

How to make a traversable wormhole from a Schwarzschild black hole

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The theoretical construction of a traversable wormhole from a Schwarzschild black hole is described, using analytic solutions in Einstein gravity. The matter model is pure phantom radiation (pure radiation with negative-energy density) and the idealization of impulsive radiation is employed.

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I. INTRODUCTION

The recently discovered acceleration of the universe [1,2] indicates that its large-scale evolution is dominated by unknown dark energy which violates at least the strong energy condition, and perhaps also the weak energy condition, where it is known as phantom energy [3]. Such phantom energy is precisely what is needed to support traversable wormholes [4–9]. This paper shows how one could, strictly according to Einstein gravity, theoretically construct such a wormhole from the most standard black hole solution due to Schwarzschild.

As the matter model, for simplicity we assume pure phantom radiation, i.e., pure radiation with negative energy density. It was recently shown that two opposing streams of such radiation support a static traversable wormhole [10]. Penrose diagrams of the wormhole space-time and the Schwarzschild space-time are shown in Fig. 1. The aim then is to find analytic solutions to the field equations of general relativity which interpolate appropriately between such space-time regions. For simplicity again, we employ the idealization of impulsive radiation, where the radiation forms an infinitely thin null shell, thereby delivering finite energy momentum in an instant. It is then possible to explicitly construct spherically symmetric solutions of the desired type using Vaidya regions, which describe space-times with a single stream of pure radiation [11]. The conformal diagram is shown in Fig. 2: one begins with a Schwarzschild region including part of the black hole region then beams in impulses of phantom radiation from both sides symmetrically. The impulses are followed by phantom radiation with constant energy profiles, forming Vaidya regions. If the energies and timing are related appropriately, the region left between the receding impulses after collision is a static traversable wormhole. The analytic details of how to match such regions using the Barrabès-Israel formalism [12] are quite complex and given in a longer

article [13]. However, it turns out that one can understand the matching in a comparatively simple way, by continuity of the area A and a jump formula for the energy E across impulses. This is the method explained in this paper.

Here A is the area of the spheres of symmetry and E is the active gravitational mass energy defined by [14,15]

$$E = [1 - g^{-1}(dr, dr)]r/2, \quad (1)$$

where $r = \sqrt{A/4\pi}$ is the area radius and g the space-time metric. Note that $E = r/2$ on a trapping horizon $g^{-1}(dr, dr) = 0$ [16,17], which includes both the Killing horizons of a stationary black hole, where dr is null, and the throat of a static wormhole, where dr vanishes [8]. Thus for a Schwarzschild black hole with mass M , $r = 2M$ on the horizons, while $r = a$ on a wormhole throat with area $4\pi a^2$.

II. BASIC SOLUTIONS

The required metrics are as follows. (i) The Schwarzschild metric is given by

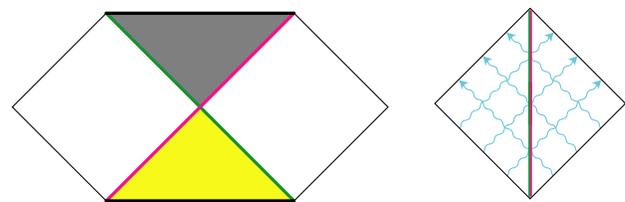


FIG. 1 (color online). Penrose diagrams of a Schwarzschild black hole and a traversable wormhole [10]. The bold diagonal or vertical lines (magenta and green) represent the trapping horizons, $\partial_+ A = 0$ and $\partial_- A = 0$. They constitute the Killing horizons of the black hole and the throat of the wormhole. Bottom (yellow) and top (gray) quadrants represent past trapped and future trapped regions, respectively. Wavy (cyan) lines represent the constant-profile phantom radiation supporting the wormhole structure.

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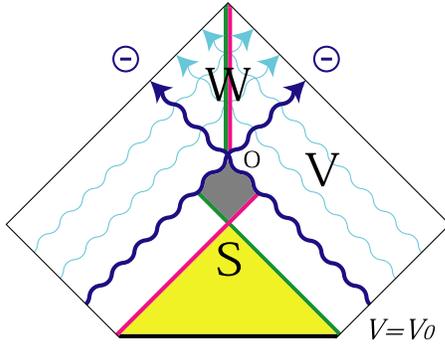


FIG. 2 (color online). Penrose diagram of the wormhole-construction model. The wavy bold (blue) lines represent impulsive radiation with negative-energy density. The region S is Schwarzschild, V is Vaidya, and W is static-wormhole.

$$ds^2 = r^2 d\Omega^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 - \left(1 - \frac{2M}{r}\right) dt^2, \quad (2)$$

where $d\Omega^2$ refers to the unit sphere and the constant M is the Schwarzschild mass, which coincides with the energy $E = M$.

(ii) The metric of the Vaidya solutions is given by

$$ds^2 = r^2 d\Omega^2 - dV \left[\left(1 - \frac{2m}{r}\right) dV + 2\zeta dr \right], \quad (3)$$

where ζ is a sign factor, with $\zeta = 1$ for outgoing radiation and $\zeta = -1$ for ingoing radiation. The mass function $m(V)$ coincides with the energy $E = m$. The corresponding energy tensor is

$$T = -\frac{\zeta}{4\pi r^2} \frac{dm}{dV} dV \otimes dV. \quad (4)$$

The Vaidya solutions reduce to the Schwarzschild solution in the case $m = M$ (constant), as can be seen by writing the latter in terms of the Eddington-Finkelstein coordinate

$$V = t - \zeta [r + 2M \ln(1 - r/2M)], \quad (5)$$

appropriate to the interior of the black hole $r < 2M$.

(iii) The static wormholes [10] supported by opposing streams of pure phantom radiation can be written as

$$ds^2 = r^2 d\Omega^2 + 2are^{-l^2} dl^2 - \frac{2\lambda ae^{-l^2}}{r} dt^2, \quad (6)$$

where t is a static time coordinate,

$$r(l) = a(e^{-l^2} + 2l\phi), \quad \phi(l) = \int_0^l e^{-\ell^2} d\ell, \quad (7)$$

and $a > 0$ and $\lambda > 0$ are constants. The space-time is not asymptotically flat, but otherwise constitutes a Morris-Thorne wormhole [4] with doubly minimal surfaces $dr = 0$ at the throat $l = 0$ and throat radius $r = a$. The energy evaluates as $E = \epsilon$ where

$$\epsilon(l) = (e^{-l^2} + 2l\phi - 2e^{l^2}\phi^2)a/2. \quad (8)$$

In terms of dual-null coordinates

$$x^\pm = t \pm \frac{a}{2\sqrt{\lambda}} [le^{-l^2} + (1 + 2l^2)\phi], \quad (9)$$

the metric is given by

$$ds^2 = r^2 d\Omega^2 - \frac{2\lambda ae^{-l^2}}{r} dx^+ dx^-. \quad (10)$$

Then the energy tensor is found as

$$T = -\frac{\lambda}{8\pi r^2} (dx^+ \otimes dx^+ + dx^- \otimes dx^-). \quad (11)$$

This is the energy tensor of two opposing streams of pure phantom radiation, with $\lambda = -4\pi r^2 T_{tt}$ being the resulting negative linear energy density. One may also write

$$ds^2 = r^2 d\Omega^2 - \frac{\sqrt{\lambda}}{e^{l^2}\phi} dx^\pm \left[\frac{2\sqrt{\lambda}a\phi}{r} dx^\pm \mp 2dr \right], \quad (12)$$

for comparison with the Vaidya solutions.

III. JUMP IN ENERGY DUE TO IMPULSIVE RADIATION

A general spherically symmetric metric can be written in dual-null form as

$$ds^2 = r^2 d\Omega^2 - h dx^+ dx^-, \quad (13)$$

where $r \geq 0$ and $h > 0$ are functions of the future-pointing null coordinates (x^+, x^-) . Writing $\partial_\pm = \partial/\partial x^\pm$, the propagation equations for the energy are obtained from the Einstein equations as [17]

$$\partial_\pm E = 8\pi h^{-1} r^2 (T_{+-} \partial_\pm r - T_{\pm\pm} \partial_\mp r). \quad (14)$$

We consider impulsive radiation defined by

$$T = \frac{\mu_\pm dx^\pm \otimes dx^\pm}{4\pi r^2} \delta(x^\pm - x_0), \quad (15)$$

where δ is the Dirac distribution, the constant x_0 gives the location of the impulse and the constant μ_\pm is its energy. More invariantly, the vector $\varphi = -g^{-1}(\mu_\pm dx^\pm)$ is the energy momentum of the impulse. Then the jump

$$[E]_\pm = \lim_{\alpha \rightarrow 0} \int_{x_0 - \alpha}^{x_0 + \alpha} \partial_\pm E dx^\pm, \quad (16)$$

in energy across the impulse is given by the jump formula

$$[E]_\pm = c^\pm \mu_\pm, \quad c^\pm = -2(h^{-1} \partial_\mp r)|_{x^\pm = x_0}. \quad (17)$$

The vector $c = c^+ \partial_+ + c^- \partial_-$ is actually $c = g^{-1}(dr)$ and so

$$[E]_\pm = -\varphi \cdot dr, \quad (18)$$

is a manifestly invariant form of the jump formula. Note that while the energy momentum vector φ (or $\mu_\pm dx^\pm$) is

invariant, the energy μ_{\pm} depends on the choice of null coordinate x^{\pm} , reflecting the fact that a particle moving at light speed has no rest frame and no preferred energy. However, in a curved but stationary space-time, the stationary Killing vector provides a preferred frame and a preferred energy μ_{\pm} .

We need employ the jump formula only in the following cases: (i) Inside a Schwarzschild black hole, we can take future-pointing $x^{\pm} = -r^* \pm t$ where $dr^*/dr = (2M/r - 1)^{-1}$. Then $r^* = -(x^+ + x^-)/2$, $h = 2M/r - 1$, and $\partial_{\pm} r = (dr/dr^*)\partial_{\pm} r^* = (1 - 2M/r)/2$ gives $c^{\pm} = 1$. (ii) Inside a Schwarzschild white-hole, we can take future-pointing $x^{\pm} = r^* \pm t$ and similarly obtain $c^{\pm} = -1$. (iii) On the throat of a static wormhole, where $\partial_{+} r = \partial_{-} r = 0$ and h is finite [4], one finds $c^{\pm} = 0$. Then the jump formula yields

$$[E]_{\pm} = \begin{cases} \mu & \text{inside a Schwarzschild black hole} \\ -\mu & \text{inside a Schwarzschild white hole} \\ 0 & \text{on the throat of a static wormhole,} \end{cases} \quad (19)$$

where the subscripts on μ_{\pm} are now omitted.

IV. WORMHOLE CONSTRUCTION

The free data for the solution will be the initial Schwarzschild mass $M > 0$ and the energy $\mu < 0$ of the impulses. Now imagining an infinitesimal diamond-shaped box around the point O where the impulses collide as in Fig. 3, the energy E will jump by μ from the region S to V and by 0 from the region V to W, evaluated in the limit at the point. By continuity of the area $A = 4\pi r^2$ and the fact that $E = a/2$ on a wormhole throat of radius a , we therefore have $a/2 = M + \mu$. Thus we require $|\mu| < M$ and have determined a . The timing of the impulses is also determined, as follows: The tortoise coordinate r^* inside a Schwarzschild black hole can be defined as

$$r^* = -r - 2M \ln(1 - r/2M). \quad (20)$$

The symmetry of the impulses means that the intersection point O is given by $t = 0$, $r = a$, or $r^* = a^*$. Then the ingoing ($\zeta = -1$) Eddington-Finkelstein relation (5) at the point O where the impulses collide gives

$$V_0 = -a^* = 2(M + \mu) + 2M \ln(-\mu/M). \quad (21)$$

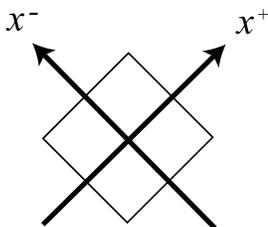


FIG. 3. An infinitesimal box around the intersection of two radiative impulses.

Finally, the mass function m of the Vaidya regions is now implicitly determined by comparing the expressions for the energy densities of the Vaidya (4) and static-wormhole (11) regions, or equivalently by using the expression (8) for E in the wormhole region and the jump formula. Explicit expressions require a comparison of the null coordinates in (3) and (12) and are given in the longer article [13], which performs the matching in detail using the Barrabès-Israel formalism [12].

The results are analogous to those obtained for a two-dimensional gravity theory where the solutions are much simpler [18,19]. The opposite process, namely, collapse of a traversable wormhole to a black hole, is easy to see for the wormhole considered here [10] and has previously been demonstrated numerically for the Ellis wormhole [20], where still unexplained critical phenomena were discovered [21]. Nonetheless, the above solution appears to be the first which describes construction of a traversable wormhole from a black hole in standard Einstein gravity.

V. WORMHOLE ENLARGEMENT

As constructed above, the area of the wormhole throat is less than the area of the initial black hole. Therefore another interesting question is how to enlarge a wormhole. Similar analytic solutions describing wormhole enlargement are given in the longer article [13] and briefly described here. As in Fig. 4, one begins with a static wormhole, beams in negative-energy impulses while simultaneously switching off the constant-profile radiation, then beams in positive-energy impulses while simultaneously switching on more constant-profile radiation. Then the regions V1 and V2 are Vaidya, the region S is

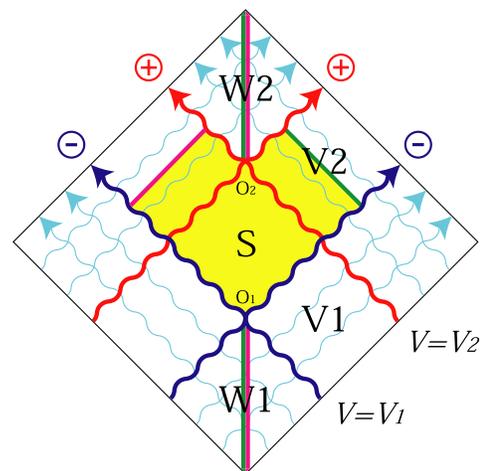


FIG. 4 (color online). Penrose diagram of the wormhole enlargement model. Wavy bold lines represent impulsive radiation with positive (red) or negative (blue) energy density. The regions W1 and W2 are static-wormhole, V1 and V2 are Vaidya, and S is Schwarzschild.

vacuum and therefore Schwarzschild, and in fact is a white-hole region and therefore expanding.

The free data can be taken as the initial throat radius a_1 and the energies $\mu_1 < 0$, $\mu_2 > 0$ of the impulses; this determines the energies of the constant-profile radiation, the timings V_1, V_2 of the impulses, similarly to (21), and the final throat radius a_2 . In particular, the energy E will jump by 0 from W1 to V1 and by $-\mu_1$ from V1 to S at the point O_1 , and by $-\mu_2$ from S to V2 and by 0 from V2 to W2 at O_2 , yielding $M = a_1/2 - \mu_1$ and $a_2/2 = M - \mu_2$, where M is the Schwarzschild mass of S. Thus the increase in throat radius is given by $a_2 - a_1 = -2(\mu_1 + \mu_2)$ and so $|\mu_1| > \mu_2$.

The results are similar to those found in the two-dimensional model [19]. Self-inflating wormholes were also recently discovered numerically [21] and wormhole inflation was previously suggested by Roman [22], but the above solutions appear to be the first analytic examples of wormhole enlargement in Einstein gravity with a specified exotic matter model. Additionally, the enlargement is not a runaway inflation but an apparently stable process, whereby the amount of enlargement can be controlled by the energy or timing of the impulses. Reducing the wormhole size can similarly be achieved by reversing the order of the positive-energy and negative-energy impulses [13].

VI. REMARKS

The solutions described here provide concrete manifestations of ideas about traversable wormholes which were explained previously [8] using general results and intuition concerning trapping horizons, particularly how they develop under strengthening or weakening positive- or negative-energy density. Some useful ideas or results were a general definition of wormhole mouth, unified with a general local definition of black hole [16]; a proof of the necessary violation of the null energy condition at a nondegenerate wormhole mouth; an invariant measure of the radial curvature or “flareout” at the mouth using a definition of surface gravity previously proposed for dynamical black holes [17]; and analogues of basic laws of black hole dynamics [16,23] for wormholes. A newer observation is that impulses of positive or negative energy, respectively, shift a trapping horizon discontinu-

ously to the past or future along the impulse. For brief conference reviews see [24].

In the wormhole-construction solution, the white-hole region of the space-time, usually regarded as nonphysical, is not essential to the argument; it could be excised and replaced with a regular region of matter, though the subsequent evolution of the matter would make analytic solutions more difficult. On the other hand, the spatial topology of the maximally extended Schwarzschild solution is relevant, since topology change is classically forbidden unless causal loops exist [25,26]. Black holes formed by gravitational collapse of supernova remnants would presumably not have the appropriate topology.

For this issue one may turn to Wheeler’s space-time foam picture [27], where by a reasonable application of quantum principles and the geometric nature of general relativity, Planck-sized virtual black holes are expected to continually form and disappear. To quote Morris and Thorne [4]: “One could *imagine* an exceedingly advanced civilization pulling a wormhole out of this...space-time foam and enlarging it.” We now have exact solutions in Einstein gravity describing how a Schwarzschild black hole, perhaps formed in space-time foam, may be converted into a traversable wormhole and enlarged to usable size. This might also be relevant in the quantum-gravity epoch presumed to begin the universe, where primordial black holes and wormholes might both be formed. It is tempting to speculate whether primordial wormholes might still survive; clearly it would depend on the nature of the dark energy.

The fascinating potential of traversable wormholes, as short cuts across the universe and even as time machines, has already passed into popular culture as science fiction. As science, they now appear to be less speculative than much of theoretical physics by recent standards. We have assumed only well-proven Einstein gravity with an idealized model of phantom energy. If the mysterious cosmological dark energy is or can be phantom in nature, one could argue that traversable wormholes are as much a prediction of general relativity as black holes.

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