Bouncing universes and their perturbations: A simple model reexamined

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We reconsider the toy model studied by Gordon and Turok of a spatially closed Friedmann-Lemaître universe, driven by a massive scalar field, which deflates quasiexponentially, bounces, and then enters a period of standard inflation. We find that the equations for the matter density perturbations can be solved analytically, at least at lowest order in some "slow-roll" parameter. We can therefore give, in that limit, the explicit spectrum of the postbounce perturbations in terms of their prebounce initial spectrum. Our result is twofold. *If* the prebounce growing and decaying modes are of the same order of magnitude at the bounce, then the spectrum of the prebounce growing mode is carried over to the postbounce *decaying* mode. On the other hand, if, more likely, the prebounce growing mode dominates, then resolution at next order indicates that its spectrum is nicely carried over, with reduced amplitude, to the postbounce *growing* mode.

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I. INTRODUCTION

In the wake of the developments of the string-inspired "pre-big bang" [1] and "ekpyrotic" or "cyclic" [2] cosmological scenarios, four-dimensional, general relativistic, and bouncing Friedmann-Lemaître models, even if they have little or no relevance to the aforementioned scenarios, have recently attracted renewed interest, and an issue still under debate is how the spectrum of initial, prebounce, matter fluctuations is modified by the bounce (see, e.g., [3]).

A toy model of a bouncing universe was, in particular, studied in [4] (see also [5]): a spatially closed Friedmann-Lemaître model, driven by a massive scalar field, which deflates quasiexponentially, bounces and then enters a period of standard inflation. Unfortunately, no definite prediction on the postbounce spectrum of perturbations was reached, the main reason being the singular behavior of the evolution equation in the bounce region.

In this paper, we reconsider this simple model and rewrite the evolution equation for the matter perturbations in a well-behaved form. Having done so, we are able to solve it analytically, at least at lowest order in some "slow-roll" parameter, that is, when the pre- and post-bounce quasi-de Sitter periods are long enough. We shall hence obtain the explicit spectrum of the perturbations when they exit the Hubble radius during postbounce inflation, in terms of their initial spectrum when they enter the Hubble radius during prebounce deflation.

As we shall see, two cases will arise. If the prebounce growing and decaying modes are of the same order of magnitude at the bounce, then the prebounce spectrum of the cosmologically interesting growing mode is carried over to the postbounce *decaying* mode, and hence soon lost. (As for the postbounce surviving growing mode, it inherits the prebounce decaying mode spectrum.)

On the other hand, if the prebounce decaying mode has become negligible at the bounce, then resolution at next order of the perturbation equation indicates that the spectrum of the only left prebounce growing mode is nicely carried over to the surviving, cosmologically interesting, postbounce *growing* mode with no modification, apart from an overall reduction factor.

We shall conclude with a few remarks on the genericity of the result and the validity of the slow-roll approximation which was made to yield it.

II. THE BACKGROUND

Consider a spatially closed, homogeneous, and isotropic universe with line element $ds^2 = -dt^2 + S^2(t)d\Omega_3^2$, where t is cosmic time, $S(t)$ the scale factor, and $d\Omega_3^2$ the line element of a three-dimensional unit sphere. If matter is just a scalar field φ with mass m the Einstein equations reduce to the homogeneous Klein-Gordon and Friedmann-Lemaître equations:

$$\ddot{\varphi} + 3H\dot{\varphi} + m^2\varphi = 0;$$
$$3\left(H^2 + \frac{1}{S^2}\right) = \kappa\left(\frac{1}{2}\dot{\varphi}^2 + \frac{1}{2}m^2\varphi^2\right), \tag{2.1}$$

where a dot denotes differentiation with respect to cosmic time, where $H \equiv \dot{S}/S$ is the Hubble parameter, and where κ is Einstein's constant.

The system of Eqs. (2.1) has been thoroughly studied, in particular, in [6–8]. We shall retain here that there exist ranges of initial conditions for which the scale factor has a minimum. We shall restrict ourselves to the case when such a bounce occurs, at $t = 0$ without loss of generality, and set the initial conditions there as $\varphi(0) = \varphi_0$ and

$\dot{\varphi}(0) = \dot{\varphi}_0$ [the initial condition for S follows from the fact that $\dot{S}(0) = 0$]. Introducing the rescaled initial condition, time, scalar field, and scale factor as

$$\phi_0 \equiv \sqrt{\frac{\kappa}{6}}\varphi_0, \qquad \tau \equiv \phi_0 m t, \qquad \phi \equiv \frac{\varphi}{\varphi_0}, \qquad a \equiv \phi_0 m S, \tag{2.2}$$

as well as the auxiliary function $z(\tau) \equiv \frac{1}{m}\sqrt{\frac{\kappa}{6}}\dot{\varphi}$, the system (2.1) becomes

$$\frac{d\phi}{d\tau} = \frac{z}{\phi_0^2}, \qquad \frac{dz}{d\tau} = -\frac{3z}{a}\frac{da}{d\tau} - \phi, \qquad \frac{da}{d\tau} = \pm\sqrt{a^2\left(\phi^2 + \frac{z^2}{\phi_0^2}\right) - 1}, \tag{2.3}$$

where the plus sign holds after the bounce ($\tau \geq 0$) and the minus sign before. As for the initial conditions, they become

$$\phi(0) = 1, \qquad z(0) = z_0, \qquad a(0) = \frac{1}{\sqrt{1 + z_0^2/\phi_0^2}} \tag{2.4}$$

(with $z_0 \equiv \frac{1}{m}\sqrt{\frac{\kappa}{6}}\dot{\varphi}_0$). As one can see from (2.3), the solution for $\tau < 0$ corresponding to the set of initial conditions (ϕ_0, z_0) can be obtained from the solution for $\tau > 0$ corresponding to the set $(\phi_0, -z_0)$ by means of the transformation

$$a(\tau, z_0) = a(-\tau, -z_0), \qquad \phi(\tau, z_0) = \phi(-\tau, -z_0), \qquad z(\tau, z_0) = -z(-\tau, -z_0) \quad \phi_0 \text{ fixed}. \tag{2.5}$$

Now, if the standard[1] conditions for postbounce inflation are imposed, that is if

$$\phi_0 \gg 1 \qquad \text{and} \qquad |z_0| \ll \phi_0, \tag{2.6}$$

then the solution of the system (2.3) can be approximated, at zeroth order in the slow-roll parameter $1/\phi_0^2$, by (see [4] who limited themselves to the case $z_0 = 0$)

$$a \simeq \cosh\tau, \qquad \phi \simeq 1, \qquad z \simeq \frac{z_0}{\cosh^3\tau} - \frac{1}{3}\frac{\sinh\tau}{\cosh^3\tau}(\cosh^2\tau + 2). \tag{2.7}$$

By comparison with the direct numerical integration of (2.3), one sees that (2.7) is a good approximation to the exact solution for ϕ and a as long as $|\tau| \ll \phi_0$, and a good approximation for z on the much wider range $|\tau| \ll \phi_0^2$, that is as long as the universe is well within the two, pre- and postbounce, dustlike eras.[2]

III. THE EVOLUTION EQUATION FOR THE SCALAR PERTURBATIONS

We consider now the perturbed metric $ds^2 = -(1 + 2\Phi)dt^2 + S^2(t)(1 - 2\Psi)d\Omega_3^2$ and the perturbed scalar field $\varphi(t) + \chi$. In Fourier space, the "scalar" perturbations Φ_n, Ψ_n, and χ_n are functions of time and of the eigenvalues n of the Laplacian on the 3-sphere [defined as $\Delta f_n = -n(n+2)f_n$, $n \in N$, and $n \geq 2$]. The (kl), $(0k)$, and (00) components of the linearized Einstein equations then are, respectively, (see, e.g., [9])

$$\Phi_n = \Psi_n, \qquad \dot{\Psi}_n + H\Phi_n = \frac{\kappa\dot{\varphi}}{2}\chi_n,$$
$$-3H\dot{\Psi}_n - \frac{k^2\Psi_n}{S^2} + \frac{3K\Phi_n}{S^2} = \frac{\kappa}{2}\left(\dot{\varphi}\dot{\chi}_n + \chi_n\frac{dV}{d\varphi} + 2V\Phi_n\right), \tag{3.1}$$

where $k^2 \equiv n(n+2) - 3K$, with $K = 1$, and where, in the toy model we consider here $V(\varphi) \equiv \frac{1}{2}m^2\varphi^2$. As is well known [9], the last equation can be rewritten, using the two constraints, as

$$\ddot{\Phi}_n + \left(7H + \frac{2}{\dot{\varphi}}\frac{dV}{d\varphi}\right)\dot{\Phi}_n + \left[\frac{k^2 - 5K}{S^2} + 2\left(\kappa V + \frac{H}{\dot{\varphi}}\frac{dV}{d\varphi}\right)\right]\Phi_n = 0. \tag{3.2}$$

Another form of that equation is easily found to be (see [9])

$$u_n'' + [k^2 - W(\eta)]u_n = 0, \quad \text{with } u_n \equiv \frac{a}{\varphi'}\Phi_n \quad \text{and} \quad W(\eta) = -\frac{\varphi'''}{\varphi'} + 2\frac{\varphi''^2}{\varphi'^2} + \kappa\frac{\varphi'^2}{2}, \tag{3.3}$$

where a prime denotes differentiation with respect to conformal time η—related to cosmic time t by $Sd\eta = dt$.

It is clear that none of the forms (3.1), (3.2), and (3.3) is suitable for integration when $\dot{\varphi}$ goes through zero [which is necessarily the case if there is to be quasi-de Sitter regimes before and after the bounce, see, e.g., Eq. (2.7) for $z \propto \dot{\varphi}$].[3] Now, and this is in fact the foundation of this paper, it is easy to check that, at least when $V(\varphi) \equiv \frac{1}{2}m^2\varphi^2$,[4] they can be put into the strictly equiva-

[1] See [7] for fine-tuned values of ϕ_0 of order 1 which also yield inflation. See [8] for a proof that all models recollapse. Generic values of ϕ_0 of order 1 yield small universes which soon recollapse. Hence the toy model considered here is inappropriate to describe a postbounce universe which does not inflate and immediately enters a radiation era.

[2] The analytical solution at next order in $1/\phi_0^2$ can easily be obtained, see [4], but will not be used in this paper.

[3] The authors of Ref. [4] solved the perturbation Eq. (3.2) (in the particular case $z_0 = 0$) using delicate numerical matching techniques.

[4] The result can easily be extended to any potential of the form $V(\varphi) \equiv c_0 + c_1\varphi^n$.

lent, well-behaved form:

$$\begin{cases} \frac{d(a\Phi_n)}{d\tau} & = \frac{z}{a^2} A_n, \\ & \text{with } A_n \equiv a^3 (z g_n - \frac{k^2 \Phi_n}{a^2 \phi}), \\ \frac{dg_n}{d\tau} & = \frac{k^2}{a^5 \phi} A_n + \frac{\Phi_n}{\phi_0^2} (3 - \frac{k^2}{a^2 \phi^2}), \end{cases} \tag{3.4}$$

where $a(\tau)$, $\phi(\tau)$, and $z(\tau)$ solve the background equations (2.3). [The first equation is nothing but the $(0k)$ linearized equation and the second is a rewriting of the (00) one in terms of the suitably chosen auxiliary function g_n.[5]] Once (3.4) is solved and Φ_n and g_n known, then the other scalar perturbations are given by

$$\Psi_n = \Phi_n \quad \text{and} \quad \sqrt{\frac{3\kappa}{2}} \frac{\chi_n}{\phi_0} = \frac{A_n}{a^3} = \left(z g_n - \frac{k^2 \Phi_n}{a^2 \phi} \right). \tag{3.5}$$

Note that, in view of the symmetry properties of the background solution, see (2.5), the solution of (3.4) is such that

$$\Phi_n[\tau, z_0, \Phi_n(0), g_n(0)] = \Phi_n[-\tau, -z_0, \Phi_n(0), -g_n(0)]. \tag{3.6}$$

IV. RELATING THE PRE- AND POSTBOUNCE SPECTRA AT LOWEST ORDER IN $1/\phi_0^2$

When the conditions (2.6) on the initial conditions are met, the term in $1/\phi_0^2$ in the system (3.4) can, at lowest order, be ignored.[6] The evolution equation for the scalar perturbations thus simplifies into

$$\frac{d(a\Phi_n)}{d\tau} = \frac{z}{a^2} A_n, \qquad \frac{dg_n}{d\tau} \simeq \frac{k^2}{a^5 \phi} A_n$$
$$\text{with, recall } A_n \equiv a^3 \left(z g_n - \frac{k^2 \Phi_n}{a^2 \phi} \right), \tag{4.1}$$

where, at the same approximation, the background functions $a(\tau)$, $\phi(\tau)$, and $z(\tau)$ are given by (2.7). Differentiating A_n once, one finds

$$\frac{dA_n}{d\tau} \simeq -g_n \cosh^3 \tau. \tag{4.2}$$

Differentiating again, one gets the following, closed, equation for A_n (or, equivalently for χ_n):

$$\frac{d^2 A_n}{d\tau^2} - 3 \tanh\tau \frac{dA_n}{d\tau} + \frac{k^2}{\cosh^2 \tau} A_n \simeq 0. \tag{4.3}$$

[5]In the late-time dustlike era when $\dot{\varphi}$ and $\varphi \propto \phi$ go periodically through zero, but H remains positive, another well-behaved form is required and was given in [10].

[6]Indeed, the potentially dangerous term $(\Phi_n/\phi_0^2)(k^2/a^2\phi^2)$ remains small in comparison with $(k^2/a^5\phi)A_n \approx (k^4/a^4\phi^2)\Phi_n$ for large k^2.

Recalling that $k^2 = (n-1)(n+3)$, the general solution of Eq. (4.3) is a sum of even and odd functions:

$$A_n = \alpha_n A_n^{(1)} + \beta_n A_n^{(2)}, \tag{4.4}$$

where (α_n, β_n) are constants of integration, and for $\tau > 0$:

$$A_n^{(1)} = (\cosh^3 \tau) F\left[-\frac{n+2}{2}, \frac{n}{2}, -\frac{1}{2}, \frac{1}{\cosh^2 \tau} \right],$$
$$A_n^{(2)} = F\left[\frac{1-n}{2}, \frac{n+3}{2}, \frac{5}{2}, \frac{1}{\cosh^2 \tau} \right], \tag{4.5}$$

for $\tau < 0$:

$$A_n^{(1)} = \pm (\cosh^3 \tau) F\left[-\frac{n+2}{2}, \frac{n}{2}, -\frac{1}{2}, \frac{1}{\cosh^2 \tau} \right],$$
$$A_n^{(2)} = \mp F\left[\frac{1-n}{2}, \frac{n+3}{2}, \frac{5}{2}, \frac{1}{\cosh^2 \tau} \right], \tag{4.6}$$

where the upper vs the lower signs hold for n even vs n odd, and where $F[a, b, c, x]$ is the hypergeometric function (usually denoted $2F1[a, b, c, x]$).

The function A_n being known, the scalar perturbation Φ_n follows from (4.1) and (4.2) and the approximate background solution (2.7). It reads

$$\Phi_n = \alpha_n \Phi_n^{(1)} + \beta_n \Phi_n^{(2)} \quad \text{with}$$
$$\Phi_n^{(1,2)} \simeq -\frac{1}{(n-1)(n+3)} \frac{1}{\cosh\tau} \left(A_n^{(1,2)} + z \frac{dA_n^{(1,2)}}{d\tau} \right) \tag{4.7}$$
$$= -\frac{1}{(n-1)(n+3)} \sqrt{x} \left\{ A_n^{(1,2)} + \frac{2}{3} \times \frac{dA_n^{(1,2)}}{dx} x\sqrt{1-x} [\sqrt{1-x}(1+2x) \mp 3 z_0 x^{3/2}] \right\},$$

where $x \equiv a^{-2} \simeq 1/\cosh^2 \tau$ and where the upper vs the lower signs hold for $(\tau > 0)$ vs $(\tau < 0)$. At the bounce:

$$\Phi_{n|\tau=0}^{(1)} \simeq -\frac{1}{(n-1)(n+3)} \times \left[(n+1) \cos\frac{n\pi}{2} + z_0 n(n+2) \sin\frac{n\pi}{2} \right],$$
$$\Phi_{n|\tau=0}^{(2)} \simeq -\frac{3}{(n-1)(n+3)} \left[\frac{\sin\frac{n\pi}{2}}{n(n+2)} - \frac{z_0}{n+1} \cos\frac{n\pi}{2} \right], \tag{4.8}$$

$$g_{n|\tau=0}^{(1)} = -n(n+2) \sin\frac{n\pi}{2}, \qquad g_{n|\tau=0}^{(2)} = \frac{3}{n+1} \cos\frac{n\pi}{2}. \tag{4.9}$$

Writing down the explicit expression of Φ_n in terms of hypergeometric functions is not particularly illuminating: suffice it to say that it is a good approximation of the numerical solution of the exact Eqs. (2.3) and (3.4) in the range $|\tau| \ll \phi_0$, and that it tends to constants for large $|\tau|$ (in practice $|\tau|$ bigger than a few unities) and oscillates in the bouncing region more and more as n grows bigger.

We are now in a position to relate the pre- and postbounce spectra, at the order considered, that is the lowest in the slow-roll parameter $1/\phi_0^2$.

Using the following asymptotic expansions of hypergeometric functions,

$$F\left[\frac{1-n}{2}, \frac{n+3}{2}, \frac{5}{2}, x\right] = 1 + \mathcal{O}(x),$$
$$F\left[-\frac{n+2}{2}, \frac{n}{2}, -\frac{1}{2}, x\right] = 1 + \frac{n(n+2)}{2}x + \mathcal{O}(x^2), \tag{4.10}$$

the asymptotic behaviors of Φ_n (in practice for $|\tau|$ bigger than a few unities) are readily obtained from (4.7) (recalling that $x \simeq a^{-2}$):

$$\text{postbounce region}: \Phi_n \sim G^n_{\text{post}} + \frac{D^n_{\text{post}}}{a} \quad \text{with}$$
$$\begin{cases} G^n_{\text{post}} &= -\frac{\alpha_n}{3}, \\ D^n_{\text{post}} &= -\frac{\beta_n + 3z_0\alpha_n}{(n-1)(n+3)}, \end{cases} \tag{4.11}$$

$$\text{prebounce region}: \Phi_n \sim D^n_{\text{pre}} + \frac{G^n_{\text{pre}}}{a} \quad \text{with}$$
$$\begin{cases} D^n_{\text{pre}} &= \mp\frac{\alpha_n}{3}, \\ G^n_{\text{pre}} &= \pm\frac{\beta_n + 3z_0\alpha_n}{(n-1)(n+3)}, \end{cases} \tag{4.12}$$

where the upper vs the lower signs hold for even vs odd n.[7]

At lowest order in the slow-roll parameter $1/\phi_0^2$, the pre- and postbounce spectra are thus very simply related:

$$\begin{pmatrix} G^n_{\text{post}} \\ D^n_{\text{post}} \end{pmatrix} = \begin{pmatrix} 0 & \pm 1 \\ \mp 1 & 0 \end{pmatrix} \begin{pmatrix} G^n_{\text{pre}} \\ D^n_{\text{pre}} \end{pmatrix} + \mathcal{O}(1/\phi_0^2). \tag{4.13}$$

In the toy model and at the approximation considered here, the prebounce spectrum of the growing mode is carried over to the postbounce decaying mode (and vice versa). In other words, the prebounce spectrum of the cosmologically interesting growing mode is carried over to the postbounce decaying mode, and hence soon lost. On the other hand, the postbounce surviving growing mode inherits the prebounce decaying mode spectrum, which, usually, is unfortunately blue [1,2,4].

There is however a case when the result does not hold, that is $\alpha_n = 0$ when the prebounce decaying mode has become insignificant when reaching the bouncing region. This is a case of physical interest, to which we now turn.

[7]Note that the notation used in (4.11) and (4.12) takes into account the fact that the constant, D^n_{pre}, to which the perturbation tends when $t \to -\infty$ is the prebounce decaying mode, whereas the constant, G^n_{post}, to which the perturbation tends when $t \to +\infty$ is the postbounce growing mode.

V. VALIDITY OF THE APPROXIMATION AND NEXT ORDER IN $1/\phi_0^2$

Let us first look at the structure of the "transfer matrix" at next order in the slow-roll parameter. Including $1/\phi_0^2$ corrections will yield

$$\begin{pmatrix} G^n_{\text{post}} \\ D^n_{\text{post}} \end{pmatrix} \simeq \begin{pmatrix} c(n)/\phi_0^2 & \pm 1 \\ \mp 1 & d(n)/\phi_0^2 \end{pmatrix} \begin{pmatrix} G^n_{\text{pre}} \\ D^n_{\text{pre}} \end{pmatrix} \tag{5.1}$$

and will not change significantly the zeroth order result, if D^n_{pre} and G^n_{pre} are of the same order of magnitude. If, now, the initial conditions on D^n_{pre} and G^n_{pre} are such that

$$D^n_{\text{pre}} \ll \frac{c(n)}{\phi_0^2} G^n_{\text{pre}}, \tag{5.2}$$

which, in view of (4.12), is the case when α_n is vanishingly small so that the prebounce decaying modes have become insignificant when reaching the bouncing region, then the pre- and postbounce spectra become related by

$$G^n_{\text{post}} \simeq \frac{c(n)}{\phi_0^2} G^n_{\text{pre}}, \qquad D^n_{\text{post}} \simeq \mp G^n_{\text{pre}}. \tag{5.3}$$

The question is then to find the dependence of $c(n)$ on n: if $c(n)$ turns out NOT to depend on n, then the spectrum of the prebounce growing mode goes through the bounce unmodified. If, on the other hand, $c(n)$ turns out to depend on n, then the spectrum is modified by the bounce. This is the physically relevant problem studied numerically in [4] where G^n_{post} is found to be nonzero even when $D^n_{\text{pre}} = 0$. The authors of [4] did not however give the n dependence of the constant $c(n)$ and hence left open the question of how G^n_{post} was related to G^n_{pre}.

The constant $c(n)$ can however be estimated as follows.

The exact equations for the perturbations are (3.4) where the background functions solve (2.3). In the previous Section we solved them at zeroth order in $1/\phi_0^2$; that is we ignored the $1/\phi_0^2$ term in (3.4) and used for the background functions the zeroth order approximation (2.7). To consistently iterate them at next order one should

(1) keep the $1/\phi_0^2$ term in (3.4), replacing Φ_n by the zeroth order solution;
(2) use for the background functions the first order approximation of (2.3).

To estimate $c(n)$, we shall however ignore step 2, for the following reason: treating the background at first order in the slow-roll parameter introduces logarithmic corrections in the solutions of the perturbation equation which render more difficult the numerical extraction of the postbounce growing mode. Of course, the difficulty can be overcome,[8] but is perhaps not worth the effort as it seems a reasonable guess that the inclusion of the $1/\phi_0^2$

[8]Indeed, the task of integrating numerically the background and the well-behaved perturbation equations (3.4) is in principle straightforward.

correction for the background should not affect the n dependence of the perturbation spectrum.

We therefore integrated numerically the set of Eqs. (3.4) where, in the $1/\phi_0^2$ term, we replaced Φ_n by the prebounce purely growing zeroth order solution, that is $\Phi_n^{(2)}$, see (4.5), (4.6), and (4.7), but where we used for the background functions the zeroth order approximation (2.7).

We chose the initial conditions at the bounce, given by (4.8) and (4.9) with $\alpha_n = 0$, $\beta_n = 1$. With those initial conditions, integration yields a solution $\Phi_n^{\rm iter}$ which, when $\phi_0^2 \to \infty$, is nothing but the analytical solution $\Phi_n^{(2)}$ obtained in the previous Section; that is a mode which is exponentially growing before the bounce and exponentially decaying after the bounce. For a large but finite value for ϕ_0^2 on the other hand, $\Phi_n^{\rm iter}$ no longer vanishes in the asymptotic regions but tends to small constants, $(c_{\rm post}^n, c_{\rm pre}^n)$, which scale, as they should, as $1/\phi_0^2$. Let us therefore introduce the rescaled constants ($C_{\rm post}^n \equiv \phi_0^2 c_{\rm post}^n$, $C_{\rm pre}^n \equiv \phi_0^2 c_{\rm pre}^n$) which are independent of ϕ_0^2.

Consider now the linear combination

$$\Phi_n^{\rm shooting} = \Phi_n^{\rm iter} \pm 3\frac{C_{\rm pre}^n}{\phi_0^2}\Phi_n^{(1)}, \tag{5.4}$$

where the upper vs the lower signs hold for n even vs n odd. Since $\Phi_n^{(1)} \to \mp 1/3$ in the prebounce asymptotic region [see (4.12)], we have

$$\text{prebounce region}: \Phi_n^{\rm shooting} \to 0 \tag{5.5}$$

so that $\Phi_n^{\rm shooting}$ is the purely prebounce growing mode at the approximation considered. In the postbounce region on the other hand [see (4.11)]:

$$\text{postbounce region}: \Phi_n^{\rm shooting} \to \frac{C_{\rm post}^n \mp C_{\rm pre}^n}{\phi_0^2}. \tag{5.6}$$

Therefore, from (4.12) and (5.3):

$$G_{\rm post}^n \simeq T_n G_{\rm pre}^n \quad \text{with } T_n = \pm\frac{k^2}{\phi_0^2}(C_{\rm post}^n \mp C_{\rm pre}^n). \tag{5.7}$$

Numerical integration gives, per each n [and each value of the parameter z_0 entering the background zeroth order solution (2.7)] the values of the constants $C_{\rm post}^n$ and $C_{\rm pre}^n$ and it turns out, quite remarkably, that, for large n, T_n does not depend on n. Hence the prebounce spectrum encoded in $G_{\rm pre}^n$ is nicely carried over to the postbounce growing mode $G_{\rm post}^n$, although with an amplitude reduced by the overall factor $1/\phi_0^2$.

VI. CONCLUDING REMARKS

In the very simple toy model of a bouncing universe that we studied in this paper, and at lowest order in the slow-roll parameter, the spectrum of the prebounce growing mode is carried over through the bounce to the decaying postbounce mode. That analysis however breaks down if the prebounce decaying mode is negligible in the prebounce region. In that case the analysis must be pushed to next order in the slow-roll parameter with the neat indication that the large n postbounce growing mode inherits without distortion the prebounce growing mode spectrum.

These results do not depend on the value of $\dot\phi$ at the bounce. (Indeed numerical integration indicates that the value of z_0 affects only the overall amplitude of the transfer factor T_n.)

It would be surprising if they depended crucially on the particular potential [$V(\varphi) = \frac{1}{2}m^2\varphi^2$] chosen for the scalar field, as long as there exist quasi-de Sitter regimes before and after the bounce. They should not be spoiled either when treating more carefully the logarithmic corrections, but that point certainly deserves further attention.

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[1] G. Veneziano, Phys. Lett. B **265**, 287 (1991); M. Gasperini and G. Veneziano, Phys. Rep. **373** (2003), and references therein.

[2] J. Khoury, B. A. Ovrut, P. J. Steinhardt, and N. Turok, Phys. Rev. D **64**, 123522 (2001); J. Khoury, B. A. Ovrut, N. Seiberg, P. J. Steinhardt, and N. Turok, Phys. Rev. D **65**, 086007 (2002); N. Turok and P. J. Steinhardt, Phys. Rev. D **65**, 126003 (2002); J. Khoury, B. A. Ovrut, P. J. Steinhardt, and N. Turok, Phys. Rev. D **66**, 0406005 (2002), and references therein.

[3] R. Brustein, M. Gasperini, M. Giovaninni, V. Mukhanov, and G. Veneziano, Phys. Rev. D **51**, 6744 (1995); M. Gasperini, N. Giovaninni, and G. Veneziano, Phys. Lett. B **259**, 113 (2003); R. Brandenberger and F. Finelli, J. High Energy Phys. 11 (2001) 056; R. F. Finelli and R. Brandenberger, Phys. Rev. D **65**, 103522 (2002); A. J. Tolley and N. Turok, Phys. Rev. D **66**, 106005 (2002); D. H. Lyth, Phys. Lett. B **524**, 1 (2002); J. C. Hwang, Phys. Rev. D **65**, 063514 (2002); S. Tsujikawa, Phys. Lett. B **526**, 179 (2002); P. Peter and N. Pinto-Neto, Phys. Rev. D **65**, 123513 (2001); **66**, 063509 (2002); J. Martin, P. Peter, N. Pinto-Neto, and D. J. Schwarz, Phys. Rev. D **65**, 123513 (2002); **67**, 028301 (2003); hep-th/0204227; J. C. Fabris, R. G. Furtado, P. Peter, and N. Pinto-Neto, Phys. Rev. D **67**, 124003 (2003); P. Peter and N. Pinto-Neto, hep-th/0306005; J. Martin and P. Peter, Phys. Rev. D **68**, 103517 (2003); R. Durrer and F. Vernizzi, hep-th/

0203375; L. Allen and D. Wands, astro-ph/0404441 [Phys. Rev. D (to be published)].
[4] C. Gordon and N. Turok, Phys. Rev. D **67**, 123508 (2003).
[5] J. Hwang and H. Noh, Phys. Rev. D **65**, 124010 (2002).
[6] H. Nariai, Prog. Theor. Phys. **46**, 433 (1971); H. Nariai and K. Tomita, Prog. Theor. Phys. **46**, 776 (1971); L. Parker and S. A. Fulling, Phys. Rev. D **7**, 2357 (1973); H. Nariai, Prog. Theor. Phys. **51**, 613 (1974); A. A. Starobinski, Pis'ma Astron. Zh. **4**, 155 (1978); V. N. Melnikov and S.V. Orlov, Phys. Lett. **70A**, 263 (1979); J. D. Barrow and R. A. Matzner, Phys. Rev. D **21**, 336 (1980); J. B. Hartle and S.W. Hawking, Phys. Rev. D **28**, 2960 (1983); S.W. Hawking and J. C. Luttrell, Nucl. Phys. **B247**, 250 (1984); D. N. Page, Classical Quantum Gravity **1**, 417 (1984); V. A. Belinsky, L. P. Grishchuk, Ya. B. Zel'dovich, and I. M. Khalatnikov, J. Exp. Theor. Phys. **89**, 346 (1985); M. Visser, Phys. Lett. B **349**, 443 (1995); A. Yu Kamenshchik, I. M. Khalatnikov, and A.V. Toporensky, Int. J. Mod. Phys. D **6**, 673 (1997); **7**, 129 (1998); I. M. Khalatnikov and A. Yu Kamenshchik, Phys. Rep. **288**, 513 (1997); C. Molina-Paris and M. Visser, Phys. Lett. B **455**, 90 (1999); S. Gratton, A. Lewis, and N. Turok, Phys. Rev. D **65**, 043513 (2002).
[7] N. J. Cornish and E. P. S. Shellard, Phys. Rev. Lett. **81**, 3571 (1998).
[8] H. J. Schmidt, Astron. Nachr. **311**, 99 (1990).
[9] J. M. Bardeen, Phys. Rev. D **22**, 1882 (1980); M. Sasaki, Prog. Theor. Phys. **70**, 394 (1983); V. F. Mukhanov, H. A. Feldman, and R. H. Brandenberger, Phys. Rep. **215**, 203 (1992); J. M. Stewart, Classical Quantum Gravity **7**, 1169 (1990).
[10] N. Deruelle, C. Gundlach, and D. Polarski, Classical Quantum Gravity **9**, 137 (1992).