Well-posedness of the Baumgarte-Shapiro-Shibata-Nakamura formulation of Einstein's field equations

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We give a well posed initial value formulation of the Baumgarte-Shapiro-Shibata-Nakamura form of Einstein's equations with gauge conditions given by a Bona-Massó–like slicing condition for the lapse and a frozen shift. This is achieved by introducing extra variables and recasting the evolution equations into a first order symmetric hyperbolic system. We also consider the presence of artificial boundaries and derive a set of boundary conditions that guarantee that the resulting initial-boundary value problem is well posed, though not necessarily compatible with the constraints. In the case of dynamical gauge conditions for the lapse and shift we obtain a class of evolution equations which are strongly hyperbolic and so yield well posed initial value formulations.

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I. MOTIVATION

Most numerical evolutions of Einstein's field equations try to approximate solutions on a generically infinite (noncompact) three-space by computations on a truncated finite (compact) domain. For this, artificial boundaries and corresponding boundary conditions have to be introduced. Mathematically, this immediately poses the question of well-posedness of the initial-boundary value problem (IBVP) for the evolution equations and compatibility with the constraints. In addition, governed by causality, the solution on the finite domain is very likely to differ from the solution on the infinite domain, after disturbances from the boundaries enter the computational domain. This makes the choice of the boundary condition crucial for the physical interpretation of the results, especially if one thinks of integrated quantities like masses, charges and momenta. Physically, it could be even argued that ultimately only estimates of the deviation of the numerical solution from the actual solution are significant and that the mathematical concepts of well-posedness of the IBVP including compatibility with the constraints and avoidance of reflections from artificial boundaries are only steps towards achieving this goal.

Removing the influence of the boundaries could be achieved by enlarging the computational domain to a size such that, according to causality, disturbances from boundaries cannot have reached the domain of physical interest. Note that this requires knowledge of the causality structure of the spacetime, which presupposes estimates on the solution to be found. In addition, increasing the size of the computational domain goes at the cost of resolution, because of finite computational resources, which is particularly restricting in three dimensional settings although this problem can be alleviated by using adaptive or fixed mesh refinement techniques [1]. So, both restrictions in computational resources and the demand for higher resolution lead us to attempt to minimize the influence of artificial boundaries on the numerical solution. This is tried by so called "outgoing boundary conditions" meant to make those boundaries appear as "transparent" as possible. For instance, one such approach is given by Endquist and Majda [2] using a hierarchy of conditions which gradually decrease reflections at the boundary. Alternative approaches are the methods of characteristic [3] or perturbative matching [4]. See [5,6] for an approach trying to avoid the introduction of artificial boundaries altogether by a suitable compactification of spacetime.

II. INTRODUCTION

In this article we analyze the IBVP of the Baumgarte-Shapiro-Shibata-Nakamura (BSSN) [7,8] formulation of Einstein's vacuum equations, which is currently used by several groups in numerical relativity with applications to the binary black hole and binary neutron star problem, see [9] for a review. Since the BSSN equations are first order in time, but mixed first/second order in space, their type (elliptic, parabolic, hyperbolic or mixed) is *a priori* not clear. Here, we analyze the well-posedness of their (nonlinear) Cauchy problem with and without boundaries. Well-posedness means that the Cauchy problem has a unique solution local in time and that the solution depends continuously on the initial data. The last property is important in view of obtaining convergent discretizations since in general numerical simulations introduce small errors in the initial data. If violated, this can lead to errors at a later time which grow exponentially with increasing resolution (see [10,11] for examples of this phenomenon).

We find that the BSSN system with a large family of gauge conditions for the lapse, including Bona-Massó–like slicing conditions, and an *a priori* specified shift yields a well posed initial value problem. This is achieved by introducing extra fields that make it possible to recast the system into a first order quasilinear symmetric hyperbolic form for which standard well-posedness results are known [12]. The introduction of extra fields brings additional constraints, and the original BSSN system and the first order symmetric hyperbolic system derived in this article are only equivalent if these constraints are satisfied. However, we show that the associated constraint variables obey a closed evolution system that is *independent* of the other constraints. This means that the additional constraints are satisfied everywhere at later times if satisfied initially, even if the other constraints are violated. This implies that the (original) BSSN system is well posed; in particular, unique solutions local in time exist, and depend continuously on the initial data.

Our first order symmetric hyperbolic reduction also facilitates the analysis of characteristic modes which is particularly useful when constructing boundary conditions. Here we construct maximally dissipative boundary conditions that guarantee the well-posedness of the resulting IBVP [13]. These conditions assume that the shift is tangential to the boundary. For nonsmooth boundaries this implies that the shift vanishes at corners. For smooth boundaries on the other hand this should not be a too severe restriction. For example, it would still allow for the use of corotating shift conditions. Although in general our boundary conditions are not compatible with the constraints, they are consistent with the evolution equations and constitute a first step towards improving numerical evolutions of the BSSN system. In particular, the present analysis offers the possibility to construct constraint-preserving boundary conditions [14] in the linearized case, following the lines of [15–17].

The techniques used in this article are the same used in [18] where well-posedness of the BSSN system with an explicitly given shift and an algebraic gauge condition is found by considering an auxiliary first order system. A different technique which makes use of pseudodifferential calculus has recently been applied in order to show well-posedness for a closely related formulation [19]. More recently, in [17,20] a definition of symmetric hyperbolicity based on energy estimates for second order systems was presented which was verified for the BSSN system and the formulation in [19] for the case of an algebraic lapse and an explicitly given shift. Nevertheless the connection of their definition and existence of solutions is open.

The remainder of this work is organized as follows. In Sec. III we review the BSSN equations, specify the gauge conditions we are considering, and discuss the evolution system for the constraint variables. In Sec. IV we introduce extra fields and derive a first order symmetric hyperbolic system that reflects the dynamics of the original BSSN system. The characteristic fields with nontrivial speeds are constructed in Sec. V and are used to write down maximally dissipative boundary conditions. In Sec. VI we find using pseudodifferential calculus that the BSSN system with a "K-driver" and a "Gamma-freezing" condition as defined in [21] but with a different time coordinate is strongly hyperbolic according to the definition in [19,22] and so yields a well posed initial value formulation. Conclusions are drawn in Sec. VII.

III. THE BSSN EQUATIONS

Since our results depend crucially on the principal part of the equations, we write down the BSSN system explicitly in this section. The system of equations is the one that has been used in Ref. [21] for numerical simulations, but it might differ from the one used by other groups. Decomposing the three metric and the extrinsic curvature according to

$$\gamma_{ij} = e^{4\phi}\tilde{\gamma}_{ij}, \tag{1}$$

$$K_{ij} = e^{4\phi}\left(\tilde{A}_{ij} + \frac{1}{3}\tilde{\gamma}_{ij}K\right), \tag{2}$$

where $\tilde{\gamma}_{ij}$ has unit determinant and $K = \gamma^{ij}K_{ij}$ is the mean curvature, the evolution equations are obtained from

$$\hat{\partial}_0\phi = -\frac{\alpha}{6}K + \frac{1}{6}\partial_k\beta^k, \tag{3}$$

$$\hat{\partial}_0\tilde{\gamma}_{ij} = -2\alpha\tilde{A}_{ij} + 2\tilde{\gamma}_{k(i}\partial_{j)}\beta^k - \frac{2}{3}\tilde{\gamma}_{ij}\partial_k\beta^k, \tag{4}$$

$$\begin{aligned}\hat{\partial}_0 K = {}& -e^{-4\phi}[\tilde{D}^i\tilde{D}_i\alpha - 2\partial_i\phi\cdot\tilde{D}^i\alpha] \\ & + \alpha\left(\tilde{A}^{ij}\tilde{A}_{ij} + \frac{1}{3}K^2\right) - \alpha S,\end{aligned} \tag{5}$$

$$\begin{aligned}\hat{\partial}_0\tilde{A}_{ij} = {}& e^{-4\phi}[\alpha\tilde{R}_{ij} + \alpha R^{\phi}_{ij} - \tilde{D}_i\tilde{D}_j\alpha - 4\partial_{(i}\phi\cdot\tilde{D}_{j)}\alpha]^{TF} \\ & + \alpha K\tilde{A}_{ij} - 2\alpha\tilde{A}_{ik}\tilde{A}^k_j + 2\tilde{A}_{k(i}\partial_{j)}\beta^k - \frac{2}{3}\tilde{A}_{ij}\partial_k\beta^k \\ & - \alpha e^{-4\phi}\hat{S}_{ij},\end{aligned} \tag{6}$$

$$\hat{\partial}_0\tilde{\Gamma}^i = \tilde{\gamma}^{kl}\partial_k\partial_l\beta^i + \frac{1}{3}\tilde{\gamma}^{ij}\partial_j\partial_k\beta^k + \partial_k\tilde{\gamma}^{kj}\cdot\partial_j\beta^i - \frac{2}{3}\partial_k\tilde{\gamma}^{ki}\cdot\partial_j\beta^j - 2\tilde{A}^{ij}\partial_j\alpha + 2\alpha\Big[(m-1)\partial_k\tilde{A}^{ki} - \frac{2m}{3}\tilde{D}^iK + m(\tilde{\Gamma}^i_{kl}\tilde{A}^{kl} + 6\tilde{A}^{ij}\partial_j\phi)\Big] - S^i, \tag{7}$$

where we have introduced the operator $\hat{\partial}_0 = \partial_t - \beta^j\partial_j$. Here, all quantities with a tilde refer to the conformal three metric $\tilde{\gamma}_{ij}$, and the latter is used in order to raise and lower their indices. In particular, $\tilde{D}_i$ and $\tilde{\Gamma}^k_{ij}$ refer to the covariant derivative and the Christoffel symbols, respectively, with respect to $\tilde{\gamma}_{ij}$. The expression $[\ldots]^{TF}$ denotes the traceless part (with respect to the metric $\tilde{\gamma}_{ij}$) of the expression inside the parentheses, and

$$\tilde{R}_{ij} = -\frac{1}{2}\tilde{\gamma}^{kl}\partial_k\partial_l\tilde{\gamma}_{ij} + \tilde{\gamma}_{k(i}\partial_{j)}\tilde{\Gamma}^k - \tilde{\Gamma}_{(ij)k}\partial_j\tilde{\gamma}^{jk} + \tilde{\gamma}^{ls}(2\tilde{\Gamma}^k_{l(i}\tilde{\Gamma}_{j)ks} + \tilde{\Gamma}^k_{is}\tilde{\Gamma}_{klj}),$$

$$R^{\phi}_{ij} = -2\tilde{D}_i\tilde{D}_j\phi - 2\tilde{\gamma}_{ij}\tilde{D}^k\tilde{D}_k\phi + 4\tilde{D}_i\phi\cdot\tilde{D}_j\phi - 4\tilde{\gamma}_{ij}\tilde{D}^k\phi\cdot\tilde{D}_k\phi.$$

The parameter m, which was introduced in [23], controls how the momentum constraint is added to the evolution equations for the variable $\tilde{\Gamma}^i$. The system in Ref. [21] corresponds to the choice $m = 1$. However, in order to obtain a first order symmetric hyperbolic reduction, we will see later that we need m to be a specific function of the lapse and the mean curvature. The source terms S, $\hat{S}_{ij}$, and S^i are defined in terms of the four Ricci tensor, $R^{(4)}_{ij}$, and the constraint variables

$$H \equiv \frac{1}{2}(\gamma^{ij}R^{(3)}_{ij} + K^2 - K^{ij}K_{ij}), \tag{8}$$

$$M_i \equiv \tilde{D}^j\tilde{A}_{ij} - \frac{2}{3}\tilde{D}_iK + 6\tilde{A}_{ij}\tilde{D}^j\phi, \tag{9}$$

$$C^i_\Gamma \equiv \tilde{\Gamma}^i + \partial_j\tilde{\gamma}^{ij}, \tag{10}$$

as

$$S = \gamma^{ij}R^{(4)}_{ij} - 2H, \tag{11}$$

$$\hat{S}_{ij} = [R^{(4)}_{ij} + \tilde{\gamma}_{k(i}\partial_{j)}C^k_\Gamma]^{TF}, \tag{12}$$

$$S^i = 2\alpha m\tilde{\gamma}^{ij}M_j - \hat{\partial}_0C^i_\Gamma. \tag{13}$$

The vacuum equations consist of the evolution Eqs. (3)–(7) with $S = 0$, $\hat{S}_{ij} = 0$, $S^i = 0$ and the constraints $H = 0$, $M_i = 0$, and $C^i_\Gamma = 0$.

Using the Bianchi identities, $2\nabla^\mu R^{(4)}_{\mu\nu} - \nabla_\nu R^{(4)} = 0$ and imposing the evolution equations, it can be shown that the constraint variables obey the following propagation system:

$$\hat{\partial}_0H = -\frac{1}{\alpha}D^j(\alpha^2M_j) - \alpha e^{-4\phi}\tilde{A}^{ij}\tilde{\gamma}_{ki}\partial_jC^k_\Gamma + \frac{2\alpha}{3}KH, \tag{14}$$

$$\hat{\partial}_0M_j = \frac{\alpha^3}{3}D_j(\alpha^{-2}H) + \alpha KM_j + D^i(\alpha[\tilde{\gamma}_{k(i}\partial_{j)}C^k_\Gamma]^{TF}), \tag{15}$$

$$\hat{\partial}_0C^k_\Gamma = 2\alpha m\tilde{\gamma}^{kl}M_l. \tag{16}$$

By introducing the further constraint variable $Z^k_j = \partial_jC^k_\Gamma$ which satisfies $\partial_{[i}Z^k_{j]} = 0$ one can reduce Eqs. (14)–(16) to a first order symmetric hyperbolic system provided that $m > 1/4$. In the absence of boundaries, this implies that the constraints are preserved, i.e., trivial initial data for the constraints variables lead to zero constraint variables at later times as well. If timelike boundaries are present, the constraints are only preserved if suitable boundary conditions are specified. Such constraint-preserving boundary conditions are discussed in [14–17]; but for the (nonlinear) BSSN system it is not yet understood if they lead to a well posed IBVP.

In order to evolve the system (3)–(7) we have to specify conditions on the lapse α and the shift β^i. The simplest possibility is to set $\alpha = 1$ (or any other fixed function) and $\beta^i = 0$. However, this leads to a formulation that is not strongly hyperbolic [this will follow from the results in Sec. VI if we set the function f defined below in Eq. (21) to zero]. This can be avoided by "densitizing" the lapse. More generally, we can require [24] that the lapse

$$\alpha = \alpha(\phi, x^\mu), \tag{17}$$

is a smooth strictly positive function of the conformal factor (or the determinant of the three metric) and spacetime coordinates with the restriction that $\sigma = (12\alpha)^{-1}\partial\alpha/\partial\phi$ is strictly positive. Taking a time derivative of this, assuming that $\partial\alpha/\partial t = 0$ and using Eq. (3) we obtain

$$\frac{d}{dt}\alpha = -2\alpha^2\sigma\Big(K - \frac{1}{\alpha}D_k\beta^k\Big), \tag{18}$$

which is the modification of the Bona-Massó condition [25] proposed in [26,27]. The advantage of this gauge is that it is compatible with a time-independent lapse in a time slicing that is adapted to stationarity if ∂_t is a Killing field. It follows from the calculations of Ref. [18] that in this case the BSSN system is strongly hyperbolic if one chooses $m > 1/4$ and symmetric hyperbolic if the parameter m is adjusted such that $4m = 6\sigma + 1$ with $\sigma > 1/2$ [28]. We mention here that the special case $\alpha = e^{6\phi}Q(x^\mu)$, where $Q(x^\mu)$ is an *a priori* specified function, has been observed to lead to more stable numerical evolutions of a single black hole with the BSSN system [29].

Here, we are interested in live gauge conditions which allow lapse and shift to react on changes of the fields. Such conditions can be useful, for instance, to evade singularities. In this article, we consider two cases of gauge conditions:

(a) The following evolution equation for the lapse

$$\hat{\partial}_0\alpha = -\alpha F(\alpha, K, x^\mu), \quad (19)$$

where F is a smooth function of α, K, and x^μ with the restriction that

$$\sigma \equiv \frac{1}{2\alpha}\frac{\partial F}{\partial K} > 0. \quad (20)$$

This condition generalizes the Bona-Massó gauges. The shift is frozen, that is, assumed to be an *a priori* specified function of spacetime. Symmetric hyperbolic formulations of the vacuum field equations with these gauge conditions were obtained in [24].

(b) The gauge conditions of Ref. [21] which, for the lapse, require the "hyperbolic K-driver" condition

$$\hat{\partial}_0\alpha = -\alpha^2 f(\alpha, \phi, x^\mu)[K - K_0(x^\mu)], \quad (21)$$

where the function $f(\alpha, \phi, x^\mu)$ is smooth and strictly positive, and $K_0(x^\mu)$ is an arbitrary smooth function. For the shift, the "hyperbolic Gamma driver" [21] type condition

$$\hat{\partial}_0\beta^i = \alpha^2 G(\alpha, \phi, x^\mu)B^i, \quad (22)$$

$$\hat{\partial}_0 B^i = e^{-4\phi}H(\alpha, \phi, x^\mu)\hat{\partial}_0\tilde{\Gamma}^i - \eta(B^i, \alpha, x^\mu) \quad (23)$$

is imposed, where $G(\alpha, \phi, x^\mu)$ and $H(\alpha, \phi, x^\mu)$ are smooth, strictly positive functions, and $\eta(B^i, \alpha, x^\mu)$ is a smooth function. Notice that Eq. (21) is a special case of Eq. (19). Note also that the conditions (21)–(23) differ from the ones considered in [21] by the replacement $\partial_t \mapsto \hat{\partial}_0$ which simplifies the analysis in the present article.

In the next section, we show that the gauge conditions (a) lead to a well posed initial value problem provided that the parameter m is chosen such that $4m = 6\sigma + 1$. In the presence of boundaries, we derive boundary conditions in Sec. V that make sure that in this case the resulting IBVP is well posed. In Sec. VI we show that the initial value problem with the gauge conditions (b) is well posed provided that some specified conditions on m and the functions f, G, and H are satisfied. Symmetric hyperbolic first order formulations of Einstein's equations that incorporate gauge conditions that are similar to (b) have been worked out in [30].

IV. FIRST ORDER SYMMETRIC HYPERBOLIC FORM (FROZEN SHIFT)

In this section we recast the BSSN equations with the gauge conditions (a) into a first order symmetric hyperbolic system. In order to do so we introduce the extra variables

$$d_k = 12\partial_k\phi, \qquad \tilde{d}_{kij} = \partial_k\tilde{\gamma}_{ij}, \qquad A_k = \frac{\partial_k\alpha}{\alpha}, \quad (24)$$

and rewrite Eqs. (5)–(7) as

$$\hat{\partial}_0 K = -\alpha e^{-4\phi}\tilde{\gamma}^{ij}\partial_i A_j + l.o., \quad (25)$$

$$\hat{\partial}_0\tilde{A}_{ij} = \alpha e^{-4\phi}\left[-\frac{1}{2}\tilde{\gamma}^{kl}\partial_k\tilde{d}_{lij} + \zeta\tilde{\gamma}^{kl}C^{\tilde{d}}_{k(ij)l} + \tilde{\gamma}_{k(i}\partial_{j)}\tilde{\Gamma}^k - \frac{1}{6}\partial_{(i}d_{j)} - \partial_{(i}A_{j)}\right]^{TF} + l.o., \quad (26)$$

$$\hat{\partial}_0\tilde{\Gamma}^i = 2\alpha\left[(m-1)\partial_k\tilde{A}^{ki} - \frac{2m}{3}\tilde{D}^i K\right] + l.o., \quad (27)$$

where $l.o.$ refers to lower order terms that depend on ϕ, $\tilde{\gamma}_{ij}$, K, $\tilde{A}_{ij}$, $\tilde{\Gamma}^i$, α, d_k, $\tilde{d}_{kij}$, A_k but not their derivatives. Here, we have added the constraint variables $C^{\tilde{d}}_{lkij} = \partial_{[l}\tilde{d}_{k]ij}$ with an arbitrary parameter ζ in the equation for $\tilde{A}_{ij}$. As we will see shortly, the addition of these constraints will allow us to obtain a larger family of symmetric hyperbolic formulations. Evolution equations for the extra variables are obtained by applying the operator $\hat{\partial}_0$ on the Eqs. (24), using the commutation relation $[\hat{\partial}_0, \partial_k] = \partial_k\beta^l \cdot \partial_l$ and using the evolution Eqs. (3), (4), and (19) for ϕ, $\tilde{\gamma}_{ij}$, and α. The result is

$$\hat{\partial}_0 d_k = -2\alpha(\partial_k + A_k)K + d_l\partial_k\beta^l + 2\partial_k\partial_l\beta^l, \quad (28)$$

$$\hat{\partial}_0\tilde{d}_{kij} = -2\alpha(\partial_k + A_k)\tilde{A}_{ij} + \tilde{d}_{lij}\partial_k\beta^l + 2\tilde{d}_{kl(i}\partial_{j)}\beta^l - \frac{2}{3}\tilde{d}_{kij}\partial_l\beta^l + 2\tilde{\gamma}_{l(i}\partial_{j)}\partial_k\beta^l - \frac{2}{3}\tilde{\gamma}_{ij}\partial_k\partial_l\beta^l, \quad (29)$$

$$\hat{\partial}_0 A_k = -2\sigma\alpha\partial_k K - \alpha\frac{\partial F}{\partial\alpha}A_k - \frac{\partial F}{\partial x^k} + A_l\partial_k\beta^l. \quad (30)$$

We have rewritten the BSSN equations (with a fixed prescribed shift but a live condition for the lapse) as a first order quasilinear evolution system for the variables $u = (\phi, \tilde{\gamma}_{ij}, \alpha, K, \tilde{A}_{ij}, \tilde{\Gamma}^k, d_k, \tilde{d}_{kij}, A_k)^T$ which is given by the Eqs. (3), (4), (19), and (25)–(30). It has the form

$$\hat{\partial}_0 u = \alpha\mathbf{A}^i(u)\partial_i u + F(u), \quad (31)$$

where the matrix-valued functions $\mathbf{A}^i(u)$, $i = 1, 2, 3$, and the vector-valued function $F(u)$ depend on u but not their derivatives. An important point to notice here is that we have not added any of the constraints variables (8)–(10) to the right-hand side (RHS) of the evolution equations for the extra variables. As a consequence, the additional constraints, defined by,

$$C^d_k \equiv d_k - 12\partial_k\phi = 0, \tag{32}$$

$$C^{\tilde{d}}_{kij} \equiv \tilde{d}_{kij} - \partial_k\tilde{\gamma}_{ij} = 0, \tag{33}$$

$$C^A_k \equiv A_k - \frac{\partial_k\alpha}{\alpha} = 0, \tag{34}$$

that arise when writing the system as a first order one propagate independently of whether or not the remaining constraints are satisfied:

$$\hat{\partial}_0 C^d_k = -2\alpha K C^A_k + C^d_l\partial_k\beta^l, \tag{35}$$

$$\hat{\partial}_0 C^{\tilde{d}}_{kij} = -2\alpha\tilde{A}_{ij}C^A_k + C^{\tilde{d}}_{lij}\partial_k\beta^l + 2C^{\tilde{d}}_{kl(i}\partial_{j)}\beta^l - \frac{2}{3}C^{\tilde{d}}_{kij}\partial_l\beta^l, \tag{36}$$

$$\hat{\partial}_0 C^A_k = -\alpha\frac{\partial F}{\partial\alpha}C^A_k + C^A_l\partial_k\beta^l. \tag{37}$$

This means that if initial data is given such that $C^d_k = 0$, $C^{\tilde{d}}_{kij} = 0$, $C^A_k = 0$ (and suitable boundary conditions are chosen), these constraints will also be satisfied at later times and we obtain a solution of the BSSN Eqs. (3)–(7). This is true even if the initial data *violates* the constraints $H = 0$, $M_i = 0$, $C^i_\Gamma = 0$ of the BSSN system.

Having obtained a first order quasilinear system that yields the same solutions as the BSSN system (provided that the constraints $C^d_k = 0$, $C^{\tilde{d}}_{kij} = 0$, $C^A_k = 0$ are satisfied initially) we now analyze for what range of the parameters m, σ, and ζ the first order system is symmetric hyperbolic. Introducing the principal symbol $\mathbf{A}(\mathbf{n}) = \mathbf{A}^i n_i$ where $\mathbf{n} = n_k dx^k$ is any normalized one-form, this means that we have to find a positive definite matrix $\mathbf{H} = \mathbf{H}(u, x^\mu)$ which depends smoothly on u and the spacetime coordinates x^μ such that $\mathbf{HA}(\mathbf{n})$ is symmetric for all u, x^μ, and all normalized one-forms $\mathbf{n}$ [31]. A necessary condition for this is that each $\mathbf{A}(\mathbf{n})$ is diagonalizable and has only real eigenvalues. So we first analyze the eigenvalue problem

$$\mu u = \mathbf{A}(\mathbf{n})u. \tag{38}$$

Explicitly, we have

$$\mu\phi = 0, \tag{39}$$

$$\mu\tilde{\gamma}_{ij} = 0, \tag{40}$$

$$\mu\alpha = 0, \tag{41}$$

$$\mu K = -A_n, \tag{42}$$

$$\mu\tilde{A}_{ij} = -\frac{1}{2}\tilde{d}_{nij} + \frac{\zeta}{2}[\tilde{d}_{(ij)n}]^{TF} + e^{-4\phi}\left[n_{(i}\tilde{\Gamma}_{j)} - \frac{1}{6}n_{(i}d_{j)} - n_{(i}A_{j)} - \frac{\zeta}{2}n_{(i}\tilde{d}^k_{j)k}\right]^{TF}, \tag{43}$$

$$\mu\tilde{\Gamma}_i = 2(m-1)e^{4\phi}\tilde{A}_{ni} - \frac{4m}{3}n_i K, \tag{44}$$

$$\mu d_k = -2n_k K, \tag{45}$$

$$\mu\tilde{d}_{kij} = -2n_k\tilde{A}_{ij}, \tag{46}$$

$$\mu A_k = -2\sigma n_k K, \tag{47}$$

where $A_n \equiv A_i n^i$, $\tilde{d}_{nij} = \tilde{d}_{kij}n^k$ etc., and $\tilde{\Gamma}_i = \tilde{\gamma}_{ij}\tilde{\Gamma}^j$. Here, and in the following, we normalize n_i with respect to the three metric γ_{ij}. A convenient way for obtaining the nonzero eigenvalues is by deriving a closed equation for the extrinsic curvature. Introducing $K_{ij} = e^{4\phi}\tilde{A}_{ij} + \gamma_{ij}K/3$ we obtain

$$\mu^2 K_{ij} = K_{ij} + 2(m-1)n_{(i}K_{j)n} + (1 - 2m + 2\sigma)n_i n_j K + \frac{2}{3}(m-1)\gamma_{ij}(K - K_{nn}). \tag{48}$$

In [24] it was shown that the system is strongly hyperbolic if the operator on the RHS is diagonalizable and has only *strictly positive* eigenvalues. This is the case if and only if the squares of the eigenspeeds,

$$\mu_1^2 = 2\sigma, \qquad \mu_2^2 = \frac{4m-1}{3}, \qquad \mu_3^2 = m, \qquad \mu_4^2 = 1, \tag{49}$$

are strictly positive, that is, if and only if $m > 1/4$ and $\sigma > 0$. Notice that these conditions are independent of ζ and that for $\sigma = 1/2$ [which implies that the function F in Eq. (19) must have the form $F(\alpha, K, x^\mu) = \alpha K + F_0(\alpha, x^\mu)$ for some smooth function F_0] and $m = 1$ all speeds are one or zero.

In order to find the most general symmetrizer it is convenient to define $\hat{K}_{ij} = e^{4\phi}\tilde{A}_{ij}$, to decompose

$$\tilde{d}_{kij} = -2e^{-4\phi}e_{kij} + \frac{3}{5}\tilde{\gamma}_{k(i}b_{j)} - \frac{1}{5}\tilde{\gamma}_{ij}b_k,$$
$$b_j = \tilde{\gamma}^{ki}\tilde{d}_{kij},$$

where e_{kij} is completely trace-free, and to replace $\tilde{\Gamma}_i$, d_i, b_i by the combinations

$$v_i = \tilde{\Gamma}_i - \frac{1}{6}d_i - A_i - \frac{9\zeta + 6}{20}b_i,$$
$$w_i = \tilde{\Gamma}_i - \frac{1}{6}d_i - A_i + (m-1)b_i,$$
$$z_i = \sigma d_i - A_i.$$

Here, we assume that $\sigma > 0$ and that $20m + 9\zeta - 14 > 0$

which implies that the transformation is regular. The first condition is necessary for strong hyperbolicity, and the second one can always be achieved by choosing the parameter ζ (which does not appear in the original BSSN system) to be sufficiently large.

In terms of these variables the nontrivial block of the principal part reads

$$\mu K = -A_n, \tag{50}$$

$$\mu A_i = -2\sigma n_i K, \tag{51}$$

$$\mu \hat{K}_{ij} = e_{nij} - \zeta e_{(ij)n} + [n_{(i} v_{j)}]^{TF}, \tag{52}$$

$$\mu e_{kij} = n_k \hat{K}_{ij} - \frac{3}{5}\gamma_{k(i}\hat{K}_{j)n} + \frac{1}{5}\gamma_{ij}\hat{K}_{kn}, \tag{53}$$

$$\mu v_i = \left(2m + \frac{9\zeta - 14}{10}\right)\hat{K}_{ni} + \left(2\sigma + \frac{1-4m}{3}\right)n_i K, \tag{54}$$

(and $\mu\phi = 0$, $\mu\tilde{\gamma}_{ij} = 0$, $\mu\alpha = 0$, $\mu w_i = 0$, $\mu z_i = 0$). From this representation of the principal part it is not difficult to see that the system is symmetric hyperbolic if and only if

$$4m = 6\sigma + 1, \qquad \sigma > 0, \tag{55}$$

and that in this case a symmetrizer $\mathbf{H} = \mathbf{H}(\gamma^{ij}, \sigma, m, \zeta)$ is given by

$$\begin{aligned}(u^{(1)})^T \mathbf{H} u^{(2)} = {}& \phi^{(1)}\phi^{(2)} + \gamma^{ik}\gamma^{jl}\tilde{\gamma}^{(1)}_{ij}\tilde{\gamma}^{(2)}_{kl} + \alpha^{(1)}\alpha^{(2)} \\ &+ \gamma^{ij} w^{(1)}_i w^{(2)}_j + \gamma^{ij} z^{(1)}_i z^{(2)}_j + 2\sigma K^{(1)} K^{(2)} \\ &+ \gamma^{ij} A^{(1)}_i A^{(2)}_j + \gamma^{ik}\gamma^{jl}\hat{K}^{(1)}_{ij}\hat{K}^{(2)}_{kl} \\ &+ \gamma^{kl}\gamma^{ir}\gamma^{js}(e^{(1)}_{kij} e^{(2)}_{lrs} - \zeta e^{(1)}_{kij} e^{(2)}_{rsl}) \\ &+ \left(2m + \frac{9\zeta - 14}{10}\right)^{-1}\gamma^{ij} v^{(1)}_i v^{(2)}_j.\end{aligned}$$

In order for $\mathbf{H}$ to be positive definite we need $-2 < \zeta < 1$. (This can be seen by using the orthogonal decomposition $e_{kij} = e^s_{kij} + e^a_{kij}$, where $e^s_{kij} = e_{(kij)}$ is totally symmetric, and by noticing that $e^a_{(ij)k} = -e^a_{kij}/2$.) Therefore, we have to choose

$$\max\left\{-2, 1 - \frac{10\sigma}{3}\right\} < \zeta < 1. \tag{56}$$

Since $\sigma > 0$ this choice is always possible. Summarizing, we have shown that our first order system is symmetric hyperbolic if $4m = 6\sigma + 1 > 1$, ζ satisfies the inequality (56), and if σ and ζ depend smoothly on u and the spacetime coordinates x^μ. This implies that in those cases the corresponding initial value problem is well posed. Since the additional constraints propagate, the same result holds for the BSSN system with the gauge conditions (a) when $4m = 6\sigma + 1 > 1$ and σ depends smoothly on u and x^μ. Since in this case the evolution system for the constraint variables can be reduced to a symmetric hyperbolic system, it follows that the constraints are satisfied if satisfied initially. Notice that if $0 < \sigma \leq 1/2$, there are no superluminal speeds. In the next section, we assume the presence of artificial boundaries and discuss boundary conditions.

V. BOUNDARY CONDITIONS

Consider the BSSN system (3)–(7) on a bounded domain $\Omega \subset \mathbb{R}^3$ with smooth boundary $\partial\Omega$. Consider the slicing condition (19) with $\sigma > 0$ and choose m such that $4m = 6\sigma + 1$. We also assume that the shift is *a priori* specified, and that at the boundary, the shift is tangential to $\partial\Omega$.

From the previous section we know that the BSSN system can be reduced to a first order symmetric hyperbolic system. For such a system, the specification of maximally dissipative [32] boundary conditions yields a well posed initial-boundary value formulation [13]. Maximally dissipative boundary conditions consist in a coupling of the ingoing to the outgoing characteristic fields with respect to the normal $\mathbf{n}$ to the boundary and some free boundary data. The characteristic fields with respect to the normal to the boundary are defined as the projections of u onto the corresponding eigenspaces of $\mathbf{A}(\mathbf{n})$.

In order to find the characteristic fields for our first order system, we define

$$\begin{aligned}E_{ij} &= e_{nij} - \zeta e_{(ij)n} + [n_{(i}v_{j)}]^{TF} \\ &= -\frac{1}{2}e^{4\phi}\tilde{d}_{nij} + \frac{\zeta}{2}e^{4\phi}[\tilde{d}_{(ij)n}]^{TF} + \Big[n_{(i}\tilde{\Gamma}_{j)} - \frac{1}{6}n_{(i}d_{j)} \\ &\quad - n_{(i}A_{j)} - \frac{\zeta}{2}n_{(i}\tilde{d}^k_{j)k}\Big]^{TF},\end{aligned}$$

which is trace-free. Eqs. (52)–(54) imply that

$$\mu\hat{K}_{ij} = E_{ij}, \tag{57}$$

$$\mu E_{ij} = \hat{K}_{ij} + 2(m-1)\left(n_{(i}\hat{K}_{j)n} - \frac{1}{3}\gamma_{ij}\hat{K}_{nn}\right). \tag{58}$$

In terms of a triad e_1, e_2, e_3 which is such that $e_1^i = n^i$, it follows from this and Eqs. (50) and (51) that the characteristic fields with respect to the normal n_i that have nonzero speeds are given by

$$V^{(\pm)} = K \mp \mu_1^{-1}A_n, \tag{59}$$

$$V^{(\pm)}_{nn} = \hat{K}_{nn} \pm \mu_2^{-1}E_{nn}, \tag{60}$$

$$V^{(\pm)}_{nA} = \hat{K}_{nA} \pm \mu_3^{-1}E_{nA}, \tag{61}$$

$$V^{(\pm)}_{AB} = [\hat{K}_{AB} \pm E_{AB}]^{tf}, \tag{62}$$

where A, B refer to the triad indices 2 and 3, where $[\ldots]^{tf}$ denotes the trace-free part with respect to the two dimensional metric δ_{AB}, and where μ_1, μ_2, μ_3 are given by the positive square roots of the expressions in (49). A short calculation shows that

$$u^T\mathbf{H}\mathbf{A}(\mathbf{n})u = \sqrt{2\sigma}\sigma[(V^{(+)})^2 - (V^{(-)})^2] + \frac{3\mu_2}{4}[(V_{nn}^{(+)})^2 - (V_{nn}^{(-)})^2] + \mu_3\delta^{AB}(V_{nA}^{(+)}V_{nB}^{(+)} - V_{nA}^{(-)}V_{nB}^{(-)}) + \frac{1}{2}\delta^{AC}\delta^{BD}(V_{AB}^{(+)}V_{CD}^{(+)} - V_{AB}^{(-)}V_{CD}^{(-)}). \quad (63)$$

The maximally dissipative boundary conditions are given as follows: Let $p \in \partial\Omega$, and let n_i be the unit outward normal to $\partial\Omega$. Then, the boundary conditions at p are

$$V^{(+)} = aV^{(-)} + G, \quad (64)$$

$$V_{nn}^{(+)} = bV_{nn}^{(-)} + G_{nn}, \quad (65)$$

$$V_{nA}^{(+)} = c_A^B V_{nB}^{(-)} + G_{nA}, \quad (66)$$

$$V_{AB}^{(+)} = d_{AB}^{CD} V_{CD}^{(-)} + G_{AB}, \quad (67)$$

where a, b are smaller than 1 in magnitude and the matrices c_A^B and d_{AB}^{CD} have norm smaller than 1, and where G, G_{nn}, G_{nA}, and G_{AB} are freely specified source functions (subject to the condition $\delta^{AB}G_{AB} = 0$). In order to illustrate why these boundary conditions lead to a well posed IBVP, let us linearize the equations around an arbitrary background. The resulting equations have the form

$$\hat{\partial}_0 v = \alpha\mathbf{A}^i\partial_i v + \mathbf{B}v,$$

where v denotes the perturbation. Defining the energy norm

$$\mathcal{E} = \int_\Omega v^T\mathbf{H}v d^3x,$$

taking a time derivative, using the symmetries of the matrices $\mathbf{H}$ and $\mathbf{H}\mathbf{A}^i$ and using Gauss's theorem, we find

$$\begin{aligned}\frac{d}{dt}\mathcal{E} &= 2\int_\Omega v^T\mathbf{H}[(\alpha\mathbf{A}^i + \beta^i)\partial_i v + \mathbf{B}v]d^3x \\ &= \int_\Omega\{\partial_i(v^T\alpha\mathbf{H}\mathbf{A}^i v + v^T\mathbf{H}\beta^i v) + v^T[\mathbf{H}\mathbf{B} + \mathbf{B}^T\mathbf{H} - \partial_i(\alpha\mathbf{H}\mathbf{A}^i + \mathbf{H}\beta^i)]v\}d^3x \\ &\le \int_{\partial\Omega}\alpha v^T\mathbf{H}\mathbf{A}(\mathbf{n})v d^2x + C\mathcal{E}, \end{aligned} \quad (68)$$

where we have used the fact that the shift is tangential to the boundary at the boundary and where C is a constant that only depends on bounds for $\mathbf{B}$ and $\mathbf{H}^{-1}\partial_i(\alpha\mathbf{H}\mathbf{A}^i + \mathbf{H}\beta^i)$. If the boundary conditions are homogeneous, i.e., if $G = 0$, $G_{nn} = 0$, $G_{nA} = 0$, $G_{AB} = 0$, Eqs. (63)–(67) immediately imply that the boundary integral is negative or zero, and we obtain the energy estimate $\mathcal{E}(t) \le \exp(Ct)\mathcal{E}(0)$. If the boundary conditions are inhomogeneous one can bound $\mathcal{E}(t)$ by $\mathcal{E}(0)$ and the L^2-norm of the boundary data [10,33]. These energy estimates play a key role in proofs for well-posedness. These proofs can be generalized to quasilinear symmetric hyperbolic systems, see for instance [13].

Therefore, the boundary conditions (64)–(67) lead to a well posed initial-boundary value formulation. Since the shift is tangential to the boundary at $\partial\Omega$, the additional constraints propagate as before, and thus the same boundary conditions applied to the Eqs. (3)–(7), where we perform the replacements (24), yields a well posed initial-boundary formulation for the BSSN system. In particular, choosing $a = b = 0$, $c_A^B = 0$, $d_{AB}^{CD} = 0$, and setting the source functions G, G_{nn}, G_{nA}, G_{AB} to zero, corresponds to Sommerfeld-type boundary conditions, in the sense that these conditions are algebraic conditions for the first order systems which are perfectly absorbing for plane waves of normal incidence to the boundary in the frozen coefficient approximation. Explicitly, we obtain the six boundary conditions

$$K - \frac{1}{\sqrt{2\sigma}\alpha}n^i\partial_i\alpha = 0, \quad (69)$$

$$n^i n^j\left[e^{4\phi}\tilde{A}_{ij} + \frac{\sqrt{3}}{\sqrt{4m-1}}E_{ij}\right] = 0, \quad (70)$$

$$n^i e_A^j\left[e^{4\phi}\tilde{A}_{ij} + \frac{1}{\sqrt{m}}E_{ij}\right] = 0, \qquad A = 2, 3, \quad (71)$$

$$\left[e_A^i e_B^j - \frac{1}{2}\delta_{AB}\delta^{CD}e_C^i e_D^j\right][e^{4\phi}\tilde{A}_{ij} + E_{ij}] = 0, \qquad A, B = 2, 3, \quad (72)$$

where

$$E_{ij} = -\frac{1}{2}e^{4\phi}n^k\partial_k\tilde{\gamma}_{ij} + \left[n_{(i}\tilde{\Gamma}_{j)} - 2n_{(i}\partial_{j)}\phi - \frac{1}{\alpha}n_{(i}\partial_{j)}\alpha + \frac{\zeta}{2}(e^{4\phi}n^k\partial_{(i}\tilde{\gamma}_{j)k} - n_{(i}\tilde{\gamma}^{rs}\partial_{|r|}\tilde{\gamma}_{j)s})\right]^{TF},$$

where n^i is the unit outward normal to the boundary and the vectors e_2 and e_3 must be chosen such that n^i, e_2^i, e_3^i form a triad with respect to the three metric γ_{ij}, and $n_i = \gamma_{ij}n^j$. The vectors e_2 and e_3 are unique up to a rotation; such a rotation does not alter the boundary conditions.

The parameter ζ has to be chosen such that the inequality (56) is satisfied. The boundary conditions (69)–(72) can be generalized to inhomogeneous conditions by replacing the zeros on their right-hand sides by freely specifiable source functions G, G_{nn}, G_{nA}, G_{AB}. If the solution is known in a neighborhood of the boundary, one can compute these source functions by evaluating the left-hand sides of Eqs. (69)–(72). Notice that the occurrence in the boundary conditions of the parameter ζ, which does not appear in the BSSN system, has its origin in the ζ-dependence of the unphysical energy $\mathcal{E}$ defined by the symmetrizer.

VI. STRONG HYPERBOLICITY WITH A DYNAMICAL SHIFT

Here we consider the BSSN Eqs. (3)–(7) with the live gauge conditions (b), see Sec. III. In this case one could proceed as in the frozen shift case and introduce the shift and its first derivatives (with respect to time and space) as extra variables. One obtains a first order system that is equivalent to the original system provided that the additional constraints are satisfied. Unfortunately, we did not succeed in finding a symmetrizer for the resulting first order system. Our goal in this section, therefore, is more modest: We show that the BSSN system with the live gauge conditions is strongly hyperbolic and so prove that the resulting Cauchy problem (in the absence of boundaries) is well posed. A related analysis for a different form of the system has been performed in [34], where a complete set of characteristic fields is given. However, to our knowledge, there are no results that show that this property alone implies the well-posedness of the Cauchy problem. The known results (see, for example, Ref. [19,35]) demand, in addition, smoothness of a generalized symmetrizer.

For differential equations that are not first order, a definition of strong hyperbolicity has recently be given in [19,22] that does not require the introduction of extra variables (nor extra constraints). It is based on pseudodifferential calculus. The intuitive idea behind this definition is to freeze the coefficients in the differential equations at some fixed point and to analyze the resulting linear constant coefficient problem by means of a Fourier transformation in space. In our case, the frozen coefficient problem is given by

$$\partial_0\hat{\phi} = -\frac{\alpha}{6}\hat{K} + \frac{i}{6}\omega_k\hat{\beta}^k, \qquad \partial_0\hat{\gamma}_{rs} = -2\alpha\hat{A}_{rs} + 2i\tilde{\gamma}_{k(r}\omega_{s)}\hat{\beta}^k - \frac{2i}{3}\tilde{\gamma}_{rs}\omega_k\hat{\beta}^k,$$

$$\partial_0\hat{K} = e^{-4\phi}\tilde{\gamma}^{kl}\omega_k\omega_l\hat{\alpha} + l.o., \qquad \partial_0\hat{A}_{rs} = \alpha e^{-4\phi}\left[\frac{1}{2}\tilde{\gamma}^{kl}\omega_k\omega_l\hat{\gamma}_{rs} + i\tilde{\gamma}_{k(r}\omega_{s)}\hat{\Gamma}^k + 2\omega_r\omega_s\hat{\phi} + \omega_r\omega_s\frac{\hat{\alpha}}{\alpha}\right]^{TF} + l.o.,$$

$$\partial_0\hat{\Gamma}^s = -\tilde{\gamma}^{kl}\omega_k\omega_l\hat{\beta}^s - \frac{1}{3}\tilde{\gamma}^{rs}\omega_r\omega_k\hat{\beta}^k + 2\alpha\left[i(m-1)\omega_k\hat{A}^{ks} - \frac{2im}{3}\tilde{\gamma}^{rs}\omega_r K\right] + l.o.,$$

$$\partial_0\hat{\alpha} = -\alpha^2 f(\alpha,\phi,x^\mu)\hat{K} + l.o., \qquad \partial_0\hat{\beta}^s = \alpha^2 G(\alpha,\phi,x^\mu)\hat{B}^s,$$

$$\partial_0\hat{B}^s = e^{-4\phi}H(\alpha,\phi,x^\mu)\left\{-\tilde{\gamma}^{kl}\omega_k\omega_l\hat{\beta}^s - \frac{1}{3}\tilde{\gamma}^{rs}\omega_r\omega_k\hat{\beta}^k + 2\alpha\left[i(m'-1)\omega_k\hat{A}^{ks} - \frac{2im'}{3}\tilde{\gamma}^{rs}\omega_r K\right]\right\} + l.o.,$$

where a hat denotes the Fourier transformation in space, $\hat{\phi}(\omega) = \int \phi(x)\exp(-i\omega\cdot x)d^3x$, and $l.o.$ denotes terms that depend on lower order spatial derivatives. Here, we have also allowed for a parameter m' that is different than m in the evolution equation for B^i. We can rewrite this as a first order system in t and ω_i by writing $\omega_i = |\omega|n_i$, $|\omega| = \sqrt{\gamma^{kl}\omega_k\omega_l}$, and introducing the variables

$$\hat{\varphi} = i|\omega|\hat{\phi}, \qquad \hat{h}_{rs} = \frac{i|\omega|}{2}e^{4\phi}\hat{\gamma}_{rs}, \qquad \hat{a} = i\alpha^{-1}|\omega|\hat{\alpha},$$

$$\hat{b}_s = i\alpha^{-1}|\omega|\gamma_{rs}\hat{\beta}^r, \qquad \hat{k}_{rs} = e^{4\phi}\hat{A}_{rs}, \qquad \hat{\Gamma}_s = \tilde{\gamma}_{rs}\hat{\Gamma}^r,$$

$$\hat{B}_s = \gamma_{rs}\hat{B}^r.$$

In terms of these variables we obtain a first order pseudo-differential system of the form

$$\partial_t\hat{u} = i|\omega|[\alpha\mathbf{P}(\mathbf{n}) + \beta^i n_i]\hat{u} + l.o.,$$

where $\hat{u} = (\hat{\varphi}, \hat{h}_{ij}, \hat{K}, \hat{k}_{ij}, \hat{a}, \hat{b}_i, \hat{\Gamma}_i, \hat{B}_i)^T$. The system is strongly hyperbolic if there exists a positive definite Hermitian matrix $\mathbf{H}(x^\mu, u, \mathbf{n})$ which is smooth in all its entries such that $\mathbf{HP}$ is symmetric. A necessary condition for this is that $\mathbf{P}$ is diagonalizable and has only real eigenvalues. Therefore, we first consider the eigenvalue problem $\mu\hat{u} = \mathbf{P}(\mathbf{n})\hat{u}$; explicitly

$$\mu\hat{\varphi} = -\frac{1}{6}\hat{K} + \frac{1}{6}\hat{b}_n, \qquad \mu\hat{h}_{rs} = -\hat{k}_{rs} + [n_{(r}\hat{b}_{s)}]^{TF},$$
$$\mu\hat{K} = -\hat{\alpha},$$
$$\mu\hat{k}_{rs} = -\hat{h}_{rs} + [n_{(r}\hat{\Gamma}_{s)} - 2n_r n_s\hat{\varphi} - n_r n_s\hat{a}]^{TF},$$
$$\mu\hat{a} = -f\hat{K}, \qquad \mu\hat{b}_s = G\hat{B}_s,$$
$$\mu\hat{\Gamma}_s = \hat{b}_s + \frac{1}{3}n_s\hat{b}_n + 2(m-1)\hat{k}_{ns} - \frac{4m}{3}n_s\hat{K},$$
$$\mu\hat{B}_s = H\left[\hat{b}_s + \frac{1}{3}n_s\hat{b}_n + 2(m'-1)\hat{k}_{ns} - \frac{4m'}{3}n_s\hat{K}\right],$$

where $\hat{b}_n = \gamma^{rs}n_r\hat{b}_s$ and $\hat{k}_{nj} = \gamma^{rs}n_r\hat{k}_{sj}$. A careful analysis reveals that the matrix on the RHS has the eigenvalues $0, \pm\mu_1, \pm\mu_2, \pm\mu_3, \pm\mu_4, \pm\mu_5$ where

$$\mu_1 = \sqrt{f}, \qquad \mu_2 = \sqrt{\frac{4m-1}{3}}, \qquad \mu_3 = \sqrt{m},$$
$$\mu_4 = 1, \qquad \mu_5 = \sqrt{GH}, \qquad \mu_6 = \sqrt{\frac{4GH}{3}}.$$

Therefore, we need $m > 1/4$, $f > 0$, and $GH > 0$. (If $G = H = 0$ the equation for the shift decouples, and we are back in the case considered in the previous section.) Furthermore, it turns out that the matrix is diagonalizable only if $4GH \neq 3f$ and provided that $m' = 1$ if $m = GH$ or $4GH = 4m - 1$. In the remaining cases the system is only weakly hyperbolic which, in the nonlinear case, can lead to exponential growth with arbitrarily small growth time. Introducing the functions

$$\Omega_1 = \frac{4GH}{3f - 4GH},$$
$$\Omega_2 = \frac{6(m'-1)}{4m - 1 - 4GH},$$
if $4m - 1 \neq 4GH$ and Ω_2 arbitrary otherwise,
$$\Omega_3 = \frac{2(m'-1)GH}{m - GH},$$
if $m \neq GH$ and Ω_3 arbitrary otherwise,

the eigenfields can be expressed as

$$Z_0 = 8m\hat{\varphi} - 2(m-1)\hat{h}_{nn} - \hat{\Gamma}_n, \qquad Z_i = H[2(m-m')\hat{h}_{ni} + m'\hat{\Gamma}_i] - m\hat{B}_i, \qquad V^{(\pm)} = \hat{K} \mp \mu_1^{-1}\hat{a},$$
$$V_{nn}^{(\pm)} = \hat{k}_{nn} - \frac{2\hat{K}}{3} \mp \mu_2^{-1}\left(\hat{h}_{nn} - \frac{2}{3}\hat{\Gamma}_n + \frac{4}{3}\hat{\varphi}\right), \qquad V_{nA}^{(\pm)} = \hat{k}_{nA} \mp \mu_3^{-1}\left(\hat{h}_{nA} - \frac{1}{2}\hat{\Gamma}_A\right), \qquad V_{AB}^{(\pm)} = [\hat{k}_{AB} \mp \mu_4^{-1}\hat{h}_{AB}]^{tf},$$
$$V_A^{(\pm)} = \hat{b}_A - \Omega_3\hat{k}_{nA} \pm \mu_5^{-1}\left[G\hat{B}_A + \Omega_3\left(\hat{h}_{nA} - \frac{1}{2}\hat{\Gamma}_A\right)\right],$$
$$V_n^{(\pm)} = \hat{b}_n + \Omega_1\hat{K} - \Omega_2\left(\hat{k}_{nn} - \frac{2\hat{K}}{3}\right) \pm \mu_6^{-1}\left[G\hat{B}_n - \Omega_1\hat{a} + \Omega_2\left(\hat{h}_{nn} - \frac{2}{3}\hat{\Gamma}_n + \frac{4}{3}\hat{\varphi}\right)\right],$$

where the components $n, A = 2, 3$, refer to triad indices as described in the previous section. The matrix $\mathbf{H}$ which symmetrizes $\mathbf{P}$ can be obtained from the quadratic form which is built by summing over the square of the eigenfields:

$$u^T\mathbf{H}u = Z_0^2 + \gamma^{ij}Z_iZ_j + \sum_{\pm}\{(V^{(\pm)})^2 + (V_{nn}^{(\pm)})^2 + \delta^{AB}V_{nA}^{(\pm)}V_{nB}^{(\pm)} + \delta^{AB}\delta^{CD}V_{AC}^{(\pm)}V_{BD}^{(\pm)} + \delta^{AB}V_A^{(\pm)}V_B^{(\pm)} + (V_n^{(\pm)})^2\}.$$

This quadratic form depends smoothly on n_i. For example, the term $\delta^{AB}\hat{h}_{nA}\hat{h}_{nB}$ appearing in the expression $\delta^{AB}V_{nA}^{(\pm)}V_{nB}^{(\pm)}$ can be rewritten as $\delta^{AB}\hat{h}_{nA}\hat{h}_{nB} = n^i n^k(\gamma^{jl} - n^j n^l)\hat{h}_{ij}\hat{h}_{kl}$ which is smooth in n_i and the coefficients of the three metric γ_{ij}. $\mathbf{H}$ is also smooth in the other variables provided that m, m', f, G, and H are such that the functions Ω_1, Ω_2, and Ω_3 stay bounded and can be chosen to be smooth. A simple possibility of achieving this is by choosing $m = m' = 1$ and $f = \kappa GH$ with κ a constant that is unequal $4/3$. The pseudodifferential calculus shows that in these cases the full nonlinear Cauchy problem is well posed. Since the evolution system for the constraint variables can be reduced to a symmetric hyperbolic system if $m > 1/4$ it follows that the constraints are satisfied if satisfied initially.

VII. CONCLUSION

We discussed some mathematical aspects of the BSSN system which is currently used by several groups in numerical relativity. In particular, we derived a well posed initial-boundary value formulation of the BSSN system with a Bona-Massó–like slicing condition for the lapse and a frozen shift. This is achieved by introducing extra variables and recasting the evolution equations into a first order symmetric hyperbolic system, for which maximally dissipative boundary conditions are specified. The introduction of extra fields brings additional constraints, and the original BSSN system and the first order symmetric hyperbolic system derived in this article are only equivalent if these constraints are satisfied. However, we showed that the associated constraint variables obey a closed evolution system that is *independent* of the other constraints. Moreover, by choosing the shift to be tangential to the boundary, these additional constraints propagate tangentially to the boundary. This implies that they are satisfied everywhere at later times if satisfied initially, even if the other constraints are violated. This allows us to return to the second order system and to conclude that the BSSN system with the specified boundary conditions is well posed; in particular, unique solutions local in time exist, and depend continuously on the initial and boundary data. To our knowledge, the specified (six) boundary conditions (69)–(72) have not yet appeared in the literature. For nonsmooth boundaries the assumption of a tangential shift implies a vanishing shift at corners. For smooth boundaries on the other hand this assumption should not be too restrictive. For example, it would still allow for the use of corotating shift conditions. Nontangential shifts could also be considered, but in this case, additional boundary conditions have to be specified if the normal component of the shift is positive.

In general, the boundary conditions derived in this article are not compatible with the constraints of the BSSN system. They can feed in some constraint violating modes. Nevertheless, they are consistent with the evolution equations and constitute a first step towards improving numerical evolutions of the BSSN system. In particular, the present analysis offers the possibility to construct constraint-preserving boundary conditions [14] in the linearized case, following the lines of [15–17]. Furthermore, the derivation of the symmetrizer and the energy estimate presented in Sec. V should be useful as a guidance principle to construct discretizations schemes that guarantee numerical stability at least at the linearized level [10,36–38].

We have also considered dynamical gauge conditions for lapse and shift and obtained a class of second order evolution equations which can be shown to be strongly hyperbolic using pseudodifferential calculus. For these systems, one can show well-posedness of the initial value problem. Here, the presence of boundaries has not been considered. To derive boundary conditions in the case of a finite domain one could proceed as follows: First, derive a first order system by introducing extra variables as described at the beginning of Sec. VI. Next, consider the matrix $\mathbf{A}(\mathbf{n})$ multiplying the derivatives normal to the boundaries. In case this matrix is diagonalizable, to every strictly positive eigenvalue of $\mathbf{A}(\mathbf{n})$ there corresponds a Sommerfeld-type outgoing boundary condition given by the condition of a vanishing projection of the field vector onto the corresponding eigenspace. This corresponds to setting to zero the incoming characteristic fields with respect to the direction which is normal to the boundary. Finally, well-posedness of the initial-boundary value problem in a suitable Hilbert space has to be proved [39]. Necessary conditions for well-posedness can be obtained by using the method of Laplace transformation, see, for example, [10,33]. The derivation of boundary conditions in the dynamical shift case is beyond the scope of the present work.

The gauge conditions considered here differ from the ones used in [21] for numerical simulations only by the replacement $\partial_t \mapsto \partial_t - \beta^j \partial_j$, which leads to a simpler principal part and makes it more amendable to analyze the algebraic conditions that guarantee symmetric or strong hyperbolicity. Preliminary investigations of the problem without this replacement have been done in [40] where one of the Sommerfeld-type conditions has already been computed. The structure of this condition is more complicated than the conditions derived in this article.

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[32] Note that this terminology has its origin in semigroup theory. Maximality refers to the fact that generators of semigroups do not have proper extensions to operators that generate semigroups. Dissipativity of the operator implies that the spectrum of the generator is contained in the closed left half-plane. Using analogy from quantum theory, dissipativity is the condition that all "expectation values" of the symmetric part of the operator are negative (≤ 0). In special applications those expectation values are called "energies" although in most cases, in particular, in General Relativity, they are not energies in a physical sense, since they are coordinate, i.e., gauge dependent. Therefore, calling boundary conditions maximally dissipative has nothing to do with the boundary conditions dissipating as much energy to the outside of the computational domain as possible or the like. Even boundary conditions conserving energies are maximally dissipative, but, of course, lead to the worst reflections at the boundaries. The quality of the boundary condition concerning reflections at the boundary has to be decided by different means, for instance reflection coefficients for modes incident at the boundary. Often instead of saying that an operator is dissipative its negative is referred to as being "accretive."
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