Accelerated cosmological models in Ricci squared gravity

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Alternative gravitational theories described by Lagrangians depending on general functions of the Ricci scalar have been proven to give coherent theoretical models to describe the experimental evidence of the acceleration of the Universe at present time. In this paper we proceed further in this analysis of cosmological applications of alternative gravitational theories depending on (other) curvature invariants. We introduce Ricci squared Lagrangians in minimal interaction with matter (perfect fluid); we find modified Einstein equations and consequently modified Friedmann equations in the Palatini formalism. It is striking that both Ricci scalar and Ricci squared theories are described in the same mathematical framework and both the generalized Einstein equations and generalized Friedmann equations have the same structure. In the framework of the cosmological principle, without the introduction of exotic forms of dark energy, we thus obtain modified equations providing values of $w_{\text{eff}} < -1$ in accordance with the experimental data. The spacetime bi-metric structure plays a fundamental role in the physical interpretation of results and gives them a clear and very rich geometrical interpretation.

I. INTRODUCTION

In this paper we try to better understand and to analyze alternative theories of gravity depending on higher-order terms in the curvature invariant $R^{(\mu\nu)}R_{(\mu\nu)}$, in relation with some very interesting and possible cosmological application and, in particular, in relation with their capability to explain the cosmological acceleration of the Universe, both in early times (inflation) and in the present time Universe. Nevertheless we will focus our attention on the possible theoretical explanation of the present cosmological acceleration.

Recent astronomical observations have shown that the Universe is accelerating at present time (see [1,2] for supernova observation results; see [3] for the observations about the anisotropy spectrum of the cosmic microwave background (CMB); see [4] for the results about the power spectrum of large-scale structure). Physicists have thus to face the evidence of the acceleration of the Universe and should give a coherent theoretical explanation to these experimental results: a problem which up to now seems to be still unsolved! General relativity in interaction with a perfect-fluid-like matter and the cosmological principle, providing the standard cosmological models, fail to give by their own a theoretical framework to explain the acceleration of the Universe. We are thus forced to introduce some kind of *dark matter or dark energy*, which are responsible for the acceleration of the Universe, or to modify general relativity such that acceleration is predicted (see for example [5]).

Dark matter or dark energy models have been deeply investigated in relation with their capability of explaining the acceleration of the Universe (see [6] and references therein), however up to now there are no satisfactory experimental evidences of the presence of the predicted amount of dark energy in the Universe. The real nature of dark energy, which is required by general relativity in this cosmological context, is unknown but it is fairly well accepted that dark energy should behave like a fluid with a large negative pressure. The dark energy models with effective equation of state w_{eff} (which determines the relation between pressure p and density of matter ρ) smaller than $w_{\text{eff}} < -1$ are currently preferable, owing to the experimental results of [3].

On the other side the simplest way of obtaining accelerated expansions within general relativity is to introduce a positive cosmological constant [7], an introduction which leads however to some theoretical and experimental problems and contradictions (see for example [5,7]). We just want to stress here that models with a constant cosmological constant are not able to explain the evolution between different epochs of the Universe, characterized by different values of acceleration (deceleration).

The other possibility is to assume that we do not yet understand gravity at large scales, which means that general relativity should be modified or replaced by alternative gravitational theories of gravity when the curvature of spacetime is small (see for example [8–10] and references therein), providing modified Friedmann equations. Hints in this direction are suggested moreover from the quantization on curved spacetimes, when interactions among the quantum fields and the background geometry or the self-interaction of the gravitational field are considered. It follows that the standard Hilbert-Einstein

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Lagrangian has to be suitably modified by means of corrective terms, which are essential in order to remove divergences [8]. These corrective terms result to be higher order terms in the curvature invariants such as R, $R^{\mu\nu}R_{\mu\nu}$, $R^{\mu\nu\alpha\beta}R_{\mu\nu\alpha\beta}$, $R\square^l R$, or nonminimally coupled terms between scalar fields and the gravitational field. It is moreover interesting that such corrective terms to the standard Hilbert-Einstein Lagrangian can be predicted in higher dimensions by some time-dependent compactification in string/M-theory (see [9]) and corrective terms of this type arise surely in brane-world models with large spatial extra dimensions [10]. As a matter of fact, if these brane models are the low energy limit of string theory, it is likely that the field equations include, in particular, the Gauss-Bonnet term, which in five dimensions is the only nonlinear term in the curvature which yields second order field equations. In this framework Gauss-Bonnet corrections should be taken into account and cosmological models deriving from the Gauss-Bonnet have been recently studied; see [11] and references therein.

As an alternative to extra dimensions, it is also possible to explain the modification to Friedmann equations (which could provide a theoretical explanation for the acceleration of the Universe) by means of a modified theory of four dimensional gravity. The first attempts in this direction were performed by adding to the standard Hilbert-Einstein Lagrangian analytical terms in the Ricci scalar curvature invariant [12]. A simple task to modify general relativity, when the curvature is very small, is hence to add to the Lagrangian of the theory a piece which is proportional to the inverse of the scalar curvature $\frac{1}{R}$ or to replace the standard Hilbert-Einstein action by means of polynomial-like Lagrangians, containing both positive and negative powers of the Ricci scalar R and logarithmic-like terms. Such theories have been analyzed and studied both in the metric [13] and the Palatini formalisms [14,15]. It results that both in the metric and the Palatini formalism they provide a possible theoretical explanation to the present time acceleration of the Universe. Moreover a mechanism ruling the present dark energy dominance[1] (due to the Universe expansion) and the present cosmic acceleration has been proposed in this framework; see [17].

A discussion is open on the physical reliability of the Palatini and/or the metric formalism [18] and on the physical relevant frame both in the metric and the Palatini formalism [19]. Up to now it appears that the aforementioned metric approach[2] leads to results which are in contrast with the solar system experiments and also

that the relevant fourth order field equations suffer serious instability problems [13]. On the contrary the Palatini formalism produces second order field equations which are not afflicted by instability problems and are in acceptable accordance with the results of the solar system experiments [15]. A discussion is actually open on the accordance of the Palatini formalism with the electron-electron scattering experiments [18].

The importance of modified theories of gravity depending on general analytical functions of the Ricci scalar is also related with the possibility of avoiding singularities in these cosmological models [20] and in the interpretation of black holes entropy in this context [21]. Recently, an explanation of the present day acceleration of the Universe has been moreover formulated in the framework of nonsymmetric gravitational theories [22] and in modified theories depending on the determinant of the Ricci tensor [23].

Encouraged by recent developments of cosmological applications of alternative theories of gravity we consider in this paper Ricci squared Lagrangians in minimal interaction with matter, which have been deeply analyzed in [24] in the vacuum case. As we already said before such Lagrangians are deeply related with quantum field theory: to remove divergences one has to add counterterms to the Lagrangian which depend not only on the Ricci scalar but also on the Ricci and the Riemann tensors [8]. It was proven in [24] that Ricci squared Lagrangians provide second order field equations in the Palatini formalism, such that the universality of Einstein equations and the universality of the Komar energy-momentum complex hold in vacuum. These remarkable results have important implications also in cosmological models. They imply that, in some sense, field equations for Ricci squared Lagrangians reproduce (apart from conformal transformations) the standard Einstein field equations in the vacuum case, while in the presence of matter this equivalence might be broken. The geometrical structure of the spacetime manifold is very rich and it is endowed with an anti-Kählerian structure, deriving directly from the variational principle of Ricci squared Lagrangians (see [25]). Spacetime turns out to have a bi-metric structure, or better a so-called *metric compatible almost-product* as well as an *almost-complex structure with a Norden metric*. The geometrical structure of spacetime is moreover characterized by a scalar-valued structural equation, which is simply obtained by contracting field equations with the metric [24] and controls the solutions of field equations. Lagrangians based on higher order Ricci scalars which led to higher order metric compatible polynomial structures have been considered both in purely metric and Palatini formalism in [26].

It was moreover shown in [25,27] that Ricci squared theories in vacuum give field equations equivalent to Einstein field equations with a cosmological constant,

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[1]Taking into account the transition of the Universe from a decelerated era to an accelerated era, a scenario with w_{eff} varying from values below -1 to values above -1 is actually preferable [16].

[2]We remark that field equations are in that case fourth order field equations.

the value of which is fixed once the structural equations are solved and one particular solution of the structural equations is chosen. This fact is no longer true in the case of interaction with generic matter, where the solutions of the structural equation are dynamical (the same happens in the case of nonlinear Lagrangians in the Ricci scalar; see [14] and references therein). The equivalence with Einstein field equations is hence broken and we obtain modified field equations, depending on the stress-energy tensor of matter involved in the theory. Nevertheless, we remark that field equations are once again second order field equations in the metric field.

We remark however that the theories studied in this paper should be mainly understood as corrections to Einstein's general relativity (in the vein of Starobinsky original paper [8]), in order not to conflict with the successes of general relativity at the smaller noncosmological scales. In particular, the S^n Lagrangian term we study in detail (and more generally the $f(S)$ Lagrangians), with S the Ricci squared invariant, should be suppressed relative to the linear curvature term by the power of a very small mass scale. More general Lagrangians, involving both terms in the Ricci scalar invariant and in the Ricci squared invariant, will be studied in forthcoming papers.

For cosmological applications we consider the physical metric g to be a Robertson-Walker metric and the stress-energy tensor of matter to be a perfect-fluid one. In this particular framework deriving from the cosmological principle we obtain that the Levi-Civita metric h is *conformal* to the physical metric g, apart for a rescaling factor of the cosmological time. From our construction it follows however that the signature of h can be arbitrarily chosen (it can be either Riemannian, or Lorentzian, or Kleinian, apart from some restrictions deriving from field equations). We are consequently able to introduce a generalized Hubble constant and modified Friedmann equations. We remark that cosmological models of this type, providing similar field equations, have been already studied in different frameworks in the past, see, e.g., [28].

A comparison with the $f(R)$ theories is immediate. It is striking to notice that modified Friedmann equations are once more first order field equations, which prevent the appearing of instabilities as it has already been shown in the case of Ricci scalar theories in [15]. This is an important consideration giving the Palatini formalism a deeper physical significance, in view of cosmological applications.

An explicit example dealing with power Lagrangians in the Ricci squared invariant $f(S) = \beta S^n$ is analyzed in detail and the Hubble constant is derived. It results that the deceleration parameter can be negative if particular values of n are chosen. Moreover we obtain values of w_{eff} that can be suitably fitted to the experimental results of

[1]. Considerations are exposed about the *frame changing*, which means choosing h to be a Friedmann-Robertson-Walker (FRW) metric instead of g. Field equations and cosmological parameters are obtained and discussed also in that alternative (Jordan) frame.

The paper is organized as follows: we start in Sec. II by considering the case of $f(R)$ Lagrangians in a new matrix formalism, with the introduction of an operator P modifying the Einstein field equations [14]. We pass in Sec. III to the more complicated case of Ricci squared $f(S)$ Lagrangians and we analyze the field equations and the structure of spacetime in the case of interaction with matter. We proceed in Sec. IV with cosmological applications and we obtain modified Friedmann equations. In Sec. V we discuss the relevant example of polynomial Lagrangians in the Ricci squared invariant. In Sec. VI we consider the theory in the alternative Jordan frame, where h is assumed to be *a priori* the FRW physical metric.

II. COSMOLOGICAL MODELS IN $f(R)$ GRAVITY

We start considering nonlinear Lagrangians in the Ricci scalar invariant $f(R)$, already treated and developed in [14] and in [29] in the vacuum case. We think that it is worth summarizing those theories in order to have a comparison with Ricci squared theories here developed and analyzed in detail. We moreover modify the formalism introduced in [14] to treat both Ricci scalar and Ricci squared theories in the same mathematical framework. The action for $f(R)$ gravity is introduced to be:

$$A = A_{\text{grav}} + A_{\text{mat}} = \int [\sqrt{\det g} f(R) + 2\kappa L_{\text{mat}}(\Psi)] d^4 x \quad (1)$$

where $R \equiv R(g, \Gamma) = g^{\alpha\beta} R_{\alpha\beta}(\Gamma)$ is the *generalized Ricci scalar* and $R_{\mu\nu}(\Gamma)$ is the Ricci tensor of a torsionless connection Γ. The gravitational part of the Lagrangian is controlled by a given real analytic function of one real variable $f(R)$, while $\sqrt{g}$ denotes the scalar density $|\det\| g_{\mu\nu} \||^{1/2}$ of weight 1. The total Lagrangian contains also a matter part L_{mat} in minimal interaction with the gravitational field, depending on matter fields Ψ together with their first derivatives and equipped with a gravitational coupling constant $\kappa = 8\pi G$. Equations of motion, ensuing from the first order á la Palatini formalism are (we assume the spacetime manifold to be a Lorentzian manifold M with $\dim M = 4$; see [29]):

$$f'(R) R_{(\mu\nu)}(\Gamma) - \frac{1}{2} f(R) g_{\mu\nu} = \kappa T_{\mu\nu} \quad (2)$$

$$\nabla_\alpha^\Gamma [\sqrt{\det g} f'(R) g^{\mu\nu}] = 0 \quad (3)$$

where $T_{\mu\nu} = -\frac{2}{\sqrt{g}} \frac{\delta L_{\text{mat}}}{\delta g_{\mu\nu}}$ denotes the matter source stress-energy tensor and ∇^Γ means covariant derivative with respect to Γ.

We shall use the standard notation denoting by $R_{(\mu\nu)}$ the symmetric part of $R_{\mu\nu}$, i.e., $R_{(\mu\nu)} \equiv \frac{1}{2}(R_{\mu\nu} + R_{\nu\mu})$. In order to get (3) one has to additionally assume that L_{mat} is functionally independent of Γ; however it may contain metric covariant derivatives ∇^g of fields. This means that the matter stress-energy tensor $T_{\mu\nu} = T_{\mu\nu}(g, \Psi)$ depends on the metric g and some matter fields denoted here by Ψ, together with their derivatives. From (3) one sees that $\sqrt{\det g} f'(R) g^{\mu\nu}$ is a symmetric twice contravariant tensor density of weight 1, so that if not degenerate one can use it to define a metric $h_{\mu\nu}$ such that the following holds true

$$\sqrt{\det g} f'(R) g^{\mu\nu} = \sqrt{\det h} h^{\mu\nu}. \qquad (4)$$

This means that both metrics h and g are conformally equivalent. The corresponding conformal factor can be easily found to be $f'(R)$ (in $\dim M = 4$) and the conformal transformation results to be:

$$h_{\mu\nu} = f'(R) g_{\mu\nu} \qquad (5)$$

Therefore, as it is well-known, Eq. (3) implies that $\Gamma = \Gamma_{LC}(h)$ and $R_{(\mu\nu)}(\Gamma) = R_{\mu\nu}(h) \equiv R_{\mu\nu}$. Let us now introduce a $(1, 1)$-tensorfield P by

$$P^\mu_\nu = g^{\mu\alpha} R_{\alpha\nu}(h) \qquad (6)$$

so that (2) rewrites as

$$f'(R) P^\nu_\mu - \frac{1}{2} f(R) \delta^\nu_\mu = \kappa \hat{T}^\nu_\mu \qquad (7)$$

where, with an abuse of notation, $\hat{T} = \hat{T}^\nu_\mu = g^{\mu\alpha} T_{\alpha\nu}$ and from (6) we obtain that $R = \text{tr} P$. Equation (7) can be supplemented by the scalar-valued equation obtained by taking the trace of (7), (we define $\tau = \text{tr}\hat{T}$)

$$f'(R) R - 2f(R) = \kappa g^{\alpha\beta} T_{\alpha\beta} \equiv \kappa\tau \qquad (8)$$

which controls solutions of (7). We shall refer to this scalar-valued equation as the *structural equation* of spacetime. The structural Eq. (5), if explicitly solvable, provides an expression of $R = F(\tau)$ and consequently both $f(R)$ and $f'(R)$ can be expressed in terms of τ. More precisely, for any real solution $R = F(\tau)$ of (8) one has that the operator P can be obtained from the matrix Eq. (7):

$$P = \frac{f[F(\tau)]}{2f'[F(\tau)]} I + \frac{\kappa}{f'[F(\tau)]} \hat{T}. \qquad (9)$$

Now we are in position to introduce the generalized Einstein equations under the form

$$g_{\mu\alpha} P^\alpha_\nu = R_{\mu\nu}(h) \qquad (10)$$

where $h_{\mu\nu}$ is given by (5) and P^μ_ν is obtained from the algebraic Eqs. (8) and (9) (for a given $g_{\mu\nu}$ and $T_{\mu\nu}$); see also [14,29]. For the matter-free case we find that $R = F(0)$ becomes a constant implying that the two metrics are proportional and the operator P is proportional to the

Kronecker delta. Equation (10) is hence nothing but Einstein equation for the metric g, almost independently on the choice of the function $f(R)$, as already obtained in [29]. Also the standard Einstein equation with a cosmological constant Λ can be recast into the form (10). It corresponds to the choice $f(R) = R - \Lambda$. These properties justify the name of generalized Einstein equation given to (10). In the presence of matter Eq. (10) expresses a deviation for the metric g to be an Einstein metric as it was discussed in [14]. It can be otherwise interpreted as an Einstein equation with additional stress-energy contributions deriving from the modified gravitational Lagrangian [15], or possibly as a modified theory of gravity with a time-dependent cosmological constant.

A. Cosmological applications of first order nonlinear gravity

We give here a brief summary of the results obtained in [14], where we refer the reader for further details. We assume g to be the FRW metric which (in spherical coordinates) takes the standard form:

$$g = -dt^2 + a^2(t)\left\{\frac{1}{1 - Kr^2} dr^2 + r^2[d\theta^2 + \sin^2(\theta)d\varphi^2]\right\} \qquad (11)$$

where $a(t)$ is the so-called *scale factor* and K is the space curvature ($K = 0, 1, -1$). We further choose a perfect-fluid stress-energy tensor for matter:

$$T_{\mu\nu} = (\rho + p)u_\mu u_\nu + p g_{\mu\nu} \qquad (12)$$

where p is the pressure, ρ is the density of matter, and u^μ is a comoving fluid vector, which in a comoving frame $[u^\mu = (1, 0, 0, 0)]$ becomes simply:

$$T_{\mu\nu} = \begin{pmatrix} \rho & 0 & 0 & 0 \\ 0 & \frac{pa^2(t)}{1 - Kr^2} & 0 & 0 \\ 0 & 0 & pa^2(t)r^2 & 0 \\ 0 & 0 & 0 & pa^2(t)r^2\sin^2(\theta) \end{pmatrix}. \qquad (13)$$

The metric h turns out to be conformal to the FRW metric g by means of the conformal factor $f'(R)$, which can be moreover expressed in terms of τ by means of (8) and finally as a function of time $b(t) = f'[R(\tau)]$, by an abuse of notations. From (10) we can obtain an analogue of the Friedmann equation under the form

$$\hat{H} = \left(\frac{\dot{a}}{a} + \frac{\dot{b}}{2b}\right)^2 = \frac{\kappa}{3b}\left[\rho + \frac{f(\tau) + \kappa\tau}{2\kappa}\right] - \frac{K}{a^2} \qquad (14)$$

which can be seen as a generalized definition of a *modified Hubble constant* $\hat{H} = (\frac{\dot{a}}{a} + \frac{\dot{b}}{2b})$, taking into account the presence of the conformal factor $b(t)$ which enters into the definition of the conformal metric h (see [14] for details). This equation reproduces, as expected, the standard Einstein equations in the case $f(R) = R$.

Considering the particular example $f(R) = \beta R^n$ we have obtained that the Hubble constant for the metric g can be locally calculated to be:

$$H^2 = \varepsilon r(n, w) a^{-3(w+1)/n} - s(n, w) \frac{K}{a^2} \quad (15)$$

where:

$$\begin{cases} r(n, w) = \frac{2n}{3(3w-1)[3w(n-1)+(n-3)]} \left[\frac{-\kappa(3w-1)}{\beta(2-n)} \right]^{1/n} \\ s(n, w) = \left[\frac{2n}{3w(n-1)+(n-3)} \right]^2 \end{cases}$$

are functions of the power n and of the equation state of matter, through w. We remark that $\varepsilon = \text{sgn}R = 1$ for odd values of n and, on the contrary, $\varepsilon = \pm 1$ for even values of n; see [14] for details. The deceleration parameter can be obtained from the Hubble constant by means of the following relation:

$$q(t) := -\left(1 + \frac{\dot{H}(t)}{H^2(t)} \right) = -\left(\frac{\ddot{a}(t)}{a(t)H^2(t)} \right) \quad (16)$$

and from (15) it turns out to be formally equal to:

$$q(t, w, n) = -1 \\ + \frac{\frac{3(1+w)}{2n} \epsilon r(n, w) a^{-[3(1+w)/n]} - s(n, w) K a^{-2}}{\epsilon r(n, w) a^{-[3(1+w)/n]} - s(n, w) K a^{-2}}. \quad (17)$$

It follows that when the a^{-2} term dominates over $a^{-[3(1+w)/2n]}$ the deceleration parameter results to be positive, i.e., $q(t, w, n) \to 0$. On the contrary, when the term $a^{-[3(1+w)/n]}$ dominates over a^{-2} (or in the case $K = 0$ corresponding to spatially flat spacetime) the deceleration parameter results to be:

$$q(w, n) = -1 + \frac{3(1+w)}{2n} \quad (18)$$

which is negative for $n < 0$ or $n > \frac{3(1+w)}{2} > 0$ owing to the positivity of $(1 + w)$ for standard matter; see [14]. This implies that the accelerated behavior of the Universe is predicted in a suitable limit. In particular it follows that superacceleration ($q < -1$) can be achieved only for $n < 0$. The effective w_{eff} can be obtained (as in [12]) by means of simple calculations from (15) and (18). It results to be, for this theory:

$$w_{\text{eff}} = \frac{2}{3} q(n, w) - \frac{1}{3} = -1 + \frac{(w+1)}{n}. \quad (19)$$

We remark that the range of $-1.45 < w_{\text{eff}} < -0.74$ for dark energy, stated in [3], can be easily recovered in this theory by choosing suitable and admissible values[3] of n.

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[3]As already explained in [14] the parameter n should not be an integer, it can be any real number satisfying some reliability conditions; see [14] for further discussions and details.

We refer to [14] for physical considerations and for more detailed discussions and examples concerning polynomial-like Lagrangians in the generalized Ricci scalar.

III. RICCI SQUARED LAGRANGIANS IN MINIMAL INTERACTION WITH MATTER FIELDS

We consider now the action functional:

$$A = A_{\text{grav}} + A_{\text{mat}} = \int [\sqrt{\det g} f(S) + 2\kappa L_{\text{mat}}(\Psi)] d^4 x \quad (20)$$

where $S \equiv S(g, \Gamma) = g^{\mu\alpha} R_{(\alpha\nu)}(\Gamma) g^{\nu\beta} R_{(\beta\mu)}(\Gamma)$ and $R_{\mu\nu}(\Gamma)$ is, as above, the Ricci tensor of a torsionless connection Γ [see discussion after formula (1)]. The gravitational part of the Lagrangian is controlled by a given real function of one real variable $f(S)$; see [24]. Under the same assumptions of [24] and in four dimensional spacetimes M ($\dim M = 4$) equations of motion ensuing from the variational principle in the Palatini formalism are [24]:

$$2f'(S) g^{\alpha\beta} R_{(\mu\alpha)}(\Gamma) R_{(\beta\nu)}(\Gamma) - \frac{1}{2} f(S) g_{\mu\nu} = \kappa T_{\mu\nu} \quad (21)$$

$$\nabla_\sigma^\Gamma [\sqrt{\det g} f'(S) g^{\mu\alpha} R_{(\alpha\beta)}(\Gamma) g^{\beta\nu}] = 0 \quad (22)$$

where $T_{\mu\nu} = -\frac{2}{\sqrt{g}} \frac{\delta L_{\text{mat}}}{\delta g_{\mu\nu}}$ denotes again the matter source stress-energy tensor. The above system of equations splits, as before, into an algebraic part (21) and a differential one (22) for the unknown variables g (the metric) and Γ (the connection).

Following the general strategy elaborated for the matter-free case [24,29] (see also [14]) let us notice that $\sqrt{\det g} f'(S) g^{\mu\alpha} R_{(\alpha\beta)} g^{\beta\nu}$ is a symmetric $(2, 0)$-rank tensor density of weight 1 which we additionally assume to be nondegenerate. This assumption entitles us to introduce a new metric $h_{\mu\nu}$ by the following definition

$$\sqrt{\det h} h^{\mu\nu} = \sqrt{\det g} f'(S) g^{\mu\alpha} R_{(\alpha\beta)}(\Gamma) g^{\beta\nu}. \quad (23)$$

The metric h is hence called a Levi-Civita metric, as it follows from the field Eqs. (22), and consequently Γ is the Levi-Civita connection of it: $\Gamma = \Gamma_{\text{LC}}(h)$. The Ricci tensor of h can be simply defined as $R_{(\mu\nu)}(\Gamma) = R_{\mu\nu}(h) \equiv R_{\mu\nu}$. It should be easily recognized that Eq. (22) defines $h_{\mu\nu}$ only up to multiplicative constant. Therefore the metric h is not a good candidate for a physically meaningful object.

The algebraic Eq. (21) can be easily converted into the matrix form

$$P^2 = \frac{1}{4f'(S)} f(S) I + \frac{\kappa}{2f'(S)} \hat{T} \quad (24)$$

by using the endomorphisms P and $\hat{T}$ [i.e., $(1, 1)$-tensorfields] as defined before:

$$P = P^\mu_\nu = g^{\mu\alpha}R_{(\alpha\nu)} \qquad \hat{T} = \hat{T}^\mu_\nu = g^{\mu\alpha}T_{\alpha\nu} \qquad (25)$$

and $I = \delta^\mu_\nu$ denotes the identity endomorphism, i.e., a Kronecker delta in dimension 4. In matrix notation one can also write $P = g^{-1}R$ and $\hat{T} = g^{-1}T$.

Equation (24) can be supplemented by the scalar-valued equation obtained by taking the trace of (21) or of (24)[4]

$$f'(S)S - f(S) = \frac{\kappa}{2}g^{\alpha\beta}T_{\alpha\beta} \equiv \frac{\kappa}{2}\tau \qquad (26)$$

which governs solutions of the matrix Eqs. (21) and (24) and we will define it as the *structural equation* of space-time under analysis. We remark that in the vacuum case, as much as in the particular case of radiating matter ($\tau = 0$), we have that (26) gives constant solutions for the values of S, so that the universality property of Einstein field equations still holds [24]. In the more general case of interaction of the gravitational field with matter we are considering, we will have that solutions of (26) are no longer constants, but they are related with the values of τ. This means that the solutions of (26) are dependent on the choice of the stress-energy tensor for matter (at least on the trace of the stress-energy tensor) and moreover these solutions are dynamical, since τ is generally time-dependent. The structural Eq. (26) can be formally (and hopefully explicitly) solved expressing $S = F(\tau)$. This allows to reinterpret both $f(S)$ and $f'(S)$ as functions of τ in the expressions:

$$\begin{cases} f(S) = f[F(\tau)] = f(\tau) \\ f'(S) = f'[F(\tau)] = f'(\tau) \end{cases} \qquad (27)$$

where, for convenience, we will use in the following the abuse of notation $f[F(\tau)] = f(\tau)$ and $f'[F(\tau)] = f'(\tau)$. For any real solution $S = F(\tau)$ of (26) it is hence possible to compute the operator P by solving the matrix Eq. (24):

$$P^2 = \frac{f(\tau)}{4f'(\tau)}I + \frac{\kappa}{2f'(\tau)}\hat{T} \qquad (28)$$

by simply calculating a square root of the endomorphism on the right-hand side (RHS) [under the condition $f'(\tau) \neq 0$]. The P tensorfield results consequently to be a function of $P = P(\tau)$, due to (27). Owing to the cosmological principle it results that τ and consequently the operator P will be simply functions of time, once the stress-energy tensor for matter is chosen.

We remark that the solution proposed above for the matrix Eq. (28) is just one of the solutions of (28) and precisely it represents the simplest diagonal solution in the set of all possible solutions of (28). The definition of P

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[4]We remark that in this context $S = \text{tr}P^2$.

given in (25) should satisfy some restrictions, deriving directly from field equations of the theory (21), as much as h should satisfy them. These conditions can be read in a differential form in (25), which is however unsolvable, or translated into an algebraic expression (28). This equation thus selects operators which are meaningful in the theory we are constructing.

Off-diagonal solutions can also be found (as much as in [27]), but in the four dimensional case under analysis they are very difficult to be explicitly calculated. For our purposes we restrict then overselves to the diagonal solution; more complicated solutions of these equations, in relation with the geometrical structure of spacetime, will be possibly analyzed in forthcoming papers.

On the other hand Eq. (23) tells us the metric h is conformal to a symmetric bilinear form; i.e., in matrix notations:

$$h \simeq (g^{-1}Rg^{-1})^{-1} = P^{-1}g. \qquad (29)$$

Now we are in position to calculate the conformal factor, which results to be $\Omega = \frac{\sqrt{\det h}}{\sqrt{\det g}}[f'(S)]^{-1}$ and we will have in matrix notations that $h = \Omega P^{-1}g$. Owing to the Eqs. (23) and (25), respectively, it is then possible to set:

$$\det R = \det g \cdot \det P \qquad f'(S)^4 \det R = \det h. \qquad (30)$$

If we consider together the above Eqs. (30) it results we see that the conformal factor can be calculated to be $\Omega = f'(S)\sqrt{\det P}$, where $\det P$ can be simply obtained from the solution of Eq. (28), once the structural Eqs. (26) are solved. At this point we stress again that the conformal factor Ω is defined only up to an irrelevant multiplicative constant which has no influence on the physically measurable quantities g and Γ.

We are thus able to express the metric h in terms of the operator P and the physical metric g from (23), as:

$$h_{\mu\nu} = h_{\mu\nu}(\tau) = f'(\tau)\sqrt{\det P(\tau)}g_{\mu\alpha}(P^{-1})^\alpha_\nu \qquad (31)$$

where we have stressed the dependence of h from τ, which follows from (26). Once again to obtain this expression for $h_{\mu\nu}$ explicitly we should require that (27) can be solved analytically. Having finally calculated P^μ_ν and $\det P$ from the algebraic Eqs. (28) and (30) (for a given $g_{\mu\nu}$ and $T_{\mu\nu}$) the generalized Einstein equation ensuing from (21) takes the simple form:

$$R_{\mu\nu}(h) = P^\alpha_\nu g_{\mu\alpha} \qquad (32)$$

with $h_{\mu\nu}$ given by (31) and now the physical metric g does not need to be Einstein. This expression for the generalized Einstein equations is formally the same obtained for nonlinear Lagrangians in the generalized Ricci scalar in (10). Differences arise in the definition of the operator P [compare expressions (9) and (28)] and the metric h [compare expressions (5) and (31)], which in this last case results, in general, to be no longer conformal to g.

For the same reasoning as before one should easily realize that for the matter-free case Eq. (32) becomes just Einstein equation for the metric h, with a cosmological constant depending on the analytical form of $f(S)$. We remark once again that in the vacuum case we have that P is proportional to the identity, and solutions of (26) are constants. In the case of interaction with matter both P and $f(S) = f(\tau)$ depend on the stress-energy tensor of matter, i.e., they are both dynamical. We thus skip from a *static* model equivalent to a standard Einstein theory with cosmological constant to a more complicated *dynamical* model, which is no longer analogous to Einstein gravity.

IV. FRW COSMOLOGY IN RICCI SQUARED GRAVITY

For cosmological applications (as already explained in Sec. II A) one has first to choose the physical metric, which is assumed to be g for the moment, to be the Friedmann-Robertson-Walker metric, which (in spherical coordinates) takes the standard form (11), i.e.,

$$g = -dt^2 + a^2(t)\left\{\frac{1}{1 - Kr^2}dr^2 + r^2[d\theta^2 + \sin^2(\theta)d\varphi^2]\right\}. \tag{33}$$

Another main ingredient of the cosmological model is to choose the perfect-fluid stress-energy tensor for matter, introduced in (12) and in a comoving frame in (13). From the conservation law of the energy-momentum $\nabla^\mu T_{\mu\nu} = 0$ the consequent continuity equation takes the form:

$$\dot{\rho} + 3H(\rho + p) = 0 \tag{34}$$

where $H = \frac{\dot{a}}{a}$ is the *Hubble constant*. The above continuity equation imposes standard relations between the pressure p, the matter density ρ, and the expansion factor $a(t)$ [30], namely,

$$p = w\rho, \qquad \rho = \eta a^{-3(1+w)} \tag{35}$$

with a positive constant $\eta > 0$. As it is well-known the particular values of the parameter $w \in \{-1, 0, \frac{1}{3}\}$ will correspond to the vacuum, dust, or radiation dominated universes. Exotic matters, which are up to now under investigation as possible models for dark energy, admit instead values of $w < -1$ which are supported actually by experimental data [3]. We remark that the above expressions (35) imply that both ρ and p depend just on time, while they do not depend on space coordinates as an immediate consequence of the cosmological principle. This implies that the variable τ is an implicit function of the cosmic time t. In order to find its explicit dependence of time one has to solve the Friedmann equation.

From (33) and (13) it follows that $\hat{T}$ results to be, using the definition (25):

$$\hat{T}^\mu_{\ \nu} = \begin{pmatrix} -\rho & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{pmatrix}. \tag{36}$$

All diagonal solutions of (28) can be thus calculated, using expressions (33) and (36):

$$P^\nu_{\ \mu} = \frac{1}{2}\sqrt{\frac{f(\tau) + 2\kappa p}{f'(\tau)}}\text{Diag}\left(\epsilon_0\sqrt{\frac{f(\tau) - 2\kappa\rho}{f(\tau) + 2\kappa p}}, \epsilon_1, \epsilon_2, \epsilon_3\right) \tag{37}$$

where we have formally expressed $S = F(\tau)$ from (27), where $\tau = 3p - \rho$. We introduce moreover $\epsilon_\mu = \pm 1; \mu = 0, \ldots, 3$, ensuing from the square root of the operator P^2. Notice that all possible choices of ϵ_μ give rise to all possible diagonal solutions of the matrix equation but still correspond to the same solution $S = F(\tau)$. This exhibits a phenomenon of signature change in $f(S)$ theories (see below and [27]). Reality condition forces us to assume that all three terms

$$f'(\tau) \neq 0, \quad f(\tau) - 2\kappa\rho \neq 0 \text{ and } f(\tau) + 2\kappa p \neq 0 \tag{38}$$

have to have at the same time the same (negative or positive) sign. In what follows we denote $\varepsilon = \text{sgn}f'(\tau) = \text{sgn}[f(\tau) - 2\kappa\rho] = \text{sgn}[f(\tau) + 2\kappa p]$. It is hence possible to calculate the Levi-Civita metric h, which from (23) turns out to be:

$$h_{\mu\nu} = \frac{1}{2}\varepsilon\sqrt{\epsilon|f'(\tau)|\{[f(\tau) + 2\kappa p][f(\tau) - 2\kappa\rho]\}^{1/2}}$$

$$\times\text{Diag}\left[-\epsilon_0\sqrt{\frac{f(\tau) + 2\kappa p}{f(\tau) - 2\kappa\rho}}, \frac{\epsilon_1 a^2}{1 - Kr^2},\right.$$

$$\left.\times\epsilon_2 r^2 a^2, \epsilon_3 r^2 a^2 \sin^2(\theta)\right] \tag{39}$$

where we denoted $\epsilon = \epsilon_0\epsilon_1\epsilon_2\epsilon_3$. Neglecting an irrelevant multiplicative constant factor (which can be in general complex or imaginary) the above expression can be suitably rewritten, for convenience, as:

$$h_{\mu\nu} = b(\tau)\text{Diag}\left[-\epsilon_0 c(\tau), \frac{\epsilon_1 a^2}{1 - Kr^2}, \epsilon_2 r^2 a^2, \epsilon_3 r^2 a^2 \sin^2(\theta)\right] \tag{40}$$

where:

$$\begin{cases} b(\tau) = \sqrt{|f'(\tau)|}\{[f(\tau) + 2\kappa p][f(\tau) - 2\kappa\rho]\}^{1/4} \\ c(\tau) = \sqrt{\frac{f(\tau) + 2\kappa p}{f(\tau) - 2\kappa\rho}} \end{cases} \tag{41}$$

and $b(t)$ results to be a *generalized conformal factor*[5] between the two metrics g and h, while $c(t)$ describes a

---

[5]It is evident from the above expression that the two metrics h and g are no more conformal than they were in the case of the $f(R)$ Lagrangians, apart from the very particular case of $c(t) = $ const. However a suitable redefinition of the cosmic time variable restores the conformal relation between h and g.

rescaling factor for the cosmological time t. We notice that both the generalized conformal factor $b(t)$ and the rescaling factor $c(t)$ are positive definite by definition. The change of signature is related with coefficients $\epsilon = \pm 1$ and the freedom in their choice produces a multiplying of the Friedmann-Robertson-Walker manifold, which could be related with quantum cosmology phenomena.

From the above expression (40) it is possible to notice that some choices of the value of ϵ_μ, which is up to now completely free, will change the signature of the metric h,

so that a signature change process appears as much as in [27]. If we choose all $\epsilon_\mu = \pm 1$ to be equal we will obtain again a Lorentzian metric, at most with a different convention in signs. If any other choice is performed we will possibly have different signatures for the metric (corresponding to Euclidean, Lorentzian or Kleinian signatures) and the t coordinate may then loose its preferred physical significance.

The Ricci tensor of the metric h can be calculated from the expression (40); it results to be diagonal with the following components:

$$
\begin{aligned}
R_{00} &= \frac{3}{4}\left[-2\frac{\dot{a}}{a}\frac{\dot{b}}{b} + 2\frac{\dot{a}}{a}\frac{\dot{c}}{c} + \frac{\dot{b}}{b}\frac{\dot{c}}{c} + 2\left(\frac{\dot{b}}{b}\right)^2 - 2\frac{\ddot{b}}{b} - 4\frac{\ddot{a}}{a}\right] \\
R_{11} &= \frac{\epsilon_0\epsilon_1}{c}\frac{a^2}{4(1-Kr^2)}\left\{\left[10\frac{\dot{a}}{a}\frac{\dot{b}}{b} - 2\frac{\dot{a}}{a}\frac{\dot{c}}{c} - \frac{\dot{b}}{b}\frac{\dot{c}}{c} + 8\left(\frac{\dot{a}}{a}\right)^2 + 2\frac{\ddot{b}}{b} + 4\frac{\ddot{a}}{a}\right] + \frac{8K\epsilon_0\epsilon_1 c}{a^2}\right\} \\
R_{22} &= \epsilon_2\frac{a^2 r^2}{4}\left\{\frac{\epsilon_0}{c}\left[10\frac{\dot{a}}{a}\frac{\dot{b}}{b} - 2\frac{\dot{a}}{a}\frac{\dot{c}}{c} - \frac{\dot{b}}{b}\frac{\dot{c}}{c} + 8\left(\frac{\dot{a}}{a}\right)^2 + 2\frac{\ddot{b}}{b} + 4\frac{\ddot{a}}{a}\right] + \epsilon_1\frac{8K}{a^2}\right\} + \frac{\epsilon_1 - \epsilon_2}{\epsilon_1} \\
R_{33} &= \epsilon_3\frac{a^2 r^2\sin^2(\theta)}{4}\left\{\frac{\epsilon_0}{c}\left[10\frac{\dot{a}}{a}\frac{\dot{b}}{b} - 2\frac{\dot{a}}{a}\frac{\dot{c}}{c} - \frac{\dot{b}}{b}\frac{\dot{c}}{c} + 8\left(\frac{\dot{a}}{a}\right)^2 + 2\frac{\ddot{b}}{b} + 4\frac{\ddot{a}}{a}\right] + \epsilon_1\frac{8K}{a^2}\right\} + \sin^2(\theta)\epsilon_3\frac{\epsilon_1 - \epsilon_2}{\epsilon_1\epsilon_2}.
\end{aligned}
\tag{42}
$$

The RHS of the generalized Einstein Eqs. (32) is obtained from (37):

$$
P_\mu^\alpha g_{\alpha\nu} = \frac{1}{2}\sqrt{\frac{f(\tau) + 2\kappa p}{f'(\tau)}}\mathrm{Diag}\left[-\epsilon_0\sqrt{\frac{f(\tau) - 2\kappa\rho}{f(\tau) + 2\kappa p}}, \frac{\epsilon_1 a^2}{1 - Kr^2}, \epsilon_2 a^2 r^2, \epsilon_3 a^2 r^2\sin^2(\theta)\right].
\tag{43}
$$

Comparing expressions (42) and (43) we obtain that we must impose that $\epsilon_1 = \epsilon_2$, which derives from simple algebraic consistent conditions on the generalized Einstein Eqs. (32). It follows that, like in the standard cosmological models, we have only two relevant field equations, corresponding to the 00 and the ii components. The values of ϵ_0 and ϵ_3 are completely arbitrary. We remark however that the choice of the value of ϵ_3 does not affect field equations as it cancels from field equations, as we will see later. Field equations are fixed once we have chosen the values of ϵ_0 and ϵ_1 which modify, respectively, the 00 and the ii components of field equations.

To obtain modified Friedmann equations, we have to take into account the relevant generalized Einstein equations, which are for the 00 component of (32):

$$
\begin{aligned}
&\left[2\frac{\dot{a}}{a}\frac{\dot{b}}{b} - 2\frac{\dot{a}}{a}\frac{\dot{c}}{c} - \frac{\dot{b}}{b}\frac{\dot{c}}{c} - 2\left(\frac{\dot{b}}{b}\right)^2 + 2\frac{\ddot{b}}{b} + 4\frac{\ddot{a}}{a}\right] \\
&= \frac{2\epsilon_0}{3}\sqrt{\frac{f(\tau) - 2\kappa\rho}{f'(\tau)}}
\end{aligned}
\tag{44}
$$

and for the generic the ii component[6]:

---

[6]We stress again that with the assumption $\epsilon_1 = \epsilon_2$ each ii component provides the same field equation, as it should be expected.

$$
\begin{aligned}
&\left\{\left[10\frac{\dot{a}}{a}\frac{\dot{b}}{b} - 2\frac{\dot{a}}{a}\frac{\dot{c}}{c} - \frac{\dot{b}}{b}\frac{\dot{c}}{c} + 8\left(\frac{\dot{a}}{a}\right)^2 + 2\frac{\ddot{b}}{b} + 4\frac{\ddot{a}}{a}\right] + \frac{\epsilon_0}{\epsilon_1}\frac{8Kc}{a^2}\right\} \\
&= 2c\epsilon_0\sqrt{\frac{f(\tau) + 2\kappa p}{f'(\tau)}}.
\end{aligned}
\tag{45}
$$

Subtracting the first Eq. (44) from the second Eq. (45), we obtain that the second derivatives of the scale factor a and of the conformal factor b both disappear and we get the modified Friedmann equations in the form:

$$
\begin{aligned}
\left[\frac{\dot{a}}{a} + \frac{\dot{b}}{2b}\right]^2 &= \frac{c\epsilon_0}{4}\sqrt{\frac{f(\tau) + 2\kappa p}{f'(\tau)}} - \frac{\epsilon_0}{12}\sqrt{\frac{f(\tau) - 2\kappa\rho}{f'(\tau)}} \\
&\quad - \epsilon_0\epsilon_1\frac{Kc}{a^2}
\end{aligned}
\tag{46}
$$

where the expression on the left-hand side can be defined as a modified Hubble constant {which is moreover analogue to (14); see also [14]}, which rules the dynamical evolution of the Universe:

$$
\hat{H}^2 = \left[\frac{\dot{a}}{a} + \frac{\dot{b}}{2b}\right]^2.
\tag{47}
$$

The RHS of the modified Friedmann equations for Ricci squared theories differs however from the RHS of (14) for Ricci scalar theories, as it should be reasonably expected. The evolution of the model is just dependent on the evolution of the scale factor $a(t)$ and of the modified

conformal factor $b(t)$ (i.e., on $\hat{H}$), while derivatives of the factor $c(t)$ disappear. This fact is strongly analogous with the case of Ricci scalar theories.

We remark that, as already observed before, the expression (46) and field equations depend only on the values of ϵ_0 and ϵ_1. The sign factor ϵ_0 appears as a constant in front of the RHS of modified Friedmann equations, which can be rewritten as

$$\hat{H}^2 = \epsilon_0 \left[\frac{f(\tau) + \kappa\tau + 2\kappa\rho}{6\sqrt{f'(\tau)[f(\tau) - 2\kappa\rho]}} - \epsilon_1 \frac{Kc}{a^2} \right] \quad (48)$$

so that a suitable choice of ϵ_0 allows the RHS to be always positive as expected. In fact ϵ_0 has to be chosen in accordance with the prescription $\epsilon_0 = \mathrm{sgn}f'(\tau) \times [f(\tau) + \kappa\tau + 2\kappa\rho]$, provided the conditions (38) are satisfied. However, we see that on the other hand ϵ_1 appears only in the term related with the curvature K of the spacelike hypersurface. As it is obvious also from the explicit expression of h (40), choosing different values of ϵ_1 is equivalent to changing the sign of the spatial curvature. We remark finally that the choice of ϵ_3 is irrelevant for field equations.

V. POLYNOMIAL LAGRANGIANS IN THE RICCI SQUARED INVARIANT

We choose, as a relevant example to deal with, polynomial Lagrangians in S. In strict analogy with what has already been done for the Ricci scalar theories, polynomial Lagrangians can be considered as approximations[7] of any analytical expression in S in the suitable limit [14]. It is hence worth investigating the behavior of cosmological solutions of Ricci squared theories described by means of Lagrangians which are pure powers of S:

$$f(S) = \beta S^n. \quad (49)$$

As a matter of fact polynomial Lagrangians can be approximated to pure-power Lagrangians if the asymptotical behavior is considered and just the first leading term is taken into account. From the structural Eqs. (26) we obtain that for the above pure-power Lagrangian (49) in the Ricci squared invariant:

$$\begin{cases} f(\tau) = \frac{\kappa}{2(n-1)}\tau \\ f'(\tau) = \frac{n\kappa\tau}{2(n-1)}\left[\frac{\kappa\tau}{2\beta(n-1)}\right]^{-(1/n)} \end{cases} \quad (50)$$

where we have to require $n \neq 1$ to avoid singularities in

the theory (this implies that the case $f(S) \simeq S$ is not allowed[8]). Since in the physically interesting cases one has $\tau \leq 0$ we see that for generic n we have to assume

$$\beta(n - 1) < 0.$$

However, for odd n we can allow $\beta(n - 1) > 0$ (see also [14] in this context). Taking into account the standard relations:

$$\tau = (3w - 1)\rho, \qquad p = w\rho, \qquad \rho = \eta a^{-3(1+w)} \quad (51)$$

and performing straightforward calculations we obtain from (41) the generalized conformal factor (up to multiplicative constant):

$$b(t) \simeq a^{-3(1+w)[1-(1/2n)]}. \quad (52)$$

Performing further calculations by means of (47) it is simple to obtain:

$$\hat{H}^2 = \left[\frac{(3w + 1)(2n - 1) - 2}{4n} \right]^2 H^2. \quad (53)$$

We remark that in the particular case of $w = -1$ the expression (53) implies that $\hat{H}^2 = H^2$, independently on the value of n. Using the same relations, the RHS of Eq. (46) results to be:

$$\hat{H}^2 = \frac{\epsilon_0}{6} \frac{(3w + 1)(2n - 1) - 2}{\sqrt{n(3w - 1)(3w + 3 - 4n)}}$$
$$\times \left[\frac{\kappa(3w - 1)}{2\beta(n - 1)} \right]^{1/2n} \rho^{1/2n} - \frac{\epsilon_0}{\epsilon_1} \frac{K}{a^2}$$
$$\times \sqrt{\frac{4nw - (w + 1)}{3(1 + w) - 4n}}. \quad (54)$$

We stress that the rescaling factor $c(t)$ is, in this particular example, not dependent on time:

$$c(t) = \sqrt{\frac{4wn - w - 1}{3w + 3 - 4n}}. \quad (55)$$

Combining Eqs. (53) and (54) together we obtain that the Hubble H^2 constant for the physical metric g is:

$$H^2 = \left(\frac{\dot{a}}{a}\right)^2 = \epsilon_0 P(n, w) a^{-[3(1+w)/2n]} - \frac{\epsilon_0}{\epsilon_1} Q(n, w) K a^{-2} \quad (56)$$

where we have defined:

---

[7]We are particularly interested in the cases of very small and large values of S, reproducing the cases of large and small curvatures of the Universe, owing to the (linear) quadratic relation between S and $R_{\mu\nu}$.

[8]This is similar to the presence of a methodological singularity for $f(R) \simeq R^2$ in nonlinear theories of gravity depending on the Ricci scalar; see , e.g., [8,14].

$$\begin{cases} P(n, w) = \dfrac{8n^2}{3\sqrt{n(3w-1)(3w+3-4n)}[(3w+1)(2n-1)-2]}\left[\dfrac{\kappa(3w-1)}{2\beta(n-1)}\eta\right]^{1/2n} \\ Q(n, w) = \sqrt{\dfrac{4nw-(w+1)}{3(1+w)-4n}}\left[\dfrac{4n}{(3w+1)(2n-1)-2}\right]^2 \end{cases} \tag{57}$$

From the above expression the deceleration parameter can be calculated by means of the standard formula already introduced in (16) and it can be formally calculated from (56) under the form[9]:

$$q(t, w, n) = -1$$
$$+ \frac{\frac{3(1+w)}{4n}\epsilon_1 P(n, w)a^{-[3(1+w)/2n]} - Q(n, w)Ka^{-2}}{\epsilon_1 P(n, w)\eta^{1/2n}a^{-[3(1+w)/2n]} - Q(n, w)Ka^{-2}}. \tag{58}$$

We obtain consequently that when the a^{-2} term dominates over $a^{-[3(1+w)/2n]}$ the deceleration parameter results to be positive, i.e., $q(t, w, n) \to 0^+$, while when the term $a^{-[3(1+w)/2n]}$ dominates over a^{-2} or in the physically very important case $K = 0$, the deceleration parameter will be:

$$q(w, n) = -1 + \frac{3(1+w)}{4n}. \tag{59}$$

This implies that $q(w, n)$ is negative for $n < 0$ or $n > \frac{3(1+w)}{4} > 0$, owing to the positivity of $(1 + w) > 0$ for standard matter. Taking also into account restriction on parameters (w, n) coming from (38) the whole situation can be visualized on the phase diagram Fig. 1.

Comparing expression (56) and the standard relation which derives from general relativity (see [12,14]) it is easy to obtain that the effective value of w_{eff} deriving from Ricci squared alternative theories of gravity is:

$$w_{\text{eff}}(w, n) = -1 + \frac{(1+w)}{2n}. \tag{60}$$

Both limiting values for $w_{\text{eff}}(\frac{1}{3}, n)$ and $w_{\text{eff}}(0, n)$ are marked on the phase diagram Fig. 1.

We can compare the values of $q(w, n)$ and $w_{\text{eff}}(w, n)$ obtained in this case with the values of $q(w', n')$ (18) and $w_{\text{eff}}(w', n')$ (19) obtained for alternative theories of gravity with pure-power Lagrangians of the Ricci scalar $F(R) = \beta' R^{n'}$, already treated in [14]. It turns out that they differ just for a factor if the value of w is assumed to be fixed, which is equivalent to state that we are dealing with the same kind of matter. It is simple to see that:

$$\begin{cases} q(w, n) = q(w', n') \\ w_{\text{eff}}(w, n) = w_{eff}(w', n') \end{cases} \Longleftrightarrow n = \frac{n'}{2}. \tag{61}$$

These results generalize and confirm the results already

---

[9]We say that the deceleration parameter can be formally calculated, as we do not know *a priori* if any physical solution exists in all cases considered.

obtained in [31] in the very particular case of quadratic Lagrangians.

A. Polynomial Lagrangians

As we already stated before, pure-power Lagrangians in the Ricci squared invariant can be considered as *approximations* of more physical polynomial-like Lagrangians of the type:

$$f(S) = S + \frac{\alpha}{(1 + n)S^n} + \frac{\beta}{1 - m}S^m \tag{62}$$

(here both $n > 0$ and $m > 0$, with $m \neq 1$).

We just consider for simplicity the case of flat Universe $K = 0$. In the limit of small or large curvatures, corresponding to the cases of present time Universe and early time Universe, we obtain from the structural equations that the leading terms are respectively:

$$\begin{cases} S \to 0 \Rightarrow \frac{-\alpha}{S^n} = \kappa\tau \\ S \to \infty \Rightarrow -\beta S^m = \kappa\tau \end{cases}.$$

From (59) we deduce that polynomial Lagrangians provide an explanation for early time inflation assuming that $m > \frac{3}{4}$ and they can provide an explanation for present time cosmic acceleration assuming that some inverse
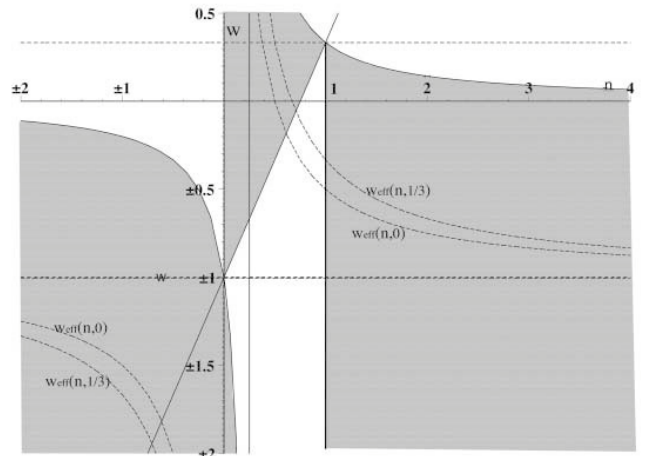

FIG. 1. Phase portrait for the plane (w, n), where w is the fluid parameter characterizing matter and n the exponent of S in the Lagrangian. The shaded areas represent physically and mathematically admissible pairs. We notice the following: for $n > 1$ we cannot have radiation ($w = \frac{1}{3}$) but dust is allowed; for dust matter ($w = 0$) and $n > 1$, $w_{\text{eff}} \to -1^+$, i.e., w_{eff} can be approached only from above; on the contrary, negative powers ($n < 0$) do not allow dust, although any dustlike matter can be allowed for large enough $|n|$. Moreover in this case $w_{\text{eff}} \to -1^-$ is possible from below (superacceleration).

power of the generalized Ricci squared curvature invariant is also present in the Lagrangian (i.e., $\alpha \neq 0$).

This result generalizes previous results which have been obtained for Ricci scalar alternative theories of gravity [13–15] and they are related to the so-called Starobinsky inflation [8].

VI. CHANGING FRAME

We have developed up to now a first order á la Palatini theory which after appropriate reduction turns out to be based, as we remarked before, on a bi-metric spacetime with an almost-complex structure. In Sec. IV we have assumed g to be the "physical" FRW metric. However we do not know *a priori* which is the most appropriate frame in the bi-metric structured spacetime we have constructed in Sec. III. This is the same problem already studied and examined in [14,18,19] in the case of Lagrangians depending on the Ricci scalar, where different frames result to be somehow inequivalent. We shall not comment here on this equivalence problem and refer the reader to a recent interesting discussion by Flanagan ([32] and Refs. quoted therein). In our understanding this important problem should be analyzed in more detail, also in relation with the physical consistency requirements and the presence of instabilities; we plan to treat it in a forthcoming paper [33].

In this framework it is thus worth considering also the case when the "Jordan" h is chosen to be the physical FRW metric, as we have already done in [14]. More precisely, according to (40) we set:

$$
h = -\epsilon_0 d\tilde{t}^2 + A^2(\tilde{t})\Big\{\frac{\epsilon_1}{1 - Kr^2}dr^2 + r^2[\epsilon_1 d\theta^2
$$
$$
+ \epsilon_3 \sin^2(\theta)d\varphi^2]\Big\} \tag{63}
$$

to be, modulo signature, FRW metric with a new cosmic time $\tilde{t}$ and a new scale factor A. The generalized Einstein Eq. (10) can be also calculated in $(\tilde{t}, x^i)$-coordinates. This is equivalent to the assumption that the metric h is the physical one (i.e., that we can use conformal Jordan frame instead of the original Einstein frame). In this case, one has to restore the standard Lorentzian signature by setting $\epsilon_0 = \epsilon_i = \pm 1$. We consequently obtain:

$$
3\frac{\ddot{A}}{A} = \frac{\epsilon_0}{c(\tau)b(\tau)}\Big(\frac{f(\tau) - 2\kappa\rho}{4f'(\tau)}\Big)^{1/2}
$$
$$
= \frac{\epsilon_0}{2|f'(\tau)|}\Big(\frac{f(\tau) - 2\kappa\rho}{f(\tau) + 2\kappa p}\Big)^{3/4}
$$
$$
= \frac{\epsilon_0}{2|f'(\tau)|}c(\tau)^{-(3/2)} \tag{64}
$$

for the 00 component while for the 11 component we find

$$
\frac{\ddot{A}}{A} + 2\Big(\frac{\dot{A}}{A}\Big)^2 + 2\frac{K}{A^2} = \frac{\epsilon_0}{b(\tau)}\Big(\frac{f(\tau) + 2\kappa p}{4f'(\tau)}\Big)^{1/2}
$$
$$
= \frac{\epsilon_0}{2|f'(\tau)|}\Big(\frac{f(\tau) + 2\kappa p}{f(\tau) - 2\kappa\rho}\Big)^{1/4}
$$
$$
= \frac{\epsilon_0}{2|f'(\tau)|}c(\tau)^{1/2} \tag{65}
$$

where $\dot{A}$ denotes now the differentiation with respect to the new cosmic time $\tilde{t}$. We have also taken into account that in this case $d\tilde{t}^2 = c(\tau)b(\tau)dt^2$ and $A^2 = b(\tau)a^2$.

Now the analogue of the Friedmann equation takes the form

$$
\tilde{H}^2 = \frac{\epsilon_0 c^{1/2}}{12|f'(\tau)|}(3 - c^{-2}) - \frac{K}{A^2} \tag{66}
$$

with $\tilde{H} = \frac{\dot{A}}{A}$ being the Hubble constant of the conformal metric h. Thus up to now arbitrary sign factor $\epsilon_0 = \pm 1$ can be adjusted as $\epsilon_0 = \text{sgn}(3 - c^{-2})$ in order to preserve the positivity. We specialize now to the case of pure-power Lagrangians in the Ricci squared curvature invariant and in the meanwhile, as already stated before, to the case of polynomial Lagrangians in some suitable limit. Choosing, as already done in (49), the Lagrangian to be:

$$
f(S) = \beta S^n \tag{67}
$$

we obtain respectively $f(\tau) = \frac{\kappa}{2(n-1)}\tau$ and $f'(\tau) = n\beta[\frac{\kappa\tau}{2\beta(n-1)}]^{(n-1)/n}$. It follows from Eq. (66) that the modified Friedmann equation in this frame is:

$$
\tilde{H}^2 = \tilde{P}(w, n)A^\lambda - \frac{K}{A^2} \tag{68}
$$

where for convenience sake we have defined the coefficient $\tilde{P}(\eta, w, n)$ as:

$$
\tilde{P}(w, n) = \frac{|2n(1 + 3w) - 3(w + 1)|}{6\beta|n||4wn - w - 1|^{3/4}|3w + 3 - 4n|^{1/4}}
$$
$$
\times \Big[\frac{2\beta(n - 1)}{\kappa\eta(3w - 1)}\Big]^{(n-1)/n} \tag{69}
$$

and the exponent λ as

$$
\lambda = \frac{12(w + 1)(n - 1)}{3(w + 1) - 2n(3w + 1)}. \tag{70}
$$

We can consequently obtain the deceleration parameter by means of formula (58):

$$
\tilde{q}(\tilde{t}) = -\frac{(1 + \frac{\lambda}{2})\tilde{P}A^\lambda}{\tilde{P}A^\lambda - KA^{-2}}. \tag{71}
$$

This implies that, in the limit when the term A^{-2} is dominating over A^λ we will have $\tilde{q}(t) \to 0^+$. Otherwise in the limit when the term A^λ is dominating over A^{-2} or in the case $K = 0$, we will have

$$\tilde{q}(\tilde{t}) \rightarrow \tilde{q}(w, n) = -\frac{4n - 3(w + 1)}{4n - 3(w + 1)(2n - 1)}.$$

Thus

$$\tilde{w}_{\text{eff}}(w, n) = \frac{2n(w - 1) + w + 1}{4n - 3(w + 1)(2n - 1)} \qquad (72)$$

which for big n behaves as

$$\tilde{w}_{\text{eff}}(w, \pm\infty) = \frac{1 - w}{3w + 1}.$$

Comparing Eq. (72) and equation (37) in [14], obtained for $f(R) = \beta' R^{n'}$ theories, it follows that also in the so-called *Jordan frame* the rule $n = \frac{n'}{2}$ turns out to be valid too. This rule has already been obtained in the so-called *Einstein frame* in (61). This result entitles us to conjecture that for k-order Ricci scalars, already studied in [26], one should have $n = \frac{n'}{k}$, as we will study in future papers.

It has been recently shown that the best fit with experimental data for R^n theories corresponds to values of $n \sim 2.25$ (as it has been proven in [34]). This implies that for S^n theories the best fit should be realized for values of $n \sim 1.125$. It is interesting to stress that both values $n \sim 2.25$ and $n \sim 1.125$ are respectively close to the critical values $n = 2$ and $n = 1$.

VII. CONCLUSIONS AND PERSPECTIVES

In this paper we have analyzed alternative theories of gravity depending on a Lagrangian assumed to be a general function of the generalized Ricci squared curvature invariant S constructed out of a dynamical metric g and a dynamical (torsionless) connection Γ. The Palatini formalism provides first order field equations for the metric and the connection Γ. A structural metric h is introduced, so that the connection turns out to be the Levi-Civita connection of h and h is consequently a Levi-Civita metric. A convenient spacetime bi-metric geometry is thus defined by means of generalized Einstein equations and it is controlled by means of structural equations; signature changing phenomena appear. This implies that the metric h can be either a Lorentzian, a Euclidean, or a Norden metric, giving us an immediate and natural insight into quantum cosmology theories.

To treat explicitly cosmological models we choose g to be a Robertson-Walker metric and the stress-energy tensor to be the stress-energy tensor of a perfect fluid. This allowed us to obtain modified Friedmann field equations and a modified Hubble constant related to a (generalized) conformal transformation factor $b(t)$ between g and h and to a rescaling factor for the cosmological time $c(t)$. The metric h can be considered to be FRW, too, so that it can be conveniently considered as a physical metric in place of the original g. Generalized Friedmann equations are obtained also in this framework.

If we moreover specialize to the pure-power case $f(S) = \beta S^n$ (with n an arbitrary real exponent) we have seen that, with suitable choices of the parameters involved, these models are able to explain the current acceleration of the Universe. We obtain moreover that polynomial Lagrangians in the generalized Ricci squared invariant provide an explanation for the inflation of the Universe in suitable limits [9].

This paper was thus devoted to analyze the geometrical structure of spacetimes described by means of Ricci squared Lagrangians in interaction with matter; cosmological applications of this model have been analyzed, following the ideas of [8] and generalizing the effort to understand current acceleration of the Universe in alternative theories of gravity [12,14]. The relation between the geometrical bi-metric structure of spacetime (and, in particular, the signature change phenomena) and its cosmological implications is very rich in mathematical and physical significance and will form the subject of future investigations.

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