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Classical model of an elementary particle with a Bertotti-Robinson core and extremal black holes

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We discuss the question, whether the Reissner-Nordström (RN) metric can be glued to other solutions of Einstein-Maxwell equations in such a way that (i) the singularity at $r = 0$ typical of the RN metric is removed, and (ii) matching is smooth. Such a construction could be viewed as a classical model of an elementary particle balanced by its own forces without support by an external agent. One choice is the Minkowski interior that goes back to the old Vilenkin and Fomin's idea who claimed that in this case the bare deltalike stresses at the horizon vanish if the RN metric is extremal. However, the relevant entity here is the integral of these stresses over the proper distance which is infinite in the extremal case. As a result of the competition of these two factors, the Lanczos tensor does not vanish and the extremal RN cannot be glued to the Minkowski metric smoothly, so the elementary-particle model as an empty ball inside fails. We examine the alternative possibility for the extremal RN metric—gluing to the Bertotti-Robinson (BR) metric. For a surface placed outside the horizon there always exist bare stresses but their amplitude goes to zero as the radius of the shell approaches that of the horizon. This limit realizes the Wheeler idea of "mass without mass" and "charge without charge." We generalize the model to the extremal Kerr-Newman metric glued to the rotating analog of the BR metric.

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I. INTRODUCTION

Among other remarkable things, a black hole is sometimes considered as a classical model (or analog) of an elementary particle [1,2]. For an external observer it reveals itself like an object with a few parameters such as a mass m or charge e, while it also can contain a rich structure inside. In this context there is temptation to remove singularities, typical of an inner region of black holes. In particular, the Schwarzschild singularity inside may be replaced by the de Sitter (dS) region, if the hypothesis about the limiting curvature is accepted but this demands some transition layers with deltalike stresses inside the Schwarzschild region [3,4]. Similar stresses appear on the horizon itself [5] if the whole inner region of the Schwarzschild region is replaced by the dS one [6]. Another possibility is to consider matter with the equation of state $p_r = -\varepsilon$ (p_r is the radial pressure, ε is the energy density) that can lead to regular black holes with the dS core [7,8].

In the case of a charged black hole, there exist the following variants. First, the solution can represent an usual Reissner-Nordström (RN) black hole that, as is well known, contains a singularity (hidden beyond the horizon, if $m \geq e$). Second, if sources are distributed on the sphere, one can consider matching the RN metric outside with the Minkowski region inside. In doing so, the region with a singularity inside a horizon is removed and replaced by an empty Minkowski spacetime. As far as the problem of the self-energy is concerned, the event horizon manifests itself for an external observer as a regulator due to gravitational effects, removing divergencies typical of a point particle in classical electrodynamics. If bare stresses on the shell vanish, one would obtain the pure field model of a "classical electron" in the spirit of Abraham and Lorentz. This idea found explicit realization in the paper by Vilenkin and Fomin (VF) [9] who claimed that such a field model is self-consistent for the extremal case $m = e$ only.

However, a more thorough analysis presented below does not confirm the conclusion about smooth matching between extremal RN and Minkowski regions. In the coordinate frame where all metric coefficients are continuous, the amplitude of the deltalike stress-energy tensor of the extremal configuration does tend to zero on the horizon. However, in spite of this, the Lanczos tensor (obtained by integration over the proper distance across the shell) does not vanish because of an infinite proper distance. It was stated in [9] that if the black hole is extremal ($m = e$), one can define the energy-momentum vector entirely in the outer region, whereas the region beyond the horizon contributes nothing into dynamical characteristics in any frame. We demonstrate, however, that this conclusion of [9] is not covariant: it is valid in static coordinates but are violated in the frame of a free-falling observer.

Meanwhile, in the case $m = e$ there is also the third possibility—the so-called Bertotti-Robinson (BR) solution which is not spherically symmetrical since it does not possess a center at all and is regular everywhere [10,11]. The aim of the present paper is to examine the corresponding possibility to construct a self-consistent model

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of a classical electron by gluing the external extremal RN metric to the BR one inside. Such a construction is (i) free from singularities, (ii) self-supporting in the sense that there is no external agent to maintain it in the equilibrium, and (iii) asymptotically flat. However, the important reservation is in order. Both properties (i) and (ii) should be understood in the sense of the limiting transition only. For a shell placed at the $r_0 = r_h + \varepsilon$ (r_h is a horizon radius) there always exist singular bare stresses on the shell but their amplitude tends to zero as $\varepsilon \to 0$. In other words, the construction under consideration represents a regular limit of singular configurations. Instead of a sophisticated structure inside a black hole our construction is in a sense as simple as possible there, being a direct product $AdS_2 x S_2$. In this sense it can be viewed as an alternative to the black holes with the dS core [7,8].

We discuss the energy contribution from the horizon and, as a by-product, we establish some features of quasilocal energy momentum [12,13] inherent to the generic (not necessarily extremal) spherically symmetrical horizon. We also generalize our construction to the Kerr-Newman (KN) black hole.

II. BASIC EQUATIONS

Let us consider the metric of a static spherically symmetric black hole spacetime

$$ds^2 = -b^2 dt^2 + dl^2 + r^2(l) d\omega^2. \tag{1}$$

We assume that there are two regions to be glued together along $r = r_0$. We denote these regions as "+" for $r \geq r_0 > r_h$ and "−" for $r \leq r_0$, where r_h corresponds to the horizon.

We would like the metric to be continuous across the shell but, in general, the terms with first derivatives may acquire jumps. It means that there is some effective transition layer at $r = r_0$. If we write down the Einstein equations for the system with such a layer in the form

$$G_\mu^\nu = 8\pi(T_\mu^\nu + \tilde{T}_\mu^\nu), \tag{2}$$

where $\tilde{T}_\mu^\nu$ is the deltalike contribution from the layer, the conditions of smooth matching read [14]

$$S_\mu^\nu \equiv \int_{r_0-0}^{r_0+0} dl \tilde{T}_\mu^\nu = 0, \tag{3}$$

where S_μ^ν is the so-called Lanczos tensor. Following the general formalism [14], one can write

$$8\pi S_\mu^\nu = [K_\mu^\nu] - \delta_\mu^\nu [K], \tag{4}$$

where K_μ^ν is the tensor of the extrinsic curvature, $K = K_i^i$ ($i = 0, 2, 3$) and $[\cdots] = (\cdots)_+ - (\cdots)_-$. If $[K_\mu^\nu] = 0$, then both regions match smoothly and $S_\mu^\nu = 0$.

For our spacetime the components of this tensor which do not vanish identically are equal to

$$K_0^0 = -\frac{b'}{b}, \qquad K_2^2 = -\frac{r'}{r} = K_3^3, \qquad K = -\frac{2r'}{r} - \frac{b'}{b}, \tag{5}$$

where the prime denotes differentiation with respect to the proper length l. We have

$$8\pi S_2^2 = [K_2^2] - [K] = -[K_0^0] - [K_2^2] = \frac{(br)'_+ - (br)'_-}{br}, \tag{6}$$

$$8\pi S_0^0 = [K_0^0] - [K] = -2[K_2^2] = \frac{2(r'_h - r'_-)}{r}, \tag{7}$$

$$\tilde{T}_\mu^{\ \nu} = S_\mu^\nu \delta(l - l_0), b_0 = b(r_0). \tag{8}$$

By assumption, the outer region is the RN one; the index "0" corresponds to the shell. We consider the mixed components of tensors S_μ^ν and K_μ^ν with one upper and one lower indices since in the metric (1) they correspond to the orthonormal frame, so that on the horizon, where the coordinate frame (1) fails, they remain well defined.

Then for the + region the metric can be written in the curvature coordinates like

$$ds^2 = -dt^2 f + f^{-1} dr^2 + r^2 d\omega^2, \qquad f = b^2, \tag{9}$$

$$b^2 = 1 - \frac{2m}{r} + \frac{e^2}{r^2}. \tag{10}$$

One should distinguish two cases of the metric inside the shell.

III. GLUING REISSNER-NORDSTRÖM AND MINKOWSKI METRICS

First, following [9], we consider the empty (Minkowski) spacetime inside. Then in the − region $b = 1$. In the nonextremal case one obtains, in accordance with [9] that smooth matching is impossible: on the horizon $S_0^0 \to -1/4\pi r_h \neq 0$, $S_2^2 \to (\sqrt{m^2 - e^2})/r_h^2 b \to \infty$. Much more interesting is the extremal case. Let now $m = e$. Then

$$b = 1 - \frac{m}{r}, \qquad S_2^2 = 0 = S_3^3, \qquad 8\pi S_0^0 = -\frac{2m}{r_0^2}, \tag{11}$$

$$\lim_{r_0 \to r_h} S_0^0 = -\frac{1}{4\pi r_h} \neq 0. \tag{12}$$

Our conclusion that bare stresses for the extremal case do not vanish contradicts the statement in [9]. To understand the source of discrepancy, let us use, say, isotropic coordinates [15] whose advantage consists of the continuity of the g_{11} coefficient across the shell at $\rho = a$:

$$\begin{aligned} ds^2 &= -b^2(\rho)dt^2 + c^2(\rho)(d\rho^2 + \rho^2 d\omega^2) \\ &= -b^2(\rho)dt^2 + c^2(\rho)(dx^2 + dy^2 + dz^2), \\ r &= \rho c(\rho), \end{aligned} \tag{13}$$

for $\rho \geq a$ and

$$ds^2 = -b^2(a)dt^2 + c^2(a)(d\rho^2 + \rho^2 d\omega^2) \tag{14}$$

for $\rho \leq a$,

$$b = \frac{1}{c}, \qquad c = 1 + \frac{m}{\rho}. \tag{15}$$

The tensor $\tilde{T}^{\mu}_{\nu}$ for the case under consideration looks like

$$\tilde{T}^{0}_{0} = B\delta(\rho - a), \qquad B = -\frac{ma}{4\pi(m + a)^3}, \tag{16}$$

all other components vanishing identically. If the shell approaches the horizon, $a \to 0$ and $B \to 0$. It *would seem* that in this limit matching becomes smooth. However, the point is that it is S^0_0 but not $\tilde{T}^0_0$ that is the relevant quantity that determines whether matching is smooth or not. Direct evaluation according to (4) gives nonzero S^0_0 (11) due to the factor c in the proper distance $dl = cd\rho$ in the definition (4). Moreover, the coefficient B (in contrast to S^0_0) is coordinate dependent. Using, for example, the proper distance, one would obtain instead of B the quantity $\tilde{B} = Bc = S^0_0$ that does not vanish when $\rho \to a$, $l \to \infty$. Thus, in the coordinate frame under discussion $\lim_{r_0 \to r_h} \tilde{T}^{\mu}_{\nu} = 0$ but $\lim_{r_0 \to r_h} S^{\mu}_{\nu} \neq 0$, the latter fact being independent of the frame. (As far as the quantity $\tilde{T}^0_0\sqrt{-g}$ used in [9] is concerned, it does vanish on the horizon even in the case of a nonextremal black hole due to the addition factor b. However, it cannot be used as a criterion of smooth matching.)

The impossibility of smooth matching can also be understood as follows. It is shown in [16,17] that for any timelike surface $r = r_0(t)$ interior and exterior match smoothly, only if the condition

$$m_- \equiv m(r_0 - 0) = m_+ \equiv m(r_0 + 0) \tag{17}$$

is satisfied,

$$m(r) \equiv \frac{r}{2}[1 - (\nabla r)^2]. \tag{18}$$

For the Minkowski metric $m_- = 0$. For the RN metric $m_+(r) = m - e^2/2r$. In particular, $m_+(\infty) = m$, $m_+(r_h) = m - e^2/2r_h = [(2m^2 - e^2 + 2m\sqrt{m^2 - e^2})/2(m + \sqrt{m^2 - e^2})]$. Both in the nonextremal and extremal cases $m_+ \neq 0$. Thus, one cannot sew smoothly both regions contrary to what was stated in [9,15].

The fact that $\lim_{r_0 \to r_h} S^0_0 \neq 0$ actually means that the proper mass m_p of the layer is finite. Indeed,

$$m_p = 4\pi \int_{r_0 - 0}^{r_0 + 0} dl r^2(-\tilde{T}^0_0) = -4\pi r_h^2 S^0_0 = r_h = m \neq 0. \tag{19}$$

Thus, the *proper* mass of the layer is equal to the active mass.

On the other hand, the contribution of the layer to the active mass $m = 4\pi \int dr(-\tilde{T}^0_0)r^2 dr$ vanishes in the horizon limit due to an additional factor $b(r_0) \to 0$.

IV. GLUING REISSNER-NORDSTRÖM AND BERTOTTI-ROBINSON METRICS

We saw that the classical model of an elementary particle based on matching RN and Minkowski metrics suffers from discontinuity in the geometry. Meanwhile, there is another possibility due to the fact that in the case $m = e$ there is a special branch of solutions of field equations, apart from the extremal RN metric. This is nothing else than the Bertotti-Robinson metric [10,11] that is characterized by the property $r(l) = \text{const}$. In particular, as is well known, in the near-horizon region the metric of the extremal RN tends to that of the BR. It also appears naturally in the thermodynamic context as the extremal limit of nonextremal configurations [18,19]. However, it does not entail immediately that RN and BR metrics match smoothly in the limit under discussion since the quantities like K^{ν}_{μ} involve not only the metric itself but also first derivatives. There exist different forms of the BR metrics corresponding to the nonextremal [$b = \sinh(l/r_0)$] and extremal versions [$b = \exp(l/r_0)$] and also to the absence of the horizon at all [$b = \cosh(l/r_0)$]. This is connected with the existence of three independent Killing timelike vectors, the horizons being in the case under discussion acceleration (not a black hole) horizons (see, e.g., [20] for details).

A. The horizon limit of timelike shells

It follows from the continuity of g_{00} that the only suitable candidate for matching is the extremal BR metric,

$$b = \exp(l/r_0). \tag{20}$$

Then direct calculations give us

$$S^2_2 = S^3_3 = 0, \qquad S^0_0 = \frac{b_0}{4\pi r_0}. \tag{21}$$

In the limit $r_0 \to r_h$ we have $S^{\nu}_{\mu} \to 0$. Moreover, the proper mass $m_p = 4\pi \int_{r_0-0}^{r_0+0} dr r^2 b^{-1}(-\tilde{T}^0_0) = -4\pi S^0_0 r_0^2$ of the transition layer is negative and vanishes in this limit:

$$m_p = -b_0 r_0 \to 0. \tag{22}$$

Again, the fact that both metrics match smoothly, can be understood in terms of the effective mass (18). For the BR metric $r = r_h = \text{const}$, so $(\nabla r)^2 \equiv 0$. For the extremal RN $(\nabla r)^2$ does not vanish identically but tends to zero on the horizon. As $r_0 \to r_h$, masses coincide from both sides of the surface ($m_- = m_+ = \frac{e}{2}$) and this makes smooth gluing possible.

Thus, for any $r_0 \neq r_h$ it is impossible to glue smoothly RN and BR spacetimes but, as r_0 approaches the horizon, mismatch becomes smaller and smaller and disappears in the limit $r_0 = r_h$.

B. Lightlike shells

We discussed matching along the sequence of timelike surfaces. One may ask the question, what happens to a lightlike shell if one places it on the horizon $r = r_h$ from the very beginning? In general, because of different conditions of matching, one cannot expect the result to coincide with the lightlike limit of timelike shells. In particular, if a shell is placed along the line $u = \text{const}$, where u is an isotropic coordinate, only S_{uu} can survive and S_2^2 vanish, whereas for timelike shells it remains nonzero in the nonextremal case. Matching of two different nonextremal RN black holes and nonextremal RN and Minkowski metric along lightlike surfaces was considered in [21] and it follows from the corresponding results that stresses on the shell do not vanish. Now let us discuss the case of the extremal RN. It is obvious that Minkowski spacetime cannot be glued to RN along the horizon since the surface $r = r_h = \text{const}$ is not lightlike in the Minkowski metric. Instead, we again discuss the possibility of smooth gluing between the extremal RN and BR spacetimes.

For the spherically symmetrical case it is sufficient to use the condition derived in [22] (the most general formalism for a lightlike shell is developed in [23]). Let us write the metric in the form

$$ds^2 = -H(U,V)dUdV + r^2 d\Omega^2, \tag{23}$$

where U and V are the Kruskal-like coordinates in which the metric coefficient H remains bounded on the horizon. For definiteness, consider the future horizon $U = 0$.

The condition of matching along $U = 0$ follows from Eq. (6.14) of Ref. [22] and reads

$$\left(\frac{\partial r}{\partial U}\right)_+ - \left(\frac{\partial r}{\partial U}\right)_- = 0. \tag{24}$$

The explicit construction of the Kruskal-like coordinates for the extremal case was carried out in [24], where the coordinates U and V are defined according to

$$u = -\psi(-U), \qquad v = \psi(V), \qquad u = t - r_*, \qquad v = t + r_*, \tag{25}$$

the tortoise coordinate

$$r_* = \int \frac{dr}{b^2} = r + \frac{1}{2}\psi(r - r_h), \qquad b = \left(1 - \frac{r_h}{r}\right), \qquad \psi(\xi) \equiv 4r_h\left(\ln\xi - \frac{r_h}{2\xi}\right). \tag{26}$$

In the vicinity of the future horizon $U = 0$ it follows that $U = -(r - r_h)$ [24], so on the horizon $(\partial r/\partial U)_{\text{RN}} = -1 \neq (\partial r/\partial U)_{\text{BR}} = 0$. Thus, smooth matching is impossible.

V. ENERGY ASSOCIATED WITH THE HORIZON

A. Acceleration horizons and gravitational mass defect

As shown in [18,19], there exists such a limiting transition from a near-extremal black hole to the extremal state that a black hole horizon turns into the acceleration one, typical of the BR spacetime and, in doing so, all points of manifold pick up the value $r = r_h$. The similar conclusion is valid if the RN metric is extremal from the very beginning. The mass between two values $r = r_1$ and $r = r_2$ $m(1,2) = 4\pi \int_{r_1}^{r_2}(-T_0^0)r^2 dr$. As T_0^0 is finite and $r_1 \to r_2 \to r_h = e$ in this limit, $m(1,2) \to 0$. On the other hand, the proper mass of the same region

$$m_p = 4\pi \int_{r_1}^{r_2}(-T_0^0)r^2 dl \to \frac{l(1,2)}{2}, \tag{27}$$

where $l(1,2)$ is the proper distance between points 1 and 2. If one of them corresponds to the horizon, $m_p \to \infty$ since $l \to \infty$. This is the feature inherent of the extremal horizon independent of the concrete form of the metric and is valid, in particular, for the RN and BR spacetimes.

On the other hand, the quasilocal energy [12]

$$E = 4\pi r^2 \varepsilon, \qquad \varepsilon = \frac{k - k_0}{8\pi}, \qquad k = -2b, \tag{28}$$

for the flat spacetime $k_0 = -2$. Here k is the mean curvature of the two-dimensional surface $r = r_0$ embedded into the three-metric. It was shown in [25] that such an energy appears naturally in the thermodynamic context for generic bounded self-gravitating static systems. When $r_0 \to r_h$, $k \to 0$ and $E = E_0 = \text{const}$ does not depend on l (i.e., the position of the boundary).

Thus, the BR spacetime is an example of spacetimes which give the ultimate case of the gravitational mass defect: the active mass $m = \text{const}$, the energy $E = \text{const}$. The proper mass between a horizon and any other point is infinite in the extremal case. For the nonextremal one it is finite but the total amount integrated over all manifold is infinite. This situation is typical of acceleration horizons (it is worth noting that for such horizons not only the energy but also thermodynamics becomes in some sense degenerate [26]). These properties of the energy are similar to those of the so-called T models which were analyzed carefully by Ruban [27,28]. In both cases the coefficient at the angular part of the metric r does not depend on l but T models are actually cosmological, the time dependence of $r(t)$ being essential, whereas in the BR case $r = \text{const}$. Thus, acceleration horizons give one more way of ultimate gravitation binding of an infinite amount of energy into a finite active mass and quasilocal energy.

B. VF model and Møller pseudotensor

It was one of the main statements in [9] that the extremal RN horizon contributes nothing into dynamics. This conclusion was reached on the basis of the Møller pseudotensor. Omitting details, the energy-momentum vector of the system can be written as

$$P_\mu = P_\mu^\infty - P_\mu^h, \tag{29}$$

where $P_\mu^\infty = \frac{1}{2}\oint_{r\to\infty_h} U_\mu^{\nu\sigma} d\sigma_{\nu\sigma}$ is calculated at infinity, $P_\mu^h = \frac{1}{2}\oint_{r=r_h} U_\mu^{\nu\sigma} d\sigma_{\nu\sigma}$ is the contribution from horizon, and $U_\mu^{\nu\sigma}$ is the superpotential. The integration in P_μ^h is carried out over the two-dimensional surface obtained as the intersection of the horizon $r = r_h$ and some three-dimensional spacelike surface depending on the foliation.

It is shown in [9] for the RN metric that (9) $U_0^{ik} = U_i^{0k} = 0$ $(i = 1, 2, 3)$,

$$U_0^{0k} = -\frac{x^k}{4\pi r^2} b(b-1). \tag{30}$$

$$U_i^{kl} = \frac{b}{8\pi}\left(\frac{b-1}{r} + b'\right)\left(\frac{x^k}{r}\delta_{il} - \frac{x^l}{r}\delta_{ik}\right), \tag{31}$$

x^k are quasi-Cartesian coordinates related to r, θ, and ϕ in the same manner as in the usual flat space.

For the foliation of the spacetime by spacelike hypersurfaces $t = \text{const}$, the element of the two-dimensional surface $d\sigma_{\mu\nu}$ has nonvanishing components $d\sigma_{0k}$ only; the term $U_0^{0k} d\sigma_{0k}$ vanishes on the horizon due to the factor b, so that $P_\mu^h = 0$. However, if one uses some other foliation with another spacelike surface, the terms with U_i^{kl} give rise, in general, to $P_\mu^h \neq 0$. To make it vanish and, thus, to achieve the zero contribution from the horizon independent of foliation, Vilenkin and Fomin demand that $U_0^{kl}(r_h) = 0$, whence $(b^2)'_{r=r_h} = 0$. This entails that the black hole should be extremal, $m = e$. As the quantity $U_\mu^{\nu\sigma} d\sigma_{\nu\sigma}$ is a vector (since both factors are tensor densities of opposite weights), it is concluded in [9] that for the extremal case the equality $Q_\mu \equiv U_\mu^{\nu\sigma} d\sigma_{\nu\sigma} = 0$ holds in any coordinate system.

However, the fact that the covariant components of the vector vanish in the system which itself is ill defined on the horizon should not entail the conclusion about vanishing this vector as such. To clarify the essence of matter, let us consider a more general situation when some vector Q_μ has in the coordinate system (9) the components $Q_\mu = (Q_0, 0)$, where $Q_0 = bb'A$, $A \neq 0$ is finite on the horizon. Let us consider at first the nonextremal case, when $b \sim \sqrt{r - r_h}$. It *would seem* that, as $Q_0 \to 0$ as $r \to r_h$, the vector $Q_\mu(r_h) = 0$ and, moreover, this equality holds in any coordinate system since Q_μ is a vector. However, it is easy to see that the vector norm $Q_\mu Q^\mu = g^{00} Q_0^2$ remains nonzero on the horizon because of the factor $g^{00} \sim (r - r_h)^{-1}$. In the Kruskal-like coordinates U, V in which the metric coefficients are well defined, a direct check shows that $Q_U, Q_V \neq 0$ near the point $U = 0 = V$ that corresponds to the surface $t = \text{const}$. For the extremal horizon $Q_0 \sim r - r_h$ but $g^{00} \sim (r - r_h)^{-2}$ and, again, $Q_\mu Q^\mu \neq 0$ on the horizon.

The fact that in the static frame $Q_\mu(r_h) = 0$ can be indeed interpreted as a manifestation of freezing dynamics but from the viewpoint of an external observer only. Correspondingly, the conclusion that for the extremal case P_μ can be defined in the outer region only, with the contribution from the horizon vanishing [9], retains its validity in such a frame. However, another observer, who is diving inside a black hole, will find that the horizon does contribute into the dynamics of the system.

C. Quasilocal energy of the horizon

In the modern approach, there is no necessity to resort to pseudotensors for constructing dynamic characteristics of the gravitational field. Quasilocal energy and momentum are defined on the basis of the action principle [12,13]. A reader can address these papers for a detailed formalism; here we only borrow from there some general results. Consider the spacetime region M with the boundary ∂M that consists of a timelike element $\bar{T}$ and spacelike elements Σ' and Σ'' which are leaves of foliation defined by $t = \text{const}$. The intersections of Σ leaves with $\bar{T}$ define a foliation of $\bar{T}$ into two-dimensional spacelike surfaces B.

Let $u_\mu = -N t_{,\mu}$ be the unit four-velocity for a family of Σ leaves, the lapse function N ensures the condition $u_\mu u^\mu = -1$. Let us also consider the foliation of the spacetime M by the family of timelike surfaces $s = \text{const}$ that contains $\bar{T}$ as one of its leaves and introduces the unit-normal vector $\bar{n}_\mu = \bar{M}\nabla_\mu s$, where $\bar{M}$ ensures normalization $\bar{n}^\mu \bar{n}_\mu = 1$. Also define the unit vector $\bar{u}_\mu = -\bar{N} D_\mu t$, where D_μ is the covariant derivative on $s = \text{const}$. Then, one can define the quasilocal densities for the energy ε, normal momentum $j_\vdash$, tangential momentum, and temporal stresses. We focus our attention on the first two quantities and the law of their transformation. There are two sets of observers connected by local boosts with the relative velocity v. The first one consists of those comoving with $\bar{T}$ and at rest with respect to the B foliation of $\bar{T}$ (characterized by the barred quantities), while the second one consists of those at rest with respect to the Σ foliation (characterized by the unbarred quantities).

The energy density and normal momentum $j_\vdash$ are equal to

$$\kappa\varepsilon = k, \qquad \kappa j_\vdash = -\sigma^{ij}K_{ij}, \qquad \kappa = 8\pi, \tag{32}$$

where K_{ij} is the extrinsic curvature tensor associated with the spacelike hypersurface Σ, and k is the mean curvature of the two-dimensional boundary $2B$ with the metric σ_{ab} embedded in Σ. Similar quantities are defined for barred observers with the transformation law under local boosts [13]

$$\bar{\varepsilon} = \gamma\varepsilon - \gamma v j_\vdash, \qquad \bar{j}_\vdash = \gamma j_\vdash - \gamma v\varepsilon, \tag{33}$$

$$\varepsilon = \gamma\bar{\varepsilon} + \gamma v \bar{j}_\vdash, \qquad j_\vdash = \gamma \bar{j}_\vdash + \gamma v\varepsilon, \tag{34}$$

where

$$\gamma = -u_\mu \bar{u}^\mu = (1 - v^2)^{-1/2} \tag{35}$$

is the Lorentz factor.

Let the RN metric be written in the form

$$ds^2 = -N^2dt^2 + H^2dr^2 + R^2d\Omega^2, \tag{36}$$

where all metric coefficients, which depend in general on r and t, are regular on the horizon (coordinates of the Graves and Brill types [29]). Our goal is to compare dynamic characteristics of the surface $r = r_0$ in the limit $r_0 \to r_h$ for two sets of aforementioned observers. We use the barred quantities for the observers comoving with respect to the boundary element $\bar{T}$ that in our case represents the surface $R - R_0 = 0$, $R_0 = \text{const}$. Then in the coordinates (36)

$$\bar{n}_\mu = \alpha(\dot{R}, R', 0, 0), \qquad \bar{u}_\mu = \alpha\left(-R'\frac{N}{H}, -\dot{R}\frac{H}{N}, 0, 0\right), \\ \alpha^{-2} \equiv (\nabla R)^2 = \frac{R'^2}{H^2} - \frac{\dot{R}^2}{N^2}, \tag{37}$$

$$\bar{n}_\mu = \alpha(\dot{R}, R', 0, 0), \qquad \bar{u}_\mu = \alpha\left(-R'\frac{N}{H}, -\dot{R}\frac{H}{N}, 0, 0\right), \\ \alpha^{-2} \equiv (\nabla R)^2 = \frac{R'^2}{H^2} - \frac{\dot{R}^2}{N^2}, \tag{38}$$

where $\bar{n}_\mu \bar{u}^\mu = 0$, $\bar{n}_\mu$ is a spacelike unit vector. In the curvature coordinates (9) our surface looks like $r = \text{const}$ and the same vector $\bar{u}_\mu$ written in these coordinates has the typical form $-\sqrt{f}(1, 0, 0, 0)$.

We also consider observers who are at rest with respect to the slices of the constant Graves and Brill time t (36) but move from the viewpoint of observers who use static coordinates (9). For such observers, using notations without the bar, we have in the coordinates (36):

$$u_\mu = -N(1, 0, 0, 0), \qquad n^\mu = H^{-1}(0, 1, 0, 0). \tag{39}$$

We can calculate the energy density ε by two methods—directly from (32) or on the basis of the transformation law (34).

For the foliation (39)

$$K^\theta_\theta = K^\phi_\phi = -\frac{\dot{R}}{RN}, \tag{40}$$

$$\kappa\varepsilon = -\frac{2R'}{HR}, \tag{41}$$

$$\kappa j_\vdash = -2\frac{\dot{R}}{RN}. \tag{42}$$

To find $\bar{\varepsilon}$, it is convenient to use the coordinates (9). Then $\bar{j}_\vdash = 0$,

$$\kappa\bar{\varepsilon} = -k = -2\frac{\sqrt{f}}{R}. \tag{43}$$

It follows from (35), (38), and (39) that

$$\gamma = \frac{R'}{H|\nabla R|}. \tag{44}$$

Expressing the scalar $|\nabla R| \equiv \sqrt{(\nabla R)^2}$ in curvature coordinates, where $R = r$, we see that $|\nabla R| = \sqrt{f}$. Now we may exploit the formula (34), where only radial boosts are relevant which do not touch upon the angle variable θ and ϕ. We have

$$\varepsilon = \gamma\bar{\varepsilon} = -\frac{2R'}{HR} \tag{45}$$

that again leads to (41). We see that (41) agrees with (45) and, thus, both methods of calculations (for unbarred quantities at once or for barred with the subsequent boost) give the same result. It is worth stressing that, as one approaches the horizon, $f \to 0$, $\bar{\varepsilon} \to 0$ but $\gamma \to \infty$, so that the product $\varepsilon = \gamma\bar{\varepsilon}$ remains finite. It is worth noting that the fact that $v \to 1$, $\gamma \to \infty$ means that the static system (curvature coordinates) becomes ill defined: its relative speed to comoving observers approaches that of light.

It is also instructive to calculate the invariant $M^2 \equiv (\kappa\varepsilon)^2 - (\kappa j_\vdash)^2 = \kappa^2 p_\mu p^\mu$, where $p_\mu = (\varepsilon, j_\vdash)$ [13]. Then one obtains

$$M^2 = \frac{4}{R^2}\left(\frac{R'^2}{H^2} - \frac{\dot{R}^2}{N^2}\right) = \frac{4}{R^2}(\nabla R)^2. \tag{46}$$

Thus, from the fact that near the horizon $\bar{\varepsilon} \to 0$ and $\bar{j}_\vdash = 0$ in static coordinates it does *not* follow that the vector p_μ vanishes in any frame, be the horizon an extremal or nonextremal. Rather, this vector becomes isotropic on the horizon. To probe it, an observer should use the Graves and Brill reference frame; in other words, he should fall into a black hole.

As a by-product, we see from (46) that for spherically symmetrical generic spacetimes $M^2 > 0$ in the so-called R region, $M^2 < 0$ in the T regions [30], and $M^2 = 0$ on the horizons or in regions where $(\nabla R)^2$ is isotropic. For the BR spacetimes, when $R \equiv \text{const}$, $M^2 = 0$ as well. It is also seen from (18) that $m = \frac{R}{2}(1 - M^2R^2/4)$.

VI. EXTREMAL KERR-NEWMAN GEOMETRY AND ROTATING ANALOG OF BERTOTTI-ROBINSON SPACETIME

In this section we generalize the results typical of a RN metric to the case of the KN one. Namely, we consider the extremal KN metric and sew it with the rotating analog of the BR. (There is no reason to consider also the nonextremal version since even in the nonrotational case matching under discussion is impossible, as shown in previous sections.) Next, we show that in the horizon limit this matching becomes smooth. The extremal KN metric has the general form ($x^0 = t$, $x^1 = r$, $x^2 = \theta$, $x^3 = \phi$)

$$ds^2 = -N^2dt^2 + A^2dr^2 + \rho^2d\theta^2 + D^2(d\phi + Vdt)^2, \tag{47}$$

$$A^2 \equiv B^{-2}, \qquad B = \frac{(r - r_h)}{\rho}, \qquad \rho^2 = r^2 + a^2\cos^2\theta, \tag{48}$$

where all metric coefficients do not depend on t and ϕ, the coefficient D (whose explicit form is irrelevant for us) is finite on the horizon.

The lapse and shift functions

$$V^\phi \equiv V = (r - r_h)q(x^1, x^2), \qquad N = (r - r_h)\chi(x^1, x^2), \tag{49}$$

where in the vicinity of the horizon q and χ are finite. Our frame corotates with the horizon, so that on the horizon $V \to 0$ and $V^\phi/N \to q_+/\chi_+$, where f_+ means $f(r_h, \theta)$.

One can obtain from the KN the rotating analog of the BR metric (RBR) just as BR can be obtained from the near-extremal or extremal RN. For the nonextremal case such a procedure was carried out in [31] and for the extremal case in [32]. As we are dealing with the extremal horizon, the limiting transition of [32] is now relevant. Making the coordinate transformation

$$r = r_h + \lambda\tilde{r}, \qquad t = \frac{\tilde{t}}{\lambda}, \tag{50}$$

one obtains in the limit $\lambda \to 0$ the extremal version of RBR in the form

$$ds^2 = -\chi_+^2(x^i)\tilde{r}^2d\tilde{t}^2 + \frac{d\tilde{r}^2}{\tilde{r}^2}\rho_+^2 + \rho_+^2d\theta^2 + D_+^2(d\phi + \tilde{V}d\tilde{t})^2, \tag{51}$$

$\tilde{V} = q_+\tilde{r}$, $\tilde{V}/\tilde{N} = q_+/\chi_+$, where χ_+ and q_+ do not depend on $\tilde{r}$.

Let us consider the surface $r = r_0$ such that for $r > r_0$ the metric is the KN and for $r < r_0$ it is the RBR, calculate K_μ^ν from both sides and compare the results. The unit vector orthogonal to the surface has the components $n_\mu = (0, B, 0, 0)$. The extrinsic curvature tensor reads

$$K_{ij} = -n_{i;j}, \tag{52}$$

where $i, j = 0, 2, 3$ and the covariant derivative is calculated on the hypersurface $r = r_0$. We must consider the limiting transition $r_0 \to r_h$ and compare the extrinsic curvature tensor for the extremal KN metric with that for the RBR one. On the horizon the coordinate frame (47)–(49) becomes ill defined but we overcome this difficulty by using the orthonormal frame with basic vectors $h_{(a)}^\mu$, $a = 0, 1, 2, 3$ and calculating $K_{(a)(b)} = K_{\mu\nu}h_{(a)}^\mu h_{(b)}^{(\nu)}$. It is convenient to choose the standard basic [33]

$$h_{(0)}^\mu = \frac{1}{N}(1, 0, 0, -V), \tag{53}$$

$$h_{(1)}^\mu = B(0, 1, 0, 0), \tag{54}$$

$$h_{(2)}^\mu = \frac{1}{\rho}(0, 0, 1, 0), \tag{55}$$

$$h_{(3)}^\mu = \frac{1}{D}(0, 0, 0, 1). \tag{56}$$

Its typical feature consists of the fact that it corresponds to local observers with the zero angular momentum: the point with the four-velocity u^μ which has the angular velocity $\omega = -V$ in the coordinate frame (47) is at rest in the locally inertial frame, so that $u^{(3)} = u_{(3)} = 0$. In the coordinate basic the formula (52) gives us

$$K_{ik} = -\frac{B}{2}\frac{\partial g_{ik}}{\partial r}. \tag{57}$$

Then it follows from (53)–(57) that (the prime here denotes differentiation with respect to r) for the KN metric

$$K_{(2)(2)}^{KN} = -B\frac{\rho'}{\rho}, \tag{58}$$

$$K^{KN}_{(3)(3)} = -B\frac{D'}{D}, \quad (59)$$

$$K_{(0)(0)} = B\left(\frac{N'}{N} + 2\frac{V^2DD'}{N^2}\right), \quad (60)$$

$$K^{KN}_{(0)(3)} = -\frac{V'DB}{2N}. \quad (61)$$

When $r_0 \to r_h$, the coefficient $B \to 0$ and, using the properties (48) and (49), we obtain $K^{KN}_{(2)(2)} \to 0$, $K^{KN}_{(3)(3)} \to 0$, $K^{KN}_{(0)(3)} \to -q_+D_+/2\rho_+\chi_+$, and $K_{(0)(0)} \to 1/\rho_+$.

On the other hand, if we calculate the extrinsic tensor for the RBR metric (51) we obtain

$$K^{RBR}_{(2)(2)} = 0 = K^{RBR}_{(3)(3)}, \qquad K^{RBR}_{(0)(3)} = -\frac{q_+D_+}{2\rho_+\chi_+}, \qquad K^{RBR}_{(0)(0)} = \frac{1}{\rho_+}. \quad (62)$$

We see that for all components

$$\lim_{r_0\to r_h} K^{KN}_{(a)(b)} = K^{RBR}_{(a)(b)}. \quad (63)$$

Thus, in the horizon limit the extremal KN geometry goes smoothly to that of RBR spacetime.

VII. SUMMARY AND CONCLUSIONS

Pure field-theoretical models with bare stresses, vanishing in the horizon limit, proved to be realized by means of sewing the extremal RN metric with not the Minkowski, but rather the BR spacetimes. Thus, the whole spacetime reveals a nonuniform topology structure: along with a spherically symmetrical external part, it contains also a direct product of two subspaces of constant curvature inside. Our classical model of an elementary particle reveals itself for an external observer as an extremal black hole, whose horizon is situated at an infinite proper distance. According to the properties of the BR spacetime [20], it extends infinitely without hitting a singularity also beyond the horizon for an observer who dares to dive into it. Both for the nonextremal or extremal cases the region inside the horizon does in general contribute to dynamic characteristics. The energy-momentum vector, associated with the horizon, turns out to be isotropic but nonvanishing. It is of interest to generalize this result to generic isolated horizons [34].

We saw that for a family of timelike surfaces $r = r_0$ the magnitude of the Lanczos tensor tends to zero when $r_0 \to r_h$. On the other hand, if $r_0 = r_h$ exactly, the surface becomes lightlike in which case the matching conditions are qualitatively different and cannot be satisfied for the case under discussion, when the BR metric is glued to the extremal RN one. Thus, our construction should be understood in the sense of the limiting transition only, in which case it gives ”mass without mass” and ”charge without charge” [35]. The singular residue for any member of the family of configurations (before the limit is taken) can be understood as follows. The applicability of the theorem about singularities inside black holes implies that $\varepsilon \geq 0$ (weak energy condition) and $\varepsilon + \sum_j p_j \geq 0$ [36]. These conditions break down, for example, for black holes with a dS core when $p = -\varepsilon < 0$ and, thus, regularity is achieved [7,8]. However, they are satisfied for an electromagnetic field ($\varepsilon = w$, $p_r = -w$, $p_\perp = w$, $w = e^2/r^4$). Thus, had we had an everywhere regular black hole metric, this would have contradicted the singularity theorem. Our construction occupies an intermediate place between singular and regular models: the ideal purely field classical Abraham-Lorentz model of the electron remains unattainable but one may approach it as nearly as one likes.

Our consideration was purely classical. As far as the role of quantum backreaction is concerned, it was checked directly that the backreaction of quantum massive fields changes the condition of extremality in such a way that $m \neq e$ but does not prevent the existence of extremal horizons as such; it also leaves intact the general geometrical character of acceleration horizons as direct products of two spheres of constant curvature (but, in contrast to the BR metric, their radii in the quantum-corrected case no longer coincide) [37]. Therefore, one can glue the quantum-corrected extremal RN metric to the quantum-corrected analog of the BR one in much the same way as was done above classically. Bearing also in mind that extremal horizons have the Hawking temperature $T_H = 0$ and do not radiate, one can expect our static self-supported solutions as a whole obtained pure classically survive also on the semiclassical level.

It is a common belief that extremal black holes can be suitable candidates on the role of remnants after black hole evaporation. Investigations of some two-dimensional exactly solvable models with account for quantum backreaction showed that remnants can also represent semi-infinite throats corresponding to two-dimensional AdS spacetimes which are nothing else than a two-dimensional analog of BR (AdSxS_2) [38]. One is led to think that, perhaps BR spacetime can also be relevant in this context for late stages of evaporation of near-extremal black holes.

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