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Final state and thermodynamics of a dark energy universe

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As it follows from the classical analysis, the typical final state of a dark energy universe where a dominant energy condition is violated is a finite-time, sudden future singularity (a big rip). For a number of dark energy universes (including scalar phantom and effective phantom theories as well as specific quintessence models) we demonstrate that quantum effects play the dominant role near a big rip, driving the universe out of a future singularity (or, at least, moderating it). As a consequence, the entropy bounds with quantum corrections become well defined near a big rip. Similarly, black hole mass loss due to phantom accretion is not so dramatic as was expected: masses do not vanish to zero due to the transient character of the phantom evolution stage. Some examples of cosmological evolution for a negative, time-dependent equation of state are also considered with the same conclusions. The application of negative entropy (or negative temperature) occurrence in the phantom thermodynamics is briefly discussed.

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I. INTRODUCTION

Recent astrophysical data, ranging from Wilkinson Microwave Anisotropy Probe observations to high red-shift surveys of supernovae, indicate that about 70% of the total energy of the universe is to be attributed to a strange cosmic fluid with negative pressure, dark energy. From another side, the universe is accelerating currently. It is also observed that the equation of state parameter w is close to -1, most probably being below -1. (The possibility of time-dependent negative w is not excluded too.)[1]

The case with w less than -1 is often dubbed as phantom dark energy. At the moment, there is no satisfactory theoretical description of phantom dark energy (for a number of attempts in this direction, see [1,2] and for a recent review of dark energy, see [3]). The easiest current model of a phantom is motivated by quintessence [4]; it is just a scalar field with a wrong sign for the kinetic energy term. Definitely, such theory being unstable shows some weird properties caused mainly by the violation of a dominant energy condition. Indeed, the energy density grows with time in the phantom universe so that in a finite time such a universe ends up in the singularity dubbed as a big rip [5] (see also the earlier discussion of a finite-time singularity in [6]). The related phenomenon is that all black holes lose their masses to vanish exactly in a big rip [7]. Phantom thermodynamics looks also strange leading to a negative entropy of the universe [8] (and divergent entropies near big rip) or to the appearance of negative temperatures [9]. If our universe is indeed a phantom one, this all may call to revision of basic physical principles governing our reality.

In the present article aiming to discuss the final state of a (phantom) dark energy universe we show that the situation is much less dramatic than it looks from the very beginning (even in the absence of consistent phantom theory). Indeed, with the growth of phantom energy density the typical energies and curvature invariants grow as well. As a result, much before a big rip the quantum effects start to play the dominant role. In a sense, a second quantum gravity era begins. The simple account of the quantum effects, in the same way as was proposed in Refs. [10,11], demonstrates that the big rip singularity is moderated or even does not occur at all. As a result, the entropy bounds remain to be meaningful and black hole masses do not vanish to zero. This observation indicates also that the phantom stage (if it is realistic) is just a transient period in the universe evolution.

The paper is organized as follows. In the next section we start from the finite-time, sudden singularity model proposed by Barrow [12] and consider its generalization and give its Lagrangian description in terms of scalar-tensor theory. In fact, in terms of scalar-tensor theory the sudden singularity is different from the model of [12]. It is interesting that such a model where a dominant energy condition is also violated is not necessarily a phantom with a wrong sign for the kinetic term. Then we show that the account of quantum effects (using quantum energy density and pressure obtained by integration of the conformal anomaly) moderates the finite-time singularity or even prevents it. The universe presumably ends up in a

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[1]It is quite possible that what looks like dark energy is the manifestation of some unknown feature of the gravitational theory which apparently should be modified.

de Sitter phase (future inflationary era). A similar analysis has been done in [10] but with coefficient of one of the terms in conformal anomaly ($\Box R$) being equal to zero. In the present paper, due to the importance of the corresponding term at high energies (as being advocated by Hawking) the account of such an arbitrary coefficient is made. Moreover, the backreaction from the matter is included. Section III is devoted to the study of the final stage for the scalar phantom universe and effective phantom universe [13] produced by a higher derivative coupling of the scalar kinetic energy with curvature. Again, the quantum escape of a big rip occurs or, at least, the singularity is moderating permitting the evolution after big rip time. In Sec. IV the entropy bounds near a big rip are studied. Basically, the typical entropies (including the one for a negative time-dependent equation of state universe) diverge at singularity. The account of quantum effects makes the entropies finite and the entropy bounds well behaved. Section V is devoted to the study of a more general, time-dependent equation of state which may be also effectively phantom. The examples where the scale factor is accelerating are presented and the occurrence of a big rip is mentioned again. Finally, a summary and outlook are given in the Discussion. In Appendix A the entropy is written for a specific model of phantom thermodynamics. It becomes negative for positive temperatures, and positive if temperatures are negative. In Appendix B the evolution of black hole mass in the phantom universe is discussed, also in the case when the equation of state is time dependent. The same quantum effects which drive the final state out of a big rip significantly improve the evolution of black holes mass. It may increase or decrease by phantom energy accretion but eventually does not vanish.

II. LAGRANGIAN DESCRIPTION OF CLASSICAL SUDDEN FUTURE SINGULARITY AND QUANTUM EFFECTS ACCOUNT

In this section, we construct the scalar-tensor theory with a specific potential which describes a classical sudden future singularity. The account of quantum effects near to a sudden singularity (where the future quantum gravity era starts) shows that a sudden singularity most probably never occurs.

In [12], it has been shown that even if the strong energy condition

$$\rho > 0, \qquad \rho + 3p > 0 \tag{1}$$

for some kind of (exotic) matter is satisfied, the future singularity (big rip) can occur. Note that a dominant energy condition is violated [14] for such a scenario. Here ρ and p are the energy density and the pressure of the matter, respectively. We now consider the spatially flat FRW metric

$$ds^2 = -dt^2 + a(t)^2 \sum_{i=1,2,3} (dx^i)^2. \tag{2}$$

Following [12], the scale factor $a(t)$ is chosen as

$$a(t) = A + Bt^q + C(t_s - t)^n. \tag{3}$$

Here $A > 0$, $B > 0$, $q > 0$, and $t_s > 0$ are constants and $C = -At_s^{-n}$. It is assumed that $t < t_s$ and $2 > n > 1$. There is a singularity at $t \to t_s$, where $(1/a)(d^2a/dt^2) \to +\infty$. Classical FRW equations,

$$\frac{6}{\kappa^2} H^2 = \rho, \qquad \frac{2}{\kappa^2 a} \frac{d^2 a}{dt^2} = -\frac{\rho + 3p}{6}, \tag{4}$$

show that

$$\begin{aligned} \rho &\sim \frac{6q^2 B^2 t_s^{2q-2}}{\kappa^2 (A + Bt_s^q)} > 0, \\ p &\sim -\frac{Cn(n-1)(t_s - t)^{n-2}}{A + Bt_s^q} > 0, \end{aligned} \tag{5}$$

near the singularity $t \sim t_s$. In (4), $H \equiv (1/a)(da/dt)$. Thus, the energy density ρ is finite but the pressure p diverges. Nevertheless, the strong energy condition (1) is satisfied since ρ and p are positive.

Let us present the generalization of Barrow's model [12] as it was presented in Ref. [10]. In this model, the matter has been given implicitly via the FRW equations:

$$\rho = \frac{6}{\kappa^2} H^2, \qquad p = -\frac{2}{\kappa^2}\left(2\frac{dH}{dt} + 3H^2\right). \tag{6}$$

One may assume H has the following form:

$$H(t) = \tilde{H}(t) + A'|t_s - t|^\alpha. \tag{7}$$

Here $\tilde{H}$ is a smooth, differentiable (infinite number of times, in principle) function and A' and t_s are constants. Another assumption is that a constant α is not a positive integer. Then $H(t)$ has a singularity at $t = t_s$. In case α is a negative integer, the singularity is a pole. Even if α is positive, in case α is not an integer, there appears a singularity, that is, if we analytically continue t to the region $t > t_s$ from the region $t < t_s$, H might become complex and double valued due to the cut which appears when we analytically continue t to be a complex number. It is important for us that a singularity is present. If we consider the region $t > t_s$, there is no finite-time *future* singularity.

When $\alpha > 1$, one gets

$$\rho \sim -p \sim \frac{6}{\kappa^2} \tilde{H}(t_s)^2. \tag{8}$$

Hence $w = \frac{p}{\rho} = -1$, which may correspond to the positive cosmological constant.

The case $0 < \alpha < 1$ corresponds to Barrow's model and when $t \sim t_s$, we find

$$\rho \sim \frac{6}{\kappa^2}\tilde{H}(t_s)^2, \qquad p \sim \pm\frac{4A'\alpha}{\kappa^2}|t_s - t|^{\alpha-1}. \tag{9}$$

Here the plus sign in $\pm$ corresponds to the $t < t_s$ case and the -1 to $t > t_s$. In the following, the upper (lower) sign always corresponds to $t < t_s$ ($t > t_s$). The parameter of equation of state w is given by

$$w = \pm\frac{2}{3}\frac{A'\alpha|t_s - t|^{\alpha-1}}{\tilde{H}(t_s)^2}. \tag{10}$$

Hence, w is positive in two cases: one is $A' > 0$ and $t < t_s$, which directly corresponds to Barrow's model, and another is $A' < 0$ $t > t_s$. In other cases, w is negative.

When $-1 < \alpha < 0$, the energy density ρ and the pressure is given by

$$\rho = \frac{6A'^2}{\kappa^2}|t_s - t|^{2\alpha}, \qquad p \sim \pm\frac{4A'\alpha}{\kappa^2}|t_s - t|^{\alpha-1}. \tag{11}$$

The parameter of the equation of state is

$$w = \pm\frac{2\alpha}{3A'}|t_s - t|^{-\alpha-1}, \tag{12}$$

which diverges at $t = t_s$. Here w is positive when $A' > 0$ and $t < t_s$ or $A' < 0$ and $t > t_s$. The former case corresponds to a sudden future singularity even if w is positive. The singularity can be regarded as a big rip. (For the recent comparison of a phantom big rip with the above type of it, see [15].) The account of quantum effects leads to the escape from the future singularity as shown in Refs. [10,11].

The case $\alpha = -1$ gives

$$\begin{aligned}\rho &= \frac{6A'^2}{\kappa^2}|t_s - t|^{-2},\\ p &\sim -\frac{2}{\kappa^2}(\pm 2A' + 3A'^2)|t_s - t|^{-2},\end{aligned} \tag{13}$$

which may correspond to the scalar field with an exponential potential. The parameter w is given by

$$w = -1 \mp \frac{2}{A'}. \tag{14}$$

Near $t = t_s$, the universe is expanding if $A' > 0$ and $t < t_s$ or $A' < 0$ and $t > t_s$. The former case corresponds to the phantom with $w < -1$. In the latter case, if $2 > A' > 0$, the equation of state describes the usual matter with positive w and if $A' > 2$, the matter may be the quintessence with $0 > w > -1$.

If $\alpha < -1$, one obtains

$$\rho = -p = \frac{6A'^2}{\kappa^2}|t_s - t|^{2\alpha}, \tag{15}$$

which gives $w = -1$ as for the cosmological constant case. In this case, however, there is a sharp singularity at $t = t_s$ since both ρ and p diverge at $t = t_s$. This is contrary to the case $\alpha > -1$ as they tend to infinity in the limit $t \to t_s$ more rapidly than in the case $\alpha \geq -1$.

Let us construct the Lagrangian (scalar-tensor) model which contains a sudden future singularity. We start from the rather general action of the scalar field ϕ coupled with gravity:

$$S = \frac{1}{\kappa^2}\int d^4x\sqrt{-g}\left[R - \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - V(\phi)\right]. \tag{16}$$

Then the energy density ρ and the pressure p are given by

$$\rho = \frac{1}{2}\left(\frac{d\phi}{dt}\right)^2 + V(\phi), \qquad p = \frac{1}{2}\left(\frac{d\phi}{dt}\right)^2 - V(\phi). \tag{17}$$

The scalar equation of motion is

$$0 = \frac{d^2\phi}{dt^2} + 3H\frac{d\phi}{dt} + V'(\phi). \tag{18}$$

We are searching for the potential $V(\phi)$, which gives a solution

$$H = h_0 + h_1(t_s - t)^\alpha, \qquad \phi = \phi_0(t_s - t)^\beta \tag{19}$$

with constants h_0, h_1, α, ϕ_0, and β. Since

$$\begin{aligned}\frac{d\phi}{dt} &= -\beta\phi_0(t_s - t)^{\beta-1} = -\beta\phi_0^{1/\beta}\phi^{1-1/\beta},\\ \frac{d^2\phi}{dt^2} &= \beta(\beta-1)\phi_0(t_s - t)^{\beta-2}\\ &= \beta(\beta-1)\phi_0^{2/\beta}\phi^{1-2/\beta},\\ H &= h_0 + h_1\phi_0^{-\alpha/\beta}\phi^{\alpha/\beta}.\end{aligned} \tag{20}$$

From the FRW equation

$$\frac{6}{\kappa^2}H^2 = \frac{1}{2}\left(\frac{d\phi}{dt}\right)^2 + V(\phi), \tag{21}$$

it follows

$$\begin{aligned}V(\phi) &= \frac{6}{\kappa^2}H^2 - \frac{1}{2}\left(\frac{d\phi}{dt}\right)^2\\ &= \frac{6}{\kappa^2}(h_0^2 + 2h_0h_1\phi_0^{-\alpha/\beta}\phi^{\alpha/\beta} + h_1^2\phi_0^{-2\alpha/\beta}\phi^{2\alpha/\beta})\\ &\quad - \frac{\beta^2}{2}\phi_0^{2/\beta}\phi^{2-2/\beta},\end{aligned} \tag{22}$$

which gives

$$\begin{aligned}V'(\phi) &= \frac{12\alpha}{\kappa^2\beta}(h_0h_1\phi_0^{-\alpha/\beta}\phi^{\alpha/\beta-1} + h_1^2\phi_0^{-2\alpha/\beta}\phi^{2\alpha/\beta-1})\\ &\quad - (\beta-1)\beta\phi_0^{2/\beta}\phi^{1-2/\beta}.\end{aligned} \tag{23}$$

On the other hand, from the scalar equation of motion:

$$0 = \frac{d^2\phi}{dt^2} + 3H\frac{d\phi}{dt} + V'(\phi), \tag{24}$$

one obtains

$$V'(\phi) = -\frac{d^2\phi}{dt^2} - 3H\frac{d\phi}{dt} = -(\beta-1)\beta\phi_0^{2/\beta}\phi^{1-2/\beta} + 3h_0\beta\phi_0^{1/\beta}\phi^{1-1/\beta} + 3h_1\beta\phi_0^{-(\alpha/\beta)+(1/\beta)}\phi^{1+(\alpha/\beta)-(1/\beta)}. \tag{25}$$

Comparing (23) and (25), we get

$$\beta = \frac{\alpha+1}{2}, \qquad h_1 = \frac{\kappa^2(\alpha+1)^2\phi_0^2}{16\alpha}, \tag{26}$$

and h_0 can be arbitrary. Then by substituting (26) into (22), we obtain

$$V(\phi) = \frac{6h_0^2}{\kappa^2} + \frac{3h_0(\alpha+1)^2}{4\alpha}\phi_0^{2/(\alpha+1)}\phi^{2\alpha/(\alpha+1)} + \frac{3\kappa^2(\alpha+1)^4}{128\alpha^2}\phi_0^{4/(\alpha+1)}\phi^{4\alpha/(\alpha+1)} - \frac{(\alpha+1)^2}{8}\phi_0^{4/(\alpha+1)}\phi^{2[(\alpha-1)/(\alpha+1)]}. \tag{27}$$

It is interesting to investigate the (in)stability of the scalar theory (16) with potential (27). The perturbations from the solution (19) are

$$H = h_0 + h_1(t_s - t)^\alpha + \delta h, \quad \phi = \phi_0(t_s-t)^{(\alpha+1)/2} + \delta\phi. \tag{28}$$

Then from the FRW equation (21) and ϕ equation (24), it follows

$$0 = -\frac{12}{\kappa^2}H_0\delta h + \frac{d\Phi_0}{dt}\frac{d\delta\phi}{dt} + V'(\Phi_0)\delta\phi, \tag{29}$$

$$0 = \frac{d^2\delta\phi}{dt^2} + 3H_0\frac{d\delta\phi}{dt} + 3\delta h\frac{d\Phi_0}{dt} + V''(\Phi_0)\delta\phi. \tag{30}$$

Here

$$H_0 \equiv h_0 + h_1(t_s-t)^\alpha, \qquad \Phi_0 = \phi_0(t_s-t)^{(\alpha+1)/2}. \tag{31}$$

Since

$$V'(\Phi_0) = \frac{3(\alpha+1)h_0\phi_0}{2}(t_s-t)^{(\alpha-1)/2} + \frac{3\kappa^2(\alpha+1)^3\phi_0^3}{32\alpha} \times (t_s-t)^{(3\alpha-1)/2} - \frac{(\alpha^2-1)\phi_0}{4}(t_s-t)^{(\alpha-3)/2},$$
$$V''(\Phi_0) = \frac{3(\alpha-1)h_0}{2}(t_s-t)^{-1} + \frac{3\kappa^2(\alpha+1)^2(3\alpha-1)\phi_0^2}{32\alpha}(t_s-t)^{\alpha-1} - \frac{(\alpha-1)(\alpha-3)}{4}(t_s-t)^{-2}. \tag{32}$$

Hence, when $t \to t_s$, if $\alpha > -1$, the third terms of $V'(\Phi_0)$ and $V''(\Phi_0)$ dominate and if $\alpha < -1$, the second terms dominate. For the case of the Barrow model, $0 < \alpha < 1$. When $t \sim t_s$, Eqs. (29) and (30) are

$$0 \sim -\frac{12}{\kappa^2}h_0\delta h - \frac{(\alpha+1)\phi_0}{2}(t_s-t)^{(\alpha-1)/2}\frac{d\delta\phi}{dt} - \frac{(\alpha^2-1)\phi_0}{4}(t_s-t)^{(\alpha-3)/2}\delta\phi, \tag{33}$$

$$0 \sim \frac{d^2\delta\phi}{dt^2} + 3h_0\frac{d\delta\phi}{dt} - \frac{3(\alpha+1)\phi_0}{2}(t_s-t)^{(\alpha-1)/2}\delta h - \frac{(\alpha-1)(\alpha-3)}{4}(t_s-t)^{-2}\delta\phi. \tag{34}$$

By deleting δh from (33) and (34), we obtain

$$0 \sim \frac{d^2\delta\phi}{dt^2} - \frac{(\alpha-1)(\alpha-3)}{4}(t_s-t)^{-2}\delta\phi. \tag{35}$$

Its solution is given by

$$\delta\phi = \phi_1(t_s-t)^{(\alpha-1)/2} + \phi_2(t_s-t)^{(3-\alpha)/2}. \tag{36}$$

Here ϕ_1 and ϕ_2 are constants. Hence,

$$\delta h = -\frac{6(\alpha+1)(\alpha-2)}{\kappa^2 h_0}\phi_2. \tag{37}$$

In this order of the perturbation, ϕ_1 does not appear in δh. Since, of course, $(\alpha-1)/2 < (\alpha+1)/2$, when $t \to t_s$, the first term in $\delta\phi$ (36) becomes large more rapidly than the unperturbative part $\Phi_0 = \phi_0(t_s-t)^{(\alpha+1)/2}$ (30), which tells that the solution (19) which describes the sudden singularity model in the scalar-tensor theory is not stable. Already on the classical level, such instability may stop the appearance of future singularity. However, a more secure mechanism which acts against the singularity occurrence is the quantum effects account.

Near the singularity at $t = t_s$, the curvature becomes large in general. As the quantum corrections usually contain the powers of the curvature (higher derivative terms), the correction becomes important near the singularity. One may include the quantum effects by taking into account the conformal anomaly contribution as back-reaction near the singularity. The conformal anomaly T_A has the following form:

$$T_A = b(F + \tfrac{2}{3}\Box R) + b'G + b''\Box R, \tag{38}$$

where F is the square of the 4D Weyl tensor, and G is the Gauss-Bonnet invariant. In general, with N scalar, $N_{1/2}$ spinor, N_1 vector fields, N_2 ($= 0$ or 1) gravitons and $N_{\rm HD}$ higher derivative conformal scalars, b, b', and b'' are given by

$$b = \frac{N + 6N_{1/2} + 12N_1 + 611N_2 - 8N_{\rm HD}}{120(4\pi)^2}, \quad b' = -\frac{N + 11N_{1/2} + 62N_1 + 1411N_2 - 28N_{\rm HD}}{360(4\pi)^2}. \tag{39}$$

As is seen $b > 0$ and $b' < 0$ for the usual matter except the higher derivative conformal scalars. Notice that b'' can be shifted by the finite renormalization of the local counterterm R^2, so b'' can be arbitrary (in Ref. [10] it was chosen to be zero, for simplicity). In terms of the corresponding energy density ρ_A and pressure p_A, T_A is given by $T_A = -\rho_A + 3p_A$. Then by using the energy conservation law in the FRW universe

$$0 = \frac{d\rho_A}{dt} + 3H(\rho_A + p_A), \tag{40}$$

we may delete p_A as

$$T_A = -4\rho_A - \frac{1}{H}\frac{d\rho_A}{dt}, \tag{41}$$

which gives the following expression for ρ_A:

$$\begin{aligned}\rho_A &= -\frac{1}{a^4}\int dt a^4 H T_A \\ &= -\frac{1}{a^4}\int dt a^4 H\Big[-12b\Big(\frac{dH}{dt}\Big)^2 + 24b'\Big\{-\Big(\frac{dH}{dt}\Big)^2 \\ &\quad + H^2\frac{dH}{dt} + H^4\Big\} - 6\Big(\frac{2}{3}b + b''\Big)\Big\{\frac{d^3H}{dt^3} + 7H\frac{d^2H}{dt^2} \\ &\quad + 4\Big(\frac{dH}{dt}\Big)^2 + 12H^2\frac{dH}{dt}\Big\}\Big].\end{aligned} \tag{42}$$

Moreover,

$$\begin{aligned}p_A &= -\rho_A - \frac{1}{3H}\frac{d\rho_A}{dt} \\ &= \frac{T_A}{3} - \frac{1}{a^4}\int dt a^4 H T_A \\ &= \frac{1}{3}\Big[-12b\Big(\frac{dH}{dt}\Big)^2 + 24b'\Big\{-\Big(\frac{dH}{dt}\Big)^2 + H^2\frac{dH}{dt} + H^4\Big\} - 6\Big(\frac{2}{3}b + b''\Big)\Big\{\frac{d^3H}{dt^3} + 7H\frac{d^2H}{dt^2} + 4\Big(\frac{dH}{dt}\Big)^2 + 12H^2\frac{dH}{dt}\Big\}\Big] \\ &\quad - \frac{1}{a^4}\int dt a^4 H\Big[-12b\Big(\frac{dH}{dt}\Big)^2 + 24b'\Big\{-\Big(\frac{dH}{dt}\Big)^2 + H^2\frac{dH}{dt} + H^4\Big\} - 6\Big(\frac{2}{3}b + b''\Big)\Big\{\frac{d^3H}{dt^3} + 7H\frac{d^2H}{dt^2} + 4\Big(\frac{dH}{dt}\Big)^2 \\ &\quad + 12H^2\frac{dH}{dt}\Big\}\Big].\end{aligned} \tag{43}$$

As in (19), one assumes

$$H \sim h'_0 + h'_1(t_s - t)^{\alpha'}, \quad \text{or} \quad a = a_0 e^{h'_0 t - [h'_1/(\alpha'+1)](t_s - t)^{\alpha'-3}}. \tag{44}$$

We also consider the case $t \sim t_s$ and keep only the first and the last terms in $V(\phi)$ (27)

$$V(\phi) \sim \frac{6h_0^2}{\kappa^2} - \frac{(\alpha+1)^2}{8}\phi_0^{4/(\alpha+1)}\phi^{2[(\alpha-1)/(\alpha+1)]}. \tag{45}$$

These terms are dominant for the Barrow model $0 < \alpha < 1$. Then the consistent solution is given when $\alpha' > 2$. From (18), it is seen the behavior of ϕ is not so changed from the classical solution (19)

$$\phi \sim \phi_0(t_s - t)^\beta. \tag{46}$$

Now the quantum corrected FRW equations are

$$0 = -\frac{6}{\kappa^2}H^2 + \frac{1}{2}\Big(\frac{d\phi}{dt}\Big)^2 + V(\phi) + \rho_A, \tag{47}$$

$$0 = \frac{2}{\kappa^2}\Big(2\frac{dH}{dt} + 3H^2\Big) + \frac{1}{2}\Big(\frac{d\phi}{dt}\Big)^2 - V(\phi) + p_A. \tag{48}$$

Substituting (44) into (42), we obtain

$$\rho_A \sim -24b'h_0'^5 e^{-4h'_0 t}\int dt e^{4h'_0 t} = -6b'h_0'^4 + \rho_{A0}. \tag{49}$$

Here ρ_{A0} is the integration constant which may be chosen to be zero since $\rho_A \to 0$ when $b' \to 0$ (classical limit). Substituting (46) and (49) with $\rho_{A0} = 0$ into (47), we obtain

$$0 = -b'\kappa^2 h_0'^4 - h_0'^2 + h_0^2, \tag{50}$$

which can be solved as

$$h_0'^2 = \frac{1 \pm \sqrt{1 + 4b'\kappa^2 h_0^2}}{-2b'\kappa^2} > 0, \tag{51}$$

if

$$1 + 4b'\kappa^2 h_0^2 \geq 0, \tag{52}$$

which gives a nontrivial constraint since $b' < 0$ in general. In (51), the minus sign in $\pm$ corresponds to the classical case (19) in the limit of $b' \to 0$. In (48), when $t \sim t_s$, one finds

$$p = \frac{1}{2}\Big(\frac{d\phi}{dt}\Big)^2 - V(\phi) \sim \frac{(\alpha+1)^2\phi_0^2}{4}(t_s - t)^{\alpha-1}. \tag{53}$$

The first interesting case is that $\frac{2}{3}b + b''$ does not vanish. Since

$$p_A \sim -2\left(\frac{2}{3}b + b''\right)\frac{d^3 H}{dt^3} \sim 2\left(\frac{2}{3}b + b''\right)\alpha'(\alpha' - 1) \times (\alpha' - 2)h_1'(t_s - t)^{\alpha'-3}, \tag{54}$$

the classical term $\frac{2}{\kappa^2}(2\frac{dH}{dt} + 3H^2)$ in (48) can be neglected as this term behaves as $(t_s - t)^{\alpha'-1}$. Because of $p \sim -p_A$, one gets

$$\alpha' = \alpha + 2, \qquad h_1' = -\frac{(\alpha + 1)\phi_0^2}{8(\frac{2}{3}b + b'')(\alpha + 2)\alpha}. \tag{55}$$

As

$$H \sim h_0' + h_1'(t_s - t)^{\alpha+2}, \tag{56}$$

the singularity at $t = t_s$ is moderated, that is, the exponent of the power of $t_s - t$ becomes larger. When $\frac{2}{3}b + b'' > 0$, near $t = t_s$, H decreases with time, that is, the universe is decelerating. On the other hand, when $\frac{2}{3}b + b'' < 0$, H increases with time, that is, the universe is accelerating. If we may replace $(t_s - t)$ with its absolute value $|t_s - t|$, the decelerating (accelerating) universe turns to accelerate (decelerate) when $t > t_s$.

Another interesting situation corresponds to $\frac{2}{3}b + b'' = 0$ by properly choosing b''. In this case

$$p_A \sim -4(b - 2b')\left(\frac{dH}{dt}\right)^2 \sim -4(b - 2b')h_1'^2\alpha'^2(t_s - t)^{2(\alpha'-1)}. \tag{57}$$

The choice consistent with (48) is

$$\alpha' = \frac{\alpha + 1}{2}, \qquad h_1'^2 = -\frac{\phi_0^2}{4(b - 2b')}. \tag{58}$$

Since $\alpha' - \alpha = \frac{1-\alpha}{2} > 0$, the singularity is moderated, compared with the classical case (19).

One may also consider the case that the classical energy density ρ and the pressure p can be neglected since the quantum induced ρ_A and p_A become significantly dominant. In this case, combining the first FRW equation and (42)

$$\frac{6}{\kappa^2}H^2 = -\frac{1}{a^4}\int dt a^4 H\left[-12b\left(\frac{dH}{dt}\right)^2 + 24b'\left\{-\left(\frac{dH}{dt}\right)^2 + H^2\frac{dH}{dt} + H^4\right\} - 6\left(\frac{2}{3}b + b''\right)\left\{\frac{d^3H}{dt^3} + 7H\frac{d^2H}{dt^2} + 4\left(\frac{dH}{dt}\right)^2 + 12H^2\frac{dH}{dt}\right\}\right], \tag{59}$$

one has

$$\frac{12}{\kappa^2}\left(2H^2 + \frac{dH}{dt}\right) = -\left[-12b\left(\frac{dH}{dt}\right)^2 + 24b'\left\{-\left(\frac{dH}{dt}\right)^2 + H^2\frac{dH}{dt} + H^4\right\} - 6\left(\frac{2}{3}b + b''\right)\left\{\frac{d^3H}{dt^3} + 7H\frac{d^2H}{dt^2} + 4\left(\frac{dH}{dt}\right)^2 + 12H^2\frac{dH}{dt}\right\}\right]. \tag{60}$$

Notice Eq. (60) is nothing but

$$\frac{2}{\kappa^2}R = -T_A. \tag{61}$$

Equation(60) has a special solution, which gives a de Sitter space with constant H. In fact, if H is assumed to be the constant, Eq. (62) reduces to

$$\frac{24}{\kappa^2}H^2 = -24b'H^4, \tag{62}$$

which has solutions

$$H^2 = 0, \qquad H^2 = -\frac{1}{b'\kappa^2}. \tag{63}$$

The second solution describes de Sitter space.

If the curvature becomes significantly large, one may neglect the classical part, which is the left-hand side of (60). Assuming

$$H \sim \frac{h_0}{t}, \tag{64}$$

one arrives at the following algebraic equation:

$$0 = 12h_0\left\{-3\left(\frac{2}{3}b + b''\right) + \left[-b - 2b' + 9\left(\frac{2}{3}b + b''\right)\right]h_0 + \left[-2b' - 6\left(\frac{2}{3}b + b''\right)\right]h_0^2 + 2b'h_0^3\right\}. \tag{65}$$

Since the part inside {} of (65) is the third order polynomial, there is always a nontrivial solution for h_0, at least, if $\frac{2}{3}b + b''$ does not vanish. If the obtained h_0 is negative, the universe is shrinking but if we change the direction of the time by $T \to t_s - t$, we may obtain a solution describing the expanding universe. Even if $\frac{2}{3}b + b'' = 0$, one gets a nontrivial (nonvanishing) solution for h_0:

$$h_0 = \frac{1}{2} \pm \sqrt{\frac{5}{4} + \frac{b}{2b'}}. \tag{66}$$

Since b'' is arbitrary in principle, we may consider the case where the terms with b'' become dominant. Then Eq. (60) reduces to

$$\frac{12}{\kappa^2}\left(2H^2 + \frac{dH}{dt}\right) = -6b''\left\{\frac{d^3H}{dt^3} + 7H\frac{d^2H}{dt^2} + 4\left(\frac{dH}{dt}\right)^2 + 12H^2\frac{dH}{dt}\right\}, \quad (67)$$

which can be written by using the scalar curvature $R = 6(2H^2 + \frac{dH}{dt})$ as

$$\frac{2}{\kappa^2}R = -b''\left(\frac{d^2R}{dt^2} + 3H\frac{dR}{dt}\right). \quad (68)$$

Equation (68) has been found in R^2 gravity [16,17] with the purpose of describing the inflation. Thus, as in Ref. [10] (where b'' was chosen to be zero) we come to the following picture. Near to future singularity, the quantum effects become dominant and they drive (most probably) the universe to de Sitter space. Thus, the final state of such a universe is not the singularity. Rather, far in the future the new inflation era (which is supported by quantum gravity effects [11]) starts.

III. FINAL STATE OF A DARK ENERGY UNIVERSE

In the same way the singularity avoidance in other models (of a dark energy universe) may be considered. First of all, let us give the simple argument stressing that a big rip should not occur. Working in adiabatic approximation, one supposes that H is almost constant and the time derivatives of H can be neglected. Then since $a \propto e^{Ht}$, using (42), we find

$$\rho_A \sim -24b'H^4. \quad (69)$$

The first quantum corrected FRW equation looks like

$$\frac{6}{\kappa^2}H^2 = \rho + \rho_A = \rho - 24b'H^4. \quad (70)$$

The above equation can be rewritten as

$$0 = -24b'\left(H^2 + \frac{1}{4b'\kappa^2}\right)^2 + \frac{3}{2b'\kappa^2} + \rho. \quad (71)$$

Since b' and therefore $1/4b'\kappa^2$ are negative, in order that H^2 has a positive real solution, it follows the constraint for ρ

$$\rho < -\frac{3}{2b'\kappa^2}. \quad (72)$$

Thus, even if ρ includes the dark contribution from the phantom, ρ has an upper bound. In other words, it does not grow infinitely with the time, which was the disaster for phantom cosmology. An equivalent upper bound may be suggested when one uses Hawking radiation from a cosmological horizon (as communicated to us by Wang). Of course, near the big rip singularity, the time derivatives of H should be taken into account in the consistent treatment of the sort presented in the previous section.

Now we consider the big rip singularity [5] generated by the scalar field with the exponential potential:

$$S = \frac{1}{\kappa^2}\int d^4x\sqrt{-g}\left[R - \frac{\gamma}{2}\partial_\mu\phi\partial^\mu\phi - V(\phi)\right]. \quad (73)$$

When $\gamma < 0$, the scalar is a phantom with $w < -1$. By solving the ϕ equation of motion

$$0 = -\gamma\left(\frac{d^2\phi}{dt^2} + 3H\frac{d\phi}{dt}\right) - V'(\phi), \quad (74)$$

and the first FRW equation

$$\frac{6}{\kappa^2}H^2 = \rho_\phi = \frac{\gamma}{2}\left(\frac{d\phi}{dt}\right)^2 + V(\phi), \quad (75)$$

when

$$V(\phi) = V_0 e^{-2\phi/\phi_0}, \quad (76)$$

one gets a singular solution:

$$\phi = \phi_0 \ln\left|\frac{t_s - t}{t_1}\right|, \qquad H = -\frac{\gamma\kappa^2}{4(t_s - t)}, \qquad t_1^2 \equiv -\frac{\gamma\phi_0^2(1 - \frac{3\gamma\kappa^2}{4})}{2V_0}, \quad (77)$$

which gives

$$a = a_0\left|\frac{t_s - t}{t_1}\right|^{\gamma\kappa^2/4}. \quad (78)$$

Here a is singular at $t = t_s$ if $\gamma < 0$. A general solution of the above phantom system has been found in [11]. Even for the general solution, the behavior near $t = t_s$ is not qualitatively changed from that in (78). Hence, from the first look the big rip singularity seems to be inevitable.

Near the big rip singularity, since a blows up, curvature becomes large as $R \propto |t - t_s|^{-2}$. Since the quantum correction contains powers and higher derivatives of the curvatures in general, the quantum correction becomes dominant. Hence, one can apply the same reasoning as in the previous section. With the account of the quantum correction (42), the corrected FRW equation has the following form:

$$\frac{6}{\kappa^2}H^2 = \frac{\gamma}{2}\left(\frac{d\phi}{dt}\right)^2 + V(\phi) + \rho_A. \quad (79)$$

Let us assume

$$H = h_0 + \delta h, \qquad \phi = \phi_0 \ln\left|\frac{t_s - t}{t_1}\right| + \delta\phi, \quad (80)$$

and when $t \to t_s$, δh, $\delta\phi$ are much smaller than the first terms but $\frac{d\delta h}{dt}$ can be singular. Then the ϕ equation of motion (74) reduces to

$$0 = -\gamma\left(-\frac{\phi_0}{(t_s-t)^2} - \frac{3h_0}{t_s-t}\right) + \frac{2V_0t_1^2}{\phi_0(t_s-t)^2} \times \left(1 - \frac{2}{\phi_0}\delta\phi\right) + o((t_s-t)^{-1}), \quad (81)$$

which gives

$$V_0t_1^2 = -\frac{\gamma\phi_0^2}{2}, \qquad \delta\phi = -\frac{3}{2}(t_s-t). \quad (82)$$

With $\frac{2}{3}b + b'' \neq 0$, one gets

$$\rho_A \sim 6h_0\left(\frac{2}{3}b + b''\right)\frac{d^2\delta h}{dt^2}. \quad (83)$$

Substituting (82) and (83) into the quantum corrected FRW equation (79), we find

$$0 = \frac{3\gamma h_0\phi_0}{t_s-t} + 6h_0\left(\frac{2}{3}b + b''\right)\frac{d^2\delta h}{dt^2} + o((t_s-t)^{-1}), \quad (84)$$

and

$$\delta h = \frac{\gamma\phi_0}{2(\frac{2}{3}b + b'')}(t_s-t)\ln\left|\frac{t_s-t}{t_2}\right|. \quad (85)$$

Here t_2 is a constant of the integration. The scale factor a behaves as

$$a = a_0\left|\frac{t_s-t}{t_2}\right|^{\{(\gamma\phi_0)/4[(2/3)b+b'']\}(t_s-t)^2} e^{-h_0(t_s-t)-\{(\gamma\phi_0)/8[(2/3)b+b'']\}(t_s-t)^2+o((t_s-t)^2)}. \quad (86)$$

There appear logarithmic singularities in d^2a/dt^2, dH/dt but the singularity is moderated. Moreover, the universe might develop beyond $t = t_s$. Thus, quantum effects prevent from the most singular universe. In case $\frac{2}{3}b + b' = 0$, the assumption (80) seems to be inconsistent.

Another interesting dark energy model (which describes current acceleration and even current dominance of dark energy) was proposed in [13], where the matter Lagrangian density (dark energy) is coupled with the scalar curvature:

$$S = \int d^4x\sqrt{-g}\left\{\frac{1}{\kappa^2}R + R^\alpha L_d\right\}. \quad (87)$$

Here L_d is matterlike Lagrangian density. The second term may be induced by quantum effects as some nonlocal effective action. By the variation over $g_{\mu\nu}$, the equation of motion follows

$$0 = \frac{1}{\sqrt{-g}}\frac{\delta S}{\delta g_{\mu\nu}} = \frac{1}{\kappa^2}\left\{\frac{1}{2}g^{\mu\nu}R - R^{\mu\nu}\right\} + \tilde{T}^{\mu\nu}. \quad (88)$$

Here the effective energy momentum tensor $\tilde{T}_{\mu\nu}$ is defined by

$$\tilde{T}^{\mu\nu} = -\alpha R^{\alpha-1}R^{\mu\nu}L_d + \alpha(\nabla^\mu\nabla^\nu - g^{\mu\nu}\nabla^2)(R^{\alpha-1}L_d) + R^\alpha T^{\mu\nu},$$
$$T^{\mu\nu} \equiv \frac{1}{\sqrt{-g}}\frac{\delta}{\delta g_{\mu\nu}}\left(\int d^4x\sqrt{-g}L_d\right). \quad (89)$$

Let free massless scalar be a matter

$$L_d = -\tfrac{1}{2}\partial_\mu\phi\partial_\nu\phi. \quad (90)$$

The metric (2) is chosen. Assuming ϕ only depends on t, ($\phi = \phi(t)$), the solution of the scalar field equation is given by

$$\dot{\phi} = qa^{-3}R^{-\alpha}. \quad (91)$$

Here q is a constant of the integration. Hence $R^\alpha L_d = q^2/2a^6R^\alpha$, which becomes dominant when R is small (large) compared with the Einstein term $\frac{1}{\kappa^2}R$ if $\alpha > -1$ ($\alpha < -1$).

The accelerating solution of the FRW equation exists [13]

$$a = a_0t^{(\alpha+1)/3}\left(H = \frac{\alpha+1}{3t}\right),$$
$$a_0^6 \equiv \frac{\kappa^2q^2(2\alpha-1)(\alpha-1)}{3(\alpha+1)^{\alpha+1}(\frac{2}{3}(2\alpha-1))^{\alpha+2}}. \quad (92)$$

Equation (92) tells that the universe accelerates, that is, $\ddot{a} > 0$ if $\alpha > 2$.

For the matter with the relation $p = w\rho$, where p is the pressure and ρ is the energy density, from the usual FRW equation, one has $a \propto t^{2/3(w+1)}$. For $a \propto t^{h_0}$ it follows $w = -1 + \frac{2}{3h_0}$, and the accelerating expansion ($h_0 > 1$) of the universe occurs if $w < -\frac{1}{3}$. For (92), one gets

$$w = \frac{1-\alpha}{1+\alpha}. \quad (93)$$

Then if $\alpha < -1$, $w < -1$, i.e., an effective phantom.

When $\alpha < -1$, i.e., $w < -1$, the universe is shrinking in the solution (92). However, if one changes the direction of time as $t \to t_s - t$, the universe is expanding but has a big rip singularity at $t = t_s$. Since near the singularity, the curvature becomes very large again, we may include the quantum correction (42)

$$0 = -\frac{3}{\kappa^2}H^2 + \tilde{\rho} + \rho_A. \quad (94)$$

If H behaves as in (92), after changing the direction of the time as $t \to t_s - t$ in a,

$$H = \frac{1}{a}\frac{da}{dt} = \frac{\alpha+1}{3(t-t_s)}, \quad (95)$$

the quantum correction of the energy density ρ_A behaves as $\rho_A \sim (t - t_s)^{-4}$, which becomes very large when $t \sim t_s$. This shows that H cannot grow as in (95). If H is not very large, $\tilde{\rho}$ (94) becomes very small when a is large and can be neglected since $\tilde{\rho} \propto a^{-6}$. In such a situation, Eq. (95) reduces to (59). Hence, instead of future singularity, due to quantum effects a dark energy universe ends up in the de Sitter phase (63). Thus, quantum effects resolve the sudden future singularity of a dark energy universe.

IV. THERMODYNAMICS AND ENTROPY BOUNDS IN THE DARK ENERGY UNIVERSE

Thermodynamics of a dark energy universe was discussed in Ref. [8] where the appearance of negative entropies for models with an equation of state parameter less than -1 was demonstrated and entropy bounds were constructed. In the present section the entropy bounds near the big rip singularity are considered. The Hubble entropy S_H, Bekenstein entropy S_B, and Bekenstein-Hawking entropy S_{BH} are defined by

$$S_H = \frac{HV}{2G}, \qquad S_B = \frac{2\pi a E}{3}, \qquad S_{BH} = \frac{V}{2Ga}. \tag{96}$$

Here G is a gravitational constant ($\kappa^2 = 16\pi G$) and V is the volume of the universe where for the universe with a flat spatial part, it is chosen

$$V = V_0 a^3. \tag{97}$$

Near the big rip singularity, the dark energy dominates and the usual matter contribution may be neglected. Without quantum correction, $a \sim (t_s - t)^{2/3(w+1)}$ and $\rho \sim a^{-3(1+w)} \sim (t_s - t)^{-2}$ in accord with (77) and (78). The entropies behave as

$$\begin{aligned} S_B &\sim (t_s - t)^{2(1-3w)/3(w+1)}, \\ S_H &\sim (t_s - t)^{(1-w)/(w+1)}, \\ S_{BH} &\sim (t_s - t)^{4/3(w+1)}, \end{aligned} \tag{98}$$

where the exponents are related by

$$\frac{2(1-3w)}{3(w+1)} < \frac{1-w}{w+1} < \frac{4}{3(w+1)} < 0, \tag{99}$$

when $w < -1$. Hence, all the entropies are singular at $t = t_s$. Equation (99) shows that the Bekenstein entropy S_B is most singular while the Bekenstein-Hawking entropy is less singular.

In order to estimate the entropy, we consider the thermodynamical model [8], where the free energy corresponding to matter with w is given by

$$F_w = T\hat{F}(T^{1/w}V). \tag{100}$$

Here T is the temperature and V is the volume of the system. $\hat{F}$ is a function determined by the matter. The thermodynamical parameters are

$$\begin{aligned} p &= -\frac{\partial F_w}{\partial V} = -T^{1+1/w}\hat{F}'(T^{1/w}V), \\ \rho &= \frac{1}{V}\left(F_w - T\frac{\partial F_w}{\partial T}\right) = -\frac{1}{w}T^{1+1/w}\hat{F}'(T^{1/w}V), \\ S &= -\frac{\partial F_w}{\partial T} = -\hat{F}(T^{1/w}V) - \frac{1}{w}T^{1/w}\hat{F}'(T^{1/w}V). \end{aligned} \tag{101}$$

Here S is an entropy. Since ρ behaves as $\rho = \rho_0 a^{-3(1+w)} \propto \rho_0 (V/V_0)^{-(1+w)}$, from the second equation (101) it follows

$$\rho_0 V_0^{1+w} \propto -\frac{1}{w}(T^{1/w}V)^{1+w}\hat{F}'(T^{1/w}V). \tag{102}$$

Then $T^{1/w}V$ should be a constant, which indicates that the entropy S in the third equation (101) is also a constant. Since Hubble, Bekenstein, and Bekenstein-Hawking entropies (98) diverge at the big rip singularity, the following entropy bound holds near the (classical) big rip singularity

$$S < S_{BH} < S_H < S_B. \tag{103}$$

Here Eq. (99) was used. Then all the bounds are satisfied, which may be compared with the case of the brane-world dark energy model of Ref. [18], where the Hubble entropy bound is not satisfied and the Bekenstein bound is often violated.

The Hubble parameter in the expanding universe is given by $H = [2/3(w+1)]t$ if $w > -1$ and $H = [2/3(w+1)]t_s - t$ if $w < -1$, $dH/dt < 0$ when $w > -1$ and $dH/dt > 0$ if $w < -1$, which corresponds to the big rip singularity. $S_B \propto H^2 V a \propto H^2 a^4$ as found from the FRW equation. Then one may define the following quantity (which may indicate the future singularity occurrence):

$$\tilde{S} \equiv \frac{S_{BH}^2}{S_B}, \tag{104}$$

where $\tilde{S} \propto H^{-2}$ since $S_{BH} \propto a^2$. Hence, if $\tilde{S}$ decreases with time, there might occur the big rip singularity.

Let us reconsider the above entropy bounds with the account of the quantum effects. Using the solutions (80), (82), (85), and (86), we find the entropies (98) behave as

$$\begin{aligned} S_H &\to \frac{h_0 V_0 a_0^3}{2G}, \qquad S_B \to \frac{2\pi}{3}\frac{2\gamma h_0 \phi_0 V_0 a_0^4}{t_s - t}, \\ S_{BH} &\to \frac{V_0 a_0^2}{2G}. \end{aligned} \tag{105}$$

Then S_H and S_{BH} are finite and they may give meaningful entropy bound but S_B is negative since $\gamma < 0$ and diverges. Hence, the Bekenstein bound $S < S_B$ is violated. In (105), however, we have included only the classical part ρ_ϕ (75) in order to estimate S_B. With the account of the quantum correction ρ_A (42), the singularity in S_B can be canceled. Since $\rho = \rho_\phi + \rho_A = (6/\kappa^2)H^2 = 3H^2/8\pi G$, we find the following expression of the quantum corrected Bekenstein entropy S_B^q:

$$S_B^q = \frac{h_0^2 V_0 a_0^4}{4G}, \tag{106}$$

which is positive and finite. With S_B^q (106) instead of S_B (105), all the entropies are finite. We should note

$$\begin{aligned} S_B^q \gg S_H \gg S_{BH} \quad \text{if } h_0 \gg \frac{1}{a_0}, \\ S_B^q \ll S_H \ll S_{BH} \quad \text{if } h_0 \ll \frac{1}{a_0}. \end{aligned} \tag{107}$$

The parameters h_0 and a_0 are the values of the Hubble parameter and the size of the universe at $t = t_s$, which may be determined from the proper initial conditions.

In [19], the (quantum corrected) entropy bounds have been discussed. In [19], the spatial part of the universe is a sphere, where we have a relation [20]

$$S_H^2 + (S_{BH} - S_B)^2 = S_B^2, \tag{108}$$

even with the quantum correction. In case that the spatial part is flat, Eq. (108) reduces to $S_H^2 = 2S_{BH}S_B$. We should note that for S_H, S_{BH} (105) and S_B^q (106), it holds

$$S_H^2 = 2S_{BH}S_B^q. \tag{109}$$

Even for the classical case that all the entropies are singular, $S_H^2 = 2S_{BH}S_B$, which can be found from the FRW equation (75). One can rewrite the FRW equation (75) in the form $S_H^2 = 2S_{BH}S_B$ by using the definition of the entropies (98).

To conclude, it is shown that entropies near to classical singularity are singular as well. However, quantum corrected entropies are finite and give the well-defined entropy bounds. This is not surprising due to the fact that quantum effects help to escape the future singularity in a dark energy universe.

V. DARK ENERGY UNIVERSE WITH GENERAL EQUATION OF STATE

So far we concentrated mainly on the various aspects of a dark energy universe with a negative equation of state parameter which is less than -1. Nevertheless, the recent astrophysical data admit also the case of a time-dependent equation of state parameter. Let us consider several examples of such dark energy cosmology and its late time behavior. Note that several models of a dark energy universe with a time-dependent equation of state were discussed in [8,21] (see also references therein).

One starts from the general equation of state of the form

$$p = f(\rho), \tag{110}$$

instead of the equation $p = w\rho$ with constant w. In (110), f can be an arbitrary function. Imagine that solving gravitational equations, we want to construct the cosmology with time-dependent w, which describes the transition from the decelerating universe to the accelerating one. As an example, the following scale factor $a(t)$ may be considered:

$$a = a_0 e^{\lambda t} t^\alpha. \tag{111}$$

Here λ and α are some constants. Hence,

$$\frac{da}{dt} = aH = a\left(\lambda + \frac{\alpha}{t}\right), \qquad \frac{d^2a}{dt^2} = a\left\{\left(\lambda + \frac{\alpha}{t}\right)^2 - \frac{\alpha}{t^2}\right\}. \tag{112}$$

In the case that λ and α are positive, the universe is accelerating if

$$t > t_0 \equiv \frac{\sqrt{\alpha} - \alpha}{\lambda}, \tag{113}$$

and decelerating if

$$t < t_0. \tag{114}$$

That is, the decelerating universe turns into the accelerating one at $t = t_0$. Thus, if transition point t_0 occurred about 5×10^7 years ago, the solution may approximately describe our universe. Note that t_0 is positive when $0 < \alpha < 1$.

By using (6), one finds

$$\rho = \frac{6}{\kappa^2}\left(\lambda + \frac{\alpha}{t}\right)^2, \tag{115}$$

$$p = -\frac{2}{\kappa^2}\left(\frac{\alpha(3\alpha - 2)}{t^2} + \frac{6\lambda\alpha}{t} + 3\lambda^2\right). \tag{116}$$

Equation (115) can be solved as

$$t = \frac{\alpha}{\frac{\kappa^2\rho}{6} - \lambda}, \tag{117}$$

substituting (117) into (116), we obtain

$$p = -\frac{2}{\kappa^2}\left\{\left(3 - \frac{2}{\alpha}\right)\left(\frac{\kappa^2\rho}{6}\right)^2 + \frac{4}{\alpha}\frac{\kappa^2\rho}{6} - \left(6 - \frac{2}{\alpha}\right)\lambda^2\right\}. \tag{118}$$

Hence, with (118) as the equation of state, we arrive at a solution (111), where the decelerating universe turns into the accelerating one.

General case (110) may be considered as well. Using the first FRW equation (4) and the energy conservation law

$$0 = \frac{d\rho}{dt} + 3H(\rho + p), \tag{119}$$

one gets

$$\frac{d\rho}{dt} = F(\rho) \equiv -\kappa\sqrt{\frac{3\rho}{2}}(\rho + f(\rho)). \tag{120}$$

With a proper assumption about function $f(\rho)$, we can find the t dependence of ρ by solving Eq. (120). Using the obtained expression for $\rho = \rho(t)$, one can also find the t dependence of p as $p = f(\rho(t))$.

By combining (110) and (120), the pressure p can be expressed as

$$p = -\rho - \frac{1}{\kappa}\sqrt{\frac{2}{3\rho}}F(\rho). \tag{121}$$

Therefore if $F(\rho) > 0$ $[F(\rho) < 0]$, it follows $w < -1$ $(w > -1)$. We now assume $F(\rho) = 0$ at $\rho = \rho_0$, where ρ_0 is a particular value of ρ. We further assume that when $\rho \sim \rho_0$, $F(\rho)$ behaves as

$$F(\rho) \sim F_0(\rho - \rho_0)^n. \tag{122}$$

Here F_0 is a constant and n is a positive odd integer. If $n \neq 1$, by solving (120), one gets

$$\rho \sim \rho_0 + \{F_0(1 - n)(t - t_0)\}^{-1/(n-1)}, \tag{123}$$

and if $n = 1$,

$$\rho \sim \rho_0 + C\mathrm{e}^{F_0 t}. \tag{124}$$

Here t_0 or C is a constant of the integration. Then ρ goes to ρ_0 only at $|t| \rightarrow \infty$, which may indicate that the region with $w > -1$ could be disconnected with the region $w < -1$. Instead of a positive odd integer n, one may start from

$$n = \frac{m-1}{m}, \tag{125}$$

with an integer m. Then as in (123) the time-dependent energy density looks like

$$\rho \sim \rho_0 + \left\{\frac{F_0}{m}(t - t_0)\right\}^m. \tag{126}$$

Thus, the region $w > -1$ might be connected with the region $w < -1$. In this case, however, the equation of state has branches.

We now consider the case with a linear equation of state $p = w\rho$ where w depends on time as $w = w(t)$. Replacing $f(\rho)$ by $w(t)\rho$ in (120) and using the first FRW equation (4), it follows

$$H = \frac{2}{3}\left(\int [1 + w(t)]\right)^{-1}. \tag{127}$$

In order to investigate what happens when w changes the value from the one bigger than -1 to that less than -1, we now assume that near $t = t_0$ $w(t)$ behaves as

$$w(t) \sim -1 + w_0(t - t_0), \tag{128}$$

with constant w_0. Using

$$\int dt[1 + w(t)] \sim \frac{1}{2}w_0(t - t_0)^2 + w_1, \tag{129}$$

one finds

$$\rho \sim \frac{8}{3\kappa^2\{w_0(t - t_0)^2 + 2w_1\}}, \qquad H \sim \frac{32}{3\{w_0(t - t_0)^2 + 2w_1\}}. \tag{130}$$

Here w_1 is a constant of the integration. Unlike the case in (123), there is no singularity at $t = t_0$ if $w_1 \neq 0$.

As one more example, the case with another $w(t)$ may be considered

$$w(t) = -1 - \frac{a(t - t_0)}{t + b}. \tag{131}$$

Here a, b, and t_0 are positive constants. Then $w(t)$ has the following properties:

$$w(0) = w_0 \equiv -1 + \frac{at_0}{b} > -1, \qquad w(t_0) = -1, \qquad w(+\infty) = w_\infty \equiv -1 - a < -1. \tag{132}$$

Hence, $w(t)$ connects the region of $w > -1$ with that of $w < -1$. Since

$$W(t) \equiv \int dt[1 + w(t)] = a\left\{-t + (b + t_0)\ln\frac{t + b}{t_1}\right\}, \tag{133}$$

we find

$$\rho(t) = \frac{8}{3\kappa^2 W(t)^2}, \qquad H = \frac{2}{3W(t)}, \tag{134}$$

where t_1 is a constant of the integration. When $t \sim t_0$, one gets

$$W(t) \sim a\left[-t_0 + (b + t_0)\ln\frac{t_0 + b}{t_1}\right] + \frac{a(t - t_0)^2}{2(b + t_0)} + \mathcal{O}((t - t_0)^2), \tag{135}$$

which is consistent with (129). If the universe is expanding, that is $H > 0$, at $t = 0$ we find the following condition:

$$b > t_1 > 0. \tag{136}$$

$W(t)$ behaves as $W(t) \sim -at < 0$ when t is large. Thus, if the condition (136) is satisfied, $W(t)$ vanishes at finite t $(t = t_s)$, where t_s is a solution of the equation

$$0 = W(t_s) = a\left\{-t_s + (b + t_0)\ln\frac{t_s + b}{t_1}\right\}. \tag{137}$$

Hence, there appears singularity at $t = t_s > 0$ in ρ and H, which is nothing but the big rip singularity. However, even with (136), since $t_s - t_0 = (b + t_0)\ln[(t_s + b)/t_1] - t_0$ can be negative in general, the singularity may occur in the region $w > -1$.

To conclude, we presented several examples of a dark energy universe with a time-dependent (negative) equation of state. The possibility to have a naturally accelerated universe phase (sometimes, as a transition from deceleration) is shown. It is interesting that when a time-dependent equation of state parameter is negative (not only less but even bigger than -1) the finite-time future singularity occurs as a final state of such universe. Nevertheless, in the same way as discussed in the second and third sections one can show that quantum effects prevent the evolution to such final state (eventually driving the universe to the inflationary era).

VI. DISCUSSION

In summary, we discussed several aspects of phantom thermodynamics and the final state of a phantom dark energy universe. Despite the absence of consistent phantom energy theory, some general results look quite promising. In particular, it is shown that a finite-time big rip singularity remains to be a deeply theoretical possibility in classical phantom theory. The account of quantum effects (when the universe evolves to the singularity and when curvature invariants grow) is done. As a result, it is proved that quantum effects moderate the singularity or it even disappears completely. (Note also that stability analysis [11] and gravitational perturbations account [22] indicates that perturbations act against the big rip occurrence.) Hence, it is unlikely that the final state of phantom universe is a big rip. Rather, the final state is the initial state on the same time, because the inflationary era may start again in the future. The resolution of big rip singularity resolves also several related phenomena. For instance, entropy bounds which are divergent near a big rip become well defined after the quantum corrections are included. Similarly, the escape of finite-time singularity means that black holes mass evolution is less dramatic than was predicted (masses do not vanish to zero).

It is expected that soon precise observational cosmology data will give more stringent bounds for an equation of state parameter. At the moment, it is still unclear if it will lie at quintessence, or at a phantom region. Moreover, it is quite possible that the smiling universe hides a number of surprises for us. Nevertheless, the phantom universe remains to be the theoretical possibility which is not explored yet and which deserves some attention.

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APPENDIX A: THE ENTROPY OF THE PHANTOM UNIVERSE

One more unusual property of a dark energy universe with w less than -1 is the strange behavior of the entropy. In fact, it was pointed out in Ref. [8] that entropy of such a universe is negative. Another proposal came out in Ref. [9] suggesting to consider phantom fluid as a kind of cosmological quantum fluid (as a nuclear spin model, for instance) where negative temperature is admitted. (Note that the idea of negative temperature in cosmological context was discussed first by Vanzo-Klemm [23]). In this case, the entropy may be positive.

Let us describe the relation between the entropy and the energy of such a dark energy universe when the temperature is negative. Starting from the model [8], instead of (100) we consider the following free energy:

$$F_w = \gamma T \tilde{F}((\gamma T)^{1/w} V). \tag{A1}$$

If the temperature T is positive, $\gamma = 1$ and if it is negative, $\gamma = -1$. Simple calculation gives the pressure p, the energy density ρ, and the entropy S:

$$\begin{aligned} p &= -(\gamma T)^{1+1/w} \tilde{F}'((\gamma T)^{1/w} V), \\ \rho &= -\frac{1}{w}(\gamma T)^{1+1/w} \tilde{F}'((\gamma T)^{1/w} V), \\ S &= -\gamma\left\{\tilde{F}((\gamma T)^{1/w} V) + \frac{1}{w} T^{1/w} \tilde{F}'((\gamma T)^{1/w} V)\right\}. \end{aligned} \tag{A2}$$

If the energy is extensive, the energy behaves as $E = \rho V \rightarrow \lambda E$ under the rescaling the entropy and the volume as $S \rightarrow \lambda S$ and $V \rightarrow \lambda V$. In accord with [8] we consider the following free energy:

$$F_w = -f_0(\gamma T)^{1+1/w} V[1 + f_1(\gamma T)^{-2/nw} V^{-2/n}]. \tag{A3}$$

If there is no second term, the first term gives the extensive energy. It is assumed the second term is small compared with the first term. Then one gets

$$\begin{aligned} E &= \frac{pV}{w} \\ &= \frac{f_0}{w}(\gamma T)^{1+1/w} V\left[1 + \left(1 - \frac{2}{n}\right) f_1 (\gamma T)^{-2/nw} V^{-2/n}\right], \\ S &= f_0 \gamma (\gamma T)^{1/w} V\left[\left(1 + \frac{1}{w}\right)\right. \\ &\quad \left. + \left(1 + \frac{1}{w} - \frac{2}{nw}\right) f_1 (\gamma T)^{-2/nw} V^{-2/n}\right]. \end{aligned} \tag{A4}$$

The subextensive part of the energy E_C, which is called the Casimir energy [20], is given by

$$\begin{aligned} E_C &= n(E + pV - TS) = -nV^2 \frac{\partial}{\partial V}\left(\frac{F}{V}\right) \\ &= -2 f_0 f_1 (\gamma T)^{1+(1/w)-(2/nw)} V^{1-2/n}. \end{aligned} \tag{A5}$$

The extensive part of the energy E_E has the following form:

$$E_E = E - \frac{1}{2}E_C = \frac{f_0}{w}(\gamma T)^{1+1/w}V\left[1 + \left(1 - \frac{2}{n} + w\right) \times f_1(\gamma T)^{-2/nw}V^{-2/n}\right]. \tag{A6}$$

As in Sec. IV, $T^{1/w}V$ is a constant in the phantom dominated universe. Then if one neglects the second term in E_E and/or E as

$$E_E \sim E \sim \frac{f_0}{w}(\gamma T)^{1+1/w}, \tag{A7}$$

we obtain

$$S \sim A[V^w\sqrt{(2E - E_C)E_C}]^{n/(w+1)n-1}. \tag{A8}$$

Here A is a constant.

The natural assumption is $E > 0$. From the expression (A4), $f_0 < 0$ if $w < 0$. The starting condition is that the entropy S is positive. In case of the quintessence, where $-1 < w < -\frac{1}{3}$, since $1 + \frac{1}{w} < 0$, from Eq. (A4), we find $\gamma > 0$ (positive temperature) so that the entropy is positive. On the other side, in the case of the phantom, where $w < -1$, that is, $1 + \frac{1}{w} > 0$, it follows $\gamma < 0$ if the entropy S is positive. Therefore the temperature should be negative. Conversely, if we assume the temperature is positive in the phantom theory, the entropy should be negative.

Note also that in order to obtain the Cardy-Verlinde (CV) formula (A8) (for a list of references, see [24]), the Casimir energy E_C should be positive, which requires $f_1 > 0$. Hence, the entropy of the phantom-filled universe is positive when the temperature is negative. In this case, the standard CV entropy formula holds.

APPENDIX B: BLACK HOLE MASS EVOLUTION IN THE DARK ENERGY UNIVERSE

One more strange feature of the phantom universe is the black hole mass loss up to the full disappearance in the big rip singularity. The corresponding analysis [7] was performed in a classical phantomlike universe (where the dominant energy condition is broken) with the final state in the big rip. In the present Appendix, we reconsider this process taking into account the quantum effects which prevent the creation of a big rip singularity as well as a time-dependent (negative) equation of state.

As shown in an important paper [7] (see also [9]), the rate of the black hole mass change in the fluid with the energy density ρ and the pressure p is given by

$$\frac{dM}{dt} = 4\pi AM^2(\rho + p). \tag{B1}$$

Here M is the mass of the black hole and A is a dimensionless positive constant. As a background, the FRW universe with the metric (2) may be considered. Combining the first FRW equation (3) and the energy conservation law (119), one obtains

$$\rho + p = -\frac{2}{\kappa}\sqrt{\frac{2}{3}}\frac{d(\rho^{1/2})}{dt}. \tag{B2}$$

Further combining (B1) and (B2), we get

$$\frac{d}{dt}\left(\frac{1}{M}\right) = \frac{8\pi A}{\kappa}\sqrt{\frac{2}{3}}\frac{d(\rho^{1/2})}{dt}. \tag{B3}$$

The solution of the above equation is

$$M = \frac{M_0}{1 + \frac{8\pi AM_0}{\kappa}\sqrt{\frac{2\rho}{3}}}. \tag{B4}$$

Hence, if ρ increases as in the case that the fluid is phantom, M decreases. At the big rip singularity where ρ diverges, M vanishes. This is a universal property for any black hole in such a phantom universe. On the other hand, in the case of the Barrow model where ρ is finite (5), the mass M is finite even at the singularity. By using the first FRW equation (3), we may further rewrite M (B4) in the following form:

$$M = \frac{M_0}{1 + \frac{4\pi AM_0}{3}H}. \tag{B5}$$

In [7,9], the behavior of M for the phantom with constant $w < -1$ has been investigated in detail. As in (77), when $w < -1$ in the expanding universe, the Hubble parameter behaves as

$$H = -\frac{\gamma\kappa^2}{4(t_s - t)} = \frac{-\frac{2}{3(w+1)}}{t_s - t}. \tag{B6}$$

On the other hand, in case $w > -1$ it looks like

$$H = \frac{\frac{2}{3(w+1)}}{t}. \tag{B7}$$

Using Eq. (B5) one arrives at

$$M = \frac{M_0}{1 - \frac{4\pi AM_0}{3}\frac{\frac{2}{3(w+1)}}{t_s - t}}, \tag{B8}$$

when $w < -1$ and

$$M = \frac{M_0}{1 + \frac{4\pi AM_0}{3}\frac{\frac{2}{3(w+1)}}{t}}, \tag{B9}$$

when $w > -1$. In case of (B8), M decreases and vanishes at $t = t_s$. Near $t = t_s$, M behaves as

$$M \sim -\frac{9(w+1)(t_s - t)}{8\pi A}. \tag{B10}$$

This does not depend on M_0 and is universal as pointed out in [7]. On the other hand, M (B9) increases and reaches the maximal value $M = M_0$ when $t = \infty$. Even with the account of the cosmological term, the qualitative behavior does not change.

The above behavior is modified when quantum effects are taken into account because as was argued in the second and third sections they may stop the evolution to final singularity. Indeed, let us consider the case that the quantum correction is included as in (79). If $\frac{2}{3}b + b'' \neq 0$, combining (80) and (85), it follows that $H = h_0$ when $t = t_s$. Therefore M has a finite, nonvanishing value:

$$M \to \frac{M_0}{1 + \frac{4\pi A M_0}{3} h_0}. \tag{B11}$$

Since

$$\frac{dH}{dt} = \frac{d\delta h}{dt} = -\frac{\gamma\phi_0}{2(\frac{2}{3}b + b'')}\left(\ln\left|\frac{t_s - t}{t_2}\right| + 1\right), \tag{B12}$$

and $\gamma < 0$ for the phantom, near the singularity $t \sim t_s$, $\frac{dH}{dt} > 0$ ($\frac{dH}{dt} < 0$) if $\frac{2}{3}b + b'' < 0$ ($\frac{2}{3}b + b'' > 0$). Therefore if $\frac{2}{3}b + b'' < 0$, since H increases, M decreases towards the singularity although M is finite and nonvanishing there. On the other hand, if $\frac{2}{3}b + b'' < 0$, M increases. As the Hawking radiation occurs due to the quantum correction, the above type of behavior may be more realistic in the phantom universe (quantum effects have been neglected, at least in the leading order, in [7]).

Let us reconsider what happens with the black hole mass M evolution Eq. (B4) in a dark energy universe with time-dependent equation of state. For simplicity, the quantum corrections are neglected. When $p = w(t)\rho$, from (B2) it follows

$$[1 + w(t)]\rho = -\frac{2}{\kappa}\sqrt{\frac{2}{3}}\frac{d(\rho^{1/2})}{dt}. \tag{B13}$$

First we consider the case (128) and the behavior of ρ (130). Then if w_0 and w_1 are positive, ρ takes a minimum value at $t = t_0$. The black hole mass M (B4) increases when $t < t_0$ and it reaches the maximum at $t = t_0$. When $t > t_0$, the mass decreases.

As a more concrete example, $w(t)$ (131) may be discussed. When $t_s > t_0$ [at $t = t_s$, $W(t)$ (137) vanishes], $W(t)$ increases when $t < t_0$ and decreases when $t > t_0$. Then the energy density ρ decreases when $t < t_0$, increases when $t > t_0$, and diverges at $t = t_s$. Therefore the behavior of the black hole mass M (B4) is similar to $W(t)$, that is, M increases when $t < t_0$, decreases when $t > t_0$ and vanishes at $t = t_s$ (like in the classical phantom universe). Nevertheless, the account of quantum effects, as we showed above, qualitatively changes the black hole mass evolution. In other words, the same phenomenon which drives a dark energy universe out of final singularity (because of second quantum gravity era) is responsible for a much less sharp loss of black holes masses. As a big rip does not occur, initially massive black holes continue to be (may be less) massive.

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