

SU(3) Clebsch-Gordan coefficients for baryon-meson coupling at arbitrary N_c

Thomas D. Cohen*

Department of Physics, University of Maryland, College Park, Maryland 20742-4111, USA

Richard F. Lebed†

Department of Physics and Astronomy, Arizona State University, Tempe, Arizona 85287-1504, USA

(Received 2 September 2004; published 29 November 2004)

We present explicit formulas for the SU(3) Clebsch-Gordan coefficients that are relevant for the couplings of large N_c baryons to mesons. In particular, we compute the Clebsch-Gordan series for the coupling of the octet (associated with mesons, and remains the correct representation at large N_c) to the large N_c analogs of the baryon octet and decuplet representations.

DOI: 10.1103/PhysRevD.70.096015

PACS numbers: 11.15.Pg, 11.30.Hv, 13.75.Gx, 13.75.Jz

I. INTRODUCTION

The large N_c limit of QCD and the $1/N_c$ expansion have proven to be valuable methods to obtain qualitative insights into the structure and interactions of mesons [1] and baryons [2]. The approach became quantitatively valuable in the study of baryon properties when it was realized that large N_c consistency conditions impose on the low-lying baryonic states a contracted $SU(2N_f)$ spin-flavor symmetry that holds up to corrections subleading in $1/N_c$ [3,4]. This symmetry allows one to relate certain baryonic properties in a model-independent way, thereby providing real predictive power.

The best known applications of this method have occurred in the study of properties of the ground-state band of baryons—those baryons (such as the Δ) that become degenerate with the nucleon in the large N_c limit of QCD. These states all fall into a single representation of contracted $SU(2N_f)$, providing (for example) a field-theoretical explanation for group-theoretical results of the old $SU(6)$ spin-flavor symmetry. This means that matrix elements of *any* operator contributing at leading order in the $1/N_c$ expansion between *any* states in the band can be related to matrix elements of the same operator between other states in the band, by purely group-theoretical means. Corrections are typically $O(1/N_c)$, but for certain cases they are only $O(1/N_c^2)$ [4]. This allows for concrete, albeit approximate, predictions, which appear to work at the level one would estimate from the size of the neglected corrections [5]. This general formalism has been developed both for two and three flavors [6–8]. Of course, when working with three flavors, one must include the effects of SU(3) flavor breaking, which can yield corrections comparable to $1/N_c$ corrections [5].

At a technical level, the symmetry has been implemented in a number of ways. One is to solve algebraically

the commutation relations that arise from the consistency conditions [3,4]. A particularly elegant method for accomplishing this is via the formalism of induced representations [6]. An alternative, and somewhat more pragmatic approach has been to map the full commutation relations onto those of a simple constituent quark model and then to solve using quark model operators. Since it is straightforward to do the N_c counting for all of the operators that arise in the quark model, one can quickly and efficiently derive the results [7] of the symmetry. Although this approach is expressed in quark model language, it does not rely on any dynamical assumptions of the quark model.

Recently, a number of groups have developed techniques to extend this analysis to excited states of the baryons [9–17]. This is an important problem. Although the properties of excited baryons have been studied in models for a very long time, the connection of the models to QCD remains obscure. Thus, there is a need for reliable model-independent predictions tied directly to QCD; large N_c analysis provides such a tool. The methods used so far include a direct implementation of the consistency rules on excited states [10], an operator analysis in the quark model language [9,11–15,17], and a new approach based on the study of resonances in the context of model-independent relations between meson-baryon scattering amplitudes [16]. The last approach has an important advantage over the others in that it does not implicitly assume excited states to be stable at large N_c . This is significant since widths of excited baryons are generically $O(N_c^0)$, and there is no reason to assume that this generic behavior does not apply to the excited states of interest [18].

To date, this general analysis has been confined to studies of two flavors, although recently large N_c analysis has been performed for some exotic states of fixed strangeness [19] and for exotic baryons containing heavy quarks [20]. Of course, it is not completely surprising that the two-flavor case was explored before the three-flavor

*Electronic address: cohen@physics.umd.edu

†Electronic address: Richard.Lebed@asu.edu

case, since generically it is easier to work with flavor SU(2) than flavor SU(3). Indeed, this feature is particularly prominent for studies at large N_c QCD. The reason for this is rather simple: The SU(2) flavor representations that arise for low-lying states of baryons in large N_c QCD are identical to the representations that occur for $N_c = 3$, while the SU(3) flavor representations that arise for low-lying states of baryons in large N_c QCD are quite different from the representations that occur for $N_c = 3$ [6,21]. This in turn means that explicit dependence on N_c can arise in matrix elements directly from the N_c dependence of the representation and not just from the operator in question. As noted in Ref. [22] in the context of exotic states, the explicit N_c dependence in the matrix elements can be critical in obtaining the correct N_c scaling of an observable.

To make progress on the general three-flavor problem using either the direct large N_c consistency condition method of Ref. [10] or via the model-independent meson-baryon scattering amplitudes of Ref. [16], one needs to possess the relevant SU(3) Clebsch-Gordan coefficients (CGCs) for the coupling of a flavor octet meson with the representation of the baryon of interest. Such coefficients are also of use in computing matrix elements in the quark model basis. While SU(3) Clebsch-Gordan tables exist for fairly large representations [23] and computer codes exist for generating them for arbitrary (but fixed) representations [24], it is extremely useful in conducting a large N_c analysis to have explicit formulas for the relevant coefficients in a form that makes the N_c dependence manifest. The purpose of the present paper is to present such formulas to facilitate progress in the field.

As noted above, that since baryons for arbitrary N_c contain N_c valence quarks, the corresponding baryon SU(3) representations also grow in size with N_c [6,21]. We wish to identify these large N_c representations with their $N_c = 3$ counterparts. To keep our notation simple and aid in the extrapolation to $N_c = 3$, we use quotes to denote the generalized SU(3) representations familiar from $N_c = 3$:

$$\begin{aligned} \text{"1"} &\equiv [0, (N_c - 3)/2], & \text{"8"} &\equiv [1, (N_c - 1)/2], \\ \text{"10"} &\equiv [3, (N_c - 3)/2], & \overline{\text{"10"}} &\equiv [0, (N_c + 3)/2], \\ \text{"27"} &\equiv [2, (N_c + 1)/2], & \text{"35"} &\equiv [4, (N_c - 1)/2], \end{aligned} \quad (1.1)$$

while representations with that do not appear for $N_c = 3$ are left in the standard (p, q) weight form.

In this work we focus on those representations that can decay into the ordinary ground-state baryons by the emission of an octet meson. Since in the real world of $N_c = 3$, the only ground-state baryons that occur are the "8" and "10," we present only the Clebsch-Gordan coefficients for those representations that couple to the "8" and "10" via an 8.

II. CONVENTIONS AND METHOD OF CALCULATION

Following the usual convention, we write the SU(3) CGCs as products of ordinary SU(2) CGCs and *isoscalar factors*. Isoscalar factors [25] are the portions of SU(3) CGCs that do not depend upon the isospin additive quantum numbers I_z . The full SU(3) CGC is factored into a product of the pure isospin SU(2) CGC and an isoscalar factor:

$$\left(\begin{array}{cc|c} R_1 & R_2 & R_\gamma \\ I_1, I_{1z}, Y_1 & I_2, I_{2z}, Y_2 & I, I_z, Y \end{array} \right) = \left(\begin{array}{cc|c} R_1 & R_2 & R_\gamma \\ I_1, Y_1 & I_2, Y_2 & I, Y \end{array} \right) \left(\begin{array}{cc|c} I_1 & I_2 & I \\ I_{1z} & I_{2z} & I_z \end{array} \right), \quad (2.1)$$

where the label R indicates the SU(3) representation, which may be denoted using the usual weight diagram notation (p, q) . γ labels degenerate representations occurring in a given product. Since both the full SU(3) and SU(2) CGCs form unitary matrices, so do the isoscalar factors.

Presented here are tables of isoscalar factors for the products

$$\begin{aligned} \text{"8"} \otimes 8 &= \text{"27"} \oplus \text{"10"} \oplus \overline{\text{"10"}} \oplus \text{"8}_1 \oplus \text{"8}_2 \oplus \text{"1"} \oplus [2, (N_c - 5)/2], \\ \text{"10"} \otimes 8 &= \text{"35"} \oplus \text{"27"} \oplus \text{"10}_1 \oplus \text{"10}_2 \oplus \text{"8"} \oplus [5, (N_c - 5)/2] \oplus [2, (N_c - 5)/2] \oplus [4, (N_c - 7)/2]. \end{aligned} \quad (2.2)$$

Factors for product representations that do not occur for $N_c = 3$ are suppressed because they are not useful in the phenomenological analysis. As a result, in some cases the matrices presented below represent only a corner of a fully unitary matrix.

The SU(3) CGCs are computed by essentially the same procedure as SU(2) CGCs: One uses the matrix elements of the raising and lowering operators. In the case of SU(3), these are given by [26,27]

$$\begin{aligned}
U_+|(p, q)II_z Y\rangle &= +g[(p, q), I, -I_z, Y](p, q), I + 1/2, I_z - 1/2, Y + 1\rangle - g[(p, q), -(I + 1), -I_z, Y](p, q), \\
&\quad I - 1/2, I_z - 1/2, Y + 1\rangle, \\
U_-|(p, q)II_z Y\rangle &= -g[(p, q), -(I + 3/2), -(I_z + 1/2), Y - 1](p, q), I + 1/2, I_z + 1/2, Y - 1\rangle \\
&\quad + g[(p, q), I - 1/2, -(I_z + 1/2), Y - 1](p, q), I - 1/2, I_z + 1/2, Y - 1\rangle, \\
V_+|(p, q)II_z Y\rangle &= +g[(p, q), I, I_z, Y](p, q), I + 1/2, I_z + 1/2, Y + 1\rangle + g[(p, q), -(I + 1), I_z, Y](p, q), \\
&\quad I - 1/2, I_z + 1/2, Y + 1\rangle, \\
V_-|(p, q)II_z Y\rangle &= +g[(p, q), -(I + 3/2), I_z - 1/2, Y - 1](p, q), I + 1/2, I_z - 1/2, Y - 1\rangle \\
&\quad + g[(p, q), I - 1/2, I_z - 1/2, Y - 1](p, q), I - 1/2, I_z - 1/2, Y - 1\rangle,
\end{aligned} \tag{2.3}$$

where the function g is given by

$$g[(p, q)II_z Y] = \left\{ \frac{(I + I_z + 1) \left[\frac{1}{3}(p - q) + I + \frac{Y}{2} + 1 \right] \left[\frac{1}{3}(p + 2q) + I + \frac{Y}{2} + 2 \right] \left[\frac{1}{3}(2p + q) - I - \frac{Y}{2} \right]}{(2I + 1)(2I + 2)} \right\}^{1/2}. \tag{2.4}$$

One obtains recursion relations among SU(3) CGCs by combining two states (the usual Clebsch-Gordan series) and computing matrix elements of $U_{\pm} = U_{1\pm} + U_{2\pm}$ and $V_{\pm} = V_{1\pm} + V_{2\pm}$. Of course, this is entirely analogous to the manner in which one computes SU(2) CGCs, in that case using T_{\pm} matrix elements. The recursion relations in the SU(3) case are however much more involved, generically involving six CGCs rather than the three of the SU(2) case. For example, the recursion relation for the generator V_+ reads

$$\begin{aligned}
\langle (p_1, q_1)I_1 I_{1z} Y_1 | \langle (p_2, q_2)I_2 I_{2z} Y_2 | V_+ | (p, q)II_z Y \rangle &= +g[(p, q), I, I_z, Y] \begin{pmatrix} (p_1, q_1) & (p_2, q_2) \\ I_1 Y_1 & I_2 Y_2 \end{pmatrix} \left\| \begin{pmatrix} p, q \\ I + \frac{1}{2}, Y + 1 \end{pmatrix} \right. \\
&\times \begin{pmatrix} I_1 & I_2 \\ I_{1z} & I_{2z} \end{pmatrix} \left\| \begin{pmatrix} I + \frac{1}{2} \\ I_z + \frac{1}{2} \end{pmatrix} \right. + g[(p, q), -(I + 1), I_z, Y] \\
&\times \begin{pmatrix} (p_1, q_1) & (p_2, q_2) \\ I_1 Y_1 & I_2 Y_2 \end{pmatrix} \left\| \begin{pmatrix} p, q \\ I - \frac{1}{2}, Y + 1 \end{pmatrix} \right. \begin{pmatrix} I_1 & I_2 \\ I_{1z} & I_{2z} \end{pmatrix} \left\| \begin{pmatrix} I - \frac{1}{2} \\ I_z + \frac{1}{2} \end{pmatrix} \right. \\
&= +g[(p_1, q_1), -(I_1 + 3/2), I_{1z} - 1/2, Y_1 - 1] \\
&\times \begin{pmatrix} (p_1, q_1) & (p_2, q_2) \\ I_1 + \frac{1}{2}, Y_1 - 1 & I_2 Y_2 \end{pmatrix} \left\| \begin{pmatrix} p, q \\ IY \end{pmatrix} \begin{pmatrix} I_1 + \frac{1}{2} & I_2 \\ I_{1z} - \frac{1}{2} & I_{2z} \end{pmatrix} \left\| \begin{pmatrix} I \\ I_z \end{pmatrix} \right. \\
&+ g[(p_1, q_1), I_1 - 1/2, I_{1z} - 1/2, Y_1 - 1] \\
&\times \begin{pmatrix} (p_1, q_1) & (p_2, q_2) \\ I_1 - \frac{1}{2}, Y_1 - 1 & I_2 Y_2 \end{pmatrix} \left\| \begin{pmatrix} p, q \\ IY \end{pmatrix} \begin{pmatrix} I_1 - \frac{1}{2} & I_2 \\ I_{1z} - \frac{1}{2} & I_{2z} \end{pmatrix} \left\| \begin{pmatrix} I \\ I_z \end{pmatrix} \right. \\
&+ g[(p_2, q_2), -(I_2 + 3/2), I_{2z} - 1/2, Y_2 - 1] \\
&\times \begin{pmatrix} (p_1, q_1) & (p_2, q_2) \\ I_1 Y_1 & I_2 + \frac{1}{2}, Y_2 - 1 \end{pmatrix} \left\| \begin{pmatrix} p, q \\ IY \end{pmatrix} \begin{pmatrix} I_1 & I_2 + \frac{1}{2} \\ I_{1z} & I_{2z} - \frac{1}{2} \end{pmatrix} \left\| \begin{pmatrix} I \\ I_z \end{pmatrix} \right. \\
&+ g[(p_2, q_2), I_2 - 1/2, I_{2z} - 1/2, Y_2 - 1] \\
&\times \begin{pmatrix} (p_1, q_1) & (p_2, q_2) \\ I_1 Y_1 & I_2 - \frac{1}{2}, Y_2 - 1 \end{pmatrix} \left\| \begin{pmatrix} p, q \\ IY \end{pmatrix} \begin{pmatrix} I_1 & I_2 - \frac{1}{2} \\ I_{1z} & I_{2z} - \frac{1}{2} \end{pmatrix} \left\| \begin{pmatrix} I \\ I_z \end{pmatrix} \right.
\end{aligned} \tag{2.5}$$

Therefore, one must be much more judicious in the choice of quantum numbers than in the SU(2) case in order to obtain relations that contain a minimum number of unknown CGCs. In practice, one begins in the small “corners” of the SU(3) weight diagrams, where fewer states contribute to the recursion relations; this is analogous to starting with “stretched” states in the SU(2) case.

Since the recursion relations are homogeneous in CGCs, the next necessary step is to obtain normalization conditions. Here one uses the unitarity of the CGC matrices, not only to set the scale of the CGCs, but also to obtain some of the coefficients via orthogonality condi-

tions. Finally, in order to obtain a unique value for the CGCs, one must impose a phase condition. We use the standard Condon-Shortley [28] convention for the SU(2) CGCs, which amounts to

$$\begin{pmatrix} I_1 & I_2 \\ I_1 & I - I_1 \end{pmatrix} \left\| \begin{pmatrix} I \\ I \end{pmatrix} \right. > 0. \tag{2.6}$$

The phase of the SU(3) CGCs is determined by the de Swart convention [29], which amounts to

$$\begin{pmatrix} R_1 & R_2 \\ \tilde{I}_1 \tilde{Y}_1 & \tilde{I}_2, Y - \tilde{Y}_1 \end{pmatrix} \left\| \begin{pmatrix} R_Y \\ I_{\max} Y \end{pmatrix} \right. > 0. \tag{2.7}$$

TABLE I: Isoscalar factors of the form

$$\left(\begin{array}{c} \text{"8"} \\ I_1 Y_1 \end{array} \begin{array}{c} 8 \\ I_2 Y_2 \end{array} \parallel \begin{array}{c} \mu_\gamma \\ IY \end{array} \right):$$

$$Y = \frac{N_c}{3} + 1, I = 1$$

$I_1, Y_1; I_2, Y_2$	"27"	μ_γ
$\frac{1}{2}, \frac{N_c}{3}; \frac{1}{2}, +1$	+1	

$$Y = \frac{N_c}{3} + 1, I = 0$$

$I_1, Y_1; I_2, Y_2$	"10"	μ_γ
$\frac{1}{2}, \frac{N_c}{3}; \frac{1}{2}, +1$	-1	

$$Y = \frac{N_c}{3}, I = \frac{3}{2}$$

$I_1, Y_1; I_2, Y_2$	"27"	"10"	μ_γ
$\frac{1}{2}, \frac{N_c}{3}; 1, 0$	$+\sqrt{\frac{2}{N_c+1}}$	$-\sqrt{\frac{N_c-1}{N_c+1}}$	
$1, \frac{N_c}{3}-1; \frac{1}{2}, +1$	$+\sqrt{\frac{N_c-1}{N_c+1}}$	$+\sqrt{\frac{2}{N_c+1}}$	

$$Y = \frac{N_c}{3} - 1, I = 2$$

$I_1, Y_1; I_2, Y_2$	"27"	μ_γ
$1, \frac{N_c}{3}-1; 1, 0$	$+\frac{2}{\sqrt{N_c+1}}$	

$$Y = \frac{N_c}{3}, I = \frac{1}{2}$$

$I_1, Y_1; I_2, Y_2$	"27"	"10"	"8 ₁ "	"8 ₂ "	μ_γ
$\frac{1}{2}, \frac{N_c}{3}; 1, 0$	$+\frac{1}{\sqrt{2(N_c+7)}}$	$-\sqrt{\frac{3}{2(N_c+3)}}$	$+3\sqrt{\frac{(N_c-1)(N_c+3)}{2D}}$	$+\frac{N_c^2+8N_c+27}{\sqrt{2D(N_c+7)(N_c+3)}}$	
$1, \frac{N_c}{3}-1; \frac{1}{2}, +1$	$-\frac{1}{2}\sqrt{\frac{N_c-1}{N_c+7}}$	$-\frac{1}{2}\sqrt{\frac{3(N_c-1)}{N_c+3}}$	$-3\sqrt{\frac{N_c+3}{D}}$	$+\frac{2(2N_c+9)\sqrt{N_c-1}}{\sqrt{D(N_c+7)(N_c+3)}}$	
$\frac{1}{2}, \frac{N_c}{3}; 0, 0$	$+\frac{3}{\sqrt{2(N_c+7)}}$	$+\sqrt{\frac{3}{2(N_c+3)}}$	$-\sqrt{\frac{(N_c-1)(N_c+3)}{2D}}$	$+\frac{3(N_c^2+4N_c-1)}{\sqrt{2D(N_c+7)(N_c+3)}}$	
$0, \frac{N_c}{3}-1; \frac{1}{2}, +1$	$+\frac{1}{2}\sqrt{\frac{3(N_c+3)}{N_c+7}}$	$-\frac{1}{2}$	$-\sqrt{\frac{3(N_c-1)}{D}}$	$-\frac{2(N_c+2)\sqrt{3}}{\sqrt{D(N_c+7)}}$	

$$Y = \frac{N_c}{3} - 1, I = 1$$

$I_1, Y_1; I_2, Y_2$	"27"	"10"	"10"	"8 ₁ "	"8 ₂ "	μ_γ
$\frac{1}{2}, \frac{N_c}{3}; \frac{1}{2}, -1$	$+2\sqrt{\frac{2}{(N_c+7)(N_c+1)}}$	$+\frac{2}{\sqrt{(N_c+3)(N_c+1)}}$	$-2\sqrt{\frac{2(N_c-1)}{3(N_c+5)(N_c+1)}}$	$-\sqrt{\frac{6(N_c+3)}{D}}$	$+\frac{2(2N_c+9)\sqrt{2(N_c-1)}}{\sqrt{3D(N_c+7)(N_c+3)}}$	
$\frac{1}{2}, \frac{N_c}{3}-2; \frac{1}{2}, +1$	$+\sqrt{\frac{2(N_c+3)(N_c-1)}{3(N_c+7)(N_c+1)}}$	$-\sqrt{\frac{N_c-1}{3(N_c+1)}}$	$+\frac{2}{3}\sqrt{\frac{2(N_c+3)}{(N_c+5)(N_c+1)}}$	$-2N_c\sqrt{\frac{2}{D(N_c-1)}}$	$-\frac{7N_c+9}{3}\sqrt{\frac{2}{D(N_c+7)}}$	
$1, \frac{N_c}{3}-1; 1, 0$	0	$-\sqrt{\frac{2(N_c-1)}{(N_c+3)(N_c+1)}}$	$+\sqrt{\frac{N_c+1}{3(N_c+5)}}$	$+(N_c-3)\sqrt{\frac{3(N_c+3)}{D(N_c-1)}}$	$+(N_c+9)\sqrt{\frac{N_c+7}{3D(N_c+3)}}$	
$1, \frac{N_c}{3}-1; 0, 0$	$+\sqrt{\frac{6(N_c-1)}{(N_c+7)(N_c+1)}}$	$+\sqrt{\frac{3(N_c-1)}{(N_c+3)(N_c+1)}}$	$+2\sqrt{\frac{2}{(N_c+5)(N_c+1)}}$	$-(N_c-7)\sqrt{\frac{N_c+3}{2D(N_c-1)}}$	$+\frac{(N_c-3)(3N_c+13)}{\sqrt{2D(N_c+7)(N_c+3)}}$	
$0, \frac{N_c}{3}-1; 1, 0$	$+\sqrt{\frac{2(N_c+3)}{(N_c+7)(N_c+1)}}$	$-\frac{1}{\sqrt{N_c+1}}$	$-\sqrt{\frac{2(N_c+3)(N_c-1)}{3(N_c+5)(N_c+1)}}$	$+(N_c+1)\sqrt{\frac{3}{2D}}$	$+(N_c-3)\sqrt{\frac{(N_c-1)}{6D(N_c+7)}}$	

$$Y = \frac{N_c}{3} - 1, I = 0$$

$I_1, Y_1; I_2, Y_2$	"27"	"8 ₁ "	"8 ₂ "	"1"	μ_γ
$\frac{1}{2}, \frac{N_c}{3}; \frac{1}{2}, -1$	$+2\sqrt{\frac{3}{(N_c+7)(N_c+5)}}$	$+\sqrt{\frac{6(N_c-1)}{D}}$	$+\frac{2(N_c+2)\sqrt{6}}{\sqrt{D(N_c+7)}}$	$+\sqrt{\frac{N_c-1}{N_c+5}}$	
$\frac{1}{2}, \frac{N_c}{3}-2; \frac{1}{2}, +1$	$-\sqrt{\frac{(N_c+3)(N_c-1)}{(N_c+7)(N_c+5)}}$	$-6\sqrt{\frac{2}{D(N_c+3)}}$	$+5\sqrt{\frac{2(N_c+3)(N_c-1)}{D(N_c+7)}}$	$-2\sqrt{\frac{3}{(N_c+5)(N_c+3)}}$	
$1, \frac{N_c}{3}-1; 1, 0$	$-\sqrt{\frac{N_c-1}{(N_c+7)(N_c+5)}}$	$-\frac{3(N_c+1)}{\sqrt{2D}}$	$-(N_c-3)\sqrt{\frac{N_c-1}{2D(N_c+7)}}$	$+\sqrt{\frac{3}{N_c+5}}$	
$0, \frac{N_c}{3}-1; 0, 0$	$+3\sqrt{\frac{N_c+3}{(N_c+7)(N_c+5)}}$	$-(N_c+9)\sqrt{\frac{N_c-1}{2D(N_c+3)}}$	$+3(N_c-3)\sqrt{\frac{N_c+3}{2D(N_c+7)}}$	$-\sqrt{\frac{3(N_c-1)}{(N_c+5)(N_c+3)}}$	

$$Y = \frac{N_c}{3} - 2, I = \frac{3}{2}$$

$I_1, Y_1; I_2, Y_2$	"27"	"10"	μ_γ
$\frac{1}{2}, \frac{N_c}{3}-2; 1, 0$	$+\sqrt{\frac{10(N_c+3)}{3(N_c+7)(N_c+1)}}$	$-\sqrt{\frac{2}{N_c+1}}$	
$1, \frac{N_c}{3}-1; \frac{1}{2}, -1$	$+2\sqrt{\frac{5}{(N_c+7)(N_c+1)}}$	$+2\sqrt{\frac{3}{(N_c+3)(N_c+1)}}$	

$$Y = \frac{N_c}{3} - 2, I = \frac{1}{2}$$

$I_1, Y_1; I_2, Y_2$	"27"	"10"	"8 ₁ "	"8 ₂ "	μ_γ
$\frac{1}{2}, \frac{N_c}{3}-2; 1, 0$	$-2\sqrt{\frac{(N_c+3)(N_c-1)}{3(N_c+7)(N_c+5)(N_c+1)}}$	$+\frac{2N_c\sqrt{2}}{3\sqrt{(N_c+5)(N_c+1)}}$	$-\frac{(N_c^2+4N_c+15)}{\sqrt{2D(N_c+3)(N_c-1)}}$	$-\frac{N_c-33}{3}\sqrt{\frac{N_c+3}{2D(N_c+7)}}$	
$1, \frac{N_c}{3}-1; \frac{1}{2}, -1$	$+2\sqrt{\frac{2(N_c-1)}{(N_c+7)(N_c+5)(N_c+1)}}$	$+2\sqrt{\frac{N_c+3}{3(N_c+5)(N_c+1)}}$	$+\frac{2N_c\sqrt{3}}{\sqrt{D(N_c-1)}}$	$+\frac{7N_c+9}{\sqrt{3D(N_c+7)}}$	
$\frac{1}{2}, \frac{N_c}{3}-2; 0, 0$	$+2\sqrt{\frac{3(N_c+3)(N_c-1)}{(N_c+7)(N_c+5)(N_c+1)}}$	$+2\sqrt{\frac{2}{(N_c+5)(N_c+1)}}$	$-\frac{(N_c^2+8N_c-21)}{\sqrt{2D(N_c+3)(N_c-1)}}$	$-(3N_c-19)\sqrt{\frac{N_c+3}{2D(N_c+7)}}$	
$0, \frac{N_c}{3}-1; \frac{1}{2}, -1$	$+2\sqrt{\frac{6(N_c+3)}{(N_c+7)(N_c+5)(N_c+1)}}$	$-2\sqrt{\frac{N_c-1}{(N_c+5)(N_c+1)}}$	$-\frac{6}{\sqrt{D(N_c+3)}}$	$+5\sqrt{\frac{(N_c+3)(N_c-1)}{D(N_c+7)}}$	

$$Y = \frac{N_c}{3} - 3, I = 1$$

$I_1, Y_1; I_2, Y_2$	"27"	μ_γ
$\frac{1}{2}, \frac{N_c}{3}-2; \frac{1}{2}, -1$	$+4\sqrt{\frac{10(N_c+3)}{3(N_c+7)(N_c+5)(N_c+1)}}$	

$$Y = \frac{N_c}{3} - 3, I = 0$$

$I_1, Y_1; I_2, Y_2$	"10"	μ_γ
$\frac{1}{2}, \frac{N_c}{3}-2; \frac{1}{2}, -1$	$+2\sqrt{\frac{2}{N_c+5}}$	

TABLE II: Isoscalar factors of the form

$$\begin{pmatrix} "10" & 8 \\ I_1 Y_1 & I_2 Y_2 \end{pmatrix} \mu_\gamma$$

$Y = \frac{N_c}{3} + 1, I = 2$					$Y = \frac{N_c}{3} + 1, I = 1$					$Y = \frac{N_c}{3}, I = \frac{5}{2}$				
$I_1, Y_1; I_2, Y_2$	"35"	μ_γ	$I_1, Y_1; I_2, Y_2$	"27"	μ_γ	$I_1, Y_1; I_2, Y_2$	"35"	μ_γ	$I_1, Y_1; I_2, Y_2$	"35"	μ_γ	$I_1, Y_1; I_2, Y_2$	"35"	μ_γ
$\frac{3}{2}, \frac{N_c}{3}; \frac{1}{2}, +1$	+1		$\frac{3}{2}, \frac{N_c}{3}; \frac{1}{2}, +1$	-1		$\frac{3}{2}, \frac{N_c}{3}; 1, 0$	$+\sqrt{\frac{2}{N_c-1}}$		$\frac{3}{2}, \frac{N_c}{3}; 1, 0$	$+\sqrt{\frac{2}{N_c-1}}$		$\frac{3}{2}, \frac{N_c}{3}; 1, 0$	$+\sqrt{\frac{2}{N_c-1}}$	

$Y = \frac{N_c}{3}, I = \frac{3}{2}$				
$I_1, Y_1; I_2, Y_2$	"35"	"27"	"10 ₂ "	μ_γ
$\frac{3}{2}, \frac{N_c}{3}; 1, 0$	$+\frac{1}{2}\sqrt{\frac{3}{N_c+9}}$	$-\frac{1}{2}\sqrt{\frac{5}{N_c+1}}$	$+\frac{N_c^2+8N_c+27}{\sqrt{2D(N_c+9)(N_c+1)}}$	
$\frac{3}{2}, \frac{N_c}{3}; 0, 0$	$+\frac{1}{2}\sqrt{\frac{15}{N_c+9}}$	$+\frac{3}{2\sqrt{N_c+1}}$	$+\frac{(N_c^2+4N_c-9)\sqrt{5}}{\sqrt{2D(N_c+9)(N_c+1)}}$	
$1, \frac{N_c}{3}-1; \frac{1}{2}, +1$	$+\frac{1}{4}\sqrt{\frac{15(N_c+5)}{N_c+9}}$	$-\frac{1}{4}\sqrt{\frac{N_c+5}{N_c+1}}$	$-N_c\sqrt{\frac{10(N_c+5)}{D(N_c+9)(N_c+1)}}$	

$Y = \frac{N_c}{3}, I = \frac{1}{2}$					$Y = \frac{N_c}{3}-1, I = 2$				
$I_1, Y_1; I_2, Y_2$	"27"	"8"	μ_γ		$I_1, Y_1; I_2, Y_2$	"35"	"27"	μ_γ	
$\frac{3}{2}, \frac{N_c}{3}; 1, 0$	$-\sqrt{\frac{2}{N_c+7}}$	$-\sqrt{\frac{N_c+5}{N_c+7}}$			$1, \frac{N_c}{3}-1; 1, 0$	$+\frac{3}{2}\sqrt{\frac{N_c+5}{(N_c+9)(N_c-1)}}$	$-\frac{1}{2}\sqrt{\frac{N_c+5}{(N_c+1)(N_c-1)}}$		
$1, \frac{N_c}{3}-1; \frac{1}{2}, +1$	$-\sqrt{\frac{N_c+5}{N_c+7}}$	$+\sqrt{\frac{2}{N_c+7}}$			$\frac{3}{2}, \frac{N_c}{3}; \frac{1}{2}, -1$	$+\sqrt{\frac{6}{(N_c+9)(N_c-1)}}$	$+\sqrt{\frac{6}{(N_c+1)(N_c-1)}}$		

$Y = \frac{N_c}{3} - 1, I = 1$					
$I_1, Y_1; I_2, Y_2$	"35"	"27"	"10 ₂ "	"8"	μ
$1, \frac{N_c}{3} - 1; 1, 0$	$+\frac{1}{2}\sqrt{\frac{5(N_c+5)}{(N_c+9)(N_c+7)}}$	$-\frac{3}{2}\sqrt{\frac{N_c+5}{(N_c+7)(N_c+1)}}$	$+\frac{(N_c^2+4N_c+27)\sqrt{5}}{2\sqrt{3D(N_c+9)(N_c+1)}}$	$-\frac{N_c+1}{\sqrt{6(N_c+7)(N_c-1)}}$	
$1, \frac{N_c}{3} - 1; 0, 0$	$+\sqrt{\frac{15(N_c+5)}{2(N_c+9)(N_c+7)}}$	$+\sqrt{\frac{3(N_c+5)}{2(N_c+7)(N_c+1)}}$	$+\frac{(N_c+3)(N_c-3)\sqrt{5}}{\sqrt{2D(N_c+9)(N_c+1)}}$	$-\frac{2}{\sqrt{(N_c+7)(N_c-1)}}$	
$\frac{1}{2}, \frac{N_c}{3} - 2; \frac{1}{2}, +1$	$+\sqrt{\frac{5(N_c+5)(N_c+3)}{6(N_c+9)(N_c+7)}}$	$-\sqrt{\frac{(N_c+5)(N_c+3)}{6(N_c+7)(N_c+1)}}$	$-\frac{4N_c}{3}\sqrt{\frac{10(N_c+3)}{D(N_c+9)(N_c+1)}}$	$+\frac{2}{3}\sqrt{\frac{N_c+3}{(N_c+7)(N_c-1)}}$	
$\frac{3}{2}, \frac{N_c}{3}; \frac{1}{2}, -1$	$+\sqrt{\frac{10}{(N_c+9)(N_c+7)}}$	$+\sqrt{\frac{2}{(N_c+7)(N_c+1)}}$	$+2N_c\sqrt{\frac{10(N_c+5)}{3D(N_c+9)(N_c+1)}}$	$+2\sqrt{\frac{N_c+5}{3(N_c+7)(N_c-1)}}$	

$Y = \frac{N_c}{3}-1, I = 0$				
$I_1, Y_1; I_2, Y_2$	"27"	"8"	μ_γ	
$1, \frac{N_c}{3}-1; 1, 0$	$-\frac{2}{\sqrt{N_c+7}}$	$-\sqrt{\frac{N_c+3}{N_c+7}}$		
$\frac{1}{2}, \frac{N_c}{3}-2; \frac{1}{2}, +1$	$-\sqrt{\frac{N_c+3}{N_c+7}}$	$+\frac{2}{\sqrt{N_c+7}}$		

$Y = \frac{N_c}{3}-2, I = \frac{3}{2}$				
$I_1, Y_1; I_2, Y_2$	"35"	"27"	μ_γ	
$\frac{1}{2}, \frac{N_c}{3}-2; 1, 0$	$+\sqrt{\frac{5(N_c+5)(N_c+3)}{2(N_c+9)(N_c+7)(N_c-1)}}$	$-\sqrt{\frac{5(N_c+5)(N_c+3)}{6(N_c+7)(N_c+1)(N_c-1)}}$		
$1, \frac{N_c}{3}-1; \frac{1}{2}, -1$	$+\sqrt{\frac{15(N_c+5)}{(N_c+9)(N_c+7)(N_c-1)}}$	$+\sqrt{\frac{5(N_c+5)}{(N_c+7)(N_c+1)(N_c-1)}}$		

$Y = \frac{N_c}{3}-2, I = \frac{1}{2}$					
$I_1, Y_1; I_2, Y_2$	"35"	"27"	"10 ₂ "	"8"	μ
$\frac{1}{2}, \frac{N_c}{3}-2; 1, 0$	$+\frac{1}{2}\sqrt{\frac{5(N_c+3)}{(N_c+9)(N_c+7)}}$	$-\frac{7}{2}\sqrt{\frac{N_c+3}{3(N_c+7)(N_c+1)}}$	$+\frac{(N_c^2+27)\sqrt{5}}{3\sqrt{2D(N_c+9)(N_c+1)}}$	$-\frac{2N_c}{3\sqrt{(N_c+7)(N_c+1)}}$	
$\frac{1}{2}, \frac{N_c}{3}-2; 0, 0$	$+\frac{3}{2}\sqrt{\frac{5(N_c+3)}{(N_c+9)(N_c+7)}}$	$+\frac{1}{2}\sqrt{\frac{3(N_c+3)}{(N_c+7)(N_c+1)}}$	$+\frac{(N_c^2-4N_c-9)\sqrt{5}}{\sqrt{2D(N_c+9)(N_c+1)}}$	$-\frac{2}{\sqrt{(N_c+7)(N_c-1)}}$	
$0, \frac{N_c}{3}-3; \frac{1}{2}, +1$	$+\frac{1}{2}\sqrt{\frac{5(N_c+3)(N_c+1)}{2(N_c+9)(N_c+7)}}$	$-\frac{1}{2}\sqrt{\frac{3(N_c+3)}{2(N_c+7)}}$	$-2N_c\sqrt{\frac{5}{D(N_c+9)}}$	$+\sqrt{\frac{2(N_c+1)}{(N_c+7)(N_c-1)}}$	
$1, \frac{N_c}{3}-1; \frac{1}{2}, -1$	$+\sqrt{\frac{30}{(N_c+9)(N_c+7)}}$	$+\sqrt{\frac{2}{(N_c+7)(N_c+1)}}$	$+4N_c\sqrt{\frac{5(N_c+3)}{3D(N_c+9)(N_c+1)}}$	$+\sqrt{\frac{2(N_c+3)}{3(N_c+7)(N_c-1)}}$	

$Y = \frac{N_c}{3}-3, I = 1$				
$I_1, Y_1; I_2, Y_2$	"35"	"27"	μ_γ	
$0, \frac{N_c}{3}-3; 1, 0$	$+\sqrt{\frac{5(N_c+3)(N_c+1)}{2(N_c+9)(N_c+7)(N_c-1)}}$	$-\sqrt{\frac{5(N_c+3)}{2(N_c+7)(N_c-1)}}$		
$\frac{1}{2}, \frac{N_c}{3}-2; \frac{1}{2}, -1$	$+\sqrt{\frac{30(N_c+3)}{(N_c+9)(N_c+7)(N_c-1)}}$	$+\sqrt{\frac{10(N_c+3)}{3(N_c+7)(N_c+1)(N_c-1)}}$		

$Y = \frac{N_c}{3}-3, I = 0$					$Y = \frac{N_c}{3}-4, I = \frac{1}{2}$				
$I_1, Y_1; I_2, Y_2$	"35"	"10 ₂ "	μ_γ		$I_1, Y_1; I_2, Y_2$	"35"	μ_γ		
$0, \frac{N_c}{3}-3; 0, 0$	$+\sqrt{\frac{15(N_c+1)}{(N_c+9)(N_c+7)}}$	$+(N_c-9)\sqrt{\frac{5(N_c+1)}{2D(N_c+9)}}$			$0, \frac{N_c}{3}-3; \frac{1}{2}, -1$	$+2\sqrt{\frac{15(N_c+1)}{(N_c+9)(N_c+7)(N_c-1)}}$			
$\frac{1}{2}, \frac{N_c}{3}-2; \frac{1}{2}, -1$	$+2\sqrt{\frac{15}{(N_c+9)(N_c+7)}}$	$+2N_c\sqrt{\frac{10}{D(N_c+9)}}$							

Here, I_{\max} is the highest value of isospin in the multiplet R_γ (and its selection uniquely fixes Y), \tilde{I}_1 is the largest value of isospin in R_1 that couples to this highest-weight state (and has corresponding hypercharge \tilde{Y}_1), and \tilde{I}_2 is the largest value of isospin in R_2 that couples (\tilde{I}_1, \tilde{Y}_1) to $(I_{\max} Y)$. However, even this level of description does not completely specify all phases in the case of degenerate product representations (multiple allowed values of γ). In that case, the convention used here [23,24] is defined as follows: If there are Γ values of γ , then $R_{\gamma=1}$ is defined as the representation for which

$$\left(\begin{array}{cc} R_1 & R_2 \\ \tilde{I}_1 \tilde{Y}_1 & I_2, Y - \tilde{Y}_1 \end{array} \parallel \begin{array}{c} R_\gamma \\ I_{\max} Y \end{array} \right) = 0 \quad (2.8)$$

for the $\Gamma - 1$ highest allowed values of I_2 , namely, $\tilde{I}_2, \tilde{I}_2 - 1, \dots, \tilde{I}_2 - \Gamma + 2$, but is positive for $I_2 = \tilde{I}_2 - \Gamma + 1$; this is sufficient to determine uniquely the isoscalar factors for $R_{\gamma=1}$. Next, $R_{\gamma=2}$ is defined to be the representation orthogonal to $R_{\gamma=1}$ such that Eq. (2.8) holds for the highest $\Gamma - 2$ allowed values of I_2 but is positive for $I_2 = \tilde{I}_2 - \Gamma + 2$, and so on. Only for $R_{\gamma=\Gamma}$ in the degenerate case does Eq. (2.7) hold at face value.

Note that 10_1 defined in this way vanishes for $N_c = 3$. Therefore, the tables contain only the representation 10_2 .

The tables are designed to resemble as closely as possible those of de Swart [29]; hence, we use there the notation μ_γ instead of R_γ . Indeed, they have been checked against tables in this reference for $N_c = 3$, and against the results from a computer program [24] for $N_c = 5$. Finally, in the standard convention for distinguishing degenerate representations, the current products induce a couple of more complicated factors, which we abbreviate:

$$D \equiv 5N_c^2 + 22N_c + 9, \quad (2.9)$$

$$\tilde{D} \equiv 3N_c^2 + 14N_c - 9. \quad (2.10)$$

In alternate choices for distinguishing “ $8_{1,2}$ ” and “ $10_{1,2}$,” different factors appear.

III. SU(3) ISOSCALAR FACTORS

Figures 1 and 2 in this section contain tables that compile the relevant isoscalar factors.

ACKNOWLEDGMENTS

T.D.C. was supported by the D.O.E. through Grant No. DE-FGO2-93ER-40762; R.F.L. was supported by the N.S.F. through Grant No. PHY-0140362.

-
- [1] G. 't Hooft, Nucl. Phys. **B72**, 461 (1974); G. Veneziano, Nucl. Phys. **B117**, 519 (1976).
 - [2] E. Witten, Nucl. Phys. **B160**, 57 (1979).
 - [3] J.-L. Gervais and B. Sakita, Phys. Rev. Lett. **52**, 87 (1984); Phys. Rev. D **30**, 1795 (1984).
 - [4] R. F. Dashen and A. V. Manohar, Phys. Lett. B **315**, 425 (1993); Phys. Lett. B **315**, 438 (1993).
 - [5] E. Jenkins and R. F. Lebed, Phys. Rev. D **52**, 282 (1995); J. Dai, R. F. Dashen, E. Jenkins, and A. V. Manohar, *ibid.* **53**, 273 (1996); E. Jenkins and R. F. Lebed, *ibid.* **62**, 077901 (2000); A. J. Buchmann and R. F. Lebed, *ibid.* **67**, 016002 (2003); R. F. Lebed and D. R. Martin, *ibid.* **70**, 016008 (2004); Phys. Rev. D **70**, 057901 (2004).
 - [6] R. F. Dashen, E. Jenkins, and A. V. Manohar, Phys. Rev. D **49**, 4713 (1994); **51**, 2489(E) (1995).
 - [7] R. F. Dashen, E. Jenkins, and A. V. Manohar, Phys. Rev. D **51**, 3697 (1995).
 - [8] M. A. Luty and J. March-Russell, Nucl. Phys. **B426**, 71 (1994); C. D. Carone, H. Georgi, and S. Osofsky, Phys. Lett. B **322**, 227 (1994); M. A. Luty, J. March-Russell, and M. J. White, Phys. Rev. D **51**, 2332 (1995).
 - [9] C. D. Carone, H. Georgi, L. Kaplan, and D. Morin, Phys. Rev. D **50**, 5793 (1994).
 - [10] D. Pirjol and T.-M. Yan, Phys. Rev. D **56**, 5483 (1997); **57**, 5434 (1998).
 - [11] J. L. Goity, Phys. Lett. B **414**, 140 (1997).
 - [12] C. E. Carlson, C. D. Carone, J. L. Goity, and R. F. Lebed, Phys. Lett. B **438**, 327 (1998).
 - [13] C. E. Carlson, C. D. Carone, J. L. Goity, and R. F. Lebed, Phys. Rev. D **59**, 114008 (1999).
 - [14] C. E. Carlson and C. D. Carone, Phys. Lett. B **441**, 363 (1998); Phys. Rev. D **58**, 053005 (1998); Phys. Lett. B **484**, 260 (2000).
 - [15] J. L. Goity, C. L. Schat, and N. N. Scoccola, Phys. Rev. Lett. **88**, 102002 (2002); Phys. Rev. D **66**, 114014 (2002); Phys. Lett. B **564**, 83 (2003).
 - [16] T. D. Cohen and R. F. Lebed, Phys. Rev. Lett. **91**, 012001 (2003); Phys. Rev. D **67**, 096008 (2003); **68**, 056003 (2003).
 - [17] D. Pirjol and C. L. Schat, Phys. Rev. D **67**, 096009 (2003).
 - [18] T. D. Cohen, D. C. Dakin, R. F. Lebed, and A. Nellore, Phys. Rev. D **69**, 056001 (2004).
 - [19] T. D. Cohen and R. F. Lebed, Phys. Lett. B **578**, 150 (2004).
 - [20] D. Pirjol and C. L. Schat, hep-ph/0408293; M. E. Wessling, hep-ph/0408263; E. Jenkins and A. V. Manohar, Phys. Rev. Lett. **93**, 022001 (2004); J. High Energy Phys. **06** (2004) 039.
 - [21] Z. Dulinski and M. Praszalowicz, Acta Phys. Pol. B **18**, 1157 (1988).
 - [22] T. D. Cohen, Phys. Lett. B **581**, 175 (2004); Phys. Rev. D **70**, 014011 (2004).
 - [23] T. A. Kaeding, nucl-th/9502037.
 - [24] T. A. Kaeding and H. T. Williams, Comput. Phys. Commun. **98**, 398 (1996).
 - [25] A. R. Edmonds, Proc. R. Soc. London A **268**, 436 (1963).

- [26] L. C. Biedenharn, Phys. Lett. **3**, 69 (1962); in *Proceedings of the Summer Institute for Theoretical Physics, University of Colorado, 1962*, edited by W. E. Brittin and J. Downs (Interscience Publishers, New York, 1962), Vol. 5.
- [27] K. T. Hecht, Nucl. Phys. **62**, 1 (1965).
- [28] E. U. Condon and G. H. Shortley, *The Theory of Atomic Spectra* (Cambridge University Press, Cambridge, 1935).
- [29] J. J. de Swart, Rev. Mod. Phys. **35**, 916 (1963).