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Quasilocalized gravity without asymptotic flatness

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We present a toy model of a generic five-dimensional warped geometry in which the 4D graviton is not fully localized on the brane. Studying the tensor sector of metric perturbation around this background, we find that its contribution to the effective gravitational potential is of 4D type $(1/r)$ at the intermediate scales and that at the large scales it becomes $1/r^{1+\alpha}$, $0 < \alpha \leq 1$ being a function of the parameters of the model ($\alpha = 1$ corresponds to the asymptotically flat geometry). Large-distance behavior of the potential is therefore not necessarily five-dimensional. Our analysis applies also to the case of quasilocalized massless particles other than graviton.

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I. INTRODUCTION

The idea that we could live on a brane embedded in a higher-dimensional space dates back to Refs. [1,2] where it was proposed that the universe be modeled as a topological defect (a domain wall or a string) in a higher-dimensional space-time. The key feature of these models is the localization of matter on the brane, which makes the theory look four-dimensional at low energies. In the setup of Ref. [1] this was achieved by means of Yukawa interaction between the domain wall and the fermion field. The latter has a zero-mode localized on the brane, as well as the continuum modes separated from the zero-mode by a gap. At low energies, only the zero-mode remains in the spectrum and plays the role of a four-dimensional particle. It was suggested later [3,4] that the gravitational attraction may also trap particles on the brane.

This idea has been the subject of renewed interest since it has been shown [5] that the graviton itself can be trapped on the brane by the gravitational force, leading to effectively four-dimensional gravity. Considering a brane embedded in a 5D anti-de Sitter (AdS_5) space-time, the authors of Ref. [5] have found that in this "warped" geometry the 5D graviton has a zero-energy bound state, which, being localized on the brane, can be interpreted as the ordinary 4D graviton. Even though there is no gap separating the zero-mode of graviton from the continuum, the law of gravity on the brane is essentially Newtonian, the corrections yielded by light continuum modes modifying this behavior only at very short distances.

Soon after, a model was proposed [6] in which 4D Newton's law is reproduced only at intermediate scales and gets modified at very large scales. The geometry considered consisted of a flat 5D space glued onto a portion of AdS_5 space. It has been shown that, even though there is no localized graviton, Newton's gravity can still be recovered on the brane at intermediate scales. At short distances, the gravitational potential receives corrections similar to those encountered in the model of Ref. [5]. At very large scales, the extra dimensions "open up" and the gravitational potential receives "five-dimensional" contributions from the tensor modes, that is contributions scaling as r^{-2}, r being the distance between sources. This behavior was related [7,8] to the presence of a resonant mode—a "quasibound state"—at zero energy, with lifetime proportional to the long-distance scale of the model (see also Ref. [6] itself). Subsequent works [8–11] made manifest the importance of the scalar sector (and, namely, the radion mode) in the description of the effective gravity on the brane in models with metastable gravitons. Indeed, being constructed from the massive modes, the propagator of the effective 4D graviton is likely to have an incorrect tensor structure, leading to predictions inconsistent with the observations [8,11]. Coupling to the trace of the energy-momentum tensor, radion contributes to the graviton propagator and thus makes it possible to recover Einstein gravity at the intermediate distances [10,11]. The model presented in [6] has, however, a serious drawback: it requires the use of the negative tension branes—or, in other words, violation of the condition of positivity of energy [12]. It was shown [11] that this forces radion to become a ghost [9,11] and causes the effective gravity at very large distances to become scalar antigravity [10]. While this behavior occurs at the experimentally inaccessible scales, presence of negative norm states certainly raises doubts concerning the consistency of the model.

An alternative brane model producing an infrared modification of gravity was constructed in [13] using a thin brane embedded in five-dimensional Minkowski space. The action included a four-dimensional Einstein term on the brane, induced by radiative corrections due to the matter fields localized there. This term was shown to dominate the gravitational interactions of the brane matter at intermediate scales, giving four-dimensional gravity at intermediate scales, while at large scales it became five-dimensional. This setup was subsequently extended to higher dimensions [14] and its generalizations to smooth branes were also considered [15,16]. Recently, the idea of induced gravity was applied to the warped space-times [17–19]. As yet, the setups with induced gravity are not entirely satisfactory: most of the models suffer from ghosts and/or strong gravity problems [19–21]. Nevertheless, the idea of infrared modification of gravity remains very appealing as it is thought [12,22] that it could be a solution to the cosmological constant problem.

The feature shared by these models is that at ultralarge scales gravity becomes higher-dimensional, that is the gravitational potential scales as $r^{-(1+N)}$, r being the distance between sources and N the number of extra dimensions (at least as far as the tensor sector is concerned). It is natural to wonder whether this feature is inherent to the quasilocalized gravity on branes embedded in higher dimension space-times or if it could exhibit some different, more general, behaviors. More generally, one may ask what kinds of large-distance modifications of gravity are in principle possible in this setup and under what conditions they can be realized.

In order to cast some light on this issue, we study in this paper tensor perturbations of the metric around a fairly generic five-dimensional warped background, in which the graviton is not fully localized on the brane (becoming a metastable state). To some extent, our model can be considered as a toy model of quasilocalized gravity. Of course, as exemplified by the model of [6], studying only the tensor sector of metric perturbations is unlikely to give a full description of effective gravity on branes and we do not aspire to present a realistic solution for quasilocalization of gravity. The aim of this work is rather to signal a peculiar behavior of the potential yielded by the tensor perturbations in such kinds of backgrounds. For the sake of simplicity, throughout the paper we will loosely refer to the contribution of the tensor modes to the gravitational potential as to the potential itself, bearing in mind that in any concrete model one should account for the contribution from the scalar sector as well. Same applies to our description of gravitational waves.

We find that our model gives 4D gravitational potential at intermediate scales—quite similarly to the situation encountered in the model of Ref. [6]. This "quasilocalization" of gravity is a generic feature of the geometry which we consider—whether or not the embedding space-time is asymptotically flat. At large distances, we observe a novel behavior: the potential gets modified and is of type $1/r^{1+\alpha}$, $0 < \alpha \leq 1$ being a function of the parameters of the model. It becomes 5D ($\alpha = 1$) only when the geometry is asymptotically flat. Thus, the gravitational potential does not necessarily exhibit 5D behavior at very large distances.

Although throughout this paper we use the example and the language of gravity, our analysis is in fact quite universal, in that it is not constrained to the gravity sector, but can apply to the matter sector as well. Indeed, our considerations are based on studying the properties of a Shrödinger-like equation governing the behavior of the metric perturbations and, by consequence, the effective physics on the brane. Actually, irrespective of the spin of the matter fields present in the bulk, the mass spectrum of the effective four-dimensional theory is determined by a Schrödinger-like equation of some type (or a system of such equations). Using this fact, several authors have studied the localization of different matter fields in the AdS_5 background of Ref. [5] and its variations: It was shown that, similarly to gravitons, the spectrum of massless bulk scalars [23,24] consists of a localized zero-mode, followed by the continuum of arbitrarily light states. Massless fermions [23] can also be localized on the brane and mechanisms to localize gauge fields were proposed [25–27]. Localization of massive scalars and fermions was discussed in Ref. [28] (see also Ref. [29]), where it was shown that while there are no truly localized states (massive or massless), the setup allows for metastable massive particles on the brane, having a small, but finite, probability of tunneling into the bulk. In the scalar case, these resonant states were shown to give $1/r$ contribution to the effective static potential on the brane, while the light states of the continuum give a power-law behavior at large distances. Our analysis completes the picture, revealing that, under certain conditions, in the absence of true bound states and massive resonances, it is possible to have an "almost localized" massless state, inducing a $1/r$ potential on the brane evolving towards a (fractional) power-law behavior at large distances.

II. GENERAL SETUP

Let us begin with the general setup. We consider 5D warped space-times preserving 4D Poincaré symmetry:

$$ds^2 = e^{-A(z)}(\eta_{\alpha\beta}dx^\alpha dx^\beta - dz^2), \tag{1}$$

where the "warp factor" $A(z)$ in an even and nondecreasing function of z.

We suppose that the ordinary matter is localized on the brane centered at $z = 0$. In order to study the nature of the gravitational interactions experienced by this matter, we need to study perturbations around the background metric. We will consider exclusively the tensor perturbation $h_{\alpha\beta}(x, z) = g(z)^{3/2}\tilde{h}_{\alpha\beta}(x)u(z)$, α and β being the four-dimensional tensor indices and $g(z)$ metric determinant. The behavior of this perturbation is governed by the Schrödinger-like equation

$$-\frac{d^2u_\mu(z)}{dz^2} + V(z)u_\mu(z) = \mu^2 u_\mu(z), \tag{2}$$

where $-\partial^\rho\partial_\rho\tilde{h}_{\alpha\beta}(x) = \mu^2\tilde{h}_{\alpha\beta}(x)$ and the potential is given in terms of the warp factor:

$$V(z) = \frac{9}{16}A'(z)^2 - \frac{3}{4}A''(z), \tag{3}$$

which is a form familiar from the supersymmetric quantum mechanics with the "superpotential" $W(z) = \frac{3}{4}A'(z)$. The factorization of the Hamiltonian:

$$-\frac{d^2}{dz^2} + V(z) = \left[-\frac{d}{dz} + W(z)\right]\left[\frac{d}{dz} + W(z)\right] \tag{4}$$

ensures that the zero-energy mode

$$u_0(z) = \exp\left[-\frac{3}{4}A(z)\right] \tag{5}$$

is the ground state of the system.

In this way, studying the influence of the tensor modes on the effective gravity on the brane is reduced to studying the properties of the spectrum of the one-dimensional quantum mechanical system (2). The quantity of particular interest is the spectral density $\rho(\mu) = |u_\mu(0)|^2$, as it enters the retarded Green's function $G_R(x, z = 0; x', z' = 0)$ which determines the gravitational interactions on the brane and, consequently, the observable physical quantities, such as the induced gravitational potential and radiation of gravitational waves. Let us remind that the (five-dimensional) retarded Green's function $G_R(x, z = 0; x', z' = 0)$ of the tensor sector of the linearized Einstein equations can be constructed from the full set of eigenmodes of Eq. (2) via spectral decomposition:

$$G_R(x, z; x', z') = \int_0^\infty d\mu\; u_\mu(z)u_\mu(z') \times \int \frac{d^4p}{(2\pi)^4}\frac{e^{-ip\cdot(x-x')}}{\mu^2 - p^2 - i\varepsilon p^0}. \tag{6}$$

III. MODEL

The purpose of this work is not to present any particular realistic setup for the quasilocalized gravity, but rather to study general properties of warped space-times in which this phenomenon can arise and determine what are the possible large-distance behaviors of the gravitational potential (induced by the tensor modes). With this aim in mind, we construct and study a toy model for a generic geometry allowing (quasi)localization of gravity. Concretely, we consider a class of potentials $V(z)$ with "volcano" shape, possibly with tails—for which we choose $1/z^2$ fall-off, familiar from the Randall-Sundrum model [5]. Knowing that the potential in question necessarily follows from a superpotential, we start our study by defining one of the following form:

$$W(z) = -W(-z) = \begin{cases} a\tan(az) & z < z_0, \\ b\tanh[b(z_2 - z)] & z_0 \le z \le z_c, \\ c(z - z_1)^{-1} & z \ge z_c. \end{cases} \tag{7}$$

The motivation for choosing this particular form of superpotential is twofold: not only does it yield a simple potential for which an exact solution of the Schrödinger equation can be determined, but also allows a unified description of the (quasi)localization, naturally including regimes already discussed in the literature [5,7]. Our choice of superpotential corresponds in general to geometries which are not asymptotically flat—except when the limit $c = 0$ is considered. To ensure that the resulting potential does not have singularities stronger than finite jumps we demand that $W(z)$ be continuous. This requirement entails relations between the constants:

$$\begin{cases} b\tanh[b(z_2 - z_0)] = a\tan(az_0), \\ b\tanh[b(z_2 - z_c)] = c(z_c - z_1)^{-1}. \end{cases} \tag{8}$$

The potential

$$V(z) = -W'(z) + W(z)^2$$

generated by the superpotential (7) is symmetric and has the desired "volcano-like" shape (see Fig. 1). It is given by:

$$V(z) = \begin{cases} -a^2 & |z| < z_0, \\ b^2 & z_0 < |z| < z_c, \\ (c + c^2)(|z| - z_1)^{-2} & |z| > z_c. \end{cases} \tag{9}$$

The parameters of the potential must satisfy the condition following from the constraints (8) by eliminating z_2:

$$c\left[1 - \frac{a}{b}\tan(az_0)\tanh(b(z_c - z_0))\right] = (z_c - z_1)[a\tan(az_0) - b\tanh(b(z_c - z_0))]. \tag{10}$$

In order for (8) to remain valid we must also have

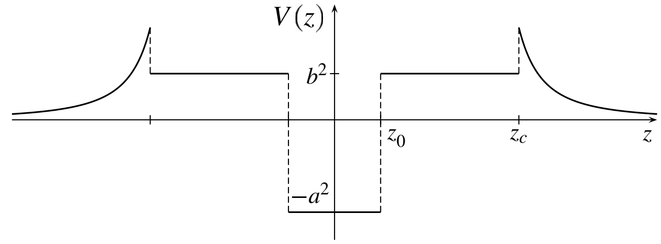

FIG. 1. Potential $V(z)$.

$$b \geq a\tan(az_0) \qquad \text{and} \qquad b \geq \frac{c}{z_c - z_1}.$$

In spite of its simplicity, the potential $V(z)$ presents a range of behaviors, depending on the choice of parameters, and thus generates different localization schemes for gravity. To start, in the limit $z_c \to \infty$, $V(z)$ becomes a simple rectangular potential well, allowing a localized symmetric zero-mode of the form:

$$u_0^\infty(z) = C_0^\infty \begin{cases} \cos(az) & z < z_0 \\ \cos(az_0)e^{-b(z-z_0)} & z \geq z_0 \end{cases} \tag{11}$$

where

$$(C_0^\infty)^{-2} = z_0 + \frac{1}{a}\cot(az_0), \tag{12}$$

followed by the continuum spectrum starting at $\mu = b$.

For finite z_c, the asymptotics of the potential makes the continuum descend towards $\mu = 0$. Condition (10) guarantees the absence of growing contribution to the $\mu = 0$ wave-function in the region $z > z_c$, where it consequently falls off as $(z - z_1)^{-c}$. When $c > 1/2$, it is therefore still normalizable and, at the bottom of the continuum, we still have a localized zero-mode. The resulting gravity on the brane will then be of the type described in Ref. [5]. When $c \leq 1/2$, $u_0(z)$ is no longer normalizable and the corresponding state can be qualified as "quasibound". In the particular case $\nu = 1/2$, $V(z)$ is identical to the "volcano box" potential used in [7] in order to explain the quasilocalization of gravity in the asymptotically flat model [6]. We claim that quasilocalization of gravity is present for the whole family of potentials with $c \leq 1/2$. Let us from now on concentrate on this family.

IV. WAVE-FUNCTION AND SPECTRAL DENSITY

The solution for a symmetric continuum wave-function has the form:

$$u_\mu(z) = u_\mu(0)\begin{cases} \cos(kz) & z \leq z_0, \\ D_1(\mu)e^{-\kappa z} + D_2(\mu)e^{\kappa z} & z_0 < z < z_c, \\ \sqrt{z - z_1}\{B_1(\mu)J_\nu[\mu(z - z_1)] + B_2(\mu)N_\nu[\mu(z - z_1)]\} & z \geq z_c. \end{cases} \tag{13}$$

where

$$k = \sqrt{\mu^2 + a^2}, \quad \kappa = \sqrt{b^2 - \mu^2} \quad \text{and} \quad \nu = c + \frac{1}{2}. \tag{14}$$

The coefficients in (13) are determined, as usual, by the requirement of continuity of the wave-function and its derivative at $z = z_0$ and $z = z_c$:

$$B_1(\mu) = \frac{\pi}{2}\sqrt{z_c - z_1}\left\{\left[\kappa f_-(\mu) + \frac{2\nu + 1}{2(z_c - z_1)}f_+(\mu)\right]N_\nu[\mu(z_c - z_1)] - \mu f_+(\mu)N_{\nu+1}[\mu(z_c - z_1)]\right\}, \tag{15a}$$

$$B_2(\mu) = -\frac{\pi}{2}\sqrt{z_c - z_1}\left\{\left[\kappa f_-(\mu) + \frac{2\nu + 1}{2(z_c - z_1)}f_+(\mu)\right]J_\nu[\mu(z_c - z_1)] - \mu f_+(\mu)J_{\nu+1}[\mu(z_c - z_1)]\right\}, \tag{15b}$$

$$f_+(\mu) = D_1(\mu)e^{-\kappa z_c} + D_2(\mu)e^{\kappa z_c} = \cos(kz_0)\cosh[\kappa(z_c - z_0)] - \frac{k}{\kappa}\sin(kz_0)\sinh[\kappa(z_c - z_0)], \tag{15c}$$

$$f_-(\mu) = D_1(\mu)e^{-\kappa z_c} - D_2(\mu)e^{\kappa z_c} = -\cos(kz_0)\sinh[\kappa(z_c - z_0)] + \frac{k}{\kappa}\sin(kz_0)\cosh[\kappa(z_c - z_0)]. \tag{15d}$$

The remaining overall constant $u_\mu(0)$, and consequently the spectral density $\rho(\mu) = |u_\mu(0)|^2$ can be determined from the normalization condition:

$$\int_{-\infty}^{\infty} dz u_\mu(z)u^*_{\mu'}(z) = \delta(\mu - \mu'), \tag{16}$$

which gives

$$\rho(\mu) = \frac{\mu}{2}\frac{1}{|B_1(\mu)|^2 + |B_2(\mu)|^2}. \tag{17}$$

As we have already stressed in Section II, $\rho(\mu)$ is the quantity relevant to the effective gravity on the brane and it is therefore useful to examine its behavior as a function of μ in some detail.

In the following, we assume that $b(z_c - z_0) \gg 1$, that is that the tunneling probability of the light modes through the potential barriers is very small. We suppose also that the constant z_1 is adjusted in such a way that the ratio $(z_c - z_1)/(z_c - z_0) \ll 1$.

For $\mu \ll \mu_1 \equiv \min[a, b, (z_c - z_1)^{-1}]$, we can approximate $B_1(\mu)$ and $B_2(\mu)$ by their power series in μ:

$$B_1(\mu) = a_0\mu^{-\nu} + a_1\mu^{-\nu+2} + \ldots + b_0\mu^\nu + \ldots \tag{18}$$

$$B_2(\mu) = c_0\mu^\nu + \ldots \tag{19}$$

The condition of continuity of the superpotential (10) forces a_0 to vanish and guarantees that c_0 and b_0 are small (of order $\exp[-b(z_c - z_0)]$). As a consequence we have for $\rho(\mu)$:

$$\rho(\mu) \approx \frac{\sin^2(\nu\pi)}{\pi^2}\frac{1}{\mu(z_c - z_1)}\frac{A(\mu)}{\alpha_0^2 + \alpha_1^2 A(\mu) + \alpha_2^2 A(\mu)^2}, \tag{20}$$

where $A(\mu)$ stands for:

$$A(\mu) = [\mu(z_c - z_1)]^{2-2\nu} e^{2b(z_c - z_0)} \tag{21}$$

and

$$\alpha_0^2 = \frac{2^{3-2\nu}}{\Gamma(\nu)^2} \frac{a^2 b^2}{a^2 + b^2} \left[b + \frac{c}{z_c - z_1} \right]^{-2},$$
$$\alpha_1^2 = -\frac{\sin(2\nu\pi)}{2\pi b} \frac{1 + b z_0}{z_c - z_1}, \qquad \alpha_2^2 = \frac{1}{4\cos^2(\nu\pi)} \frac{\alpha_1^4}{\alpha_0^2}. \tag{22}$$

The equality (20) shows that for small μ's the spectral density $\rho(\mu)$ is a uniformly decreasing function of μ, strongly peaked at $\mu = 0$ where it has a singularity of the form $\mu^{(1-2\nu)}$. The bulk of the weight of $\rho(\mu)$ lies in the region between $\mu = 0$ and $\mu \sim \mu_1 \exp[-\frac{b}{1-\nu}(z_c - z_0)]$ where $A(\mu) \sim \mathcal{O}(1)$. We have therefore in our problem two characteristic mass scales, $\mu_1 = \min[a, b, (z_c - z_1)^{-1}]$ and $\mu_2 = \mu_1 \exp[-\frac{b}{1-\nu}(z_c - z_0)]$. The assumption of small tunneling probability ensures that these scales are well separated, $\mu_2 \ll \mu_1$.

Let us remark that in the limit $\nu = 1/2$, we recover the results of [7] and $\rho(\mu)$ assumes the Breit-Wigner form:

$$\rho(\mu) \approx \frac{\mathcal{A}}{\mu^2 + \Delta\mu^2}, \tag{23}$$

with the resonance width given by

$$\Delta\mu \approx \frac{8}{(1 + b^2/a^2)(z_0 + 1/b)} e^{-2b(z_c - z_0)}. \tag{24}$$

To conclude this chapter, let us discuss briefly the connection between the spectral density $\rho(\mu)$ and the scattering matrix of our one-dimensional problem. The relevant eigenvalue of the S-matrix is:

$$S_+(\mu) = \frac{B_1(\mu) - iB_2(\mu)}{B_1(\mu) + iB_2(\mu)} e^{-2i(\pi\nu/2 + \pi/4 + \mu z_1)} = \frac{\Phi(\mu)}{\Phi^*(\mu)}, \tag{25}$$

where the Jost function $\Phi(\mu)$ is defined in such a way that tends to one in the noninteraction limit and is given by:

$$\Phi(\mu) = \sqrt{\frac{2}{\pi\mu}} e^{-i(\pi\nu/2 + \pi/4 + \mu z_1)} [B_1(\mu) - iB_2(\mu)]. \tag{26}$$

Comparing the expression for the spectral density (17) to (25), we see that $\rho(\mu)$ can be written as

$$\rho(\mu) = \frac{\pi}{\Phi(\mu)\Phi^*(\mu)}.$$

It is known from the formal scattering theory [30] that physical properties of a quantum system are closely related with the analytic properties of its scattering matrix. In particular, poles of the S-matrix in the complex μ-plane correspond, depending on their location, to bound states, virtual states or resonances of the system. In our problem, S-matrix has in general a branch point singularity at μ=0, which it inherits from the Bessel functions $J_\nu[\mu(z_c - z_1)]$ and $N_\nu[\mu(z_c - z_1)]$ entering the expression (25). The results of the next section will demonstrate that this type of singularity can also strongly influence the physics.

V. EFFECTIVE GRAVITY

Let us now turn to the contribution of the tensor modes to the effective gravity on the brane. The static potential between two sources on the brane, separated by the distance $r \equiv |\vec{x} - \vec{x}'|$ receives Yukawa-type contributions from all the modes and is given by:

$$V_G(r) = \frac{G_5}{4\pi} \int_0^\infty d\mu \frac{e^{-\mu r}}{r} |u_\mu(0)|^2 = \frac{G_5}{4\pi} \int_0^\infty d\mu \frac{e^{-\mu r}}{r} \rho(\mu). \tag{27}$$

It is convenient to divide the integral into two parts:

$$V_G(r) = \frac{G_5}{4\pi} \int_0^\infty d\mu \frac{e^{-\mu r}}{r} \rho(\mu) = \frac{G_5}{4\pi} \int_0^{\mu_1} d\mu \frac{e^{-\mu r}}{r} \rho(\mu) + \frac{G_5}{4\pi} \int_{\mu_1}^\infty d\mu \frac{e^{-\mu r}}{r} \rho(\mu). \tag{28}$$

For distances $r \gg r_1 = \mu_1^{-1}$ the second integral is negligible and in the first one we can replace $\rho(\mu)$ by its approximate form (20), which gives us:

$$V_G(r) = G_5 \frac{\sin^2(\nu\pi)}{4\pi^3(z_c - z_1)} \times \int_0^{\mu_1} d\mu \frac{e^{-\mu r}}{\mu r} \frac{A(\mu)}{\alpha_0^2 + \alpha_1^2 A(\mu) + \alpha_2^2 A(\mu)^2}. \tag{29}$$

This integral is always saturated for $A(\mu) \sim \mathcal{O}(1)$, that is for $\mu \sim \mu_2 \ll \mu_1$. We can therefore extend the integration to infinity and evaluate the resulting integral in two regimes: $r_1 \ll r \ll r_2 = \mu_2^{-1}$ and $r \gg r_2$. In the first regime, that is for distances $r_1 \ll r \ll r_2$, the exponential in (29) can be set to one and we obtain:

$$V_G(r) = \frac{G_5}{4\pi r} \int_0^\infty \frac{d\mu}{\mu} \frac{\sin^2(\nu\pi)}{\pi^2(z_c - z_1)} \times \frac{A(\mu)}{\alpha_0^2 + \alpha_1^2 A(\mu) + \alpha_2^2 A(\mu)^2} = \frac{(C_0^\infty)^2}{4\pi} \frac{G_5}{r}, \tag{30}$$

where C_0^∞ is defined by (12). Therefore, in this (large) interval of distances, light modes of the continuum spectrum reproduce precisely the same $1/r$ potential we would get in the presence of a massless boson (or, in other words, a normalizable zero-mode).

In the second regime, that is for large distances, $r \gg r_2$, only $A \ll 1$ will give significant contributions to the integral. We can therefore neglect $A(\mu)$ and $A(\mu)^2$ terms in the denominator of the integrand in (29), thus obtaining:

$$V_G(r) = \frac{\sin^2(\nu\pi)}{4\pi^3\alpha_0^2}(z_c - z_1)^{1-2\nu}e^{2b(z_c-z_0)}\frac{1}{r} \times \int_0^\infty d\mu e^{-\mu r}\mu^{1-2\nu} = \frac{\sin^2(\nu\pi)}{4\pi^3\alpha_0^2}(z_c - z_1)^{1-2\nu}e^{2b(z_c-z_0)}\Gamma(2-2\nu)\frac{G_5}{r^{3-2\nu}}. \tag{31}$$

Hence, we have a modification of the $1/r$ law for large distances, with the potential of type $r^{-\beta}$,where $\beta \in (1, 2)$. Thus, the gravity does not in general become five-dimensional, except in an asymptotically flat geometry ($\nu = 1/2$), where the potential is:

$$V_G(r) = \frac{a^2 + b^2}{16\pi^2 a^2}e^{2b(z_c-z_0)}\frac{G_5}{r^2}. \tag{32}$$

Following [6], we can also study the propagation of gravitational waves generated by a periodic pointlike source on the brane, $T(x, z) = T(\vec{x})e^{-i\omega t}\delta(z)$ (we omit the four-dimensional indices). The field induced on the brane is given by the convolution of the source with the Green's function (33)

$$G_R(\vec{x} - \vec{x}'; \omega) = \int_{-\infty}^{\infty} d(t - t') \times G_R(x, z = 0; x', z' = 0)e^{-i\omega(t-t')}, \tag{33}$$

which after inserting (6) and simplifying becomes

$$G_R(\vec{x} - \vec{x}'; \omega) = \frac{1}{4\pi}\int_0^\infty d\mu \frac{e^{i\omega_\mu r}}{r}\rho(\mu), \tag{34}$$

where again $r \equiv |\vec{x} - \vec{x}'|$ and $\omega_\mu = \sqrt{\omega^2 - \mu^2}$ when $\mu < \omega$ and $\omega_\mu = i\sqrt{\mu^2 - \omega^2}$ when $\mu > \omega$. Only modes with $\mu < \omega$ are actually radiated; the other ones exponentially fall off from the source. Being interested in the propagation of waves, we can integrate in (34) only up to $\mu = \omega$. Considering the range of frequencies $r_2^{-1} \ll \omega \ll r_1^{-1}$ allows us to replace $\rho(\mu)$ by (20) and approximate ω_μ by $\omega - \frac{\mu^2}{2\omega}$:

$$G_R(\vec{x} - \vec{x}'; \omega) = \frac{\sin^2(\nu\pi)}{4\pi^3}\frac{e^{i\omega r}}{r}\int_0^\infty d\mu e^{-i(\mu^2/2\omega)r} \times \frac{A(\mu)}{\alpha_0^2 + \alpha_1^2 A(\mu) + \alpha_2^2 A(\mu)^2}, \tag{35}$$

where we have extended the integration to infinity, using the fact that the integral is saturated for $\mu \sim \mu_2 \ll \omega$.

For distances $r \gg 2\omega r_2/r_1$ the phase factor in (35) varies very slowly and can be set to 1. We then obtain the usual $1/r$ dependence of wave amplitude on the distance to the source:

$$G_R(\vec{x} - \vec{x}'; \omega) = \frac{(C_0^\infty)^2}{4\pi}\frac{e^{i\omega r}}{r}. \tag{36}$$

For distances $r \gg 2\omega r_2/r_1$ we can neglect $A(\mu)$ and $A(\mu)^2$ terms in the denominator of the integrand in (35), obtaining

$$G_R(\vec{x} - \vec{x}'; \omega) = \frac{\sin^2(\nu\pi)}{4\pi^3\alpha_0^2}\Gamma(1-\nu)(z_c - z_1)^{1-2\nu} \times (2\omega)^{1-\nu}e^{2b(z_c-z_0)}\frac{e^{i\omega r + i\pi(\nu-1)/2}}{r^{2-\nu}} = (z_c - z_1)^{1-2\nu}\frac{(2\omega)^{1-\nu}}{\Gamma(1-\nu)}\frac{e^{2b(z_c-z_0)}}{4\pi\cos^2(az_0)} \times \frac{e^{i\omega r + i\pi(\nu-1)/2}}{r^{2-\nu}}, \tag{37}$$

that is, the amplitude is proportional to $r^{-\sigma}$, where $\sigma \in (1, \frac{3}{2})$. This behavior indicates dissipation of the gravitational waves into the extra dimension.

VI. CONCLUSIONS

In this paper we have studied a toy model for quasilocalization of gravity on a brane embedded in an asymptotically warped space-time, restricting our attention to the tensor sector of metric perturbations. While, admittedly, our model is only of limited use as a model for gravity, it allows to uncover quite peculiar behavior of the gravitational potential (and related quantities) induced by the tensor modes. In the absence of a normalized zero-mode, the effective potential is of the 4D form $1/r$, regardless of whether or not the embedding space-time is asymptotically flat. At large scales, the presence of the extra dimensions becomes significant and the potential gets modified. The nature of this modification depends on the asymptotic geometry of the space-time and potential does not necessarily assume the 5D form $1/r^2$.

Although we did not present a concrete model which would possess asymptotically nonflat solutions with quasilocalized gravity, some features inherent to such models can be derived. By Einstein equations, the warped geometry (1) corresponds to the matter energy-momentum tensor which obeys

$$\kappa^2(\rho + p_z) = e^{A(z)}\left(\frac{3}{4}A'^2 + \frac{3}{2}A''\right), \tag{38}$$

where ρ and p_z are the energy density and pressure in the z direction, respectively. Making use of Eq. (7) at large z we find that our asymptotically nonflat ansatz requires $\rho + p_z < 0$ and thus violates the weak energy condition. In this respect, our model is quite similar to (and may share the problems of) that of Ref. [6]. It remains yet to be seen whether these problems can be overcome. This is related to an important question, namely, the impact of the scalar sector of metric perturbations on the effective gravity in a theory where the embedding space-time is not asymptotically flat and the graviton is quasilocalized. It is legitimate to suspect that it could be as important as in the model of Ref. [6] and this matter should be examined on a concrete brane setup.

Finally, it is worth pointing out that Eq. (2), which is the starting point of our analysis is quite universal, in that it describes—with a suitably adjusted potential—the localization of matter fields as well. Therefore, our conclusions are directly applicable to the case of matter fields whenever the corresponding potential can be cast in a "supersymmetric" form $V(z) = W^2(z) - W'(z)$. The latter condition is equivalent to the requirement that the four-dimensional particle is massless (the massive case was treated in Ref. [28]). In the case of matter, the potential in Eq. (2) is no longer related to the metric of space-time through the relation (3) and the problem of violation of the weak energy condition does not need to arise.

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