k_0 \sum_\sigma (N_{k,\sigma} + N^c_{k,\sigma})$ for a vector or spinor field, and $\mathcal{H} = \int d^3k k_0 \sum_{\sigma=1,2} N_{k,\sigma}$ for the electromagnetic field in Coulomb gauge quantization. Here the superscript c represents charge conjugation or antiparticles, σ represents spin or polarization. In the vacuum state, the occupation number of each mode, labeled by four-momentum k, together with being particle or antiparticle, spin or polarization, is zero. Thus in momentum space, mode entanglement trivially vanishes in the vacuum state of a free field theory. Moreover, in a Hamiltonian eigenstate with a definite number of particles in a mode, the state of this mode can be factorized out, and thus there is no entanglement between this mode and other modes. However, because of degeneracy, e.g., the four-momentum $(k_0, \mathbf{k})$ and spin may be different even though k_0 is the same, a Hamiltonian eigenstate is not necessarily nonentangled.

There is subtlety in Lorenz gauge quantization of electromagnetic field. The Hamiltonian is $\mathcal{H} = \int d^3k k_0 (\sum_{\sigma=1}^3 N_{k,\sigma} - N_{k,0}) = \int d^3k k_0 \sum_{\sigma=1,2} N_{k,\sigma}$, under the Gupta-Bleuler condition $(a_{k,0} - a_{k,3})|\Psi\rangle$ for any physical state. Consequently the nature of physical modes, with $\sigma = 1, 2$, is the same as in Coulomb gauge, as it should be. It can also been seen that the unphysical modes are entangled with each other, while they are separated from the physical modes, as they should be; if they were entangled with physical modes, the physical modes would unreasonably live in a mixed state.

In general, presence of interaction, including gauge coupling, may induce nonvanishing entanglement, as in interacting field theories and even in a pure non-Abelian gauge field, where there is self-interaction.

Now are given some constraints on the nature of entanglement, imposed by symmetry properties.

A symmetry transformation $\mathcal{T}$ induces a unitary transformation $U(\mathcal{T})$ on the quantum state $|\Psi\rangle$ of the system, i.e.,

$$|\Psi\rangle \rightarrow |\Psi'\rangle = U(\mathcal{T})|\Psi\rangle. \tag{1}$$

Under symmetry transformation, the labels of the modes are also transformed, as given by the standard transformations of the single particle basis states. This is just a relabel, no matter whether the quantum state of the system is invariant under the transformation. There are two cases, as expounded below.

The word "mode" is somewhat ambiguous. Here it really means the single particle basis state. For example, a single particle state with momentum $\mathbf{p}$ and spin σ is $|\mathbf{p}, \sigma) \equiv a^\dagger_{\mathbf{p},\sigma}|0\rangle$, while a one-particle state at coordinate x, with vector or spinor component l, is $|x, l) \equiv \phi_l^\dagger(x)|0\rangle$ [12]. The transformation of the annihilation operator or field operator can be obtained from the transformation of the corresponding single particle state. From the definition of creation operator and the fact that $|0\rangle$ is always invariant, one knows that the creation operator transforms in the same way as the single particle state [13,14].

The Case I of mode transformation, under a symmetry transformation, is that a mode α is relabeled as mode α' existing in the same basis. The single particle state $|\alpha')$ is related to single particle state $|\alpha)$ as

$$|\alpha) \rightarrow U(\mathcal{T})|\alpha) = |\alpha'),$$

which is equivalent to

$$a^\dagger_\alpha \rightarrow U(\mathcal{T}) a^\dagger_\alpha U^\dagger(\mathcal{T}) = a^\dagger_{\alpha'},$$

where $a^\dagger_\alpha|0\rangle \equiv |\alpha)$ while $a^\dagger_{\alpha'}|0\rangle \equiv |\alpha')$.

Such a transformation means that in the mode expansion of the quantum state $|\Psi\rangle$, the label α is changed to α'. The state $|\Psi\rangle$ itself is changed to $|\Psi'\rangle$ as given in (1).

Consider, in a quantum state $|\Psi\rangle$, the entanglement between mode α and its complementary subsystem, denoted as $E_{|\Psi\rangle}(\alpha)$. Clearly,

$$E_{|\Psi\rangle}(\alpha) \equiv E_{|\Psi\rangle'}(\alpha').$$

Now, *if the state* $|\Psi\rangle$ *respects a symmetry*, then $|\Psi'\rangle = |\Psi\rangle$. Such is the case of the vacuum of a quantum field theory with a symmetry. Then the nature of entanglement, as a function of the state, should also be invariant under this symmetry, i.e., $E_{|\Psi\rangle}(\alpha) = E_{|\Psi\rangle'}(\alpha)$. Thus

$$E_{|\Psi\rangle}(\alpha) = E_{|\Psi\rangle}(\alpha'). \tag{2}$$

This equality is true no matter what is the specific measure of $E_{|\Psi\rangle}(\alpha)$. But it can be confirmed for specific entanglement measures. It is now well-known that for a pure state, the entanglement between a subsystem and the rest of the system is quantified as the von Neumann entropy of the reduced density matrix of either subsystem [15]. Thus $E(\alpha)$ can be quantified as

$$E(\alpha) = -\mathrm{Tr}\rho([\alpha])\ln\rho([\alpha]),$$

where the trace is over the Hilbert space of all the system excluding mode α,

$$\rho[\alpha] = \sum_N {}_{\alpha}\langle N|\Psi\rangle\langle\Psi|N\rangle_{\alpha},$$

is the reduced density matrix of the subsystem complementary with α, obtained by tracing over the Hilbert space at α. $|N\rangle_{\alpha} \equiv (1/\sqrt{N!})a_{\alpha}^{\dagger N}|0\rangle_{\alpha}$ is the particle number state at α. Indeed, under the symmetry transformation, $|N\rangle_{\alpha} \rightarrow |N\rangle_{\alpha'} \equiv U(\mathcal{T})|N\rangle_{\alpha}$. Therefore if $|\Psi\rangle = U(\mathcal{T})|\Psi\rangle$, then $\rho([\alpha]) = \rho([\alpha'])$, and thus $E_{|\Psi\rangle}(\alpha) = E_{|\Psi\rangle}(\alpha')$.

To summarize for this point, *if the state is invariant under a symmetry transformation, then in the same state, for any two modes that can be transformed into each other under a symmetry transformation, they have the same amount of entanglement with the corresponding complementary subsystems.* The statement is of course also true if one mode is replaced as a set of modes.

Symmetries of space inversion P, time reverse T, and charge conjugation C belong to this case. So does the invariance of a scalar field under Lorentz transformation Λ (translation has no effect on momentum, so only homogeneous Lorentz transformation needs to be considered here).

Any vacuum state must be invariant under Lorentz transformation and CPT. This has consequences on the entanglement properties, as given below.

For a scalar field, the single particle state $|p\rangle$ is transformed as $U(\Lambda)|p\rangle = \sqrt{(\Lambda p)^0/p^0}|\Lambda p\rangle$ under a homogeneous Lorentz transformation Λ, as $P|p\rangle = \eta|\mathcal{P}p\rangle$ under space inversion, as $T|p\rangle = \zeta|\mathcal{P}p\rangle$ under time reversal, and as $C|p, n\rangle = \xi_n|p, n^c\rangle$ under charge conjugation, where n denotes the particle species, $\mathcal{P}p = (p_0, -\mathbf{p})$, η, ζ, and ξ_n are phase factors only dependent on particle species. Phase change of the single basis particle state does not affect the entanglement between modes. Thus in a vacuum state, for any mode p of a scalar field, $E(p) = E(\Lambda p)$ for any Λ. If it is invariant under P or T, then $E(p) = E(\mathcal{P}p)$. If it is invariant under C, then $E(p, n) = E(p, n^c)$. Consequently CPT theorem implies that $E(p, n) = E(p, n^c)$ always holds.

Now consider a vector or spinor field. For a massive field, $P|p, \sigma\rangle = \eta|\mathcal{P}p, \sigma\rangle$, $T|p, \sigma\rangle = \zeta(-1)^{j-\sigma}|\mathcal{P}p, -\sigma\rangle$, where j is the spin quantum number, σ runs from j to $-j$. For a massless field, $P|p, \sigma\rangle = \eta_{\sigma}\exp(\mp i\pi\sigma)|\mathcal{P}p, -\sigma\rangle$, $T|p, \sigma\rangle = \zeta_{\sigma}\exp(\pm i\pi\sigma)|\mathcal{P}p, \sigma\rangle$. The notations are standard [13]. The single particle phase factors have no effect on entanglement. Thus for massive field modes, P symmetry implies $E(p, \sigma) = E(\mathcal{P}p, \sigma)$, while T symmetry implies $E(p, \sigma) = E(\mathcal{P}p, -\sigma)$. For massless field modes, P symmetry implies $E(p, \sigma) = E(\mathcal{P}p, -\sigma)$, while T symmetry implies $E(p, \sigma) = E(\mathcal{P}p, \sigma)$. Note the difference between massive and massless fields. For both massless and massive fields, $C|p, \sigma, n\rangle = \xi_n|p, \sigma, n^c\rangle$, hence C symmetry means $E(p, \sigma, n) = E(p, \sigma, n^c)$. The CPT theorem implies that $E(p, \sigma, n) = E(p, -\sigma, n^c)$ always holds for both massless and massive fields.

Lorentz transformation for a vector or spinor field, which mixes modes with different spins, belongs to a different case. Let us refer to it as Case II, in which a mode is transformed to a superposition of more than one mode in the single particle basis considered, i.e.,

$$|\alpha\rangle \rightarrow U(\mathcal{T})|\alpha\rangle = \sum_i \gamma_i|\alpha_i\rangle,$$

where γ_i represents coefficients. In other words,

$$a_{\alpha}^{\dagger} \rightarrow U a_{\alpha}^{\dagger} U^{-1} = \sum_i \gamma_i a_{\alpha_i}^{\dagger}.$$

In this case, the occupation-number states at mode α transform as $|0\rangle_{\alpha} \rightarrow \prod_i |0\rangle_{\alpha_i}$ and $|N\rangle_{\alpha} \rightarrow U|N\rangle_{\alpha} = \frac{1}{\sqrt{N!}}(\sum_i \gamma_i a_{\alpha_i}^{\dagger})^N \prod_i |0\rangle_{\alpha_i}$.

Therefore, if the quantum state $|\Psi\rangle$ respects the symmetry, i.e., $U|\Psi\rangle = |\Psi\rangle$, then the reduced density matrix $\rho([\alpha])$ must satisfy

$$\rho([\alpha]) = \sum_N \frac{1}{N!}\left\langle \bar{0} \left| \left(\sum_i \gamma_i^* a_{\alpha_i}\right)^N \right| \Psi \right\rangle \times \left\langle \Psi \left| \sum_i \gamma_i a_{\alpha_i}^{\dagger}\right)^N \left| \bar{0} \right\rangle,$$

where $|\bar{0}\rangle \equiv \prod_i |0\rangle_{\alpha_i}$.

Now gauge transformation is considered, which changes the phase of the field operator, accompanied by the transformation of the gauge potential. Non-Abelian gauge transformation involves a local rotation between different components of the spinor or vector field. Consider a field operator $\phi(x)$, be it scalar, vector or spinor. It is gauge transformed as $\phi(x) \rightarrow \phi'(x) = S(x)\phi(x)$. Consequently, a creation operator $a_{p,\sigma}^{\dagger}$, obtained from the momentum-spin mode expansion of $\phi(x)$, is transformed to a new mode creation operator $a_{p,\sigma}^{\prime\dagger}$, obtained from the mode expansion of $\phi'(x)$. $a_{p,\sigma}$ and $a_{p,\sigma}^{\prime\dagger}$, however, act on the same mode (p, σ). The entanglement $E(p, \sigma)$ is thus transformed to itself. It is consistent, though no particular constraint on entanglement is obtained from this simple consideration.

Now switch to real spacetime, in which there exists entanglement even in the vacuum of a free field, as simply seen by transforming the creation operators in momentum space to field operators in real spacetime. This seems consistent with the early result about violation of Bell inequalities in vacuum states [16]. Very recently, Calabrese and Cardy made some calculations on positional entanglement in 1+1 dimensional field theory [17].

My discussion here is fully relativistic; the subsystems are spacetime points.

When the quantum states are represented in the real spacetime, there is a degree of freedom in addition to the spacetime coordinate, namely, the component index of the irreducible representation of the homogeneous Lorentz group, which defines the field operator. However, one need not explicitly consider the vector or spinor components, rather, one can use the whole vector or spinor, since in field theories, the Lagrangians can be written in terms of the whole vector or spinor. Of course, one also needs to consider all different fields in the system. In this way, one can obtain the total entanglement between different spacetime points.

One may use eigenstates of a Hermitian operator as the basis for the Hilbert space at x. For example, such a Hermitian operator can be chosen to be the local density $D(x)$, which is defined to be $\phi^\dagger(x)\phi(x)$ for a scalar field $\phi(x)$, $v^\dagger(x)v(x)$ for a vector field $v(x)$, and $\psi(x)^\dagger\psi(x)$ for a spinor field $\psi(x)$.

It can be checked that for each of these fields, $D(x)$ is a scalar under a Lorentz transformation $x \rightarrow x' = \Lambda x + l$, i.e,. $D(x) = D(x')$. The Lorentz invariance of the state $|\Psi\rangle$ means that $E(x) = E(y)$, where x and y are any two spacetime points that can be connected by a Lorentz transformation. P transforms $D(x)$ to $D(\mathcal{P}x)$, T transforms $D(x)$ to $D(-\mathcal{P}x)$. Hence P symmetry implies $E(x) = E(\mathcal{P}x)$, T symmetry implies $E(x) = E(-\mathcal{P}x)$. C transformation transforms $D(x)$ to itself, so no special constraint is given by C symmetry. Therefore CPT symmetry implies that $E(x) = E(-x)$ always holds.

Because these symmetry transformations of the entanglement are, respectively, the same for different fields, they remain the same when different fields coexist.

A global gauge transformation is merely a constant phase factor, so trivially has no effect on entanglement. The local gauge transformation only depends on the local spacetime, therefore also does not have any effect on the entanglement between different spacetime points. In fact, the underlying quantum state of the field theory remains the same under any gauge transformation.

I stress that the entanglement between different spacetime points, obtained by tracing over the spinor or vector components and over different fields, is an intrinsic physical property of the system in consideration. Although a particular momentum-spin mode defined by a free single particle basis state may not be directly measurable because of renormalization, the entanglement between spacetime regions is directly measurable in principle.

This interesting point can be illustrated by using the well-known entanglement [4,18–20] in Unruh-Hawking radiation [21,22]. As shown by Unruh [21], the Minkowski vacuum can be expressed in terms of Rindler modes, which are those in the accelerating frame, as

$$|\mathrm{vac}\rangle \propto \prod_{\omega,\mathbf{k}} \exp[e^{-2\pi\omega} a^\dagger_{1,\omega,\mathbf{k}} a^\dagger_{2,\omega,\mathbf{k}}]|0\rangle_R = \prod_{\omega,\mathbf{k}} \sum_n e^{-2\pi n\omega} |n\rangle_{1,\omega,\mathbf{k}} |n\rangle_{2,\omega,\mathbf{k}}, \tag{3}$$

where the subscripts 1 and 2 represent the two halves of the Rindler space separated by the horizon. They must appear, together with the energy-momentum, as the subscripts, because for each half of the Rindler space, there is a set of momentum-mode functions, which vanish in the other part of the Rindler space. So the momentum-mode functions in both halves are needed to make a complete set. It can be seen that the entanglement between modes $(1, \omega, \mathbf{k})$ and $(2, \omega, \mathbf{k})$, equal to the entanglement between each of them and the rest of the whole system, is $S_{\omega,\mathbf{k}} = -\sum_n p_\omega(n) \ln p_\omega(n)$, where $p_\omega(n) = e^{-4\pi n\omega}/\sum_n e^{-4\pi n\omega}$. Similarly, in the exterior of a Schwarzschild black hole, in terms of the modes on the two sides of the event horizon, the vacuum state is given in Eq. (3) with π replaced as $2\pi M$, where M is the mass of the black hole. With this replacement, the nature of entanglement is the same as that for Minkowski vacuum in terms of Rindler modes.

One can obtain the total entanglement between the two halves of Rindler space, or the entanglement across the event horizon of a black hole, as $\sum_{\omega,\mathbf{k}} S_{\omega,\mathbf{k}}$. This is the entanglement between two parts of the spacetime. The result is independent of the choice of the momentum-mode functions during the calculation.

To summarize, I present some general ideas concerning field theoretic quantum entanglement, and especially its use in characterizing quantum properties of vacuum, a key issue in fundamental physics. Field theoretic entanglement can be defined in momentum space and in real spacetime, with the (distinguishable) subsystems parametrized either the energy-momentum plus internal degrees of freedom, or by the spacetime coordinate plus the component index for a vector or spinor field, respectively. With this definition, the ideas from the theories of entanglement can be applied. I give some symmetry properties concerning the entanglement in quantum field theory, in momentum space and in real spacetime, respectively. I discussed the invariance properties of entanglement when the quantum state respects symmetries. A noteworthy notion is the entanglement between different spacetime points, which is an intrinsic physical property and is measurable in principle. This notion is illustrated in terms of the entanglement between the two halves of the Rindler space or across the event horizon of a black hole.

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