

Gauged Nambu-Jona-Lasinio model with extra dimensions: Phase structure and renormalizability

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We investigate phase structure of the $D(>4)$ -dimensional gauged Nambu-Jona-Lasinio (NJL) model with $\delta(=D-4)$ extra dimensions compactified on TeV scale, based on the improved ladder Schwinger-Dyson (SD) equation in the bulk. We assume that the bulk (dimensionless) running gauge coupling in the SD equation for the $SU(N_c)$ gauge theory with N_f massless flavors is given by the truncated Kaluza-Klein effective theory and hence has a nontrivial ultraviolet fixed point (UVFP), resulting in the walking coupling. We find the critical line in the parameter space of two couplings, the gauge coupling (value fixed at the UVFP) and the (dimensionless) four-fermion coupling, which is similar to that of the gauged NJL model with fixed (walking) gauge coupling in four dimensions. It is shown that *in the presence of such walking gauge interactions* the four-fermion interactions become nontrivial and renormalizable even in higher dimensions, similar to the four dimensional gauged NJL model. Such a renormalizability/nontriviality holds only in the restricted region of the critical line (“nontrivial window”) with the gauge coupling larger than a *nonvanishing* value (“marginal triviality” point), in contrast to the four dimensional case where such a renormalizability holds for all regions of the critical line except for the pure NJL point (without gauge coupling). In the nontrivial window the renormalized effective potential yields a nontrivial interaction which is conformal invariant. The existence of the nontrivial window implies “cutoff insensitivity” of the physics prediction in spite of the ultraviolet dominance of the dynamics. In the formal limit $D \rightarrow 4$, the nontrivial window shrinks to the pure NJL point but with a nontrivial condition which coincides with the known condition of the renormalizability/nontriviality of the four dimensional gauged NJL model, $\frac{9}{2} \frac{1}{N_c} < N_f - N_c < \frac{9}{2} N_c$.

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I. INTRODUCTION

Growing attention has recently been paid to the model buildings based on the scenario of extra dimensions compactified with TeV scale [1,2]. Along such a scenario the top quark condensate (top-mode standard model: TMSM) [3–6] was reconsidered by several authors [7–12] in the versions with compactified extra dimensions, where the *ad hoc* four-fermion couplings in the original TMSM may be replaced by the strong attractive forces due to Kaluza-Klein (KK) modes of the bulk standard model (SM) gauge bosons. In particular, Arkani-Hamed, Cheng, Dobrescu and Hall (ACDH) [9] proposed a version of the TMSM where the third generation quarks/leptons as well as the SM gauge bosons are put in the bulk, while other generations in the brane. In a series of papers [10,11] we investigated the full gauge dynamics (*without four-fermion interactions*) in the bulk, based on the ladder Schwinger-Dyson (SD) equation. We further

studied [12] phenomenological implications on the ACDH scenario and found that $D = 8$ (four extra dimensions) will be a viable case which predicts a correct top mass, $m_t = 172 - 175$ GeV, and a characteristic Higgs mass, $m_H = 176 - 188$ GeV, from the requirement that only the top can condense (“topped MAC” or tMAC requirement).

Actually, the most prominent feature of the gauge theories with extra dimensions is that the theories become strongly coupled, with the coupling growing by the power running in the ultraviolet region beyond the compactification scale. Let us take a class of $SU(N_c)$ gauge theories with massless N_f flavors. It was pointed out [10] (see also [13–15]) that the bulk gauge coupling of such gauge theories can have a (nontrivial) ultraviolet fixed point (UVFP) within the truncated KK effective theory [2]. Although existence of such a UVFP is somewhat subtle in the lattice studies [16–18] and other nonperturbative approach [19], unless it is an artifact of the truncated KK effective theory, it implies existence of a renormalizable interacting gauge theory even in dimensions higher than four. It is in sharp contrast to the conventional view (based on the perturbation) that higher dimensional gauge theories are nonrenormalizable and trivial. Once we assume that such a nontrivial UVFP does exist beyond the truncated KK effective theory, the theory behaves

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near the UVFP as a *walking gauge theory* [20–24] having a large anomalous dimension $\gamma_m = D/2 - 1$ [10,11]. In the four dimensional case, as was pointed out by Bardeen *et al.* [25], the four-fermion interaction becomes a marginal operator due to the large anomalous dimension, $\gamma_m \simeq 1$, of the walking gauge theory, and hence can be mixed with the gauge interaction. Then *the phase structure of the walking gauge theories should be analyzed in a larger parameter space including the four-fermion interactions as well as the gauge interactions.*

The gauged Nambu–Jona-Lasinio (NJL) model, a model of gauge theory plus four-fermion interaction, was first studied by Bardeen, Leung and Love [25] within the ladder SD equation for the fixed gauge coupling in four dimensions. The critical line of the four dimensional model for the fixed gauge coupling was discovered by Kondo, Mino and Yamawaki [26] and independently by Appelquist, Soldate, Takeuchi and Wijewardhana [27]. Since then the model has been offering many interesting applications for physics of dynamical symmetry breaking, such as the walking technicolor [20–24], the strong ETC technicolor [28,29]/topcolor-assisted technicolor [30], and the TMSM [3–6] or topcolor [31], etc. (See for reviews Refs. [32–34].)

The characteristic dynamical feature of the four dimensional gauged NJL model is the large anomalous dimension [28], $1 \leq \gamma_m < 2$, for the fixed (walking) gauge coupling, and the finiteness of all (renormalized) couplings at composite level as well as induced Yukawa coupling of the fermion to the composite bosons in the continuum limit (infinite cutoff limit). This implies the (nonperturbative) “renormalizability” or “nontriviality” of the model, as was pointed out in Ref. [35,36] for the (“moderately”) walking gauge coupling, with $A > 1$,¹ and was subsequently shown more systematically for the fixed (walking) gauge coupling ($A \rightarrow \infty$) in Ref. [37] as well as for the moderately walking one in Ref. [37–39]. For the fixed gauge coupling, such a phenomenon takes place along the whole critical line except for the pure NJL point (without gauge coupling and “ $\gamma_m = 2$ ”) where the above couplings vanish logarithmically in the continuum limit and hence the theory becomes nonrenormalizable or trivial.

Inclusion of four-fermion interactions into the gauge theories has not been done so far in the context of the extra dimension scenario.² The analysis of the phase

¹ $A \equiv 6C_F/(-b)$, where $C_F = (N_c^2 - 1)/2N_c$ and $-b = (11N_c - 2N_f)/3$. The parameter A measures the speed of running of the running gauge coupling [22,33]: $A < 1$ is the normal running as in the QCD, while $A \gg 1$ is the walking coupling and $A \rightarrow \infty$ corresponds to the fixed (“standing” limit of the walking) gauge coupling. Combined with the asymptotic freedom, $A > 1$ reads $\frac{1}{2} \frac{1}{N_c} < N_f - N_c < \frac{1}{2} N_c$.

²The pure NJL model with extra dimensions has been already studied, e.g., in Refs. [10,40]. See also Ref. [41].

structure of the gauged NJL model with extra dimensions for such walking gauge theories may be useful not only for the TMSM but for other strong coupling theories such as the bulk technicolor, various composite models, etc.

In this paper we investigate phase structure of the gauged NJL model in the $D(>4)$ -dimensional bulk, based on the improved ladder SD equation for the bulk fermion propagator, assuming that the (dimensionless) bulk running gauge coupling is given by the truncated KK effective theory and hence almost fixed at the nontrivial UVFP in a wide energy range above the compactified energy scale. As long as the cutoff energy Λ is large, the effect of the compactified scale is negligible and the SD equation is well approximated by that with the fixed coupling [10,11]. Thus we take the gauge coupling to be fixed at the value of UVFP for simplicity. Here we should stress that our results in this paper crucially depend on the existence of such a UVFP.

By solving the SD equation we find the critical line in the two dimensional parameter space (κ_D, g) of the coupling strength κ_D (essentially the squared gauge coupling fixed at the UVFP value) and the (dimensionless) four-fermion coupling g :

$$g^{\text{crit}} = \frac{D-1}{4} \left(1 + \sqrt{1 - \kappa_D/\kappa_D^{\text{crit}}} \right)^2, \quad (1.1)$$

which exists only for $0 \leq \kappa_D \leq \kappa_D^{\text{crit}}$, where κ_D^{crit} is the critical coupling strength of the pure gauge theory without four-fermion interaction [10,11]. The critical line separating the spontaneously broken phase [with nonzero dynamical mass $M(\neq 0)$ of the fermion] and the unbroken one (with zero dynamical mass) resembles that in the four dimensional gauged NJL model with fixed gauge coupling [26,27] except for the prefactor $D/2 - 1$ instead of 1. Note, however, that we actually consider in this paper a class of $SU(N_c)$ gauge theories (with N_f flavors) which are asymptotically free having a trivial UVFP in four dimensions, $\kappa_D \rightarrow 0$ ($D \rightarrow 4$), but do have the nontrivial one in higher dimensions, $\kappa_D \neq 0$ ($D > 4$) whose value is uniquely determined once the model parameters such as N_c and N_f are specified. Each point on the above critical line thus corresponds to a single gauged NJL model with a different gauge theory with different N_c, N_f . We cannot freely adjust its value.

We further discuss the phase structure of the model, following the manner in the four dimensional case [37]. We find that the decay constant F_π of the composite Nambu–Goldstone (NG) boson π becomes finite in the continuum limit ($\Lambda/M \rightarrow \infty$), if N_c and N_f are arranged so as to satisfy the condition

$$\kappa_D^{\text{MT}} < \kappa_D < \kappa_D^{\text{crit}}, \quad (\text{nontrivial window}), \quad (1.2)$$

where $\kappa_D^{\text{MT}}(\neq 0)$ is given by

$$\kappa_D^{\text{MT}} = \left(1 - \frac{1}{(D/2 - 1)^2}\right) \kappa_D^{\text{crit}}. \quad (1.3)$$

We may call κ_D^{MT} a “marginal triviality” (MT) point, where the decay constant F_π of the composite NG boson π diverges logarithmically, $F_\pi^2 \sim \ln \Lambda$, in the continuum limit $\Lambda/M \rightarrow \infty$ and hence resembles the pure NJL point (zero gauge coupling) in the four dimensional case. Then in the nontrivial window we find a surprising result that even in higher dimensions the four-fermion theory *in the presence of gauge interactions* becomes “renormalizable” and “nontrivial” in the sense that the induced (renormalized) Yukawa coupling of the fermion to the NG boson and the (renormalized) couplings among π and composite scalar σ all remain finite in the continuum limit ($\Lambda/M \rightarrow \infty$). This is analogous to the renormalizability and nontriviality of the four dimensional case [35–39] except that the MT point in four dimensional case corresponds to the pure NJL point, $\kappa_D^{\text{MT}} \rightarrow 0$ for $D \rightarrow 4$. We demonstrate these facts by explicit computation of F_π , the Yukawa vertex Γ_s , the scalar propagator D_σ , and the effective potential $V(\sigma, \pi)$.

Here we should emphasize one distinct feature of the inclusion of the four-fermion interaction into the gauge theory in the bulk with compactified extra dimensions: In the case of pure gauge theories (without four-fermion interactions), the nontrivial theory defined at the critical point κ_D^{crit} is only formal, since the gauge coupling κ_D at the UVFP is not a continuous parameter but a discrete one depending on N_c and N_f and hence cannot be fine-tuned arbitrarily close to the critical value [10,11]. It means that even if we assume existence of the UVFP, *the pure gauge theory in higher dimensions can only be renormalizable in a formal sense*. Thus there is no finite theory for pure gauge theories. On the other hand, in the gauged NJL model we do have a continuous parameter, the four-fermion coupling g , and hence *the nontrivial theory can be defined by fine-tuning g arbitrarily close to the criticality of the four-fermion coupling* at the critical line Eq. (1.1). Note that for $\kappa_D > \kappa_D^{\text{crit}}$ where no criticality of the four-fermion coupling exists, there is no finite theory even in the gauged NJL model.

For the theories in the nontrivial window, $\kappa_D^{\text{MT}} < \kappa_D < \kappa_D^{\text{crit}}$, we perform explicitly the renormalization of the effective potential in a way similar to the four dimensional case given in Ref. [37]. The renormalized four-fermion coupling g_R also has a UVFP $g_R(\infty) = g^{\text{crit}}$. We then find that the theory has a large anomalous dimension at the UVFP $g_R(\infty)$:

$$\gamma_m = \left(\frac{D}{2} - 1\right) \left(1 + \sqrt{1 - \kappa_D / \kappa_D^{\text{crit}}}\right), \quad (1.4)$$

which takes the same form as that of the four dimensional model [28,37] when D in the prefactor is set to be $D = 4$. Since $D/2 > \gamma_m > D/2 - 1$, the dynamical dimension of the four-fermion interaction, $\dim(\bar{\psi}\psi)^2 = 2[(D - 1) - \gamma_m]$, reads $D - 2 < \dim(\bar{\psi}\psi)^2 < D$ for $\kappa_D^{\text{MT}} < \kappa_D < \kappa_D^{\text{crit}}$,

and such an operator is a relevant operator similar to the four dimensional gauged NJL model [37]. Thus, as we were motivated by the four dimensional case, we conclude that *the phase structure of the gauge theory with extra dimensions which behaves as a walking theory should also be studied with inclusion of the four-fermion interaction*. As expected, in the renormalized effective potential there exist interactions of σ_R and π_R for $\kappa_D^{\text{MT}} < \kappa_D < \kappa_D^{\text{crit}}$ (nontriviality), which are *conformal invariant*. The renormalizability/nontriviality implies a remarkable fact that the physics prediction is *cutoff insensitive*, notwithstanding the ultraviolet dominance of the dynamics of both the walking gauge and the four-fermion interactions strongly coupled in the ultraviolet region, which may be useful for model buildings.

We further study renormalization group (RG) flows of each theory in the nontrivial window governed by a UVFP $[\kappa_D, g_R(\infty)]$ corresponding to a set of (N_c, N_f) (each point lying on the critical line g^{crit}). Then we find that our renormalization is consistently performed even off the UVFP.

For $0 \leq \kappa_D \leq \kappa_D^{\text{MT}}$, on the other hand, we find that the gauged NJL model becomes trivial and nonrenormalizable in spite of the fact that we can fine-tune the four-fermion coupling to make finite the dynamical mass of the fermion. The “renormalized effective potential” actually has no interaction terms of σ_R and π_R . This renormalization would lead to the “anomalous dimension” $\gamma_m = \frac{D}{2}$ and $\dim(\bar{\psi}\psi)^2 = D - 2$ for $0 \leq \kappa_D \leq \kappa_D^{\text{MT}}$. However, our renormalization is incomplete for $0 \leq \kappa_D \leq \kappa_D^{\text{MT}}$: There exists another marginal operator $\partial_M(\bar{\psi}\psi)\partial^M(\bar{\psi}\psi)$ whose dynamical dimension is given by $\dim(\partial_M(\bar{\psi}\psi)\partial^M(\bar{\psi}\psi)) = D$ (marginal operator!) in our renormalization scheme.

We further discuss in Sec. VII the formal limit $D \rightarrow 4$, in which $\kappa_D^{\text{MT}} \rightarrow 0$ and $\kappa_D \rightarrow 0$ and hence the UVFPs for both regions of the nontrivial window $\kappa_D^{\text{MT}} < \kappa_D < \kappa_D^{\text{crit}}$ and trivial region $0 \leq \kappa_D \leq \kappa_D^{\text{MT}}$ shrink to a single point of the pure NJL point. Nonetheless, the nontrivial window for $D \rightarrow 4$ coincides with the condition of the renormalizability/nontriviality of the four dimensional gauged NJL model with (moderately) walking gauge coupling characterized by $A > 1$ [35–39], which is clearly distinguished from $A \leq 1$ corresponding to the trivial region $0 \leq \kappa_D \leq \kappa_D^{\text{MT}}$. Thus the nontrivial window is not a peculiarity of the extra dimensions but actually exists already in the four dimensional gauged NJL model in a rather sophisticated manner.

The paper is organized as follows: In Sec. II we give our model setting, Lagrangian, and the running gauge coupling constant within the truncated KK effective theory. In Sec. III we study phase structure of the model, based on the simplest approximation, the bifurcation technique [42], to the SD equation. We find the critical line, and calculate simplified scaling relation, scalar bo-

son propagator, and the decay constant F_π of the NG boson π . We also find the MT point where F_π^2 diverges logarithmically in the continuum limit. In Sec. IV we incorporate subleading terms in the asymptotic region by more involved approximations than the bifurcation, “linearized approximation” [32,43] and “power expansion method” [44]. The subdominant term is required for the effective potential. In Sec. V we calculate the effective potential. In Sec. VI we discuss the renormalization making finite the induced Yukawa coupling, the effective potential, and the scalar boson propagator. We give detailed discussions on the RG flow in the nontrivial window. Sec. VII is devoted to summary and discussion: In particular we discuss the $D \rightarrow 4$ limit. In Appendix A, we give another detailed calculation of the composite scalar propagator. Appendix B and C are for details of the “linearized approximation” and “power expansion method,” respectively. Appendix D is for more discussions on the nonrenormalizability of the region $0 \leq \kappa_D \leq \kappa_D^{\text{MT}}$.

II. THE MODEL

Let us start with the renormalization group properties of the gauge interaction in the $D = 4 + \delta$ dimensional bulk. Extra $\delta = (D - 4)$ spatial dimensions are assumed to be compactified at scale R^{-1} . The negative mass dimension of the bulk gauge coupling implies strong interaction at high energy. Thus the higher loop effects somehow need to be taken into account. The task, however, immediately meets serious trouble with the nonrenormalizability of the bulk gauge theory $D > 4$. We therefore *define* the bulk gauge theory by using the “truncated KK” effective theory [2], which allows us to calculate the loop effects within 4D effective field theory. A remarkable feature of the truncated KK effective theory is the existence of the nontrivial ultraviolet fixed point at least for its $\overline{\text{MS}}$ coupling [10,13–15]. We briefly review the properties of the renormalization group equation (RGE) in this model.

After the decomposition of the bulk gauge field into its KK modes, the running of the four dimensional gauge coupling g_{4D} can be described by the RGE,

$$(4\pi)^2 \mu \frac{dg_{4D}}{d\mu} = N_{\text{KK}}(\mu) b' g_{4D}^3, \quad (2.1)$$

where we assumed the renormalization scale μ is sufficiently larger than the compactification scale R^{-1} , $\mu \gg R^{-1}$. The RGE coefficient b' is given by

$$b' = -\frac{22 - \delta}{6} N_c + \frac{2}{3} \cdot 2^{\delta/2} N_f, \quad (2.2)$$

for $\text{SU}(N_c)$ gauge theory with N_f bulk fermions. $N_{\text{KK}}(\mu)$ stands for the number of KK modes below the renormalization scale μ . We use an approximation

$$N_{\text{KK}}(\mu) = \frac{1}{2^n} \frac{\pi^{\delta/2}}{\Gamma(1 + \delta/2)} (\mu R)^\delta. \quad (2.3)$$

The factor $1/2^n$ in Eq. (2.3) arises from the orbifold compactification on T^δ/Z_2^n . (See Refs. [10,12].) The four dimensional gauge coupling g_{4D} can be matched with the *dimensionful* bulk gauge coupling $g_{(4+\delta)D}$,

$$g_{(4+\delta)D}^2 = (2\pi R)^\delta g_{4D}^2 / 2^n.$$

It is convenient to define the *dimensionless* bulk gauge coupling \hat{g} ,

$$\hat{g}^2 \equiv g_{(4+\delta)D}^2 \mu^\delta,$$

and thus

$$\hat{g}^2(\mu) = \frac{(2\pi R \mu)^\delta}{2^n} g_{4D}^2(\mu). \quad (2.4)$$

Combining Eqs. (2.1), (2.3), and (2.4), we obtain RGE for \hat{g} [10]

$$\mu \frac{d}{d\mu} \hat{g} = \frac{\delta}{2} \hat{g} + \left(1 + \frac{\delta}{2}\right) \Omega_{\text{NDA}} b' \hat{g}^3, \quad (\mu \gg R^{-1}), \quad (2.5)$$

with Ω_{NDA} being the loop factor of naive dimensional analysis (NDA) in D dimensions,

$$\Omega_{\text{NDA}} \equiv \frac{1}{(4\pi)^{D/2} \Gamma(D/2)}. \quad (2.6)$$

The RGE (2.5) leads to a UVFP g_* [10],

$$g_*^2 \Omega_{\text{NDA}} = \frac{1}{-(1 + 2/\delta) b'}, \quad (2.7)$$

for $b' < 0$.

So far we have shown the UVFP only within the truncated KK effective theory at one-loop level. Does such a UVFP really exist beyond the approximation we adopted? This question is, of course, a highly nonperturbative problem [16–19] and extremely difficult to be answered. Instead of solving this difficult problem, in this paper, we simply assume the existence of the UVFP and address a hopefully easier but still quite exciting question: How does the bulk gauge theory behave with such a UVFP?

Such a theory should possess an approximate conformal invariance and was shown to have a large anomalous dimension $\gamma_m = D/2 - 1$ [10,11]. The situation has a strong resemblance to the four dimensional walking gauge theories [20,21,23,24], in which the gauge coupling is assumed to be on the nontrivial UVFP and the fermion bilinear operator $\bar{\psi}\psi$ acquires a large anomalous dimension $\gamma_m = 1$. It is known that the Nambu–Jona-Lasinio (NJL)–type four-fermion interaction becomes relevant in such a walking gauge dynamics. The walking gauge theory was then analyzed in an extended coupling space including the NJL-type four-fermion interaction (gauged NJL model) [25–28,37].

This resemblance motivates us to study the gauged NJL model with extra dimensions. In this paper, we therefore focus on the dynamical chiral symmetry breaking (D χ SB) in the gauged NJL model with extra dimensions. The Lagrangian of the gauged NJL model with extra dimensions is given by

$$\mathcal{L} = \bar{\psi}i\not{D}\psi - m_0\bar{\psi}\psi + \frac{G}{2N_c}[(\bar{\psi}\psi)^2 + (\bar{\psi}i\Gamma_A\tau^i\psi)^2] - \frac{1}{2}\text{tr}(F_{MN}F^{MN}), \quad (2.8)$$

where $M, N = 0, 1, 2, 3, 5, \dots, D$ and Γ_A is the chirality matrix in D dimensions, τ^i ($i = 1, 2, 3$) the Pauli matrices, and G the four-fermion coupling. The gauge group is $SU(N_c)$ and F_{MN} denotes the field strength. The gauge coupling is assumed to be on the UVFP. For simplicity, we take the number of flavors as $N_f = 2$, i.e., $\psi = (\psi_u, \psi_d)^T$ in the flavor space. The number of dimensions D is assumed to be even, $D = 6, 8, 10, \dots$, so as to introduce chiral fermions in the bulk. Extra $\delta (= D - 4)$ spatial dimensions are compactified at a TeV scale R^{-1} .

It is convenient to rewrite the Lagrangian Eq. (2.8) by using auxiliary fields σ , π_i ,

$$\mathcal{L} = \bar{\psi}i\not{D}\psi - \bar{\psi}(\sigma + i\Gamma_A\tau^i\pi_i)\psi - \frac{N_c}{2G}(\sigma^2 + \pi_i^2) + \frac{N_c}{G}m_0\sigma - \frac{1}{2}\text{tr}(F_{MN}F^{MN}). \quad (2.9)$$

In order to see the equivalence of Eq. (2.8) and (2.9), we just need to eliminate the auxiliary fields σ and π_i through their Euler equations

$$\sigma = -\frac{G}{N_c}\bar{\psi}\psi + m_0, \quad \pi_i = -\frac{G}{N_c}\bar{\psi}i\Gamma_A\tau^i\psi, \quad (2.10)$$

in Eq. (2.9). The vacuum expectation value (VEV) of σ

$$\langle\sigma\rangle = -\frac{G}{N_c}\langle\bar{\psi}\psi\rangle + m_0 \quad (2.11)$$

is proportional to the chiral condensate in the chiral limit $m_0 = 0$. Note that the model with $m_0 = 0$ possesses global $SU(2)_+ \times SU(2)_-$ chiral symmetry,

$$\psi_{\pm} \rightarrow \psi'_{\pm} = e^{i\theta_{\pm}(\tau^i/2)}\psi_{\pm}, \quad \psi_{\pm} \equiv \frac{1 \pm \Gamma_A}{2}\psi. \quad (2.12)$$

The bare mass term m_0 will be taken to be zero in the analysis of the D χ SB in this paper.

III. PHASE STRUCTURE

A. SD Equation

We express the bulk fermion propagator as

$$iS^{-1}(p) = A(-p^2)[\not{p} - \Sigma(-p^2)]. \quad (3.1)$$

Nonvanishing $\Sigma \neq 0$ implies D χ SB, which can be investigated through the (improved) ladder Schwinger-Dyson

equation. We need to take into account the running of the bulk gauge coupling in the SD equation. In the SD equation there are three different momenta, (Euclidean) square of which are $x \equiv -p^2$, $y \equiv -q^2$, and $z \equiv -(p - q)^2$ for the external and the loop momenta of the fermion and the gauge boson momentum, respectively. Hence there exist various ways to incorporate the running effects. The simplest one [45] is to take $g_{(4+\delta)D}^2(\mu) \rightarrow g_{(4+\delta)D}^2(\max(x, y))$, which has been widely used for four dimensional models in Landau gauge where the SD equation dictates $A(-p^2) = 1$, so that it is consistent with the vector Ward-Takahashi (WT) identity. In the case of extra dimensions we also used this in Ref. [10]. However, this is not consistent with the chiral WT identity [46], and hence we adopt as in Ref. [11] an ansatz of Ref. [46], namely, the gauge boson momentum is identified as the renormalization scale of the gauge coupling strength,

$$g_{(4+\delta)D}^2(\mu) \rightarrow g_{(4+\delta)D}^2(z) = \frac{\hat{g}^2(\mu = \sqrt{z})}{z^{\delta/2}}, \quad (3.2)$$

where \hat{g} is the dimensionless bulk gauge coupling. We assume here that the dimensionless bulk gauge coupling is on its UVFP, $\hat{g} = g_*$. The assumption is justified in the case that the cutoff Λ is very large and the UVFP really exists. The SD equation is then given by

$$A(x) = 1 + \frac{\kappa_D}{x} \int_0^{\Lambda^2} dy \frac{y^{D/2-1}A(y)}{A^2(y)y + B^2(y)} \frac{\min(x, y)}{[\max(x, y)]^{D/2-1}} \times \left(\frac{(D-1)(D-4)}{D} + \xi \right), \quad (3.3)$$

$$B(x) = \sigma + (D-1 + \xi)\kappa_D \int_0^{\Lambda^2} dy \frac{y^{D/2-1}B(y)}{A^2(y)y + B^2(y)} \times \frac{1}{[\max(x, y)]^{D/2-1}}, \quad (3.4)$$

where we have introduced ultraviolet cutoff Λ and the gauge fixing parameter is denoted as ξ . $B(x)$ is defined by

$$B(x) \equiv A(x)\Sigma(x). \quad (3.5)$$

We denote the gauge coupling strength by κ_D ,

$$\kappa_D \equiv C_F\Omega_{\text{NDA}}g_*^2, \quad (3.6)$$

with C_F being the quadratic Casimir of the fundamental representation,

$$C_F = \frac{N_c^2 - 1}{2N_c}. \quad (3.7)$$

It is understood that σ stands for the VEV of the auxiliary field $\langle\sigma\rangle$ in Eq. (3.4). Hereafter, we will adopt this shorthand notation without discriminating the auxiliary fields (σ , π) from their VEVs ($\langle\sigma\rangle$, $\langle\pi\rangle$). When $G = 0$ and hence $\sigma = m_0$, the SD equations Eqs. (3.3) and (3.4) are reduced

to those of the bulk gauge theory without four-fermion interactions in a form given in [11].

We choose the gauge fixing parameter ξ as [11]

$$\xi = -\frac{(D-1)(D-4)}{D}. \quad (3.8)$$

With this choice of gauge fixing parameter, the fermion wave function factor becomes trivial, i.e., $A(x) \equiv 1$ and $B(x) \equiv \Sigma(x)$. The SD equation then reads

$$\begin{aligned} \Sigma(x) &= \sigma + \frac{4(D-1)}{D} \kappa_D \int_0^{\Lambda^2} dy \frac{y^{D/2-1} \Sigma(y)}{y + \Sigma^2(y)} \\ &\times \frac{1}{[\max(x, y)]^{D/2-1}}. \end{aligned} \quad (3.9)$$

The VEV of σ is given by Eq. (2.11). We thus obtain

$$\sigma = m_0 + \frac{g}{\Lambda^{D-2}} \int_0^{\Lambda^2} dx x^{D/2-1} \frac{\Sigma(x)}{x + \Sigma^2(x)}, \quad (3.10)$$

where we defined the *dimensionless* four-fermion coupling constant g as

$$g \equiv 2^{D/2} N_f G \Lambda^{D-2} \Omega_{\text{NDA}}, \quad (N_f = 2), \quad (3.11)$$

and used

$$\langle \bar{\psi} \psi \rangle = -N_f \text{Tr} S = -2^{D/2} N_c N_f \int \frac{d^D p}{i(2\pi)^D} \frac{\Sigma(-p^2)}{-p^2 + \Sigma^2}. \quad (3.12)$$

The SD Eq. (3.9) for the mass function is equivalent to the differential equation

$$\begin{aligned} x^2 \frac{d^2}{dx^2} \Sigma(x) + \frac{D}{2} x \frac{d}{dx} \Sigma(x) \\ + \frac{2(D-1)(D-2)}{D} \kappa_D \frac{x \Sigma(x)}{x + \Sigma^2} = 0, \end{aligned} \quad (3.13)$$

with the infrared boundary condition (IRBC)

$$x^{D/2} \frac{d}{dx} \Sigma(x) \Big|_{x=0} = 0, \quad (3.14)$$

and the ultraviolet boundary condition (UVBC)

$$\left[(D/2 - 1) + x \frac{d}{dx} \right] \Sigma(x) \Big|_{x=\Lambda^2} = (D/2 - 1) \sigma. \quad (3.15)$$

The derivative of Eq. (3.9) at the cutoff scale Λ^2 is given by

$$\begin{aligned} x \frac{d}{dx} \Sigma(x) \Big|_{x=\Lambda^2} &= -\frac{2(D-1)(D-2)}{D \Lambda^{D-2}} \kappa_D \\ &\times \int_0^{\Lambda^2} dx x^{D/2-1} \frac{\Sigma(x)}{x + \Sigma^2(x)}. \end{aligned} \quad (3.16)$$

From Eq. (3.10) and (3.16) we see

$$\sigma = m_0 - \frac{D}{2(D-1)(D-2)} \frac{g}{\kappa_D} x \frac{d}{dx} \Sigma(x) \Big|_{x=\Lambda^2}. \quad (3.17)$$

The UVBC then reads

$$\begin{aligned} \left[(D/2 - 1) + \left(1 + \frac{D}{4(D-1)} \frac{g}{\kappa_D} \right) x \frac{d}{dx} \right] \Sigma(x) \Big|_{x=\Lambda^2} \\ = (D/2 - 1) m_0. \end{aligned} \quad (3.18)$$

For $\kappa_D = 0$, the SD equation reduces to the gap equation of the pure NJL model. See the appendix of Ref. [10] for the analysis of the $D(>4)$ -dimensional pure NJL model.

B. Bifurcation Technique

Even if we start with the chiral limit $m_0 = 0$, we expect the dynamical chiral symmetry breaking $\Sigma \neq 0$ takes place if the NJL coupling exceeds certain critical value. In order to fully analyze the behavior of the dynamical chiral phase transition in this model, we need to solve the nonlinear SD equation Eq. (3.9) as it stands. Such a task turns out to be almost impossible within analytical methods, however. Instead of solving the nonlinear SD equation, here we first adopt the bifurcation technique [42], restricting the solution much smaller than the cutoff scale $\Sigma \ll \Lambda$. The bifurcation technique is justified at least for the determination of the critical coupling and the leading asymptotic behavior of the mass function $\Sigma(x)$. In this section, we employ the bifurcation technique for the sake of simplicity to avoid unnecessarily complicated expressions in discussing the decay constant, effective Yukawa coupling, and the propagator of the composite scalar, which are determined only through the dominant term of $\Sigma(x)$. On the other hand, the subdominant term is relevant to the scaling relation and the lowest interaction term of the effective potential. The bifurcation method, however, does not always lead to the correct subdominant term. Thus we will perform more sophisticated analysis later in Sec. IV to determine the subdominant term of $\Sigma(x)$.

In the bifurcation technique, the Σ^2 terms in the denominators of Eq. (3.9) and (3.10) are replaced by an infrared cutoff Σ_0

$$\Sigma(x) = \sigma + \frac{4(D-1)}{D} \kappa_D \int_{\Sigma_0^2}^{\Lambda^2} dy \frac{y^{D/2-2} \Sigma(y)}{[\max(x, y)]^{D/2-1}}, \quad (3.19)$$

with

$$\sigma = \frac{g}{\Lambda^{D-2}} \int_{\Sigma_0^2}^{\Lambda^2} dx x^{D/2-2} \Sigma(x), \quad m_0 = 0. \quad (3.20)$$

We also assume $\Sigma_0 \ll \Lambda$ and $\Sigma_0 \sim \Sigma(\Sigma_0^2)$.

The integral equation Eq. (3.19) is equivalent with a set of the *linear* differential equation

$$x^2 \frac{d^2}{dx^2} \Sigma(x) + \frac{D}{2} x \frac{d}{dx} \Sigma(x) + \frac{2(D-1)(D-2)}{D} \kappa_D \Sigma(x) = 0, \quad (3.21)$$

and boundary conditions

$$\left. \frac{d}{dx} \Sigma(x) \right|_{x=\Sigma_0^2} = 0, \quad (3.22)$$

and

$$\left[D/2 - 1 + \left(1 + \frac{D}{4(D-1)} \frac{g}{\kappa_D} \right) x \frac{d}{dx} \right] \Sigma(x) \Big|_{x=\Lambda^2} = 0. \quad (3.23)$$

The solution of Eq. (3.21) behaves differently for $\kappa_D < \kappa_D^{\text{crit}}$ and for $\kappa_D > \kappa_D^{\text{crit}}$. Here critical value κ_D^{crit} is obtained as [11]

$$\kappa_D^{\text{crit}} \equiv \frac{D}{32} \frac{D-2}{D-1}. \quad (3.24)$$

For the subcritical $\kappa_D < \kappa_D^{\text{crit}}$, the solution of Eq. (3.21) is given by a power damping form,

$$\Sigma(x) = c_1 \Sigma_0 \left(\frac{x}{\Sigma_0^2} \right)^{-(\nu/2)(1-\omega)} + d_1 \Sigma_0 \left(\frac{x}{\Sigma_0^2} \right)^{-(\nu/2)(1+\omega)}, \quad (3.25)$$

where c_1 and d_1 are real constants. We also define ν and ω as³

$$\nu \equiv \frac{D}{2} - 1, \quad (3.26)$$

and

$$\omega \equiv \sqrt{1 - \kappa_D / \kappa_D^{\text{crit}}}, \quad (3.27)$$

respectively. For the supercritical $\kappa_D > \kappa_D^{\text{crit}}$, on the other hand, the solution starts to oscillate,

$$\begin{aligned} \Sigma(x) &= \tilde{c}_1 \Sigma_0 \left(\frac{x}{\Sigma_0^2} \right)^{-(\nu/2)(1-i\tilde{\omega})} + \text{h.c.}, \\ &= 2|\tilde{c}_1| \Sigma_0 \left(\frac{x}{\Sigma_0^2} \right)^{-(\nu/2)} \sin \left(\frac{1}{2} \nu \tilde{\omega} \ln \frac{x}{\Sigma_0^2} + \theta \right), \end{aligned} \quad (3.28)$$

with \tilde{c}_1 being a complex constant. The parameter $\tilde{\omega}$ and the angle θ are defined as

$$\tilde{\omega} \equiv \sqrt{\kappa_D / \kappa_D^{\text{crit}} - 1}, \quad (3.29)$$

and

$$e^{2i\theta} = -\tilde{c}_1 / \tilde{c}_1^*, \quad (3.30)$$

respectively.

Let us first discuss the case with $\kappa_D < \kappa_D^{\text{crit}}$. The IRBC Eq. (3.22) leads to

$$d_1 = -\frac{1-\omega}{1+\omega} c_1. \quad (3.31)$$

If we normalize the mass function such as $\Sigma(\Sigma_0^2) = \Sigma_0$,

³We use notations similar to those in Ref. [37].

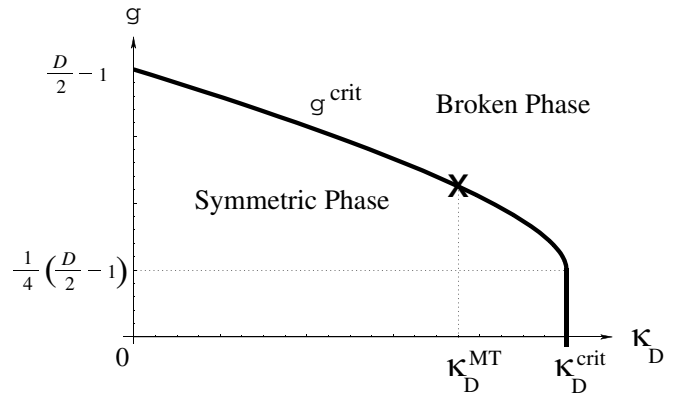


FIG. 1. The critical line in (κ_D, g) plane. The critical line is given by $g^{\text{crit}} = \nu(1+\omega)^2/4$ with $\omega = \sqrt{1 - \kappa_D / \kappa_D^{\text{crit}}}$ and $\nu = D/2 - 1$ for $0 \leq \kappa_D \leq \kappa_D^{\text{crit}}$. The chiral symmetry is unbroken below the critical line. For $\kappa_D > \kappa_D^{\text{crit}}$ the broken phase is realized irrespective of the size of g .

we can determine concretely c_1 and d_1 . It is convenient to rewrite the UVBC Eq. (3.23) in the following form,

$$\left[\nu + \left(1 + \frac{4}{1-\omega^2} \frac{g}{\nu} \right) x \frac{d}{dx} \right] \Sigma(x) \Big|_{x=\Lambda^2} = 0. \quad (3.32)$$

Combining Eq. (3.25) and (3.32), we obtain

$$\left(\frac{\Sigma_0^2}{\Lambda^2} \right)^{\nu\omega} = -\frac{(1-\omega)c_1}{(1+\omega)d_1} \frac{4g - \nu(1+\omega)^2}{4g - \nu(1-\omega)^2}. \quad (3.33)$$

Equation (3.31) then leads to the scaling relation⁴

$$\left(\frac{\Sigma_0^2}{\Lambda^2} \right)^{\nu\omega} = \frac{4g - \nu(1+\omega)^2}{4g - \nu(1-\omega)^2}. \quad (3.34)$$

The nontrivial solution $\Sigma_0 \neq 0$ exists only when the NJL coupling exceeds the critical line,

$$g^{\text{crit}} = \frac{\nu}{4} (1+\omega)^2. \quad (3.35)$$

See Fig. 1 for the phase diagram in the κ_D - g plane. Note that the negative sign in Eq. (3.31) is essentially important. If the sign of d_1 were identical to c_1 , we would not obtain positive value in Eq. (3.33) for $g > g^{\text{crit}}$, indicating the instability of the vacuum. We will be back to this problem later in Sec. V.

From Eq. (3.25) and (3.31), we also obtain the asymptotic form of the nontrivial solution,

$$\Sigma(x) \sim c_1 \Sigma_0 \left(\frac{x}{\Sigma_0^2} \right)^{-(\nu/2)(1-\omega)}, \quad \text{for } x \gg \Sigma_0^2. \quad (3.36)$$

⁴The scaling relation Eq. (3.34) is not correct near the pure NJL limit. Actually, it fails to reproduce the scaling behavior of the $D(>4)$ -dimensional pure NJL model [10] in the $\kappa_D \rightarrow 0$ limit. See Sec. IV for more detailed analysis of the scaling behavior.

We next consider the condensate σ , which can be calculated from Eq. (3.15). We obtain

$$\sigma = \frac{1 + \omega}{2} c_1 \Sigma_0 \left(\frac{\Lambda^2}{\Sigma_0^2} \right)^{-(\nu/2)(1-\omega)}, \quad (3.37)$$

neglecting the subdominant term. The asymptotic solution Eq. (3.36) can be written in terms of σ ,

$$\Sigma(x) \sim \frac{2}{1 + \omega} \sigma \left(\frac{x}{\Lambda^2} \right)^{-(\nu/2)(1-\omega)}. \quad (3.38)$$

The mass function Σ is thus proportional to the order parameter σ . We will use Eq. (3.38) later in Sec. III D to calculate the Yukawa vertex of the composite scalar field σ .

We next turn to the case with $\kappa_D > \kappa_D^{\text{crit}}$. From the IRBC Eq. (3.22) we find

$$e^{2i\theta} = \frac{1 + i\tilde{\omega}}{1 - i\tilde{\omega}}. \quad (3.39)$$

On the other hand, the UVBC Eq. (3.23) gives

$$\sin\left(\frac{1}{2} \nu \tilde{\omega} \ln \frac{\Lambda^2}{\Sigma_0^2} + \theta + \theta'\right) = 0, \quad (3.40)$$

with θ' being given by

$$e^{2i(\theta+\theta')} = \frac{4g - \nu(1 + i\tilde{\omega})^2}{4g - \nu(1 - i\tilde{\omega})^2}. \quad (3.41)$$

We note that nontrivial solution $\Sigma_0 \neq 0$ exists in Eq. (3.40), irrespective of the size of the NJL coupling. This could be easily understood in an analytical manner by looking at the region $\tilde{\omega} \ll 1$, where we can use an approximation

$$\frac{1}{\nu \tilde{\omega}} (\theta + \theta') = \frac{2}{\nu - 4g}. \quad (3.42)$$

Equation (3.40) then leads to a scaling relation (zero-node solution)

$$\frac{\Sigma_0}{\Lambda} = \exp\left(-\frac{\pi - (\theta + \theta')}{\nu \tilde{\omega}}\right) = \exp\left(\frac{2}{\nu - 4g} - \frac{\pi}{\nu \tilde{\omega}}\right). \quad (3.43)$$

For $g = \nu/4$ we find $\Sigma_0/\Lambda = \exp(\tilde{\omega}/2) \exp[-\pi/(2\nu\tilde{\omega})]$.

We next consider the continuum limit ($\Lambda \rightarrow \infty$) in the present model. As we stated before, we simply admit the existence of the UVFP in the gauge coupling strength in the present analysis. One might thus think that we should be able to obtain a finite theory even in the $\Lambda \rightarrow \infty$ limit thanks to the UVFP. This is a nontrivial problem, however, in the $D\chi\text{SB}$ vacuum. As we see in Eq. (3.34) and (3.43), the dynamical mass of the fermion (Σ_0) is proportional to the cutoff Λ and thus diverges in the $\Lambda \rightarrow \infty$ limit if both g and κ_D are fixed. In other words, in order to keep Σ_0 finite, the NJL coupling g needs to approach its critical value Eq. (3.35) in the $\Lambda \rightarrow \infty$ limit. This procedure, known as the Wilsonian renormalization, al-

lows us to define the continuum limit of the gauged NJL model. Hereafter we use the word ‘‘renormalization’’ in this sense.

Recall that the $D\chi\text{SB}$ always takes place regardless of the NJL coupling g for $\kappa_D > \kappa_D^{\text{crit}}$. There does not exist critical NJL coupling with this κ_D . What does happen in the continuum limit for $\kappa_D > \kappa_D^{\text{crit}}$ then? Instead of g , one may think about tuning κ_D to its critical value κ_D^{crit} . This procedure cannot be done, however, since κ_D is essentially given by the UVFP of the gauge coupling strength in Eq. (3.6). The value of κ_D should be fixed once the discrete model parameters such as N_c and N_f are determined. We thus conclude that there is no finite continuum limit for $\kappa_D > \kappa_D^{\text{crit}}$ at least within the ladder approximation. We probably need to introduce other higher dimensional interactions in order to define the continuum limit with $\kappa_D > \kappa_D^{\text{crit}}$.

We also note that $\kappa_D = \kappa_D^{\text{crit}}$ cannot be achieved for $D = 6, 8$ within the approximations we used.⁵ We thus concentrate upon the $\kappa_D < \kappa_D^{\text{crit}}$ case hereafter.

C. Decay Constant

There appear massless Nambu-Goldstone fields in the $D\chi\text{SB}$ vacuum. The interaction of the NG field is then described by the low energy theorem and its decay constant. In this subsection, we estimate the size of the decay constant by using an approximation proposed by Pagels and Stokar (PS) [47]. The decay constant can be written solely in terms of the mass function Σ within the PS approximation. We know that the PS approximation works reasonably well in the $D\chi\text{SB}$ of four dimensional QCD.

The D -dimensional generalization of the PS formula is given by

$$F_\pi^2 = 2^{D/2} \Omega_{\text{NDA}} \int_0^{\Lambda^2} dx x^{D/2-1} \frac{\Sigma^2(x) - \frac{x}{D} \frac{d}{dx} \Sigma^2(x)}{(x + \Sigma^2)^2}, \quad (3.44)$$

where F_π denotes the decay constant in D dimensions

$$\langle 0 | J_M^i(0) | \pi_j(q) \rangle = -i q_M F_\pi \delta_{ij}. \quad (3.45)$$

The four dimensional decay constant f_π is obtained through the matching condition

$$f_\pi^2 = \frac{(2\pi R)^\delta}{2^n} F_\pi^2. \quad (3.46)$$

It is straightforward to estimate the decay constant by using the asymptotic behavior of the mass function Eq. (3.36),

$$F_\pi^2 \sim \Sigma_0^{2+2\nu(1-\omega)} \int_{\Sigma_0^2}^{\Lambda^2} dx x^{\nu\omega-2}. \quad (3.47)$$

⁵For $D = 10$ it happens $\kappa_D = \kappa_D^{\text{crit}}$, for instance, for $N_c = 81$ and $N_f = 20$.

Note that F_π diverges in the $\Lambda \rightarrow \infty$ limit for

$$\omega \geq \frac{1}{\nu}. \quad (3.48)$$

The condition Eq. (3.48) can be rewritten in terms of κ_D ,

$$\kappa_D \leq \kappa_D^{\text{MT}}, \quad (3.49)$$

with the marginal triviality point κ_D^{MT} being defined as

$$\kappa_D^{\text{MT}} \equiv \left(1 - \frac{1}{\nu^2}\right) \kappa_D^{\text{crit}}. \quad (3.50)$$

The NG field interaction is suppressed by $1/F_\pi$ according to the low energy theorem. For $\kappa_D \leq \kappa_D^{\text{MT}}$ the NG field is therefore decoupled from the rest of system and becomes trivial in the continuum limit ($\Lambda \rightarrow \infty$).

On the other hand, surprisingly enough, the decay constant F_π remains finite for

$$\kappa_D^{\text{MT}} < \kappa_D < \kappa_D^{\text{crit}}, \quad (3.51)$$

even in the continuum limit $\Lambda \rightarrow \infty$. Once the condition Eq. (3.51) is satisfied, the NG field enjoys nontrivial interactions in the ‘‘renormalized’’ theory of the higher dimensional gauged NJL model.

The MT point κ_D^{MT} is indicated by a cross in the phase diagram Fig. 1. At this point the decay constant F_π diverges as $F_\pi^2 \sim \ln \Lambda$, while $F_\pi^2 \sim \Lambda^{2(\nu\omega-1)}$ for $0 < \kappa_D < \kappa_D^{\text{MT}}$.

In $D \rightarrow 4$ we find $\kappa_D^{\text{MT}} \rightarrow 0$, namely, the MT point coincides with the pure NJL point, where it is well-known F_π^2 diverges logarithmically. On the other hand, in the $D \rightarrow \infty$ limit $\kappa_D^{\text{MT}} \rightarrow \kappa_D^{\text{crit}}$, and hence the region of the finite F_π is squeezed out.

D. Yukawa Coupling and Scalar Propagator

In addition to the NG bosons, a composite scalar boson also appears in the spectrum of the NJL-type models. As we will show later, the auxiliary scalar field σ of Eq. (2.9) actually acquires nontrivial propagator at the loop level. In this subsection, we adopt a simple method proposed by Appelquist, Terning and Wijewardhana (ATW) [48] to calculate roughly the properties of the composite scalar. We also describe a more sophisticated method based on Ref. [49] in Appendix A. These results qualitatively agree with each other.

The auxiliary field σ propagator at $|p^2| \gg \Sigma_0^2$ is given by

$$iD_\sigma^{-1}(p) = -N_c N_f \int \frac{d^D q}{i(2\pi)^D} \text{tr} \left[\Gamma_s(q+p, q) \frac{1}{\not{q}} \frac{1}{\not{q} - \not{p}} \right] + \text{constant}, \quad (3.52)$$

where $\Gamma_s(q+p, q)$ is the Yukawa-type vertex of the auxiliary field, and q and p are the ψ and σ momenta, respectively. The basic difficulty in the calculation of Eq. (3.52) arises from our lack of knowledge of $\Gamma_s(q+p, q)$.

In Appendix A, we calculate analytically $\Gamma_s(q+p, q)$ with certain approximations. As for the effective Yukawa vertex at $p=0$, on the other hand, we easily find

$$\Gamma_s(q, q) = \frac{d}{d\sigma} \Sigma(-q^2) \sim \frac{2}{1+\omega} \left(\frac{-q^2}{\Lambda^2} \right)^{-(\nu/2)(1-\omega)}, \quad (3.53)$$

where we used Eq. (3.38). For the relation between Σ and Γ_s , see, e.g., Ref. [50].

ATW showed that Eq. (3.52) can be approximated by

$$iD_\sigma^{-1}(p) = -N_c N_f \int \frac{d^D q}{i(2\pi)^D} \text{tr} \left[\Gamma_s(-q^2) \frac{1}{\not{q}} \Gamma_s(-q^2) \frac{1}{\not{q} - \not{p}} \right] + \text{constant}, \quad (3.54)$$

within reasonable assumptions. Note that Eq. (3.54) contains Yukawa vertex only at $p=0$,

$$\Gamma_s(-q^2) = \Gamma_s(q, q), \quad (3.55)$$

and we are thus able to use Eq. (3.53) in the evaluation of the σ propagator.

After the Wick rotation and performing the angular integrals, we obtain

$$\begin{aligned} iD_\sigma^{-1}(p) - iD_\sigma^{-1}(0) &= -2^{D/2} N_c N_f \\ &\times \int \frac{d^D q}{i(2\pi)^D} \left[\Gamma_s^2(-q^2) \frac{p \cdot q - p^2}{q^2(p-q)^2} \right] \\ &= 2^{D/2} N_c N_f \Omega_{\text{NDA}} \\ &\times \int_0^{\Lambda^2} dy y^{D/2-2} \Gamma_s^2(y) K(x, y), \end{aligned} \quad (3.56)$$

where we defined

$$x \equiv -p^2, \quad y \equiv -q^2.$$

The kernel K is given by

$$\begin{aligned} K(x, y) &= \frac{2}{D} \frac{\min(x, y)}{\max(x, y)} F\left(2, 2 - D/2, D/2 + 1; \frac{\min(x, y)}{\max(x, y)}\right) \\ &\quad - \frac{x}{\max(x, y)} F\left(1, 2 - D/2, D/2; \frac{\min(x, y)}{\max(x, y)}\right). \end{aligned} \quad (3.57)$$

For even dimensions we obtain

$$\begin{aligned} K(x, y) &= - \sum_{\ell=0}^{\delta/2+1} \frac{(-\delta/2-1)_\ell}{(\delta/2+2)_\ell} \left(\frac{y}{x}\right)^\ell \theta(x-y) \\ &\quad + \sum_{\ell=1}^{\delta/2+1} \frac{(-\delta/2-1)_\ell}{(\delta/2+2)_\ell} \left(\frac{x}{y}\right)^\ell \theta(y-x), \end{aligned} \quad (3.58)$$

where $(\alpha)_\ell$ is defined as

$$(\alpha)_\ell = (\alpha + \ell - 1)(\alpha + \ell - 2) \cdots (\alpha + 1)\alpha \quad (3.59)$$

and $(\alpha)_0 = 1$. Substituting Eq. (3.53) for Eq. (3.56), we obtain the composite scalar propagator for $\omega \neq 1/\nu$ ($\kappa_D \neq \kappa_D^{\text{MT}}$):

$$iD_\sigma^{-1}(p) - iD_\sigma^{-1}(0) = 2^{D/2} N_c N_f \Omega_{\text{NDA}} \left(\frac{2}{1+\omega} \right)^2 \times \Lambda^{D-2} \left[C_{\nu\omega} \left(\frac{x}{\Lambda^2} \right)^{\nu\omega} + \sum_{\ell=1}^{\delta/2+1} C_\ell \left(\frac{x}{\Lambda^2} \right)^\ell \right], \quad (3.60)$$

where

$$C_{\nu\omega} \equiv -\frac{1}{\nu\omega} - \sum_{\ell=1}^{\delta/2+1} \frac{(-\delta/2-1)_\ell}{(\delta/2+2)_\ell} \frac{2\nu\omega}{(\nu\omega)^2 - \ell^2} \quad (3.61)$$

and

$$C_\ell \equiv \frac{(-\delta/2-1)_\ell}{(\delta/2+2)_\ell} \frac{1}{\nu\omega - \ell}, \quad (\ell \geq 1). \quad (3.62)$$

At the marginal triviality point $\omega = 1/\nu$ ($\kappa_D = \kappa_D^{\text{MT}}$), we find

$$iD_\sigma^{-1}(p) - iD_\sigma^{-1}(0) = 2^{D/2} N_c N_f \Omega_{\text{NDA}} \left(\frac{2}{1+\omega} \right)^2 \times \Lambda^{D-2} \left[C_0 \left(\frac{x}{\Lambda^2} \right) \ln \left(\frac{\Lambda^2}{x} \right) + C'_1 \left(\frac{x}{\Lambda^2} \right) + \sum_{\ell=2}^{\delta/2+1} C_\ell \left(\frac{x}{\Lambda^2} \right)^\ell \right], \quad (3.63)$$

where

$$C_0 \equiv -\frac{D-2}{D} \quad (3.64)$$

and

$$C'_1 \equiv -\frac{C_0}{2} - \frac{1}{\nu\omega} - \sum_{\ell=2}^{\delta/2+1} \frac{(-\delta/2-1)_\ell}{(\delta/2+2)_\ell} \frac{2\nu\omega}{(\nu\omega)^2 - \ell^2}. \quad (3.65)$$

We next consider the $\Lambda \rightarrow \infty$ limit. It is convenient to renormalize the auxiliary field σ

$$\sigma_R \equiv Z_\sigma^{-1/2} \sigma, \quad (3.66)$$

in such a limit. We define the renormalization constant Z_σ so as to keep the renormalized propagator finite,

$$Z_\sigma^{-1} = \begin{cases} \left(\frac{\Lambda}{\mu} \right)^{2\nu(1-\omega)}, & (\kappa_D^{\text{MT}} < \kappa_D < \kappa_D^{\text{crit}}), \\ \left(\frac{\Lambda}{\mu} \right)^{2(\nu-1)} \ln \left(\frac{\Lambda^2}{\mu^2} \right), & (\kappa_D = \kappa_D^{\text{MT}}), \\ \left(\frac{\Lambda}{\mu} \right)^{2(\nu-1)}, & (0 < \kappa_D < \kappa_D^{\text{MT}}). \end{cases} \quad (3.67)$$

The renormalized propagator is then given by

$$\frac{[iD_{\sigma(R)}^{-1}(p) - iD_{\sigma(R)}^{-1}(0)]}{2^{D/2} N_c N_f \Omega_{\text{NDA}}} \left(\frac{1+\omega}{2} \right)^2 = \begin{cases} C_{\nu\omega} \mu^{2\nu} \left(\frac{x}{\mu^2} \right)^{\nu\omega}, & (\kappa_D^{\text{MT}} < \kappa_D < \kappa_D^{\text{crit}}), \\ C_0 \mu^{2\nu} \left(\frac{x}{\mu^2} \right), & (\kappa_D = \kappa_D^{\text{MT}}), \\ C_{\ell=1} \mu^{2\nu} \left(\frac{x}{\mu^2} \right), & (0 < \kappa_D < \kappa_D^{\text{MT}}), \end{cases} \quad (3.68)$$

in the $\Lambda \rightarrow \infty$ limit.

What does happen for the Yukawa-type vertex, then? From Eq. (3.53) and (3.67) we obtain

$$\Gamma_s^{(R)}(-q^2) = Z_\sigma^{1/2} \Gamma_s(-q^2) \sim \begin{cases} \frac{2}{1+\omega} \left(\frac{-q^2}{\mu^2} \right)^{-(\nu/2)(1-\omega)}, & (\kappa_D^{\text{MT}} < \kappa_D < \kappa_D^{\text{crit}}), \\ 0 & (0 < \kappa_D \leq \kappa_D^{\text{MT}}), \end{cases} \quad (3.69)$$

in the $\Lambda \rightarrow \infty$ limit. We note that the composite scalar thus decouples from the rest of the system for $\kappa_D \leq \kappa_D^{\text{MT}}$ in the continuum limit. On the other hand, for $\kappa_D^{\text{MT}} < \kappa_D < \kappa_D^{\text{crit}}$, the composite scalar interacts with fermions in a nontrivial manner through the Yukawa vertex Eq. (3.69). These behaviors are consistent with our analysis of the NG field (decay constant) in the previous subsection.

IV. MORE ABOUT THE MASS FUNCTION

For the determination of the precise scaling behavior around the critical line, it is not enough to know the dominant asymptotic behavior of mass function Eq. (3.36). Actually, lacking the subdominant term, the bifurcation method does not reproduce exactly the same scaling relation (gap equation) as that of the pure NJL

model in the vanishing gauge coupling limit even in four dimensions [51]. We also need information of subdominant terms. The subdominant terms are also required in the calculation of the effective potential. Unfortunately, the bifurcation technique is not enough for such a purpose. In this section, we evaluate the subdominant terms first in the well-known linearized SD equation, and next in the power expansion method.

It is known that the linearized approximation works well in four dimensions. By using the linearized ansatz, we can solve analytically the ladder SD equation. We obtain the coefficient of the subdominant term as well as its exponent. The power expansion method reflects the correct asymptotic behavior of the nonlinear SD equation. However, we can determine only the exponent of the subdominant term in the method. The coefficient of the subdominant term is left as an unknown parameter. In a certain region of the gauge coupling, it turns out that the exponent of the subdominant term is different depending on the methods. We thus describe both results.

A. Linearized Approximation

We first consider the linearized SD equation [32,43], in which the Σ^2 term in the denominator is replaced by a constant $\Sigma_0^2 \equiv \Sigma^2(x=0)$. This approximation works well in the determination of the dominant asymptotic solution at $x \gg \Sigma_0^2$. It is also known that the scaling behavior determined from the linearized SD equation closely approximates to the numerical results of the nonlinear SD equation in the case of the four dimensional gauged NJL model.

The linearized SD equation is given by

$$\Sigma(x) = \sigma + \frac{4(D-1)}{D} \kappa_D \int_0^{\Lambda^2} dy \frac{y^{D/2-1} \Sigma(y)}{y + \Sigma_0^2} \times \frac{1}{[\max(x, y)]^{D/2-1}}, \quad (4.1)$$

$$\sigma = m_0 + \frac{g}{\Lambda^{D-2}} \int_0^{\Lambda^2} dx x^{D/2-1} \frac{\Sigma(x)}{x + \Sigma_0^2}, \quad (4.2)$$

and a subsidiary condition,

$$\Sigma_0 = \Sigma(x=0). \quad (4.3)$$

The integral equation Eq. (4.1) is equivalently rewritten in terms of a set of differential equation and boundary conditions,

$$x^2 \frac{d^2}{dx^2} \Sigma + (\nu+1)x \frac{d}{dx} \Sigma + \frac{\nu^2(1-\omega^2)}{4} \frac{x\Sigma}{x + \Sigma_0^2} = 0, \quad (4.4)$$

$$x^{\nu+1} \frac{d}{dx} \Sigma(x) \Big|_{x=0} = 0, \quad (4.5)$$

$$\left[\nu + x \frac{d}{dx} \right] \Sigma(x) \Big|_{x=\Lambda^2} = \nu\sigma, \quad (4.6)$$

where ν and ω are defined in Eq. (3.26) and (3.27). The UVBC Eq. (4.6) can also be expressed as

$$\left[\nu + \left(1 + \frac{4}{1-\omega^2} \frac{g}{\nu} \right) x \frac{d}{dx} \right] \Sigma(x) \Big|_{x=\Lambda^2} = \nu m_0. \quad (4.7)$$

Note that the differential Eq. (4.4) possesses three regular singular points at $x=0, -\Sigma_0^2, \infty$. The solution is then expressed by using the Gauss hypergeometric functions. It is now easy to find the solution of Eq. (4.4) satisfying the subsidiary condition Eq. (4.3) and the IRBC Eq. (4.5),

$$\Sigma(x) = \Sigma_0 F\left(\frac{\nu}{2}(1+\omega), \frac{\nu}{2}(1-\omega), \nu+1; -\frac{x}{\Sigma_0^2}\right), \quad (4.8)$$

where F is the Gauss hypergeometric function

$$F(\alpha, \beta, \gamma; z) \equiv \sum_{n=0}^{\infty} \frac{(\alpha)_n (\beta)_n}{(\gamma)_n} \frac{z^n}{n!} \quad (4.9)$$

with $(\alpha)_n, (\beta)_n, (\gamma)_n$ being defined in Eq. (3.59). We assumed here $\kappa_D < \kappa_D^{\text{crit}}$ and hence a real and positive ω .

The UVBC Eq. (4.7) determines Σ_0 and thus the scaling relation. We need to know the behavior of the mass function Eq. (4.8) in the ultraviolet region $x \gg \Sigma_0^2$. For such a purpose, it is convenient to use a well-known formula [52],

$$F(\alpha, \beta, \gamma; z) = \frac{\Gamma(\gamma)\Gamma(\beta-\alpha)}{\Gamma(\beta)\Gamma(\gamma-\alpha)} (-z)^{-\alpha} \times F\left(\alpha, \alpha-\gamma+1, \alpha-\beta+1; \frac{1}{z}\right) + \frac{\Gamma(\gamma)\Gamma(\alpha-\beta)}{\Gamma(\alpha)\Gamma(\gamma-\beta)} (-z)^{-\beta} \times F\left(\beta, \beta-\gamma+1, \beta-\alpha+1; \frac{1}{z}\right). \quad (4.10)$$

Combining Eq. (4.9) and Eq. (4.10), we expand the solution Eq. (4.8) for $x \gg \Sigma_0^2$,

$$\Sigma(x) = \Sigma_0 \left(\frac{x}{\Sigma_0^2}\right)^{-(\nu/2)(1-\omega)} \sum_{n=1}^{\infty} c_n \left(\frac{x}{\Sigma_0^2}\right)^{-n+1} + \Sigma_0 \left(\frac{x}{\Sigma_0^2}\right)^{-(\nu/2)(1+\omega)} \sum_{n=1}^{\infty} d_n \left(\frac{x}{\Sigma_0^2}\right)^{-n+1}, \quad (4.11)$$

with c_n and d_n being given by

$$c_n = \frac{(-1)^{n-1}}{(n-1)!} \frac{\Gamma(\nu+1)\Gamma(\nu\omega)}{\Gamma(\frac{\nu}{2}(1+\omega))\Gamma(\frac{\nu}{2}(1+\omega)+1)} \times \frac{(\frac{\nu}{2}(1-\omega))_{n-1} (-\frac{\nu}{2}(1+\omega))_{n-1}}{(-\nu\omega+1)_{n-1}}, \quad (4.12)$$

$$d_n = \frac{(-1)^{n-1}}{(n-1)!} \frac{\Gamma(\nu+1)\Gamma(-\nu\omega)}{\Gamma(\frac{\nu}{2}(1-\omega))\Gamma(\frac{\nu}{2}(1-\omega)+1)} \times \frac{(\frac{\nu}{2}(1+\omega))_{n-1}(-\frac{\nu}{2}(1-\omega))_{n-1}}{(\nu\omega+1)_{n-1}}. \quad (4.13)$$

The c_1 term in Eq. (4.11) gives the leading contribution to the mass function $\Sigma(x)$ in the asymptotic region $x \gg \Sigma_0^2$

$$\Sigma(x) = c_1 \Sigma_0 \left(\frac{x}{\Sigma_0^2}\right)^{-(1/2)(\nu-1)} - \frac{\nu^2-1}{4} c_1 \Sigma_0 \left(\frac{x}{\Sigma_0^2}\right)^{-(1/2)(\nu+1)} \ln \frac{x}{\Sigma_0^2} + \dots, \quad (\kappa_D = \kappa_D^{\text{MT}}). \quad (4.14)$$

In order to determine the scaling relation near the critical line, we compare the leading (c_1) and the next-to-leading (d_1 or c_1 depending on κ_D) terms in the UVBC Eq. (4.7). For $m_0 = 0$, we find

$$\left(\frac{\Sigma_0}{\Lambda}\right)^{2\nu\omega} = -\frac{(1-\omega^2)}{4\omega} \frac{c_1}{d_1} \left(1 - \frac{g^{\text{crit}}}{g}\right), \quad (4.15)$$

for $\kappa_D^{\text{MT}} < \kappa_D < \kappa_D^{\text{crit}}$, and

$$\left(\frac{\Sigma_0}{\Lambda}\right)^2 = -\frac{\nu(1-\omega^2)}{4} \frac{c_1}{c_2} \left(1 - \frac{g^{\text{crit}}}{g}\right), \quad (4.16)$$

for $\kappa_D < \kappa_D^{\text{MT}}$, where we used $g \approx g^{\text{crit}}$. Note here that the signs of c_1 and d_1 (c_2) are opposite to each other for $\kappa_D^{\text{MT}} < \kappa_D$ ($\kappa_D < \kappa_D^{\text{MT}}$). We thus find

$$\frac{\Sigma_0}{\Lambda} \sim \begin{cases} (1 - \frac{g^{\text{crit}}}{g})^{1/(2\nu\omega)}, & (\kappa_D^{\text{MT}} < \kappa_D < \kappa_D^{\text{crit}}), \\ \sqrt{1 - \frac{g^{\text{crit}}}{g}}, & (0 \leq \kappa_D < \kappa_D^{\text{MT}}). \end{cases} \quad (4.17)$$

Note here that Eq. (4.17) in $\kappa_D \rightarrow 0$ limit agrees with the scaling relation obtained in the $D(>4)$ -dimensional pure NJL model [10].

The chiral condensate is calculated from Eq. (3.15),

$$\sigma = \frac{(1+\omega)}{2} c_1 \Sigma_0 \left(\frac{\Sigma_0}{\Lambda}\right)^{\nu(1-\omega)} + \frac{(1-\omega)}{2} d_1 \Sigma_0 \left(\frac{\Sigma_0}{\Lambda}\right)^{\nu(1+\omega)} + \dots, \quad (4.18)$$

for $\kappa_D^{\text{MT}} < \kappa_D < \kappa_D^{\text{crit}}$, and

$$\sigma = \frac{(1+\omega)}{2} c_1 \Sigma_0 \left(\frac{\Sigma_0}{\Lambda}\right)^{\nu(1-\omega)} + \left(\frac{1+\omega}{2} - \frac{1}{\nu}\right) c_2 \Sigma_0 \left(\frac{\Sigma_0}{\Lambda}\right)^{\nu(1-\omega)+2} + \dots, \quad (4.19)$$

for $0 < \kappa_D < \kappa_D^{\text{MT}}$.

Further details of our analysis on the linearized SD equation are given in Appendix B.

B. Power Expansion Method

We next try to investigate the subdominant asymptotic solution directly from the nonlinear SD equation Eq. (3.9)

for $\kappa_D < \kappa_D^{\text{crit}}$. The next-to-leading contribution, on the other hand, comes from c_2 or d_1 , depending on the value of κ_D . For $\kappa_D^{\text{MT}} < \kappa_D < \kappa_D^{\text{crit}}$, d_1 term gives the next-to-leading contribution, while c_2 term becomes the next-to-leading for $\kappa_D < \kappa_D^{\text{MT}}$. At $\kappa_D = \kappa_D^{\text{MT}}$ the power of c_2 and d_1 terms goes same in the expression of Eq. (4.11). Although the coefficients $c_2 \propto 1/(-\nu\omega+1)$ and $d_1 \propto \Gamma(-\nu\omega)$ diverge at $\kappa_D = \kappa_D^{\text{MT}}$, i.e., $\omega = 1/\nu$, they cancel out each other, so that the logarithmic term appears,

without using the linearizing ansatz of the previous subsection [44]. The nonlinear differential equation Eq. (3.13) and the boundary conditions Eqs. (3.14) and (3.15) can be rewritten as

$$x^2 \frac{d^2 \Sigma}{dx^2} + (\nu+1)x \frac{d\Sigma}{dx} + \frac{\nu^2(1-\omega^2)}{4} \frac{x\Sigma}{x+\Sigma^2} = 0, \quad (4.20)$$

$$x^{\nu+1} \frac{d}{dx} \Sigma(x) \Big|_{x=0} = 0, \quad (4.21)$$

$$\left[\nu + \left(1 + \frac{4}{1-\omega^2} \frac{g}{\nu}\right) x \frac{d}{dx} \right] \Sigma(x) \Big|_{x=\Lambda^2} = \nu m_0. \quad (4.22)$$

For $x \gg \Sigma^2(x)$ the last term in the nonlinear differential equation Eq. (4.20) can be expanded as

$$\frac{x\Sigma}{x+\Sigma^2} = \Sigma - \frac{\Sigma^3}{x} + \frac{\Sigma^5}{x^2} + \dots. \quad (4.23)$$

Equation (4.23) allows us formally to expand the solution Σ of Eq. (4.20),

$$\Sigma(x) = \Sigma_1(x) + \Sigma_2(x) + \dots, \quad (4.24)$$

where $\Sigma_1(x)$ is the solution of a homogeneous linear differential equation

$$\left[x^2 \frac{d^2}{dx^2} + (\nu+1)x \frac{d}{dx} + \frac{\nu^2(1-\omega^2)}{4} \right] \Sigma_1(x) = 0, \quad (4.25)$$

while $\Sigma_2(x)$ is the inhomogeneous part of the solution of

$$\begin{aligned} & \left[x^2 \frac{d^2}{dx^2} + (\nu+1)x \frac{d}{dx} + \frac{\nu^2(1-\omega^2)}{4} \right] \Sigma_2(x) \\ &= \frac{\nu^2(1-\omega^2)}{4} \frac{\Sigma_1^3(x)}{x}. \end{aligned} \quad (4.26)$$

The higher terms in the expansion Eq. (4.24) are given in a similar manner to Eq. (4.26) just by replacing the right-hand side (RHS) to $\Sigma_1^2 \Sigma_2$, $\Sigma_1 \Sigma_2^2$, Σ_2^3 , Σ_1^5 , \dots .

What does the expansion Eq. (4.24) mean then? The solution of Eq. (4.25) is given by

$$\Sigma_1(x) = c_1 \Sigma_0 \left(\frac{x}{\Sigma_0^2} \right)^{-(1/2)\nu(1-\omega)} + d_1 \Sigma_0 \left(\frac{x}{\Sigma_0^2} \right)^{-(1/2)\nu(1+\omega)}, \quad (4.27)$$

with Σ_0 being a constant with a mass dimension. We thus obtain

$$\begin{aligned} \Sigma_2(x) = & c_2 \Sigma_0 \left(\frac{x}{\Sigma_0^2} \right)^{-(3/2)\nu(1-\omega)-1} \\ & + c'_2 \Sigma_0 \left(\frac{x}{\Sigma_0^2} \right)^{-(3/2)\nu+(1/2)\nu\omega-1} \\ & + d_2 \Sigma_0 \left(\frac{x}{\Sigma_0^2} \right)^{-(3/2)\nu(1+\omega)-1} \\ & + d'_2 \Sigma_0 \left(\frac{x}{\Sigma_0^2} \right)^{-(3/2)\nu-(1/2)\nu\omega-1}, \end{aligned} \quad (4.28)$$

with $c_2 \propto c_1^3$, $c'_2 \propto c_1^2 d_1$, $d_2 \propto d_1^3$, $d'_2 \propto c_1 d_1^2$. Note that the exponent of the power damping of $\Sigma_2(x)$ is steeper than that of $\Sigma_1(x)$ for $x \gg \Sigma_0^2$. It is also obvious that the higher terms in the expansion Eq. (4.24) fall quicker even than $\Sigma_2(x)$ for $x \gg \Sigma_0^2$. The expansion Eq. (4.24) can therefore be understood as a ‘‘power expansion’’ in this sense.

Since we are interested only in the dominant term (c_1) and in the subdominant term (d_1 or c_2 depending on κ_D), we restrict our analysis to these three terms. The power damping behavior of c_1 and d_1 terms agrees with those in the linearized SD equation. We note, however, the exponent of the power damping of the c_2 term differs from the result of the linearized SD equation. Actually, the c_2 term becomes subdominant for $\kappa_D < \kappa_D^{\text{PE}}$,

$$\kappa_D^{\text{PE}} \equiv \left[1 - \frac{1}{4} \left(1 + \frac{1}{\nu} \right)^2 \right] \kappa_D^{\text{crit}}. \quad (4.29)$$

We also note

$$\kappa_D^{\text{PE}} < \kappa_D^{\text{MT}}, \quad (4.30)$$

for $D > 4$.

We first consider the case $\kappa_D > \kappa_D^{\text{PE}}$, where the d_1 term gives the subdominant contribution. Since the power expansion cannot be adopted in the infrared region, we are not able to use the IRBC Eq. (4.21) to determine the coefficient c_1 and d_1 . If we assume that d_1 possesses opposite sign to c_1 , we find a scaling relation similar to Eq. (4.15). For $d_1/c_1 > 0$, on the other hand, the RHS of Eq. (4.15) becomes negative. Equation (4.15) thus cannot be understood as a scaling relation. This problem may be related with the vacuum instability which will be discussed in Sec. V. Hereafter we assume the result in the linearized SD equation, $d_1/c_1 < 0$, remains to be valid even in the power expansion method for $\kappa_D > \kappa_D^{\text{MT}}$.

We next turn to the $\kappa_D < \kappa_D^{\text{PE}}$ case, where the c_2 term gives the subdominant contribution. From Eq. (4.26), we find

$$c_2 = - \frac{\nu^2(1-\omega^2)}{4(\nu+1-\nu\omega)(2\nu\omega-\nu-1)} c_1^3, \quad (4.31)$$

and thus $c_2/c_1 < 0$. The scaling relation is then given by

$$\left(\frac{\Sigma_0}{\Lambda} \right)^{2\nu(1-\omega)+2} = - \frac{\nu(1-\omega^2)}{4(\nu+1-\nu\omega)} \frac{c_1}{c_2} \left(1 - \frac{g^{\text{crit}}}{g} \right), \quad (4.32)$$

with this κ_D , where we used $g \simeq g^{\text{crit}}$. We thus find

$$\frac{\Sigma_0}{\Lambda} \sim \begin{cases} (1 - \frac{g^{\text{crit}}}{g})^{1/(2\nu\omega)}, & (\kappa_D^{\text{PE}} < \kappa_D < \kappa_D^{\text{crit}}), \\ (1 - \frac{g^{\text{crit}}}{g})^{1/[2\nu(1-\omega)+2]}, & (0 \leq \kappa_D < \kappa_D^{\text{PE}}). \end{cases} \quad (4.33)$$

As in the linearized approximation, the coefficient c_2 in Eq. (4.31) diverges at $\kappa_D = \kappa_D^{\text{PE}}$, i.e., $\omega = 1/2 + 1/(2\nu)$. This implies that the logarithmic term appears at the point. We can easily confirm that $\Sigma_2(x)$ given by

$$\Sigma_2(x) = - \frac{(\nu-1)(3\nu+1)}{8(\nu+1)} c_1^3 \Sigma_0 \left(\frac{x}{\Sigma_0^2} \right)^{-(3\nu+1)/4} \ln \frac{x}{\Sigma_0^2}, \quad (4.34)$$

satisfies Eq. (4.26).

The chiral condensation is calculated from Eq. (3.15),

$$\begin{aligned} \sigma = & \frac{1}{2} (1 + \omega) c_1 \Sigma_0 \left(\frac{\Lambda^2}{\Sigma_0^2} \right)^{-(\nu/2)(1-\omega)} \\ & + \frac{1}{2} (1 - \omega) d_1 \Sigma_0 \left(\frac{\Lambda^2}{\Sigma_0^2} \right)^{-(\nu/2)(1+\omega)} + \dots, \end{aligned} \quad (4.35)$$

for $\kappa_D^{\text{PE}} < \kappa_D$, while

$$\begin{aligned} \sigma = & \frac{1}{2} (1 + \omega) c_1 \Sigma_0 \left(\frac{\Lambda^2}{\Sigma_0^2} \right)^{-(\nu/2)(1-\omega)} \\ & - \frac{1}{2} \left(1 - 3\omega + \frac{2}{\nu} \right) c_2 \Sigma_0 \left(\frac{\Lambda^2}{\Sigma_0^2} \right)^{-(3\nu/2)(1-\omega)-1} + \dots, \end{aligned} \quad (4.36)$$

for $0 < \kappa_D < \kappa_D^{\text{PE}}$.

V. EFFECTIVE POTENTIAL

We here outline our method to calculate the effective potential $V(\sigma)$ in the gauged NJL model. (See Refs. [11,53,54] for details.) For simplicity we assume $\pi_i \equiv 0$ in this analysis. It is easy to restore the π_i degree of freedom by using the chiral symmetry, i.e., $\sigma^2 \rightarrow \sigma^2 + \pi_i^2$.

We start with the partition function $W[J]$,

$$\begin{aligned} W[J] = & \frac{1}{i} \ln \int [d\sigma][d\pi_i][d\psi d\bar{\psi}] [\text{gauge}] \\ & \times \exp \left[i \int d^D x (\mathcal{L} + J\sigma) \right], \end{aligned} \quad (5.1)$$

with \mathcal{L} being the Lagrangian of the auxiliary field de-

scription of the gauged NJL model Eq. (2.9). For a constant external field J , the partition function can be written as

$$W[J] = \int d^D x w(J). \quad (5.2)$$

We note

$$\frac{d}{dJ} w(J) = \sigma, \quad (5.3)$$

with σ in the RHS being understood as the VEV of σ . Solving J as a function of σ in Eq. (5.3), we define the effective potential $V(\sigma)$,

$$V(\sigma) = J\sigma - w(J), \quad (5.4)$$

and thus obtain

$$\frac{d}{d\sigma} V(\sigma) = J \quad \text{or} \quad V(\sigma) = \int^\sigma d\sigma J. \quad (5.5)$$

We next describe how we calculate J in terms of σ . Looking at Eq. (2.9) and (5.1), we find the effect of constant J can be taken into account by replacing the

bare mass m_0 ,

$$\frac{N_c}{G} m_0 \rightarrow \frac{N_c}{G} m_0 + J. \quad (5.6)$$

in the SD equation. From Eq. (3.17) we obtain

$$\frac{G}{N_c} J = -m_0 + \sigma + \frac{4}{1-\omega^2} \frac{g}{\nu^2} x \frac{d}{dx} \Sigma(x) \Big|_{x=\Lambda^2}. \quad (5.7)$$

The derivative at the cutoff $\frac{d}{dx} \Sigma(x)|_{x=\Lambda^2}$ can be calculated from the UVBC Eq. (3.15),

$$x \frac{d}{dx} \Sigma(x) \Big|_{x=\Lambda^2} = \nu[\sigma - \Sigma(\Lambda^2)], \quad (5.8)$$

which leads to a compact expression for J ,

$$\frac{G}{N_c} J = -m_0 + \sigma + \frac{4}{1-\omega^2} \frac{g}{\nu} [\sigma - \Sigma(\Lambda^2)]. \quad (5.9)$$

We thus obtain the effective potential $V(\sigma)$ in terms of σ and $\Sigma(\Lambda^2)$:

$$(2^{D/2} N_c N_f \Lambda^D \Omega_{\text{NDA}})^{-1} V(\sigma) = -\frac{1}{g} \frac{m_0 \sigma}{\Lambda^2} + \frac{1}{2g} \frac{\sigma^2}{\Lambda^2} + \frac{4}{\nu(1-\omega^2)} \frac{1}{\Lambda^2} \int d\sigma [\sigma - \Sigma(\Lambda^2)]. \quad (5.10)$$

In the pure NJL limit $\kappa_D = 0$, i.e., $\omega = 1$, we should return to Eq. (3.9). Plugging with $\Sigma(x) = \sigma$ and Eqs. (3.10) and (5.6), we easily find

$$\begin{aligned} (2^{D/2} N_c N_f \Lambda^D \Omega_{\text{NDA}})^{-1} V(\sigma) &= -\frac{1}{g} \frac{m_0 \sigma}{\Lambda^2} + \frac{1}{2g} \frac{\sigma^2}{\Lambda^2} - \frac{1}{\Lambda^2} \int d\sigma \int_0^1 dz \frac{z^\nu \sigma}{z + \frac{\sigma^2}{\Lambda^2}} \\ &= -\frac{1}{g} \frac{m_0 \sigma}{\Lambda^2} + \frac{1}{2} \left(\frac{1}{g} - \frac{1}{\nu} \right) \frac{\sigma^2}{\Lambda^2} + \frac{1}{4(\nu-1)} \frac{\sigma^4}{\Lambda^4} + \mathcal{O}\left(\frac{\sigma^6}{\Lambda^6}\right), \quad (\kappa_D = 0). \end{aligned} \quad (5.11)$$

Since the subdominant term of $\Sigma(\Lambda^2)$ with nonvanishing gauge coupling depends on the approximations, we calculate $V(\sigma)$ separately.

A. Linearized Approximation

In the calculation of the effective potential $V(\sigma)$, we need to express $\Sigma(\Lambda^2)$ in terms of σ . Let us start with the solution of the linearized SD equation for $\kappa_D > \kappa_D^{\text{MT}}$. Combining Eq. (4.18) with

$$\Sigma(\Lambda^2) = c_1 \Sigma_0 \left(\frac{\Sigma_0}{\Lambda} \right)^{\nu(1-\omega)} + d_1 \Sigma_0 \left(\frac{\Sigma_0}{\Lambda} \right)^{\nu(1+\omega)} + \dots, \quad (5.12)$$

we obtain

$$\Sigma(\Lambda^2) = \frac{2}{1+\omega} \sigma + \frac{2\omega}{1+\omega} d_1 \Lambda \left(\frac{2c_1^{-1}}{1+\omega} \frac{\sigma}{\Lambda} \right)^{[\nu(1+\omega)+1]/[\nu(1-\omega)+1]} + \dots. \quad (5.13)$$

We are now able to calculate the effective potential from Eq. (5.5), (5.9), and (5.13),

$$(2^{D/2} N_c N_f \Lambda^D \Omega_{\text{NDA}})^{-1} V(\sigma) = -\frac{1}{g} \frac{m_0 \sigma}{\Lambda^2} + \frac{1}{2} \left(\frac{1}{g} - \frac{1}{g^{\text{crit}}} \right) \frac{\sigma^2}{\Lambda^2} + A_1 \left(\frac{\sigma}{\Lambda} \right)^{2+\{2\nu\omega/[\nu(1-\omega)+1]\}} + \dots, \quad (5.14)$$

for $\kappa_D^{\text{MT}} < \kappa_D < \kappa_D^{\text{crit}}$. Here the coefficient A_1 is defined as

$$A_1 \equiv -\frac{4}{1-\omega^2} \frac{\omega}{1+\omega} \frac{\nu(1-\omega)+1}{\nu(\nu+1)} d_1 \left(\frac{2c_1^{-1}}{1+\omega} \right)^{1+\{2\nu\omega/[\nu(1-\omega)+1]\}}. \quad (5.15)$$

It is straightforward to perform similar analysis for $0 < \kappa_D < \kappa_D^{\text{MT}}$. We obtain

$$(2^{D/2}N_cN_f\Lambda^D\Omega_{\text{NDA}})^{-1}V(\sigma) = -\frac{1}{g}\frac{m_0\sigma}{\Lambda^2} + \frac{1}{2}\left(\frac{1}{g} - \frac{1}{g^{\text{crit}}}\right)\frac{\sigma^2}{\Lambda^2} + A_2\left(\frac{\sigma}{\Lambda}\right)^{2+\{2/[\nu(1-\omega)+1]\}} + \dots, \quad (5.16)$$

for $0 < \kappa_D < \kappa_D^{\text{MT}}$. The coefficient A_2 is defined as

$$A_2 \equiv \frac{1}{2(\nu\omega - 1)} \frac{\nu(1-\omega) + 1}{\nu(1-\omega) + 2} c_1^2 \left(\frac{2c_1^{-1}}{1+\omega}\right)^{2+\{2/[\nu(1-\omega)+1]\}}. \quad (5.17)$$

The calculation at $\kappa_D = \kappa_D^{\text{MT}}$ is a little bit involved, since we need to take account of both d_1 and c_2 . By using formulas given in Appendix B, we obtain

$$(2^{D/2}N_cN_f\Lambda^D\Omega_{\text{NDA}})^{-1}V(\sigma) = -\frac{1}{g}\frac{m_0\sigma}{\Lambda^2} + \frac{1}{2}\left(\frac{1}{g} - \frac{1}{g^{\text{crit}}}\right)\frac{\sigma^2}{\Lambda^2} + A_2\left(\frac{\sigma}{\Lambda}\right)^{2+(2/\nu)} \ln\left(\frac{\Lambda}{\sigma}\right) + \dots, \quad (5.18)$$

for $\kappa_D = \kappa_D^{\text{MT}}$ and the coefficient A_2' is given by

$$A_2' \equiv \frac{1}{\nu + 1} c_1^2 \left(\frac{2\nu c_1^{-1}}{\nu + 1}\right)^{2+(2/\nu)}. \quad (5.19)$$

Using Eqs. (4.12) and (4.13), we find the coefficients A_1 , A_2 , and A_2' are all positive definite in the linearized SD equation. The potential is therefore stabilized for large value of σ even if we truncate the potential and neglect $+\dots$ terms in Eqs. (5.14), (5.16), and (5.18).

B. Power Expansion Method

The effective potential $V(\sigma)$ can also be calculated in the power expansion method. We should keep in mind, however, that the coefficients c_1 and d_1 cannot be determined with the power expansion method, unlike the calculation in the linearized SD equation.

We here summarize our results of the effective potential in the power expansion method:

$$(2^{D/2}N_cN_f\Lambda^D\Omega_{\text{NDA}})^{-1}V(\sigma) = -\frac{1}{g}\frac{m_0\sigma}{\Lambda^2} + \frac{1}{2}\left(\frac{1}{g} - \frac{1}{g^{\text{crit}}}\right)\frac{\sigma^2}{\Lambda^2} + \tilde{A}_1\left(\frac{\sigma}{\Lambda}\right)^{2+\{2\nu\omega/[\nu(1-\omega)+1]\}} + \dots, \quad (5.20)$$

for $\kappa_D^{\text{PE}} < \kappa_D < \kappa_D^{\text{crit}}$, and

$$(2^{D/2}N_cN_f\Lambda^D\Omega_{\text{NDA}})^{-1}V(\sigma) = -\frac{1}{g}\frac{m_0\sigma}{\Lambda^2} + \frac{1}{2}\left(\frac{1}{g} - \frac{1}{g^{\text{crit}}}\right)\frac{\sigma^2}{\Lambda^2} + \tilde{A}_2\frac{\sigma^4}{\Lambda^4} + \dots, \quad (5.21)$$

for $0 < \kappa_D < \kappa_D^{\text{PE}}$, and

$$(2^{D/2}N_cN_f\Lambda^D\Omega_{\text{NDA}})^{-1}V(\sigma) = -\frac{1}{g}\frac{m_0\sigma}{\Lambda^2} + \frac{1}{2}\left(\frac{1}{g} - \frac{1}{g^{\text{crit}}}\right)\frac{\sigma^2}{\Lambda^2} + \tilde{A}_2'\frac{\sigma^4}{\Lambda^4} \ln\left(\frac{\Lambda}{\sigma}\right) + \dots, \quad (5.22)$$

for $\kappa_D = \kappa_D^{\text{PE}}$. The coefficients \tilde{A}_1 , \tilde{A}_2 , and \tilde{A}_2' are defined by

$$\tilde{A}_1 \equiv -\frac{4}{1-\omega^2} \frac{\omega}{1+\omega} \times \frac{\nu(1-\omega) + 1}{\nu(\nu+1)} d_1 \left(\frac{2c_1^{-1}}{1+\omega}\right)^{[\nu(1+\omega)+1]/[\nu(1-\omega)+1]}, \quad (5.23)$$

$$\tilde{A}_2 \equiv \frac{1}{4(2\nu\omega - \nu - 1)} \left(\frac{2}{1+\omega}\right)^4, \quad (5.24)$$

$$A_2' \equiv \frac{1}{\nu+1} \left(\frac{4\nu}{3\nu+1}\right)^4, \quad (5.25)$$

respectively.

We emphasize here, due to the lack of our knowledge of c_1 and d_1 , the coefficient \tilde{A}_1 cannot be calculated in the power expansion method. On the other hand, we find $\tilde{A}_2 > 0$ for $0 < \kappa_D < \kappa_D^{\text{PE}}$, independently of c_1 .

We comment that the exponent of the lowest interaction term (\tilde{A}_1 term) in Eq. (5.14) is the same as that of A_1 in Eq. (5.20). In passing, the bifurcation solution also leads to the same exponent of the lowest interaction term. Thus the lowest interaction term in the effective potential is insensitive to the approximations of the SD equation for $\kappa_D^{\text{MT}} < \kappa_D < \kappa_D^{\text{crit}}$. Note that $\kappa_D^{\text{PE}} < \kappa_D^{\text{MT}}$. On the other hand, for $0 < \kappa_D \leq \kappa_D^{\text{MT}}$, the exponent of the lowest interaction term is different depending on the approximations.

VI. NONTRIVIAL WINDOW

In Sec. III we found renormalizability or nontriviality in the sense that the decay constant F_π and the renormalized Yukawa coupling $\Gamma_s^{(R)}$ become finite in the continuum limit for $\kappa_D^{\text{MT}} < \kappa_D < \kappa_D^{\text{crit}}$.

Here we further study the renormalization and discuss the renormalization group flow, based on the effective potential derived in Sec. V. The renormalization of the effective potential is performed in a way similar to the four dimensional gauged NJL model [37]. This method is applicable not only in the broken phase, but also in the symmetric phase. Note that we already renormalized the σ field in Eq. (3.67) so that its kinetic term is kept finite.

For $\kappa_D^{\text{MT}} < \kappa_D < \kappa_D^{\text{crit}}$ we require the bare parameters (m_0 and g) in the effective potential to depend on the cutoff [$m_0(\Lambda)$ and $g(\Lambda)$] such that

$$\frac{m_0(\Lambda)}{g(\Lambda)} \Lambda^{\nu(1+\omega)} = \text{const.}, \quad (6.1)$$

$$(2^{D/2} N_c N_f \mu^D \Omega_{\text{NDA}})^{-1} V_R(\sigma_R) = -\frac{1}{g_R} \frac{m_R \sigma_R}{\mu^2} + \frac{1}{2} \left(\frac{1}{g_R} - \frac{1}{g^{\text{crit}}} \right) \frac{\sigma_R^2}{\mu^2} + A_1 \left(\frac{\sigma_R}{\mu} \right)^{2+\{2\nu\omega/[\nu(1-\omega)+1]\}}, \quad (6.5)$$

for $\kappa_D^{\text{MT}} < \kappa_D < \kappa_D^{\text{crit}}$, and similar is Eq. (5.20) (power expansion method). Note that the “+...” terms in Eqs. (5.14) and (5.20) are decoupled in the $\Lambda \rightarrow \infty$ limit. The σ field has nontrivial self-interaction in the renormalized effective potential. Recall that $A_1 > 0$. The potential is therefore stabilized and the VEV of σ_R remains finite for $g_R > g^{\text{crit}}$. We emphasize that the form of the renormalized effective potential $V_R(\sigma_R)$ does not depend on the approximations, although the coefficient of the interaction term does.

From Eqs. (6.3) and (6.4) we obtain the beta function of the dimensionless four-fermion coupling g_R ,

$$\beta(g_R) = 2\nu\omega g_R \left(1 - \frac{g_R}{g^{\text{crit}}} \right), \quad (6.6)$$

and the anomalous dimension of the fermion mass γ_m ,

$$\gamma_m(g_R) = -\frac{\beta(g_R)}{g_R} + \nu(1+\omega) = \nu \left[1 - \omega + 2\omega \frac{g_R}{g^{\text{crit}}} \right], \quad (6.7)$$

for $\kappa_D^{\text{MT}} < \kappa_D < \kappa_D^{\text{crit}}$. They take the same form as those in the four dimensional gauged NJL model with fixed gauge coupling [37] up to the factor $\nu = D/2 - 1 \rightarrow 1$ ($D \rightarrow 4$). We note the beta function of the NJL coupling possesses a UVFP $g_R = g^{\text{crit}} [= \nu(1+\omega)^2/4]$. At the fixed point of g_R the anomalous dimension γ_m reads

$$\gamma_m = \nu(1+\omega). \quad (6.8)$$

Let us turn to the region $0 < \kappa_D < \kappa_D^{\text{MT}}$. In this region the decay constant F_π diverges as we mentioned. It is thus expected that the same renormalization procedure as in

and

$$\left(\frac{1}{g(\Lambda)} - \frac{1}{g^{\text{crit}}} \right) \Lambda^{2\nu\omega} = \text{const.} \quad (6.2)$$

The effective potential is thus kept finite even in the continuum limit $\Lambda \rightarrow \infty$ (Wilsonian renormalization). We define the renormalized parameters m_R, g_R as

$$\frac{m_R(\mu)}{g_R(\mu)} \mu^{\nu(1+\omega)} = \frac{m_0(\Lambda)}{g(\Lambda)} \Lambda^{\nu(1+\omega)}, \quad (6.3)$$

and

$$\left(\frac{1}{g_R(\mu)} - \frac{1}{g^{\text{crit}}} \right) \mu^{2\nu\omega} = \left(\frac{1}{g(\Lambda)} - \frac{1}{g^{\text{crit}}} \right) \Lambda^{2\nu\omega}, \quad (6.4)$$

with μ being the renormalization scale. The effective potential Eq. (5.14) (linearizing approximation) is thus renormalized as

the region $\kappa_D^{\text{MT}} < \kappa_D < \kappa_D^{\text{crit}}$ will break down. In fact, if we formally performed such a renormalization, we would get⁶

$$(2^{D/2} N_c N_f \mu^D \Omega_{\text{NDA}})^{-1} V_R(\sigma_R) = -\frac{1}{g_R} \frac{m_R \sigma_R}{\mu^2} + \frac{1}{2} \left(\frac{1}{g_R} - \frac{1}{g^{\text{crit}}} \right) \frac{\sigma_R^2}{\mu^2}, \quad (6.9)$$

where we defined m_R and g_R as

$$\frac{m_R(\mu)}{g_R(\mu)} \mu^{\nu+1} = \frac{m_0(\Lambda)}{g(\Lambda)} \Lambda^{\nu+1}, \quad (6.10)$$

and

$$\left(\frac{1}{g_R(\mu)} - \frac{1}{g^{\text{crit}}} \right) \mu^2 = \left(\frac{1}{g(\Lambda)} - \frac{1}{g^{\text{crit}}} \right) \Lambda^2. \quad (6.11)$$

This renormalization would imply the beta function and

⁶At $\kappa_D = \kappa_D^{\text{MT}}$ the renormalized effective potential would take the same form as Eq. (6.9). Although the definition of g_R and m_R in Eqs. (6.10) and (6.11) should be modified by logarithmic factors, the beta function and the anomalous dimension are unchanged from Eqs. (6.12) and (6.13). If one took the limit $\kappa_D \searrow \kappa_D^{\text{MT}}$ of the expression in Eq. (6.5), one would find the pole in the coefficient A_1 . This pole actually is logarithmic divergence $\ln \Lambda$ when the sub-subleading term is properly incorporated in the limit $\Lambda \rightarrow \infty$ as in the pure NJL limit of the four dimensional gauged NJL model [37]. After the renormalization of the σ field Eq. (3.67), we find the interaction term vanishes in $V_R(\sigma_R)$, consistently with Eq. (6.9).

the anomalous dimension:

$$\beta(g_R) = 2g_R \left(1 - \frac{g_R}{g^{\text{crit}}}\right), \quad (6.12)$$

and

$$\gamma_m(g_R) = -\frac{\beta(g_R)}{g_R} + \nu + 1 = \nu - 1 + 2\frac{g_R}{g^{\text{crit}}}. \quad (6.13)$$

The anomalous dimension γ_m would read

$$\gamma_m = \nu + 1 \quad (6.14)$$

at the UVFP of g_R .

However, the self-interaction term in $V_R(\sigma_R)$ Eq. (6.9) disappears at $\Lambda \rightarrow \infty$ in our renormalization procedure both in the linearized approximation and in the power expansion method. Thus the renormalized potential is not stabilized for $g_R > g^{\text{crit}}$ and hence the renormalization breaks down. The Yukawa interaction does also vanish and hence interactions of σ and π are all trivial.

Let us next consider the dynamical dimension of composite operators. Within the ladder approximation adopted in this paper, various composite operators have the dynamical dimensions,

$$\dim[(\bar{\psi}\psi)^2] = 2 \dim(\bar{\psi}\psi), \quad (6.15)$$

$$\dim[(\bar{\psi}\psi)^4] = 4 \dim(\bar{\psi}\psi), \quad (6.16)$$

$$\dim[\partial_M(\bar{\psi}\psi)\partial^M(\bar{\psi}\psi)] = 2 \dim(\bar{\psi}\psi) + 2, \quad (6.17)$$

with

$$\dim(\bar{\psi}\psi) = D - 1 - \gamma_m. \quad (6.18)$$

Using Eqs. (6.8) and (6.14), we find

$$\dim[(\bar{\psi}\psi)^2] = \begin{cases} D - 2\nu\omega, & (\kappa_D^{\text{MT}} < \kappa_D < \kappa_D^{\text{crit}}), \\ D - 2, & (0 \leq \kappa_D \leq \kappa_D^{\text{MT}}), \end{cases} \quad (6.19)$$

$$\dim[(\bar{\psi}\psi)^4] = \begin{cases} 2D - 4\nu\omega, & (\kappa_D^{\text{MT}} < \kappa_D < \kappa_D^{\text{crit}}), \\ 2D - 4, & (0 \leq \kappa_D \leq \kappa_D^{\text{MT}}), \end{cases} \quad (6.20)$$

$$\dim[\partial_M(\bar{\psi}\psi)\partial^M(\bar{\psi}\psi)] = \begin{cases} D + 2 - 2\nu\omega, & (\kappa_D^{\text{MT}} < \kappa_D < \kappa_D^{\text{crit}}), \\ D, & (0 \leq \kappa_D \leq \kappa_D^{\text{MT}}). \end{cases} \quad (6.21)$$

The four-fermion operator $(\bar{\psi}\psi)^2$ is relevant in the whole region $0 \leq \kappa_D < \kappa_D^{\text{crit}}$, which justifies the inclusion of the four-fermion interaction in the study of the phase structure of walking gauge theories.

For the region $\kappa_D^{\text{MT}} < \kappa_D < \kappa_D^{\text{crit}}$ both operators $(\bar{\psi}\psi)^4$ and $\partial_M(\bar{\psi}\psi)\partial^M(\bar{\psi}\psi)$ are irrelevant, i.e., $\dim[(\bar{\psi}\psi)^4] > D$ and $\dim[\partial_M(\bar{\psi}\psi)\partial^M(\bar{\psi}\psi)] > D$. This is consistent with our renormalization procedure without introducing such operators.

On the other hand, for $0 \leq \kappa_D \leq \kappa_D^{\text{MT}}$, the kinetic operator $\partial_M(\bar{\psi}\psi)\partial^M(\bar{\psi}\psi)$ becomes marginal, $\dim(\partial_M(\bar{\psi}\psi)\partial^M(\bar{\psi}\psi)) = D$, while the eight-fermion operator $(\bar{\psi}\psi)^4$ is irrelevant, $\dim[(\bar{\psi}\psi)^4] > D$. This is another symptom of breakdown of our renormalization in this region. We may need to incorporate the marginal operator $\partial_M(\bar{\psi}\psi)\partial^M(\bar{\psi}\psi)$ as a counterterm, which is similar to the situation in the four dimensional pure NJL model where both $\partial_\mu(\bar{\psi}\psi)\partial^\mu(\bar{\psi}\psi)$ and $(\bar{\psi}\psi)^4$ are marginal [37].

We comment on a further symptom of breakdown of our renormalization procedure for $0 \leq \kappa_D \leq \kappa_D^{\text{MT}}$. If we renormalized consistently the effective action in this region, we would expect the renormalization of $\bar{\psi}\psi$ as

$$m_R \langle (\bar{\psi}\psi)_R \rangle = m_0 \langle (\bar{\psi}\psi)_0 \rangle. \quad (6.22)$$

Note that $m_0 \sim \Lambda^{-\gamma_m} = \Lambda^{-\nu-1}$, while Eq. (3.12) yields

$$\langle (\bar{\psi}\psi)_0 \rangle \propto \int_0^{\Lambda^2} dx x^\nu \frac{\Sigma(x)}{x + \Sigma^2} \sim \Lambda^{\nu(1+\omega)}, \quad (6.23)$$

where we used the asymptotic behavior of the dominant solution in Eq. (3.36). Thus Eq. (6.22) diverges as $m_0 \langle (\bar{\psi}\psi)_0 \rangle \sim \Lambda^{\nu\omega-1}$. This can be understood by noting $(\bar{\psi}\psi)_R \propto \sigma_R$, the VEV of which is divergent due to the lack of the interaction term as stabilizer in $V_R(\sigma_R)$ Eq. (6.9). This reflects necessity of other operators such as the kinetic operator $\partial_M(\bar{\psi}\psi)\partial^M(\bar{\psi}\psi)$ as counterterms. See Appendix D for further details.

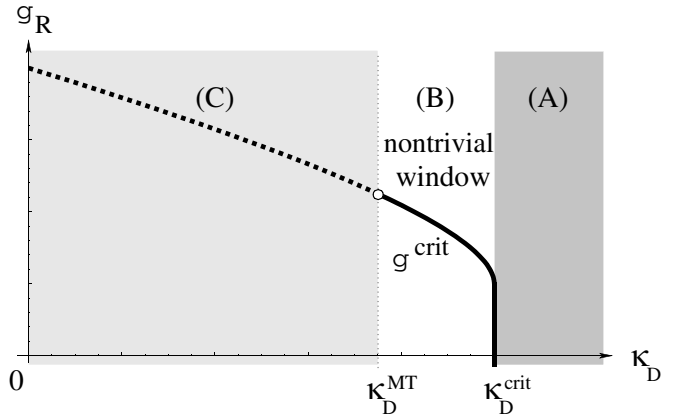


FIG. 2. Phase diagram in (κ_D, g_R) plane. The critical line g^{crit} is the same as that in Fig. 1. Each point on this line describes a different theory having different N_c, N_f . In the region (A) $\kappa_D > \kappa_D^{\text{crit}}$ the dynamical mass diverges in the continuum limit, i.e., no finite theory exists. In the region (B) $\kappa_D^{\text{MT}} < \kappa_D < \kappa_D^{\text{crit}}$ (nontrivial window), the dynamical mass can be made finite and the self-interaction of σ_R and the Yukawa interaction remain nontrivial in the continuum limit. In the region (C) $0 \leq \kappa_D \leq \kappa_D^{\text{MT}}$, the renormalized effective potential is not stabilized for the broken phase $g_R > g^{\text{crit}}$ and hence the renormalization breaks down. In this region the self-interaction of σ_R and the Yukawa interaction vanish and are trivial.

		N_f						
		1	2	3	4	5	6	7
N_c	2	C	C	B	A			
	3	C	C	C	B	A	A	A
	4	C	C	C	B	B	A	A
	5	C	C	C	C	B	B	A

(D = 6)

FIG. 3. Classification of models ($D = 6$). The entries labeled by A, B and C stand for the models in the regions (A) $\kappa_D > \kappa_D^{\text{crit}}$, (B) $\kappa_D^{\text{MT}} < \kappa_D < \kappa_D^{\text{crit}}$ and (C) $0 \leq \kappa_D \leq \kappa_D^{\text{MT}}$, respectively. No entry denotes absence of the UVFP of the gauge coupling g_* .

We depict our results mentioned above in Fig. 2 and the tables in Figs. 3–5:⁷

The region $\kappa_D > \kappa_D^{\text{crit}}$ – no finite continuum theory.

- (A) As we discussed in Sec. III, the gauge coupling strength on the UVFP κ_D is not a continuous parameter, so that κ_D cannot be fine-tuned arbitrarily close to κ_D^{crit} to make the dynamical mass finite in the continuum limit ($\Lambda \rightarrow \infty$). For example, models with $D = 6, N_c = 3, N_f = 5, 6, 7$ fall into this category.

The region $\kappa_D^{\text{MT}} < \kappa_D < \kappa_D^{\text{crit}}$ – nontrivial window.

- (B) The dynamical mass can be made finite by fine-tuning of the four-fermion coupling. Once the dynamical mass is made finite, the decay constant of the NG boson is also finite and so is the Yukawa coupling of the NG boson. The self-interaction of σ_R and the Yukawa interaction remain nontrivial. We also note that the self-interaction of σ_R is conformal symmetric. The renormalized four-

⁷There could be a possible ambiguity of the estimate of the concrete number of N_f, N_c of the nontrivial window: It could arise from the ambiguity of the estimate of κ_D^{crit} in the improved ladder SD equation with different momentum identification for the running gauge coupling [10,11]. Although, as we mentioned in Sec. III, our momentum identification Eq. (3.2) with non-local gauge fixing has much advantage over the simplest one [45], $g_{(4+\delta)D}^2(\mu) \rightarrow g_{(4+\delta)D}^2(\max(x, y))$, with Landau gauge fixing, one might use the latter. Then one could get somewhat smaller values [10]: $\kappa_6^{\text{crit}} = 0.122$, $\kappa_8^{\text{crit}} = 0.146$, and $\kappa_{10}^{\text{crit}} = 0.163$, compared with those in the present paper, $\kappa_D^{\text{crit}} = D(D-2)/[32(D-1)]$. This would shift the nontrivial window for a fixed N_c to a somewhat smaller N_f region. For example, the model with $D = 6, N_c = 3, N_f = 3$ would enter the nontrivial window.

		N_f						
		1	2	3	4	5	6	7
N_c	6	C	C	B	A	A	A	
	7	C	C	C	A	A	A	A
	8	C	C	C	B	A	A	A
	9	C	C	C	C	A	A	A
	10	C	C	C	C	B	A	A

(D = 8)

FIG. 4. Classification of models ($D = 8$). The entries labeled by A, B and C stand for the models in the regions (A) $\kappa_D > \kappa_D^{\text{crit}}$, (B) $\kappa_D^{\text{MT}} < \kappa_D < \kappa_D^{\text{crit}}$ and (C) $0 \leq \kappa_D \leq \kappa_D^{\text{MT}}$, respectively. No entry denotes absence of the UVFP of the gauge coupling g_* . For $2 \leq N_c \leq 5$ the nontrivial window is not found.

fermion coupling g_R has a UVFP $g_R(\infty) = g^{\text{crit}}$ and the anomalous dimension γ_m is very large, $\nu < \gamma_m = \nu(1 + \omega) < \nu + 1$. For example, models with $D = 6, N_c = 3, N_f = 4$ fall into this category. The nontrivial window gets closed, $\kappa_D^{\text{MT}} \rightarrow \kappa_D^{\text{crit}}$, when $D \rightarrow \infty$.

		N_f						
		1	2	3	4	5	6	7
N_c	3	A						
	4	B						
	5	C	A					
	6	C	A					
	7	C	A	A				
	8	C	B	A				

(D = 10)

FIG. 5. Classification of models ($D = 10$). The entries labeled by A, B and C stand for the models in the regions (A) $\kappa_D > \kappa_D^{\text{crit}}$, (B) $\kappa_D^{\text{MT}} < \kappa_D < \kappa_D^{\text{crit}}$ and (C) $0 \leq \kappa_D \leq \kappa_D^{\text{MT}}$, respectively. No entry denotes absence of the UVFP of the gauge coupling g_* . For $N_c = 2$ there is no UVFP.

The region $0 \leq \kappa_D \leq \kappa_D^{\text{MT}}$ —trivial or nonrenormalizable.

- (C) Even if we made the dynamical mass finite by fine-tuning the four-fermion coupling, the decay constant does diverge in the continuum limit. The self-interaction of σ_R and the Yukawa interaction vanish and are trivial. The renormalized effective potential is not stabilized for $g_R > g^{\text{crit}}$. Without introducing additional operators other than the four-fermion operator, the theory cannot be renormalized. For example, models with $D = 6$, $N_c = 3$, $N_f = 1, 2, 3$ fall into this category. Note that κ_D^{MT} is reduced to the pure NJL point in the four dimensional NJL model, $\kappa_D^{\text{MT}} \rightarrow 0$, in the limit $D \rightarrow 4$.

So far we focused our discussions on the theory with the gauge coupling exactly on the UVFP $\kappa_D = C_F g_*^2 \Omega_{\text{NDA}}$. Each point on the critical line g^{crit} describes a different theory (different N_c, N_f). Now that we have shown existence of the nontrivial window, we shall study the RG flows of $[\hat{g}^2(\mu), g_R(\mu)]$ of a particular theory specified by one point on the critical line. In order to obtain the RG flows in the vicinity of the UVFP $[\hat{g}^2(\infty) = g_*^2, g_R(\infty) = g^{\text{crit}}]$, we define variations $\delta\hat{g}^2$ and δg_R as

$$\delta\hat{g}^2(\mu) \equiv \hat{g}^2(\mu) - g_*^2, \quad (6.24)$$

$$\delta g_R(\mu) \equiv g_R(\mu) - g^{\text{crit}}. \quad (6.25)$$

It is enough to study linearized RGEs [55]

$$\frac{\partial}{\partial \ln \mu} \begin{pmatrix} \delta\hat{g}^2 \\ \delta g_R \end{pmatrix} = \mathcal{M} \begin{pmatrix} \delta\hat{g}^2 \\ \delta g_R \end{pmatrix}, \quad (6.26)$$

with

$$\mathcal{M} = \begin{pmatrix} -2(\nu - 1) & 0 \\ \Delta & -2\nu\omega \end{pmatrix}, \quad (6.27)$$

neglecting higher order terms of $\delta\hat{g}^2$ and δg_R . The diagonal components are obtained from Eqs. (2.5) and (6.6). The off-diagonal one Δ is a constant to be determined later. The eigenvalues λ_i ($i = 1, 2$) and eigenvectors $\mathbf{v}_i, \mathbf{w}_i$ are defined as

$$\mathbf{v}_i^T \mathcal{M} = \lambda_i \mathbf{v}_i^T, \quad \mathcal{M} \mathbf{w}_i = \lambda_i \mathbf{w}_i. \quad (6.28)$$

We obtain

$$\lambda_1 = -2(\nu - 1), \quad \lambda_2 = -2\nu\omega \quad (6.29)$$

and

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} \Delta \\ 2(\nu - 1 - \nu\omega) \end{pmatrix}, \quad (6.30)$$

$$\mathbf{w}_1 = \begin{pmatrix} -2(\nu - 1 - \nu\omega) \\ \Delta \end{pmatrix}, \quad \mathbf{w}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \quad (6.31)$$

We here note

$$|\lambda_1| > |\lambda_2|, \quad (6.32)$$

since $|\lambda_1| - |\lambda_2| = 2(\nu - 1 - \nu\omega) > 0$ for $D \geq 6$ ($\nu \geq 2$) in the nontrivial window ($\nu\omega < 1$). In order to estimate Δ , we need to determine the effective potential off the UVFP, $\hat{g}^2 \neq g_*^2$, which can be obtained by solving the SD equation. Instead of doing this, here we simply replace the UVFP value g_*^2 in Eq. (6.4) with the running bulk gauge coupling \hat{g}^2 :

$$\left(\frac{1}{g_R(\mu)} - \frac{1}{\frac{\nu}{4}(1 + \hat{\omega})^2} \right) \mu^{2\nu\hat{\omega}} = \text{const.}, \quad (6.33)$$

with

$$\hat{\omega} \equiv \sqrt{1 - \frac{C_F \hat{g}^2(\mu) \Omega_{\text{NDA}}}{\kappa_D^{\text{crit}}}}. \quad (6.34)$$

Differentiating Eq. (6.33) with respect to $\ln \mu$, we obtain a rough estimate of Δ as

$$\Delta = 2(\nu - 1 - \nu\omega) \frac{1 - \omega}{\omega} \frac{g^{\text{crit}}}{g_*^2}. \quad (6.35)$$

The eigenvector \mathbf{w}_1 is then parallel to the tangent of the critical line g^{crit} .

Once we choose the model, (N_c, N_f) , the UVFP value g_* is fixed and so is the critical value of the four-fermion coupling g^{crit} . RG flows for a typical (N_c, N_f) are depicted in Fig. 6 in the vicinity of the point $P_\infty \equiv (\hat{g}^2 = g_*^2, g_R = g^{\text{crit}})$. From Eq. (6.32) we can see a gross feature of RG flows independently of the value of Δ : The RG flows approach this point as increasing μ more rapidly in the direction of \mathbf{w}_1 than that of \mathbf{w}_2 . There are four regions (I,

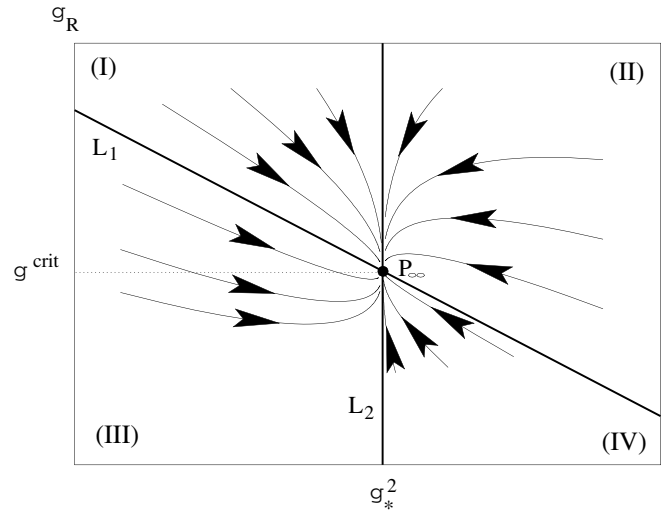


FIG. 6. RG flows of a theory with $D = 6, N_c = 3, N_f = 4$. The point P_∞ denotes the UVFP, $[\hat{g}^2(\infty) = g_*^2, g_R(\infty) = g^{\text{crit}}]$, of this theory. The arrows of the flows indicate the direction toward the UV limit. The lines L_1 and L_2 correspond to the eigenvectors \mathbf{w}_1 and \mathbf{w}_2 , respectively. L_1 coincides with the tangent of the critical line g^{crit} .

II, III, and IV) which are separated by two lines L_1 (parallel to \mathbf{w}_1) and L_2 (parallel to \mathbf{w}_2). The $D\chi$ SB takes place in the regions I and II. The line L_1 coincides with the tangent of the critical line. While the chiral symmetry is unbroken in the region III, the region IV is not precisely the symmetric phase: Although the NJL coupling in the region IV is not strong enough for the $D\chi$ SB, the gauge coupling strength grows strong in the infrared there. We thus expect $D\chi$ SB also in the region IV, with a fermion dynamical mass being of order $\Lambda \frac{(D)}{\text{MS}} (\ll \Lambda)$ [11] which is a scale parameter of this gauge theory analogously to Λ_{QCD} .⁸ Note that the theory is controlled by the UVFP P_∞ , where the composite fields σ and π enjoy nontrivial interactions. Thus the theory in the nontrivial window is consistently renormalized even off the UVFP $\hat{g}^2 \neq g_*^2$.

VII. SUMMARY AND DISCUSSION

We have discussed phase structure of the D -dimensional gauged NJL model with compactified extra $\delta (= D - 4)$ dimensions on the TeV scale, based on the improved ladder SD equation with the running gauge coupling given by the truncated KK effective theory. Such a running (dimensionless) gauge coupling has a nontrivial UV fixed point and the theory behaves as a four-fermion theory coupled to a walking gauge theory with the gauge coupling almost constant near the value of the UVFP in a wide energy region above the compactification scale. The central assumption of this paper was that the existence of the UVFP is not an artifact of the KK effective theory and may have some reality in a more elaborate nonperturbative approach.

Solving the SD equation in the bulk by setting the gauge coupling at the value of the UVFP as a reasonable approximation, we found the critical line similar to that of the four dimensional gauged NJL model:

$$g^{\text{crit}} = \frac{D-1}{4} (1 + \sqrt{1 - \kappa_D / \kappa_D^{\text{crit}}})^2, \quad (7.1)$$

with $\kappa_D^{\text{crit}} = \frac{D}{32} \frac{D-2}{D-1}$, which takes the same form as that of the four dimensional one with a fixed gauge coupling for $D \rightarrow 4$ in the prefactor. In spite of the formal resemblance to the four dimensional case, however, the UVFP value κ_D , given by Eq. (3.6) with Eqs. (2.2) and (2.7), is determined as a function of N_c and N_f and hence is not a free parameter but is a fixed quantity once we specify the model.

Here we should emphasize the following: In the pure gauge theories with extra dimensions the nontrivial the-

ory defined at the critical point is only formal, since κ_D cannot be fine-tuned arbitrarily close to the critical value [10,11]. The dynamical mass cannot be made finite in the continuum limit ($\Lambda \rightarrow \infty$). Then, even if we assume the existence of the UVFP beyond the truncated KK effective theory, the renormalizability of higher dimensional gauge theories with (N_c, N_f) such that $\kappa_D > \kappa_D^{\text{crit}}$ [region (A) in Fig. 2] is only formal.⁹ On the other hand, we do have a continuous parameter, the four-fermion coupling g , in the gauged NJL model and hence the nontrivial theory can be defined by fine-tuning g arbitrarily close to the critical line. Thus the inclusion of the four-fermion interactions may provide an interesting new possibility for higher dimensional gauge theories which were long considered nonrenormalizable and trivial theories based on the perturbation.

Remarkably enough, we in fact found the nontrivial window [region (B) in Fig. 2] for (N_c, N_f) such that

$$\kappa_D^{\text{MT}} < \kappa_D < \kappa_D^{\text{crit}}, \quad (7.2)$$

$$\kappa_D^{\text{MT}} = \left(1 - \frac{1}{(D/2 - 1)^2}\right) \kappa_D^{\text{crit}}, \quad (7.3)$$

where the four-fermion theory in the presence of walking gauge interactions becomes nontrivial and renormalizable, similar to the four dimensional gauged NJL model, in the sense that the decay constant F_π and couplings of composite π and σ as well as the induced Yukawa coupling of the fermion (with mass M) to those composites remain finite in the continuum limit $\Lambda/M \rightarrow \infty$. We explicitly performed renormalization of the kinetic term of the composite bosons and the effective potential. The renormalized four-fermion coupling has a UVFP at $g_R(\infty) = g^{\text{crit}}$, where the theory has a large anomalous dimension

$$\gamma_m = \left(\frac{D}{2} - 1\right) \left(1 + \sqrt{1 - \kappa_D / \kappa_D^{\text{crit}}}\right). \quad (7.4)$$

The renormalized effective potential indeed has nontrivial interactions which are *conformal invariant*, thanks to the large anomalous dimension.

It is rather surprising that even in higher dimensions the four-fermion operators become relevant thanks to the large anomalous dimension, $D/2 > \gamma_m > D/2 - 1$. In fact, the dynamical dimension of $(\bar{\psi}\psi)^2$ operator is relevant, $\dim[(\bar{\psi}\psi)^2] < D$. On the other hand, the $(\bar{\psi}\psi)^4$ operator and the kinetic term $\partial_M(\bar{\psi}\psi)\partial^M(\bar{\psi}\psi)$ are irrelevant, $\dim[(\bar{\psi}\psi)^4] = 2D - 2(D-2)\sqrt{1 - \kappa_D / \kappa_D^{\text{crit}}} > D$, and $\dim[\partial_M(\bar{\psi}\psi)\partial^M(\bar{\psi}\psi)] = D + 2 - (D-2) \times \sqrt{1 - \kappa_D / \kappa_D^{\text{crit}}} > D$. Thus the theory can be renormalized

⁸Thus L_1 as the border between region II and IV is not a phase boundary in the exact sense but has a similar feature in the sense that the dynamical mass grows rapidly from $\Lambda \frac{(D)}{\text{MS}}$ (instead of 0) to Λ as we cross the line from region IV to II. A similar phenomenon also takes place in the four dimensional gauged NJL model with the QCD-like gauge coupling.

⁹Nevertheless, such a theory can still be useful for model building as an effective theory with finite cutoff Λ . See, for example, the tMAC analysis of the $D = 8$ top-mode standard model with extra dimension [12].

without operators other than $(\bar{\psi}\psi)^2$. We further gave the RG flow off the UVFP in Fig. 6 which is consistent with our renormalization performed on the UVFP.

On the other hand, for (N_c, N_f) such that $0 \leq \kappa_D \leq \kappa_D^{\text{MT}}$ [region (C) in Fig. 2], the decay constant F_π diverges even if we fine-tune the four-fermion coupling close to the critical value g^{crit} to make finite the dynamical mass of the fermion. At the MT point $\kappa_D = \kappa_D^{\text{MT}}$ the divergence of F_π^2 is logarithmic, $F_\pi^2 \sim \ln \Lambda$, which is similar to the situation at the pure NJL point of the four dimensional case. The MT point is reduced to the pure NJL point $\kappa_D^{\text{MT}} \rightarrow 0$ for $D \rightarrow 4$. Without introducing other operators such as $(\bar{\psi}\psi)^4$, $\partial_M(\bar{\psi}\psi)\partial^M(\bar{\psi}\psi)$, etc., the renormalization breaks down in the region (C).

At this point it is worthwhile mentioning the formal limit $D \rightarrow 4$ of the nontrivial window. It was shown [35–39] that the four dimensional gauged NJL model becomes renormalizable and nontrivial, when the one-loop gauge coupling runs as specified by:

$$A \equiv \frac{6C_F}{-b} > 1, \quad \left(C_F = \frac{N_c^2 - 1}{2N_c}, -b = \frac{11N_c - 2N_f}{3} \right), \quad (7.5)$$

where A measures the running speed of the coupling [22,33]: $A \gg 1$ is the walking theory and $A \rightarrow \infty$ ($b \rightarrow 0$) corresponds to the nonrunning (standing limit of walking coupling) case with vanishing beta function. The condition Eq. (7.5) was first obtained [35] by evaluating the F_π^2 , taking account of the *logarithmic corrections* to the asymptotic behavior of the mass function [28]:

$$\Sigma(x) \sim c_1 \Sigma_0 \left[\ln \left(\frac{x}{\Sigma_0^2} \right) \right]^{-(A/2)}, \quad (7.6)$$

instead of Eq. (3.36), and then in the RG group analysis in the language of the equivalent gauged Yukawa model [36–38] and also in the nonperturbative RG equation [39].

Here we show that the formal limit of $D \rightarrow 4$ of the nontrivial window in this paper coincides with that characterized by precisely the same condition as Eq. (7.5), now *without logarithmic corrections* which were rather delicate factors in the previous arguments in the four dimensional model. Instead we consider $D = 4 + \delta$ with $0 < \delta \ll 1$ in the spirit of ϵ expansion [55,56]. The nontrivial window is defined by

$$\nu\omega < 1, \quad (7.7)$$

which reads

$$\frac{\kappa_D}{\kappa_D^{\text{crit}}} > \delta. \quad (7.8)$$

On the other hand, from Eqs. (2.7) and (3.6) we have

		N_f						
		1	2	3	4	5	6	7
N_c	2	C	C	C	C	B	B	B
	3	C	C	C	C	B	B	B
	4	C	C	C	C	C	B	B
	5	C	C	C	C	C	B	B
	6	C	C	C	C	C	B	B

(D = 4)

FIG. 7. Classification of models ($D = 4$). The entries labeled by B and C stand for the models in the regions (B) $A = 18C_F/(11N_c - 2N_f) > 1$ and (C) $A \leq 1$, respectively.

$$\frac{\kappa_D}{\kappa_D^{\text{crit}}} = \frac{6C_F}{-b} \delta = A\delta. \quad (7.9)$$

Then we obtain Eq. (7.5), $A > 1$. The nontrivial window in the formal limit $D \rightarrow 4$ ($\delta \rightarrow 0$) thus coincides with the condition of the renormalizability/nontriviality of the four dimensional gauged NJL models, namely, those coupled to the (moderately) walking gauge theory with $A > 1$. Combining the condition of asymptotic freedom $N_f < 11N_c/2$ and $A > 1$, we thus have theories within the nontrivial window in $D \rightarrow 4$:

$$\frac{9}{2} \frac{1}{N_c} < N_f - N_c < \frac{9}{2} N_c. \quad (7.10)$$

See the table in Fig. 7 for the theories satisfying Eq. (7.10) [or Eq. (7.5)]. On the other hand, the trivial region (C) ($0 \leq \kappa_D \leq \kappa_D^{\text{MT}}$) coincides with the condition of the triviality/nonrenormalizability, $A \leq 1$, in the $D \rightarrow 4$ limit. Thus in the $D \rightarrow 4$ limit where $\kappa_D^{\text{MT}} \rightarrow 0$ and $\kappa_D \rightarrow 0$ (for $A < \infty$),¹⁰ the UVFPs for both regions (C) ($0 \leq \kappa_D \leq \kappa_D^{\text{MT}}$) and (B) ($\kappa_D^{\text{MT}} < \kappa_D < \kappa_D^{\text{crit}}$) in the phase diagram of Fig. 2 shrink to a single point of the pure NJL point and hence the distinction among them is not obviously visible compared with the case with extra dimensions. In other words, the renormalizability/nontriviality of the four dimensional gauged NJL model with $A > 1$ gauge theories [35–39] is a four dimensional manifestation of the nontrivial window, in sharp contrast to the theory with $A \leq 1$. Thus *the nontrivial window is not a peculiarity nor an*

¹⁰If one formally changes the theory (N_c, N_f) depending upon the dimension $\delta(\rightarrow 0)$ in such a way that $A \sim 4\lambda/\delta \rightarrow \infty$, one could have a limit arriving at the four dimensional gauged NJL model with fixed (standing) gauge coupling λ : $\kappa_D/\kappa_D^{\text{crit}} \rightarrow 4\lambda(0 < \lambda < \lambda^{\text{crit}} = 1/4)$.

artifact of the extra dimensions but is rather a universal feature of the gauged NJL model.

The RG flows of the nontrivial window in this limit are only in regions II and IV of Fig. 6 and are similar to Fig. 6 but with a crucial difference: Since $|\lambda_1| < |\lambda_2|$ in this limit, the flows first approach L_1 instead of L_2 and then converge toward the pure NJL point (Fig. 8). This is actually consistent with the gross feature obtained in the four dimensional model.

In conclusion, we have shown that the four-fermion theories in $D > 4$ dimensions are renormalizable and nontrivial, *when coupled to D -dimensional walking gauge theories*, with the parameters (N_c, N_f) being in the nontrivial window, which we assumed have a nontrivial ultraviolet fixed point as given by the truncated KK effective theory. The fact that the four-fermion theory in dimensions $D < 4$, *lower than four*, is renormalizable and nontrivial has been known for long time since Wilson [56] (For a proof on the $D = 3$ theory, see Ref. [58]) and was also known for the arbitrary $D(<4)$ dimensions [37,59,60]. Further in $D = 4$ it was known [35–39] that the four-fermion theory when coupled with certain walking gauge theories is renormalizable and nontrivial. Here we have shown for the first time that the four-fermion theory in $D > 4$, *higher than four*, also shares the same feature when coupled to walking gauge theories characterized by the nontrivial window.

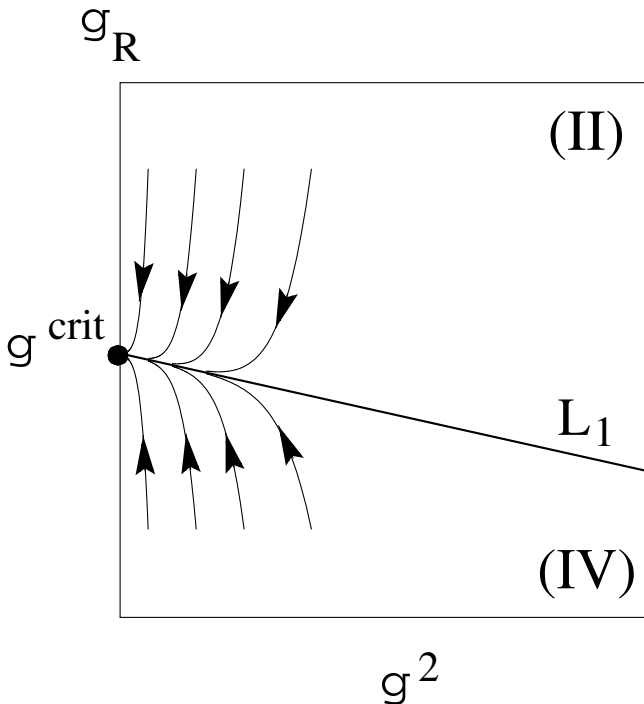


FIG. 8. RG flows of a theory for $D = 4$. The arrows of the flows indicate the direction toward the UV limit. The line L_1 corresponds to the eigenvector \mathbf{w}_1 (tangent of the critical line of the four dimensional gauged NJL model with fixed gauge coupling [26,27]). The black circle is the pure NJL point.

This renormalizability/nontriviality is not just a formal matter, but implies *cutoff insensitivity* of the physics prediction, which may be useful for model buildings even with a finite cutoff. This is rather surprising, since the theory is dominated by the dynamics in the ultraviolet region where both the gauge and the four-fermion interactions become strongly coupled. Explicit model buildings based on this observation were in fact attempted for the four dimensional case (“top-mode walking grand unified theories) [33,61].

The phase structure of the gauged NJL model in the bulk with compactified extra dimensions may be useful for model buildings such as the top-mode standard model, bulk technicolor, etc.. For instance, the $D = 6$ standard model gauge interactions are not enough to trigger the top quark condensate [10–12], while the gauged NJL model can do work due to the additional attractive interactions as in the case of the original top-mode standard model. Although introduction of such an *ad hoc* four-fermion interaction may be less attractive than the scenario without it, the KK modes of the top quark still naturally reduce the top quark mass prediction of the original model as was emphasized in Ref. [8]. Since the electroweak symmetry breaking is still a central mystery of the modern particle theory, it would be useful to consider all possible dynamical scenarios before the LHC will take off. The gauged NJL model with extra dimensions may become one of the dormant volcanos to give an explosion into rich TeV physics.

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APPENDIX A: PROPAGATOR OF THE COMPOSITE SCALAR

We have calculated the auxiliary field propagator in Sec. III D by using the resummation technique [48]. We may take yet another choice proposed in Ref. [49] in which we obtain the scalar propagator as the lowest order of the Chebyshev expansion. While the resummation technique is operative in a weak gauge coupling region, the Chebyshev expansion method is valid in the whole region. In four dimensions the results of the Chebyshev expansion method are similar to those of the resummation technique [49]. We here demonstrate that the wave function renormalization constant for the auxiliary field agrees with the result of the resummation technique shown in Eq. (3.67) even in extra dimensions.

The composite scalar propagator is written in terms of the vacuum polarization function Π_S for the composite

scalar:

$$iD_\sigma^{-1}(q) = -\frac{N_c}{G} + \Pi_S(q), \quad (\text{A1})$$

where the vacuum polarization function is given by

$$\Pi_S(q) = N_c N_f \int \frac{d^D k}{i(2\pi)^D} \text{tr}[S(k+q)\chi_S(k+q,k)S(k)], \quad (\text{A2})$$

with χ_S being the Bethe-Salpeter (BS) amplitude for the scalar. Neglecting the explicit breaking of the D dimensional Lorentz symmetry owing to the compactification, we can decompose generally the BS amplitude χ_S into

$$S(p) = S_0(p) = \frac{i}{\not{p}}. \quad (\text{A4})$$

We then obtain the vacuum polarization function as

$$2^{-(D/2)} N_c^{-1} N_f^{-1} \Pi_S(q) = \int \frac{d^D k}{i(2\pi)^D} \frac{-k \cdot (k+q)}{k^2(k+q)^2} F_1(k+q,k) + 2 \int \frac{d^D k}{i(2\pi)^D} \frac{(k \cdot q)^2 - k^2 q^2}{k^2(k+q)^2} F_2(k+q,k). \quad (\text{A5})$$

On the other hand, the BS equation under the ladder approximation is

$$\chi_S(p+q,p) = \mathbf{1} + \int \frac{d^D k}{(2\pi)^D} [-iT^a \Gamma^M] S(k+q) \chi_S(k+q,k) S(k) [-iT^a \Gamma^N] g_{(4+\delta)D}^2 D_{MN}(p-k), \quad (\text{A6})$$

where $\mathbf{1}$ is the identity matrix and D_{MN} is the propagator of the gauge boson. In the symmetric phase the BS equation is obviously decomposed into two simultaneous equations for $F_{1,2}$ and for $F_{3,4}$. Since the structure functions F_3 and F_4 do not contribute to Π_S as shown in Eq. (A5), we neglect the two hereafter. The BS equations for F_1 and F_2 are given by

$$F_1(p+q,p) = 1 + (D-1+\xi) C_F g_*^2 \int \frac{d^D k}{i(2\pi)^D} \frac{-k \cdot (k+q)}{k^2(k+q)^2 [-(p-k)^2]^{D/2-1}} F_1(k+q,k) + 2(D-1+\xi) C_F g_*^2 \int \frac{d^D k}{i(2\pi)^D} \frac{(k \cdot q)^2 - k^2 q^2}{k^2(k+q)^2 [-(p-k)^2]^{D/2-1}} F_2(k+q,k), \quad (\text{A7})$$

$$F_2(p+q,p) = \frac{C_F g_*^2}{2[(p \cdot q)^2 - p^2 q^2]} \int \frac{d^D k}{i(2\pi)^D} \frac{\Xi(p,q,k)}{k^2(k+q)^2 [-(p-k)^2]^{D/2-1}} F_1(k+q,k) + \frac{C_F g_*^2}{(p \cdot q)^2 - p^2 q^2} \int \frac{d^D k}{i(2\pi)^D} \frac{-k \cdot (k+q) \Xi(p,q,k)}{k^2(k+q)^2 [-(p-k)^2]^{D/2-1}} F_2(k+q,k) \quad (\text{A8})$$

with

$$\Xi(p,q,k) \equiv (D-3+\xi)[q^2(p \cdot k) - (p \cdot q)(q \cdot k)] + \frac{2(1-\xi)}{-(p-k)^2} \{-q^2[k \cdot (p-k)][p \cdot (p-k)] - [q \cdot (k-p)][p^2(q \cdot k) - k^2(p \cdot q)]\}, \quad (\text{A9})$$

where the bulk gauge coupling was replaced as in Eq. (3.2) with $\hat{g}^2 = g_*^2$. The gauge fixing parameter ξ should be taken as

$$\xi = -\frac{(D-1)(D-4)}{D} \quad (\text{A10})$$

for consistency with the ladder approximation.

Let us expand F_1 , F_2 , and the integral kernels of Π_S and the BS equations by series of Gegenbauer polynomials, which is generalization of the Chebyshev poly-

four Lorentz scalar functions F_i ($i = 1, 2, 3, 4$),

$$\chi_S(p+q,p) = F_1(p+q,p) + (\not{q}\not{p} - \not{p}\not{q})F_2(p+q,p) + (\not{p} + \not{q})F_3(p+q,p) + \not{p}F_4(p+q,p), \quad (\text{A3})$$

where p and q denote momenta of the fermion and the scalar bound state, respectively. Although one might suspect that more structure functions are needed in $D(>4)$ -dimensions, other terms are zero or reduced into the above four, because the BS amplitude includes only two momenta. We analyze the scalar propagator and the BS equation in the symmetric phase, i.e.,

At this stage the BS Eqs. (A7) and (A8) depend on three angles α , β , γ defined by

$$\cos\alpha = \frac{p_E \cdot q_E}{|p_E||q_E|}, \quad \cos\beta = \frac{p_E \cdot k_E}{|p_E||k_E|}, \quad (\text{A11})$$

$$\cos\gamma = \frac{q_E \cdot k_E}{|q_E||k_E|},$$

as well as an infinite chain of harmonics f_n and g_n for expansion of the Gegenbauer polynomials $C_n^\nu(\cos\alpha)$, where

$$F_1(p + q, p) = \sum_{n=0}^{\infty} f_n(p_E^2, q_E^2) C_n^\nu(\cos\alpha) \quad (\text{A12})$$

and

$$F_2(p + q, p) = \sum_{n=0}^{\infty} g_n(p_E^2, q_E^2) C_n^\nu(\cos\alpha), \quad (\text{A13})$$

so that it is quite difficult to solve analytically the equations. We here note that only the BS equation for the harmonics f_0 contains an inhomogeneous term (the constant unity), because the Gegenbauer polynomials satisfy

$$\int_0^\pi d\theta \sin^{2\nu}\theta C_n^\nu(\cos\theta) C_m^\nu(\cos\theta) = w_n \delta_{mn} \quad (\text{A14})$$

with

$$w_n \equiv \frac{\pi \Gamma(n + 2\nu)}{2^{2\nu-1} (n + \nu) n! [\Gamma(\nu)]^2}, \quad (\text{A15})$$

and

$$C_0^\nu(\cos\theta) = 1. \quad (\text{A16})$$

The inhomogeneous term certainly gives main contributions to the BS equations. Hence we assume that the BS equation for f_0 is well approximated by the closed form of f_0 , neglecting the effects of $f_n (n \geq 1)$ and $g_n (n \geq 0)$. The harmonics $f_n (n \geq 1)$ and $g_n (n \geq 0)$ are iteratively determined after f_0 is computed. We then obtain the BS equation for f_0 as

$$f_0(s, t) = 1 + \frac{4(D-1)}{D} \kappa_D \int_0^{\Lambda^2} du u^{(D/2)-2} K_0(s, t, u) \times f_0(u, t), \quad (\text{A17})$$

$$(2^{D/2} N_c N_f \Lambda^{D-2} \Omega_{\text{NDA}})^{-1} i D_\sigma^{-1}(t) = -\frac{1}{g} + \frac{1}{\frac{4(D-1)}{D} \kappa_D} [F_{\text{UV}}(\Lambda^2, t) - 1], \quad (\text{A24})$$

where we divided the zeroth order harmonics f_0 into two parts of the IR and UV regions,

$$f_0(s, t) \equiv F_{\text{IR}}(s, t) \theta(t - s) + F_{\text{UV}}(s, t) \theta(s - t). \quad (\text{A25})$$

The BS amplitude in zero momentum transfer, i.e., the effective Yukawa coupling Γ_s , is also obtained through F_{UV} ,

$$\Gamma_s(p_E^2) = \chi_S(p, p) \simeq F_{\text{UV}}(p_E^2, q_E^2 = 0). \quad (\text{A26})$$

Let us solve the zeroth order BS Eq. (A17) which is equivalent to a set of the second order differential equation,

$$\frac{\partial^2}{\partial s^2} f_0(s, t) + \frac{D}{2s} \frac{\partial}{\partial s} f_0(s, t) + \frac{\tilde{\kappa}_D}{s^2} a_0(s, t) f_0(s, t) = 0, \quad (\text{A27})$$

and the IRBC

where $s \equiv p_E^2$, $t \equiv q_E^2$, $u \equiv k_E^2$, and

$$K_0(s, t, u) \equiv a_0(u, t) b_0(s, u) \quad (\text{A18})$$

with

$$a_0(u, t) \equiv \frac{1}{w_0} \int_0^\pi d\gamma \sin^{2\nu}\gamma C_0^\nu(\cos\gamma) \frac{k_E \cdot (k_E + q_E)}{(k_E + q_E)^2}, \quad (\text{A19})$$

$$b_0(s, u) \equiv \frac{1}{w_0} \int_0^\pi d\beta \sin^{2\nu}\beta \frac{C_0^\nu(\cos\beta)}{[(p_E - k_E)^2]^{(D/2)-1}}. \quad (\text{A20})$$

We can easily perform the angular integrations and find

$$a_0(u, t) = \frac{1}{2} \left[\frac{u-t}{\max(u, t)} F\left(1, 2 - \frac{D}{2}; \frac{D}{2}; \frac{\min(u, t)}{\max(u, t)}\right) + 1 \right], \quad (\text{A21})$$

$$b_0(s, u) = \frac{1}{[\max(s, u)]^{(D/2)-1}}. \quad (\text{A22})$$

The vacuum polarization function is written in terms of f_0 as

$$\Pi_S(t) = 2^{D/2} \Omega_{\text{NDA}} N_c N_f \int_0^{\Lambda^2} du u^{(D/2)-2} a_0(u, t) f_0(u, t). \quad (\text{A23})$$

Noting that the integral of RHS in the zeroth order BS Eq. (A17) at $s = \Lambda^2$ is just the same as in Eq. (A23), we can rewrite the composite scalar propagator without the integral as follows:

$$s^{D/2} \frac{\partial}{\partial s} f_0(s, t) \Big|_{s \rightarrow 0} = 0, \quad (\text{A28})$$

and the UVBC

$$s \frac{\partial}{\partial s} f_0(s, t) + (D/2 - 1) [f_0(s, t) - 1] \Big|_{s \rightarrow \Lambda^2} = 0, \quad (\text{A29})$$

with

$$\tilde{\kappa}_D \equiv \frac{2(D-1)(D-2)}{D} \kappa_D = \frac{(D-2)^2(1-\omega^2)}{16}. \quad (\text{A30})$$

We now replace the hypergeometric function in $a_0(u, t)$ by unity, i.e.,

$$a_0(u, t) \rightarrow \frac{1}{2} \left[\frac{u-t}{\max(u, t)} + 1 \right]. \quad (\text{A31})$$

We confirm later on that the approximation works well in $D = 6$. Within the approximation the differential equations for the UV and IR parts are

$$\frac{\partial}{\partial s^2} F_{\text{UV}} + \frac{\nu+1}{s} \frac{\partial}{\partial s} F_{\text{UV}} + \frac{\tilde{\kappa}_D}{s^2} \left(1 - \frac{t}{2s}\right) F_{\text{UV}} = 0, \quad (\text{A32})$$

and

$$\frac{\partial}{\partial s^2} F_{\text{IR}} + \frac{\nu+1}{s} \frac{\partial}{\partial s} F_{\text{IR}} + \frac{\tilde{\kappa}_D}{2st} F_{\text{IR}} = 0, \quad (\text{A33})$$

respectively, where we used

$$\nu = D/2 - 1. \quad (\text{A34})$$

We can solve analytically Eqs. (A32) and (A33) and find

$$F_{\text{UV}}(s, t) = \left(\frac{t}{s}\right)^{\nu/2} \left[c_1 I_{-\nu\omega} \left(\sqrt{2\tilde{\kappa}_D \frac{t}{s}} \right) + c_2 I_{\nu\omega} \left(\sqrt{2\tilde{\kappa}_D \frac{t}{s}} \right) \right], \quad (\text{A35})$$

for the UV part, and

$$F_{\text{IR}}(s, t) = \left(\frac{t}{s}\right)^{\nu/2} \left[c_3 J_\nu \left(\sqrt{2\tilde{\kappa}_D \frac{s}{t}} \right) + c_4 Y_\nu \left(\sqrt{2\tilde{\kappa}_D \frac{s}{t}} \right) \right], \quad (\text{A36})$$

for the IR part, where J_ν , Y_ν , and $I_{\pm\nu\omega}$ represent the Bessel functions of first and second kind, and modified one, respectively. For $\omega = 1/\nu$ we should use the modified Bessel function of the third kind, K_1 , instead of I_{-1} . The IRBC (A28) leads to

$$c_4 = 0. \quad (\text{A37})$$

Other coefficients c_1 , c_2 , and c_3 are obtained from the UVBC (A29) and smoothness conditions for $f_0(s, t)$ at $s = t$,

$$F_{\text{IR}}(s \rightarrow t, t) = F_{\text{UV}}(s \rightarrow t, t), \quad (\text{A38})$$

and

$$\left. \frac{\partial}{\partial s} F_{\text{IR}}(s, t) \right|_{s \rightarrow t} = \left. \frac{\partial}{\partial s} F_{\text{UV}}(s, t) \right|_{s \rightarrow t}, \quad (\text{A39})$$

as follows:

$$c_1 \equiv c_1 \left(\frac{t}{\Lambda^2}, \omega \right) = \frac{\pi \gamma(\omega)}{2 \sin(\pi \nu \omega)} \eta^{-1} \left(\frac{t}{\Lambda^2}, \omega \right), \quad (\text{A40})$$

$$c_2 = c_1 \left(\frac{t}{\Lambda^2}, -\omega \right), \quad (\text{A41})$$

$$c_3 = \eta^{-1} \left(\frac{t}{\Lambda^2}, \omega \right), \quad (\text{A42})$$

where we defined

$$\eta \left(\frac{t}{\Lambda^2}, \omega \right) \equiv \frac{\pi}{2 \sin(\pi \nu \omega)} \left[\gamma(\omega) \zeta \left(\frac{t}{\Lambda^2}, -\omega \right) - \gamma(-\omega) \zeta \left(\frac{t}{\Lambda^2}, \omega \right) \right], \quad (\text{A43})$$

$$\gamma(\omega) \equiv \sqrt{2\tilde{\kappa}_D} [J_\nu(\sqrt{2\tilde{\kappa}_D}) I'_{\nu\omega}(\sqrt{2\tilde{\kappa}_D}) + J'_\nu(\sqrt{2\tilde{\kappa}_D}) I_{\nu\omega}(\sqrt{2\tilde{\kappa}_D})], \quad (\text{A44})$$

$$\zeta \left(\frac{t}{\Lambda^2}, \omega \right) \equiv \frac{1}{2\nu} \left(\frac{t}{\Lambda^2} \right)^{\nu/2} \left[\nu I_{\nu\omega} \left(\sqrt{\frac{2\tilde{\kappa}_D t}{\Lambda^2}} \right) - \sqrt{\frac{2\tilde{\kappa}_D t}{\Lambda^2}} I'_{\nu\omega} \left(\sqrt{\frac{2\tilde{\kappa}_D t}{\Lambda^2}} \right) \right]. \quad (\text{A45})$$

The prime $X'(z)$ denotes dX/dz . We also note the relation

$$I'_\rho(z) I_{-\rho}(z) - I_\rho(z) I'_{-\rho}(z) = \frac{2 \sin(\pi \rho)}{\pi z}. \quad (\text{A46})$$

The BS amplitude is thus given by

$$F_{\text{IR}}(s, t) = \eta^{-1} \left(\frac{t}{\Lambda^2}, \omega \right) \left(\frac{t}{s} \right)^{\nu/2} J_\nu \left(\sqrt{\frac{2\tilde{\kappa}_D s}{t}} \right), \quad (\text{A47})$$

$$F_{\text{UV}}(s, t) = \frac{\pi}{2 \sin(\pi \nu \omega)} \eta^{-1} \left(\frac{t}{\Lambda^2}, \omega \right) \left(\frac{t}{s} \right)^{\nu/2} \times \left[\gamma(\omega) I_{-\nu\omega} \left(\sqrt{\frac{2\tilde{\kappa}_D t}{s}} \right) - \gamma(-\omega) I_{\nu\omega} \left(\sqrt{\frac{2\tilde{\kappa}_D t}{s}} \right) \right]. \quad (\text{A48})$$

The effective Yukawa coupling Γ_s is also obtained as

$$\Gamma_s(p_E^2) = F_{\text{UV}}(p_E^2, q_E^2 = 0) = \frac{2}{1 + \omega} \left(\frac{p_E^2}{\Lambda^2} \right)^{-(\nu/2)(1-\omega)}, \quad (\text{A49})$$

which agrees with Eq. (3.53) calculated through the resummation technique.

In the case of $D = 6$, we can solve the BS equation without using the approximation for a_0 represented in Eq. (A31). The BS equations for F_{IR} and F_{UV} are

$$\frac{\partial}{\partial s^2} F_{\text{IR}}(s, t) + \frac{3}{s} \frac{\partial}{\partial s} F_{\text{IR}}(s, t) + \frac{\tilde{\kappa}_6}{6} \left(\frac{4}{st} - \frac{1}{t^2} \right) F_{\text{IR}}(s, t) = 0, \quad (\text{A50})$$

$$\frac{\partial}{\partial s^2} F_{\text{UV}}(s, t) + \frac{3}{s} \frac{\partial}{\partial s} F_{\text{UV}}(s, t) + \frac{\tilde{\kappa}_6}{2s^2} \left(2 - \frac{4t}{3s} + \frac{t^2}{3s^2} \right) F_{\text{UV}}(s, t) = 0, \quad (\text{A51})$$

respectively. Noting that Eq. (A51) is rewritten as

$$\frac{\partial}{\partial \bar{s}^2} F_{UV} - \frac{1}{\bar{s}} \frac{\partial}{\partial \bar{s}} F_{UV} + \tilde{\kappa}_6 \left(\frac{1}{\bar{s}^2} - \frac{2t}{3\bar{s}} + \frac{t^2}{6} \right) F_{UV} = 0 \quad (\text{A52})$$

with $\bar{s} \equiv 1/s$, we obtain the solutions

$$\begin{aligned} F_{UV}(s, t) = & C_1 \left(\frac{s}{t} \right)^{\rho_+} \\ & \times \exp\left(-i\sqrt{\frac{\tilde{\kappa}_6 t}{6s}}\right) {}_1F_1\left(\alpha_+, \gamma_+; 2i\sqrt{\frac{\tilde{\kappa}_6 t}{6s}}\right) \\ & + C_2 \left(\frac{s}{t} \right)^{\rho_-} \\ & \times \exp\left(-i\sqrt{\frac{\tilde{\kappa}_6 t}{6s}}\right) {}_1F_1\left(\alpha_-, \gamma_-; 2i\sqrt{\frac{\tilde{\kappa}_6 t}{6s}}\right), \end{aligned} \quad (\text{A53})$$

$$\begin{aligned} F_{IR}(s, t) = & C_3 \exp\left(-\sqrt{\frac{\tilde{\kappa}_6 s}{6t}}\right) {}_1F_1\left(\alpha_{IR}, 3; 2\sqrt{\frac{\tilde{\kappa}_6 s}{6t}}\right) \\ & + C_4 \exp\left(-\sqrt{\frac{\tilde{\kappa}_6 s}{6t}}\right) \Psi\left(\alpha_{IR}, 3; 2\sqrt{\frac{\tilde{\kappa}_6 s}{6t}}\right), \end{aligned} \quad (\text{A54})$$

where ${}_1F_1(\alpha, \gamma; z)$ and $\Psi(\alpha, \gamma; z)$ denote the confluent hypergeometric function and its Ψ function, respectively. We here defined

$$\rho_{\pm} \equiv -1 \pm \sqrt{1 - \tilde{\kappa}_6}, \quad \gamma_{\pm} \equiv 1 \mp 2\sqrt{1 - \tilde{\kappa}_6}, \quad (\text{A55})$$

$$\alpha_{\pm} \equiv \frac{\gamma_{\pm}}{2} - 2i\sqrt{\frac{\tilde{\kappa}_6}{6}}, \quad \alpha_{IR} \equiv \frac{3}{2} - 2\sqrt{\frac{\tilde{\kappa}_6}{6}}. \quad (\text{A56})$$

We can confirm that F_{UV} is real, i.e., $F_{UV}^* = F_{UV}$, by using the Kummer's transformation ${}_1F_1(\alpha, \gamma; z) = e^z {}_1F_1(\gamma - \alpha, \gamma; -z)$. The IRBC (A28) leads to $C_4 = 0$ owing to the behavior of the confluent hypergeometric Ψ function around zero

$$\begin{aligned} \Psi(\alpha, 1+n; z) \simeq & \frac{(n-1)!}{\Gamma(\alpha)} z^{-n} \\ & + \frac{(-1)^{n-1}}{n! \Gamma(\alpha-n)} {}_1F_1(\alpha, n+1; z) \ln z, \quad (z \sim 0). \end{aligned} \quad (\text{A57})$$

Other coefficients C_1 , C_2 , and C_3 are obtained by using the UVBC (A29) and continuous conditions (A38) and (A39). We show the difference between the solutions with or without the approximation (A31). (See Figs. 9 and 10.) The approximation works well within a few percent. We expect that the approximation is also valid in $D = 8, 10, \dots$

We now determine the wave function renormalization constant for the auxiliary field σ . We expand the function $F_{UV}(\Lambda^2, q_E^2)$ in Eq. (A48) over q_E^2/Λ^2 . Depending on the values of $\nu\omega$, three expressions are obtained:

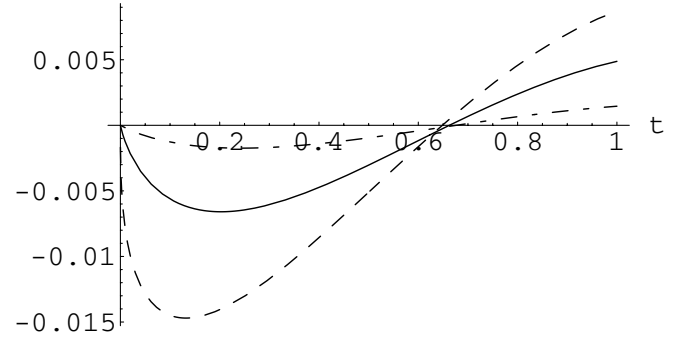


FIG. 9. Error of the zeroth order BS amplitude f_0 arising from the approximation (A31). The curves show $(f_0^{\text{app}} - f_0)/f_0$ at $s = \Lambda^2$ in the unit of $\Lambda = 1$, where f_0 and f_0^{app} denote the solutions in Eqs. (A53) and (A54) without the approximation (A31) and the approximate expressions in Eqs. (A47) and (A48), respectively. The dot-dashed, solid, and dashed lines represent graphs for $\tilde{\kappa}_6 = \kappa_6/\kappa_6^{\text{crit}} = 0.02, 20/33, 0.98$, respectively. (The ACDH model in $D = 6$, $N_c = 3$, $N_f = 2$, corresponds to $\tilde{\kappa}_6 = 20/33$.)

$$\begin{aligned} iD_{\sigma}^{-1}(q_E^2) = & -2^{D/2} N_c N_f \Omega_{\text{NDA}} \Lambda^{D-2} \left[\frac{1}{g} - \frac{1}{g^{\text{crit}}} \right. \\ & \left. + A_1 \left(\frac{q_E^2}{\Lambda^2} \right)^{\nu\omega} \right], \end{aligned} \quad (\text{A58})$$

for $0 < \omega < 1/\nu$, and

$$\begin{aligned} iD_{\sigma}^{-1}(q_E^2) = & -2^{D/2} N_c N_f \Omega_{\text{NDA}} \Lambda^{D-2} \left[\frac{1}{g} - \frac{1}{g^{\text{crit}}} \right. \\ & \left. + A_2 \frac{q_E^2}{\Lambda^2} \ln\left(\frac{\Lambda^2}{q_E^2}\right) \right], \end{aligned} \quad (\text{A59})$$

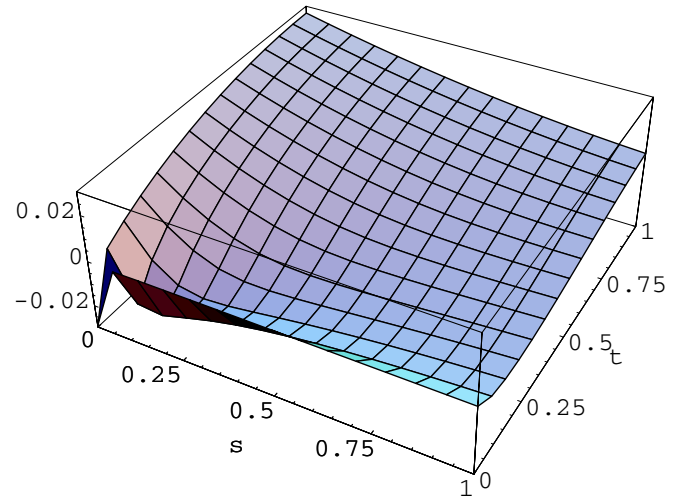


FIG. 10 (color online). Error of the zeroth order BS amplitude f_0 arising from the approximation (A31). The curved surface shows $(f_0^{\text{app}} - f_0)/f_0$ for $\tilde{\kappa}_6 = 20/33$ ($N_c = 3, N_f = 2$) in the unit of $\Lambda = 1$, where f_0 and f_0^{app} denote the solutions in Eqs. (A53) and (A54) without the approximation (A31) and the approximate expressions in Eqs. (A47) and (A48), respectively.

for $\omega = 1/\nu$, and

$$iD_\sigma^{-1}(q_E^2) = -2^{D/2} N_c N_f \Omega_{\text{NDA}} \Lambda^{D-2} \left[\frac{1}{g} - \frac{1}{g^{\text{crit}}} + A_3 \frac{q_E^2}{\Lambda^2} \right], \quad (\text{A60})$$

for $1/\nu < \omega \leq 1$, where we defined

$$A_1 \equiv \frac{\gamma(-\omega)}{\gamma(\omega)} \frac{\Gamma(1-\nu\omega)}{\Gamma(1+\nu\omega)} \frac{16\omega}{\nu(1+\omega)^3(1-\omega)} \left(\frac{\tilde{\kappa}_D}{2} \right)^{\nu\omega}, \quad (\text{A61})$$

$$A_2 \equiv \frac{2\nu^2}{(\nu+1)^2}, \quad (\text{A62})$$

and

$$A_3 \equiv \frac{2}{(1+\omega)^2(1-\nu\omega)}. \quad (\text{A63})$$

For $\omega = 1/\nu$ we also used the behavior of the Bessel function $K_1(z)$ around $z \sim 0$,

$$K_1(z) \sim \frac{1}{z} + \frac{z}{2} \ln z, \quad z \sim 0. \quad (\text{A64})$$

Although we have calculated in the symmetric phase so far, we expect that the momentum dependence of $D_\sigma^{-1}(q)$ is unchanged even in the broken phase, as long as neglecting effects of the mass term. We thus obtain the wave function renormalization constant for σ as

$$\begin{cases} Z_\sigma^{-1} = \left(\frac{\Lambda}{\mu}\right)^{2\nu(1-\omega)} & \text{for } 0 < \omega < \frac{1}{\nu} \\ Z_\sigma^{-1} = \left(\frac{\Lambda}{\mu}\right)^{2(\nu-1)} \ln\left(\frac{\Lambda^2}{\mu^2}\right) & \text{for } \omega = \frac{1}{\nu} \\ Z_\sigma^{-1} = \left(\frac{\Lambda}{\mu}\right)^{2(\nu-1)} & \text{for } \frac{1}{\nu} < \omega \leq 1 \end{cases} \quad (\text{A65})$$

with

$$\sigma_R = Z_\sigma^{-1/2} \sigma. \quad (\text{A66})$$

The results are consistent with the analysis of the resummation technique in Sec. III D.

APPENDIX B: LINEARIZED SD EQUATION

In this appendix, we summarize our results on the linearized SD equation including the cases with $\kappa_D \geq \kappa_D^{\text{crit}}$ and $\kappa_D = \kappa_D^{\text{MT}}$. We obtain the asymptotic behavior of the mass function as follows:

$$\frac{\Sigma(x)}{\Sigma_0} = \begin{cases} 2|\tilde{c}_1| \left(\frac{x}{\Sigma_0^2}\right)^{-(\nu/2)} \sin\left(\frac{1}{2}\nu\tilde{\omega} \ln\frac{x}{\Sigma_0^2} + \theta\right), & \text{for } \kappa_D > \kappa_D^{\text{crit}}, \\ \frac{\Gamma(D/2)}{\Gamma(\nu/2)\Gamma(1+\nu/2)} \left(\frac{x}{\Sigma_0^2}\right)^{-(\nu/2)} (\ln\frac{x}{\Sigma_0^2} + h_0), & \text{for } \kappa_D = \kappa_D^{\text{crit}}, \\ c_1 \left(\frac{x}{\Sigma_0^2}\right)^{-(\nu/2)(1-\omega)} + d_1 \left(\frac{x}{\Sigma_0^2}\right)^{-(\nu/2)(1+\omega)}, & \text{for } \kappa_D^{\text{MT}} < \kappa_D < \kappa_D^{\text{crit}}, \\ c_1 \left(\frac{x}{\Sigma_0^2}\right)^{-(1/2)(\nu-1)} \left[1 - \frac{1}{4}(\nu^2 - 1) \frac{\Sigma_0^2}{x} \ln\frac{x}{\Sigma_0^2}\right], & \text{for } \kappa_D = \kappa_D^{\text{MT}}, \\ c_1 \left(\frac{x}{\Sigma_0^2}\right)^{-(\nu/2)(1-\omega)} + c_2 \left(\frac{x}{\Sigma_0^2}\right)^{-(\nu/2)(1-\omega)-1}, & \text{for } 0 < \kappa_D < \kappa_D^{\text{MT}}, \\ 1, & \text{for } \kappa_D = 0, \end{cases} \quad (\text{B1})$$

where

$$\tilde{c}_1 \equiv \frac{\Gamma(D/2)\Gamma(i\nu\tilde{\omega})}{\Gamma[\nu(1+i\tilde{\omega})/2]\Gamma[1+\nu(1+i\tilde{\omega})/2]}, \quad \tilde{d}_1 = \tilde{c}_1^*, \quad e^{2i\theta} = -\tilde{c}_1/\tilde{d}_1, \quad \tilde{\omega} \equiv \sqrt{\frac{\kappa_D}{\kappa_D^{\text{crit}}} - 1}, \quad (\text{B2})$$

$$c_1 = \frac{\Gamma(D/2)\Gamma(\nu\omega)}{\Gamma[\nu(1+\omega)/2]\Gamma[1+\nu(1+\omega)/2]}, \quad d_1 = \frac{\Gamma(D/2)\Gamma(-\nu\omega)}{\Gamma[\nu(1-\omega)/2]\Gamma[1+\nu(1-\omega)/2]}, \quad c_2 = -\frac{\nu^2(1-\omega^2)}{4(\nu\omega-1)} c_1, \quad (\text{B3})$$

and

$$h_0 \equiv -2\gamma - 2\psi(\nu/2) - \frac{2}{\nu}, \quad \psi(z) \equiv \frac{d}{dz} \ln\Gamma(z). \quad (\text{B4})$$

In Eq. (B4), γ is the Euler's constant, $\gamma \simeq 0.5772$.

Substituting Eq. (B1) for the UVBC (3.18) with $m_0 = 0$, we obtain the gap equation

$$\left\{ \begin{array}{ll} \sin(\nu \tilde{\omega} \ln \frac{\Lambda}{\Sigma_0} + \theta + \theta') = 0, & \text{for } \kappa_D > \kappa_D^{\text{crit}}, \\ \ln \frac{\Sigma_0^2}{\Lambda^2} = h_0 + \frac{2}{\nu} - \frac{16}{\frac{4}{\nu} - 1}, & \text{for } \kappa_D = \kappa_D^{\text{crit}}, \\ \frac{g^{\text{crit}}}{g} = 1 + \frac{4\omega d_1}{1-\omega^2} \frac{(\frac{\Sigma_0}{\Lambda})^{2\nu\omega}}{c_1 + \frac{1-\omega}{1+\omega} d_1 (\frac{\Sigma_0}{\Lambda})^{2\nu\omega}}, & \text{for } \kappa_D^{\text{MT}} < \kappa_D < \kappa_D^{\text{crit}}, \\ \frac{g^{\text{crit}}}{g} = 1 - \nu g^{\text{crit}} \left(1 - \frac{(\nu-1)^2}{4\nu} \frac{1}{g'} \frac{\Sigma_0^2}{\Lambda^2} \ln \frac{\Lambda^2}{\Sigma_0^2}\right), & \text{for } \kappa_D = \kappa_D^{\text{MT}}, \\ \frac{g^{\text{crit}}}{g} = 1 + \frac{4c_2}{\nu(1-\omega^2)} \frac{\frac{\Sigma_0^2}{\Lambda^2}}{c_1 + (1 - \frac{2}{\nu(1+\omega)}) c_2 \frac{\Sigma_0^2}{\Lambda^2}}, & \text{for } 0 < \kappa_D < \kappa_D^{\text{MT}}, \\ \frac{g^{\text{crit}}}{g} = 1 - \frac{\nu}{\nu-1} \frac{\Sigma_0^2}{\Lambda^2}, & \text{for } \kappa_D = 0. \end{array} \right. \quad (\text{B5})$$

with

$$\tan \theta' = \frac{\nu(1 + \tilde{\omega}^2) + 4g}{\nu(1 + \tilde{\omega}^2) - 4g} \tilde{\omega}. \quad (\text{B6})$$

The scaling relation is then found as

$$\frac{\Sigma_0}{\Lambda} = \left\{ \begin{array}{ll} \exp(\frac{\theta+\theta'}{\nu\tilde{\omega}}) \exp(-\frac{\pi}{\nu\tilde{\omega}}), & \text{for } \kappa_D > \kappa_D^{\text{crit}}, \\ \exp(\frac{h_0}{2} + \frac{1}{\nu}) \exp(-\frac{8}{\frac{4}{\nu} - 1}), & \text{for } \kappa_D = \kappa_D^{\text{crit}}, \\ [\frac{c_1}{-d_1} \frac{1-\omega^2}{4\omega} (1 - \frac{g^{\text{crit}}}{g})]^{1/(2\nu\omega)}, & \text{for } \kappa_D^{\text{MT}} < \kappa_D < \kappa_D^{\text{crit}}, \\ \sqrt{(\omega - \frac{1}{\nu})(1 - \frac{g^{\text{crit}}}{g})}, & \text{for } 0 \leq \kappa_D < \kappa_D^{\text{MT}}, \end{array} \right. \quad (\text{B7})$$

where we used $g \simeq g^{\text{crit}}$. For $\kappa_D = \kappa_D^{\text{MT}}$ the scaling relation cannot be written by elementary functions:

$$\frac{\Sigma_0^2}{\Lambda^2} \ln \frac{\Lambda^2}{\Sigma_0^2} = \frac{4}{(1+\nu)^2} \frac{g - \frac{(\nu+1)^2}{4\nu}}{g - \frac{(\nu-1)^2}{4\nu}}. \quad (\text{B8})$$

The chiral condensation is calculated from Eq. (3.15):

$$\frac{\sigma}{\Sigma_0} = \left\{ \begin{array}{ll} |\tilde{c}_1| \sqrt{1 + \tilde{\omega}^2} (\frac{\Sigma_0}{\Lambda})^\nu \sin(\nu \tilde{\omega} \ln \frac{\Lambda}{\Sigma_0} + \theta + \arctan \tilde{\omega}), & \text{for } \kappa_D > \kappa_D^{\text{crit}}, \\ \frac{\Gamma(D/2)}{\Gamma(\nu/2)\Gamma(1+\nu/2)} (\frac{\Sigma_0}{\Lambda})^\nu (\ln \frac{\Lambda}{\Sigma_0} + \frac{h_0}{2} + \frac{1}{\nu}), & \text{for } \kappa_D = \kappa_D^{\text{crit}}, \\ \frac{(1+\omega)}{2} c_1 (\frac{\Sigma_0}{\Lambda})^{\nu(1-\omega)} + \frac{(1-\omega)}{2} d_1 (\frac{\Sigma_0}{\Lambda})^{\nu(1+\omega)}, & \text{for } \kappa_D^{\text{MT}} < \kappa_D < \kappa_D^{\text{crit}}, \\ \frac{(1+\nu)}{2\nu} c_1 (\frac{\Sigma_0}{\Lambda})^{\nu-1} [1 - \frac{1}{4}(\nu^2 - 1) \frac{\Sigma_0^2}{\Lambda^2} \ln \frac{\Lambda^2}{\Sigma_0^2}], & \text{for } \kappa_D = \kappa_D^{\text{MT}}, \\ \frac{(1+\omega)}{2} c_1 (\frac{\Sigma_0}{\Lambda})^{\nu(1-\omega)} + \frac{(1+\omega)}{2} - \frac{1}{\nu} c_2 (\frac{\Sigma_0}{\Lambda})^{\nu(1-\omega)+2}, & \text{for } 0 < \kappa_D < \kappa_D^{\text{MT}}, \\ 1, & \text{for } \kappa_D = 0. \end{array} \right. \quad (\text{B9})$$

Equation (B9) reads

$$\Sigma(\Lambda^2) = \left\{ \begin{array}{ll} 2\sigma - \frac{2(\nu+1)}{\nu} \frac{\sigma}{\ln(\frac{\Lambda}{\Sigma_0})}, & \text{for } \kappa_D = \kappa_D^{\text{crit}}, \\ \frac{2}{1+\omega} \sigma + \frac{2\omega}{1+\omega} d_1 \Lambda (\frac{2c_1^{-1}}{1+\omega} \frac{\sigma}{\Lambda})^{1+\{2\nu\omega/[\nu(1-\omega)+1]\}}, & \text{for } \kappa_D^{\text{MT}} < \kappa_D < \kappa_D^{\text{crit}}, \\ \frac{2\nu}{1+\nu} \sigma - \frac{\nu-1}{\nu} c_1 \Lambda (\frac{2\nu c_1^{-1}}{1+\nu} \frac{\sigma}{\Lambda})^{1+(2/\nu)} \ln \frac{\Lambda}{\Sigma_0}, & \text{for } \kappa_D = \kappa_D^{\text{MT}}, \\ \frac{2}{1+\omega} \sigma + \frac{2}{\nu(1+\omega)} c_2 \Lambda (\frac{2c_1^{-1}}{1+\omega} \frac{\sigma}{\Lambda})^{1+\{2/[\nu(1-\omega)+1]\}}, & \text{for } 0 < \kappa_D < \kappa_D^{\text{MT}}, \\ \sigma, & \text{for } \kappa_D = 0. \end{array} \right. \quad (\text{B10})$$

For $\kappa_D > \kappa_D^{\text{crit}}$ we do not obtain a handy formula.

APPENDIX C: POWER EXPANSION SOLUTION

We summarize results on the power expansion solution for $\kappa_D < \kappa_D^{\text{crit}}$. The asymptotic behavior of the mass function is given by

$$\frac{\Sigma(x)}{\Sigma_0} = \begin{cases} c_1 \left(\frac{x}{\Sigma_0}\right)^{-(1/2)\nu(1-\omega)} + d_1 \left(\frac{x}{\Sigma_0}\right)^{-(1/2)\nu(1+\omega)}, & \text{for } \kappa_D^{\text{PE}} < \kappa_D < \kappa_D^{\text{crit}}, \\ c_1 \left(\frac{x}{\Sigma_0}\right)^{-(1/4)(\nu-1)} - \frac{(\nu-1)(3\nu+1)}{8(\nu+1)} c_1^3 \left(\frac{x}{\Sigma_0}\right)^{-[(3\nu+1)/4]} \ln \frac{x}{\Sigma_0}, & \text{for } \kappa_D = \kappa_D^{\text{PE}}, \\ c_1 \left(\frac{x}{\Sigma_0}\right)^{-(1/2)\nu(1-\omega)} + c_2 \left(\frac{x}{\Sigma_0}\right)^{-(3/2)\nu(1-\omega)-1}, & \text{for } 0 < \kappa_D < \kappa_D^{\text{PE}}, \end{cases} \quad (\text{C1})$$

with

$$c_2 = -\frac{\nu^2(1-\omega^2)}{4(\nu+1-\nu\omega)(2\nu\omega-\nu-1)} c_1^3, \quad (\text{C2})$$

where c_1 and d_1 cannot be determined in the power expansion method.

Substituting Eq. (C1) for the UVBC (3.18) with $m_0 = 0$, we obtain the gap equation

$$\begin{cases} \frac{g^{\text{crit}}}{g} = 1 + \frac{4\omega d_1}{1-\omega^2} \frac{\left(\frac{\Sigma_0}{\Lambda}\right)^{2\nu\omega}}{c_1 + \frac{1-\omega}{1+\omega} d_1 \left(\frac{\Sigma_0}{\Lambda}\right)^{2\nu\omega}}, & \text{for } \kappa_D^{\text{PE}} < \kappa_D < \kappa_D^{\text{crit}}, \\ \frac{g^{\text{crit}}}{g} = 1 - \frac{2\nu^2 c_1^2 g^{\text{crit}}}{\nu+1} \left(1 - \frac{(\nu-1)^2}{16\nu^2 g}\right) \left(\frac{\Sigma_0}{\Lambda}\right)^{\nu+1} \ln \frac{\Lambda^2}{\Sigma_0^2}, & \text{for } \kappa_D = \kappa_D^{\text{PE}}, \\ \frac{g^{\text{crit}}}{g} = 1 + \frac{4(\nu+1-\nu\omega)c_2}{\nu(1-\omega^2)} \frac{\left(\frac{\Sigma_0}{\Lambda}\right)^{2\nu(1-\omega)+2}}{c_1 + \frac{\nu(3\omega-1)-2}{\nu(1+\omega)} c_2 \left(\frac{\Sigma_0}{\Lambda}\right)^{2\nu(1-\omega)+2}}, & \text{for } 0 < \kappa_D < \kappa_D^{\text{PE}}. \end{cases} \quad (\text{C3})$$

The scaling relation is then found as

$$\frac{\Sigma_0}{\Lambda} = \begin{cases} \left[\frac{c_1}{-d_1} \frac{1-\omega^2}{4\omega} \left(1 - \frac{g^{\text{crit}}}{g}\right) \right]^{1/(2\nu\omega)} & \text{for } \kappa_D^{\text{PE}} < \kappa_D < \kappa_D^{\text{crit}}, \\ \left[\frac{2\nu\omega - \nu - 1}{\nu c_1^2} \left(1 - \frac{g^{\text{crit}}}{g}\right) \right]^{1/[2\nu(1-\omega)+2]} & \text{for } 0 < \kappa_D < \kappa_D^{\text{PE}}, \end{cases} \quad (\text{C4})$$

where we used $g \simeq g^{\text{crit}}$. For $\kappa_D = \kappa_D^{\text{PE}}$ the scaling relation cannot be written by elementary functions:

$$\left(\frac{\Sigma_0}{\Lambda}\right)^{\nu+1} \ln \frac{\Lambda^2}{\Sigma_0^2} = \frac{\nu+1}{2\nu^2 c_1^2 g^{\text{crit}}} \frac{g - \frac{(3\nu+1)^2}{16\nu}}{g - \frac{(\nu-1)^2}{16\nu^2}}. \quad (\text{C5})$$

The chiral condensation is calculated from Eq. (3.15):

$$\frac{\sigma}{\Sigma_0} = \begin{cases} \frac{(1+\omega)}{2} c_1 \left(\frac{\Sigma_0}{\Lambda}\right)^{\nu(1-\omega)} + \frac{(1-\omega)}{2} d_1 \left(\frac{\Sigma_0}{\Lambda}\right)^{\nu(1+\omega)}, & \text{for } \kappa_D^{\text{PE}} < \kappa_D < \kappa_D^{\text{crit}}, \\ \frac{(3\nu+1)}{4\nu} c_1 \left(\frac{\Sigma_0}{\Lambda}\right)^{(\nu-1)/2} - \frac{(\nu-1)^2(3\nu+1)}{32\nu(\nu+1)} c_1^3 \left(\frac{\Sigma_0}{\Lambda}\right)^{(3\nu+1)/2} \ln \frac{\Lambda^2}{\Sigma_0^2}, & \text{for } \kappa_D = \kappa_D^{\text{PE}}, \\ \frac{(1+\omega)}{2} c_1 \left(\frac{\Sigma_0}{\Lambda}\right)^{\nu(1-\omega)} + \left(\frac{3\omega-1}{2} - \frac{1}{\nu}\right) c_2 \left(\frac{\Sigma_0}{\Lambda}\right)^{3\nu(1-\omega)+3}, & \text{for } 0 < \kappa_D < \kappa_D^{\text{PE}}. \end{cases} \quad (\text{C6})$$

Equation (C6) reads

$$\Sigma(\Lambda^2) = \begin{cases} \frac{2}{1+\omega} \sigma + \frac{2\omega}{1+\omega} d_1 \Lambda \left(\frac{2c_1^{-1}}{1+\omega} \frac{\sigma}{\Lambda}\right)^{1+\{2\nu\omega/[\nu(1-\omega)+1]\}}, & \text{for } \kappa_D^{\text{PE}} < \kappa_D < \kappa_D^{\text{crit}}, \\ \frac{4\nu}{3\nu+1} \sigma - \frac{\nu-1}{\nu+1} \Lambda \left(\frac{4\nu}{3\nu+1} \frac{\sigma}{\Lambda}\right)^3 \ln \frac{\Lambda}{\sigma}, & \text{for } \kappa_D = \kappa_D^{\text{PE}}, \\ \frac{2}{1+\omega} \sigma - \frac{\nu(1-\omega)}{2(2\nu\omega-\nu-1)} \Lambda \left(\frac{2}{1+\omega} \frac{\sigma}{\Lambda}\right)^3, & \text{for } 0 < \kappa_D < \kappa_D^{\text{PE}}. \end{cases} \quad (\text{C7})$$

These formulas are useful for calculation of the effective potential.

APPENDIX D: MORE ON NONRENORMALIZABLE REGION

In Sec. IV, we studied the renormalization of the effective potential, keeping the kinetic term of the auxiliary field finite. It should be so in the viewpoint of renormalization of the effective action. For $0 \leq \kappa_D \leq \kappa_D^{\text{MT}}$, however, the theory cannot get into the ‘‘broken phase’’ $g_R > g^{\text{crit}}$, because the renormalized potential (6.9) becomes unbounded from below. In this appendix, we attempt another procedure in which the VEV of σ_R remains finite independently of the value of κ_D . For such a purpose

we may admit to incorporate any required operators. Can be found a necessary and sufficient operator set? Before details, we jump to conclusions: It depends on approximation methods of the ladder SD equation.

The dynamical mass should be finite, so that Eq. (3.15) yields the wave function renormalization constant of σ as

$$Z_\sigma^{-1} = \left(\frac{\Lambda}{\mu}\right)^{2\nu(1-\omega)}, \quad (0 \leq \kappa_D < \kappa_D^{\text{crit}}) \quad (\text{D1})$$

for the whole region. [Compare Eq. (D1) with Eq. (3.67).] By using Eq. (D1), we obtain the renormalized Yukawa vertex

$$\Gamma_s^{(R)}(-q^2) \propto \left(\frac{-q^2}{\mu^2}\right)^{-(\nu/2)(1-\omega)}, \quad (0 \leq \kappa_D < \kappa_D^{\text{crit}}), \quad (\text{D2})$$

in the continuum limit $\Lambda \rightarrow \infty$. [See also Eq. (3.69).] On the other hand, the renormalized propagator of σ behaves as

$$iD_{\sigma(R)}^{-1}(p) - iD_{\sigma(R)}^{-1}(0) \propto \left(\frac{\Lambda}{\mu}\right)^{2(\nu\omega-1)} \mu^{2\nu} \left(\frac{x}{\mu^2}\right) \quad (\text{D3})$$

for $0 \leq \kappa_D < \kappa_D^{\text{MT}}$ and

$$iD_{\sigma(R)}^{-1}(p) - iD_{\sigma(R)}^{-1}(0) \propto \mu^{2\nu} \left(\frac{x}{\mu^2}\right) \ln\left(\frac{\Lambda^2}{x}\right) \quad (\text{D4})$$

for $\kappa_D = \kappa_D^{\text{MT}}$. Thus the kinetic term of σ diverges in the continuum limit $\Lambda \rightarrow \infty$ for $0 \leq \kappa_D \leq \kappa_D^{\text{MT}}$. We need to add the operator $\partial_M(\bar{\psi}\psi)\partial^M(\bar{\psi}\psi)$ in the effective action in order to absorb the divergence. Moreover, the decay constant is also divergent in the region, so that the interactions of the NG boson vanish in $\Lambda \rightarrow \infty$.

We investigate the effective potential for $0 \leq \kappa_D < \kappa_D^{\text{MT}}$ separately in two approximations.

1. Linearized Approximation

The scaling relation for $0 \leq \kappa_D < \kappa_D^{\text{MT}}$ is given by

$$\Sigma_0 \propto \Lambda \sqrt{1 - \frac{g^{\text{crit}}}{g}}. \quad (\text{D5})$$

[See Eq. (4.17).] The four-fermion coupling g is thus renormalized as

$$\left(\frac{1}{g(\Lambda)} - \frac{1}{g^{\text{crit}}}\right) \Lambda^2 = \left(\frac{1}{g_R(\mu)} - \frac{1}{g^{\text{crit}}}\right) \mu^2, \quad (\text{D6})$$

so as to make the dynamical mass Σ_0 finite. This is the very same as Eq. (6.11).

The bare effective potential has been already obtained in Eq. (5.16). We rewrite Eq. (5.16) in terms of σ_R and g_R :

$$\begin{aligned} V(\sigma_R) \propto \mu^D & \left[-\frac{1}{g} \frac{m_0 \sigma_R}{\mu^2} \left(\frac{\Lambda}{\mu}\right)^{\nu(1+\omega)} \right. \\ & + \frac{1}{2} \left(\frac{1}{g_R} - \frac{1}{g^{\text{crit}}}\right) \frac{\sigma_R^2}{\mu^2} \left(\frac{\Lambda}{\mu}\right)^{2(\nu\omega-1)} \\ & \left. + A_2 \left(\frac{\sigma_R}{\mu}\right)^{2+\{2/[\nu(1-\omega)+1]\}} \left(\frac{\Lambda}{\mu}\right)^{2(\nu\omega-1)} + \dots \right]. \end{aligned} \quad (\text{D7})$$

We renormalize the bare mass term, i.e., the first line of Eq. (D7) as follows:

$$\frac{m_0(\Lambda)}{g(\Lambda)} \Lambda^{\nu(1+\omega)} = \frac{m_R(\mu)}{g_R(\mu)} \mu^{\nu(1+\omega)}, \quad (\text{D8})$$

which is the same as Eq. (6.3). The beta function and anomalous dimensions are then found,

$$\beta(g_R) = 2g_R \left(1 - \frac{g_R}{g^{\text{crit}}}\right), \quad (\text{D9})$$

and

$$\begin{aligned} \gamma_m(g_R) &= -\frac{\beta(g_R)}{g_R} + \nu(1+\omega) \\ &= \nu(1+\omega) - 2 + 2\frac{g_R}{g^{\text{crit}}}, \end{aligned} \quad (\text{D10})$$

respectively. At the UVFP of g_R the anomalous dimension is $\gamma_m = \nu(1+\omega)$. Therefore the mismatch of the Λ dependence between m_0 and $\langle(\bar{\psi}\psi)_0\rangle$ is resolved. [See also Eq. (6.23).]

Troublesome are the σ_R^2 term and interaction term (A_2 term) in Eq. (D7). The divergence still remains after the renormalization of the four-fermion term. In addition, the operator having the fractional power of $(\bar{\psi}\psi)$ is required.

2. Power Expansion Method

The scaling relation is given by

$$\Sigma_0 \sim \begin{cases} \Lambda \left(1 - \frac{g^{\text{crit}}}{g}\right)^{1/(2\nu\omega)}, & (\kappa_D^{\text{PE}} < \kappa_D < \kappa_D^{\text{crit}}), \\ \Lambda \left(1 - \frac{g^{\text{crit}}}{g}\right)^{1/[2\nu(1-\omega)+2]}, & (0 < \kappa_D < \kappa_D^{\text{PE}}). \end{cases} \quad (\text{D11})$$

[See Eq. (4.33).] In order to keep Σ_0 finite, we renormalize the gap equation,

$$\left(\frac{1}{g(\Lambda)} - \frac{1}{g^{\text{crit}}}\right) \Lambda^{2\nu\omega} = \left(\frac{1}{g_R(\mu)} - \frac{1}{g^{\text{crit}}}\right) \mu^{2\nu\omega}, \quad (\text{D12})$$

for $\kappa_D^{\text{PE}} < \kappa_D < \kappa_D^{\text{crit}}$ and

$$\left(\frac{1}{g(\Lambda)} - \frac{1}{g^{\text{crit}}}\right) \Lambda^{2\nu(1-\omega)+2} = \left(\frac{1}{g_R(\mu)} - \frac{1}{g^{\text{crit}}}\right) \mu^{2\nu(1-\omega)+2}, \quad (\text{D13})$$

for $0 < \kappa_D < \kappa_D^{\text{PE}}$. By rewriting the bare effective potential in Eqs. (5.20) and (5.21) in terms of σ_R and g_R , we obtain

$$\begin{aligned} V(\sigma_R) \propto \mu^D & \left[-\frac{1}{g} \frac{m_0 \sigma_R}{\mu^2} \left(\frac{\Lambda}{\mu}\right)^{\nu(1+\omega)} \right. \\ & + \frac{1}{2} \left(\frac{1}{g_R} - \frac{1}{g^{\text{crit}}}\right) \frac{\sigma_R^2}{\mu^2} + \tilde{A}_1 \left(\frac{\sigma_R}{\mu}\right)^{2+\{2\nu\omega/[\nu(1-\omega)+1]\}} \\ & \left. + \dots \right], \end{aligned} \quad (\text{D14})$$

for $\kappa_D^{\text{PE}} < \kappa_D < \kappa_D^{\text{crit}}$, and

$$\begin{aligned} V(\sigma_R) \propto \mu^D & \left[-\frac{1}{g} \frac{m_0 \sigma_R}{\mu^2} \left(\frac{\Lambda}{\mu}\right)^{\nu(1+\omega)} \right. \\ & + \frac{1}{2} \left(\frac{1}{g_R} - \frac{1}{g^{\text{crit}}}\right) \frac{\sigma_R^2}{\mu^2} \left(\frac{\Lambda}{\mu}\right)^{2(2\nu\omega-\nu-1)} \\ & \left. + \tilde{A}_2 \left(\frac{\sigma_R}{\mu}\right)^4 \left(\frac{\Lambda}{\mu}\right)^{2(2\nu\omega-\nu-1)} + \dots \right], \end{aligned} \quad (\text{D15})$$

for $0 < \kappa_D < \kappa_D^{\text{PE}}$. The bare mass term is then renormalized

$$\frac{m_0(\Lambda)}{g(\Lambda)} \Lambda^{\nu(1+\omega)} = \frac{m_R(\mu)}{g_R(\mu)} \mu^{\nu(1+\omega)}, \quad (\text{D16})$$

for $0 < \kappa_D < \kappa_D^{\text{crit}}$. The beta function and anomalous dimensions are

$$\beta(g_R) = \begin{cases} 2\nu\omega g_R(1 - \frac{g_R}{g_R^{\text{crit}}}), & (\kappa_D^{\text{PE}} < \kappa_D < \kappa_D^{\text{crit}}) \\ 2[\nu(1 - \omega) + 1]g_R(1 - \frac{g_R}{g_R^{\text{crit}}}), & (0 < \kappa_D < \kappa_D^{\text{PE}}) \end{cases} \quad (\text{D17})$$

and

$$\gamma_m(g_R) = -\frac{\beta(g_R)}{g_R} + \nu(1 + \omega) = \begin{cases} \nu[1 - \omega + 2\omega \frac{g_R}{g_R^{\text{crit}}}], & (\kappa_D^{\text{PE}} < \kappa_D < \kappa_D^{\text{crit}}) \\ 3\nu\omega - \nu - 2 + 2[\nu(1 - \omega) + 1] \frac{g_R}{g_R^{\text{crit}}}, & (0 < \kappa_D < \kappa_D^{\text{PE}}) \end{cases}. \quad (\text{D18})$$

Similarly to the linearizing approximation, this γ_m at the UVFP of g_R is consistent with the renormalization of $\bar{\psi}\psi$.

After the renormalization, the effective potential for $\kappa_D^{\text{PE}} < \kappa_D < \kappa_D^{\text{crit}}$ has no divergent term. For $0 < \kappa_D < \kappa_D^{\text{PE}}$, on the other hand, the four-fermion (σ_R^2) and eight-fermion (σ_R^4) terms diverge in the continuum limit. Therefore the corresponding counterterms are required.

In passing, the bifurcation solution leads to the same results as above ones for $\kappa_D^{\text{PE}} < \kappa_D < \kappa_D^{\text{crit}}$ in the whole region. Thus only the counterterm of the auxiliary field propagator is needed in the nonrenormalizable region $0 \leq \kappa_D \leq \kappa_D^{\text{MT}}$.

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