Enhanced worldvolume supersymmetry and intersecting domain walls in $\mathcal{N} = 1$ supersymmetric QCD

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We study the worldvolume dynamics of 1/2-supersymmetric domain walls in $\mathcal{N} = 1$ supersymmetric QCD with $N_f = N$ flavors, and exhibit an enhancement of supersymmetry for the reduced moduli space associated with broken flavor symmetries. We provide an explicit construction of the worldvolume superalgebra which corresponds to an $\mathcal{N} = 2$ Kähler sigma model in 2 + 1D deformed by a potential, given by the norm squared of a U(1) Killing vector, resulting from the flavor symmetries broken by unequal quark masses. This framework leads to a worldvolume description of novel two-wall-junction configurations, which are 1/4-supersymmetric objects, but nonetheless preserve two supercharges when viewed as kinks on the wall worldvolume.

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I. INTRODUCTION

One of the more profound features of supersymmetric field theories is that solitonic field configurations are often endowed with a special status, namely, they are annihilated by a certain number of supercharges and thus lie in shortened, or Bogomol'nyi-Prasad-Sommerfield (BPS), representations [1]. This feature has far-reaching consequences due to the ensuing nonrenormalization theorems which affect the mass (or tension) and spectrum of these solitons, some of which may play an important role in the dynamics. In general, soliton configurations exhibit a moduli space of solutions, and much insight can be gleaned from a study of the low energy collective coordinate dynamics on this space and its induced metric [2]. This is particularly true in cases where the moduli space is nontrivial in the sense that it includes components beyond that associated with the broken translation generators; the latter component is always present on the grounds that a soliton is a localized configuration.

In the supersymmetric context, the moduli space $\mathcal{M}$ *locally* admits the general decomposition,

$$\mathcal{M} \simeq \mathcal{M}_{\rm SUSY} \times \tilde{\mathcal{M}}, \tag{1}$$

where $\mathcal{M}_{\rm SUSY}$ refers to the sector associated with bosonic generators in the supersymmetry (SUSY) algebra which are broken by the soliton, and in flat space always includes a translational component $\mathbb{R}^d \subset \mathcal{M}_{\rm SUSY}$, where d is the codimension. The realization of supersymmetry in this sector, associated with the unbroken generators, is then fixed by the kinematics of the bulk superalgebra.

In contrast, $\tilde{\mathcal{M}}$—the "reduced moduli space"—is not directly associated with broken generators in the superalgebra. This has the important consequence that in certain cases the realization of worldvolume supersymmetry is less constrained by the bulk kinematics. In particular, we will argue here that there are situations in which the number of supercharges which act trivially on the reduced moduli space of a BPS soliton can be *larger* than one would infer directly from the preserved fraction of bulk supersymmetry. The origin of this supersymmetry enhancement is that not all of the supercharges which are realized on the worldvolume of the soliton lift to supercharges in the full theory. The additional *supernumerary* supercharges arise due to special geometric features of the reduced moduli space, e.g., a Kähler or hyper-Kähler structure, which are not present within the full theory.

The primary aim of this paper is to illustrate how this novel feature plays an important role in the dynamics of 1/2-BPS domain walls in $\mathcal{N} = 1$ supersymmetric QCD (SQCD). In particular, we will focus on the theory with gauge group SU(N) accompanied by $N_f = N$ fundamental flavors with masses which are small relative to the dynamical scale, Λ_N, of the theory. This theory has a low energy description on the Higgs branch, in terms of meson and baryon chiral superfield moduli, where it reduces to a massive perturbation of a Kähler sigma model on the manifold determined by the quantum constraint [3],

$$\det M - B\tilde{B} = \Lambda_N^{2N}. \tag{2}$$

The massive theory possesses N quantum vacua which, with a hierarchical structure for the quark mass matrix, are in the weak coupling regime. On decoupling N flavors, these N vacua tend smoothly to the N quantum vacua of pure $\mathcal{N} = 1$ supersymmetric Yang-Mills (SYM) [4–8].

The N distinct vacua of this theory allow for domain wall solutions which interpolate between them. The corresponding central charge is present in the superalgebra [9] and such solitons are 1/2-BPS saturated. In previous work [10], we studied the BPS wall spectrum in this theory, following earlier work on BPS walls in other variants of $\mathcal{N} = 1$ SQCD [9,11–17]. The vacuum structure is illustrated in the plane of the superpotential in

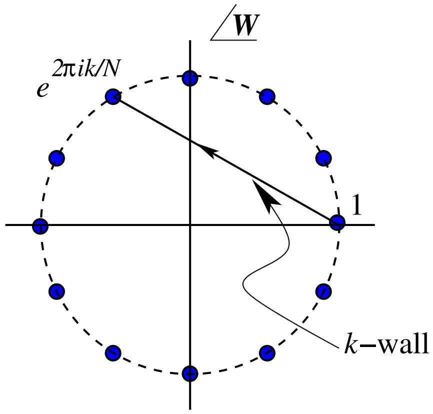

FIG. 1 (color online). A schematic representation of the N vacua, and a k-wall, for $\mathcal{N} = 1$ SQCD with N flavors.

Fig. 1, which also provides a graphical definition of a k-wall, namely, a BPS wall which interpolates between vacua differing in phase by $2\pi k/N$. In [10] we argued, as reviewed below, that k-walls exhibit a nontrivial classical reduced moduli space $\tilde{\mathcal{M}}_k$ due to localized Goldstone modes associated with the flavor symmetries which are broken by the wall solution. The corresponding coset is a complex Grassmannian [10],

$$\tilde{\mathcal{M}}_k = G(k, N) \equiv \frac{\mathrm{U}(N)}{\mathrm{U}(k) \times \mathrm{U}(N-k)}. \tag{3}$$

One can then formally deduce that the multiplicity of k-walls, ν_k, is given by the worldvolume Witten index for this Grassmannian sigma model, which depends only on the topology of the space, and is given by the Euler characteristic,

$$\nu_k = \chi[G(k, N)] = \frac{N!}{k!(N-k)!}. \tag{4}$$

This was the primary result of [10], which interestingly was consistent with an alternative string-theoretic picture of BPS walls in pure $\mathcal{N} = 1$ SYM [18].

In the present paper, we wish to study the worldvolume dynamics in more detail, and resolve some of the puzzles which arise from a closer inspection of the above result. One of these is the statement that the reduced moduli space is a Kähler manifold. Since the worldvolume theory lives in 2 + 1D, the dual constraints of (i) a Kähler target space, and (ii) Lorentz invariance, imply that the low energy dynamics must preserve $\mathcal{N} = 2$ supersymmetry, namely, four supercharges! Since only two bulk supercharges act trivially on the soliton solution, this conclusion clearly requires some justification. A seemingly related paradox was in fact noted some time ago in considering the Kähler moduli space of lumps in Kähler sigma models [19]. However, in the latter case, the problem dissipates once one realizes that Lorentz invariance places no constraint and one can consistently realize just two supercharges in terms of one-component fermions [20]. The situation here allows for no such resolution and, as alluded to above, in this case there is indeed an enhancement of supersymmetry, at least at the two-derivative level. This enhancement does not of course apply to the (decoupled) translational sector, but only to the reduced moduli space. We will provide an explicit example of how this can occur, and then apply it to k-walls in SQCD and more specifically to the simplest case of 1-walls in the case of an SU(2) gauge group. We note that the mechanism appears likely to apply more widely for other solitons in $\mathcal{N} = 1$ theories.

A second issue that we aim to resolve is to understand what happens to the flavor moduli parametrizing $\tilde{\mathcal{M}}_k$ when we explicitly break some of the flavor symmetries by putting the quark mass matrix in a hierarchical form. In such a maximally asymmetric regime, the wall no longer breaks any additional global symmetries, and one anticipates that the moduli space should be lifted. We will provide evidence that this is indeed the case. In particular, by considering the realization of the worldvolume supercharges for the SU(2) $N_f = 2$ theory, we show that, for a linear order perturbation in the mass matrix, the effect is to introduce a potential on the moduli space which geometrically is the norm squared of a U(1) Killing vector. Such a "real mass" deformation in 2 + 1D is known to be consistent with $\mathcal{N} = 2$ SUSY [21]. Moreover, one important consistency check is that the result one obtains via this linear deformation is in fact perfectly compatible with the opposite limit in which the second flavor is integrated out.

The third and final aim of this work is to explore the realization of other bulk solitons within the worldvolume theory of domain walls. The example we focus on corresponds to a novel class of two-wall 1/4-BPS junctions which are possible by virtue of the degeneracy (4) of BPS walls interpolating between the same two vacua. We will provide evidence that these configurations can be identified with 1/2-BPS kinks in the worldvolume Grassmannian sigma model. These configurations thus preserve two worldvolume supercharges, only one of which can be identified with the unbroken bulk supercharge. As evidence for this identification, we will verify for the SU(2) case that there is a direct match for the tension between the bulk result obtained in the hierarchical mass regime, and the appropriate limit of the kink tension for the massive sigma model.

This paper is organized as follows. In the next section we consider the moduli space of BPS solitons, and discuss in some generality the worldvolume realization of supersymmetry. We argue that the reduced moduli space may in certain cases exhibit supersymmetry enhancement and present a simple sigma model where this arises for the worldline dynamics of BPS kinks. We then turn to the specific case of BPS walls in $N_f = N$ flavor SQCD in

Sec. III, recalling the structure of the wall moduli space [10] and then describing the worldvolume realization of supersymmetry, which on the reduced moduli space is enhanced to $\mathcal{N} = 2$. We describe the structure explicitly for the SU(2) case with unequal quark masses, as is required to remain at weak coupling. This viewpoint is applied in Sec. IV to consider novel 1/4-BPS two-wall junctions from the viewpoint of the wall worldvolume. We finish with some concluding remarks on other worldvolume solitons, including lumps, in Sec. V. In an appendix, we review the structure of tensorial central charges in $D = 2, 3$ and 4, noting a subtlety with vectorial string charges.

II. SUPERSYMMETRY AND WORLDVOLUME MODULI

In this section we will discuss some aspects of the matching between bosonic and fermionic moduli for BPS solitons. We distinguish the translational sector, which is essentially fixed on kinematic grounds, from the remainder of the moduli—the reduced moduli space—which we argue, by way of an explicit example, can in certain situations exhibit an "enhancement" of supersymmetry, in the sense that the associated dynamics preserves more supercharges than one would infer from the bulk superalgebra. The additional supercharges act only on the reduced moduli space and are not present in the bulk theory.

A. Counting moduli and the translational sector

We begin with a simple physical perspective on the matching between bosonic and fermionic moduli of BPS solitons. Recall that on general grounds the bosonic moduli space for a configuration of solitons in flat space locally admits the decomposition

$$\mathcal{M} \simeq \mathcal{M}_{\mathrm{SUSY}} \times \tilde{\mathcal{M}}, \tag{5}$$

where $\mathcal{M}_{\mathrm{SUSY}}$ is the sector associated with broken (bosonic) symmetry generators in the superalgebra. The second factor in (5), $\tilde{\mathcal{M}}$, encodes any other modes associated with broken global symmetries, e.g., relative translations or, as will be more relevant here, flavor symmetries.

Consider a bosonic soliton configuration $S(x)$ in D-dimensional Minkowski space which has finite mass (or tension)—large relative to the scales of the underlying theory—and is localized in $d \leq D - 1$ spatial dimensions. Within a Lorentz invariant field theory, it is clear that this configuration possesses d localized bosonic zero modes as it spontaneously breaks translational invariance. It follows that the minimal content of the moduli space takes the form

$$\mathcal{M}^{\mathrm{min}}_{\mathrm{SUSY}} = \mathbb{R}^d. \tag{6}$$

For solitons within theories of extended supersymmetry, $\mathcal{M}_{\mathrm{SUSY}}$ may acquire additional bosonic dimensions, due to the enforced Kähler or hyper-Kähler structure.

We would now like to argue that there are *at least d* fermionic zero modes of the soliton configuration if the bulk theory possesses linearly realized supersymmetry. More precisely, we will consider a soliton in a globally[1] supersymmetric field theory. Furthermore, to simplify the discussion, we will assume a real representation for the superalgebra.[2] Since two SUSY variations commute to a translation, $[\delta_1, \delta_2]S(x) \propto \gamma^\mu \partial_\mu S(x)$, it follows that if there are d broken translational generators, there are at least d broken supersymmetry generators, and thus d fermionic zero modes. In practice, the number may of course be larger on account of Lorentz invariance or extended SUSY.

While this matching is essentially enforced by the representation theory of the worldvolume superalgebra in many examples, it is interesting that this simple argument also applies even if the worldvolume of the soliton is $0 + 1$-dimensional, for which supersymmetric quantum mechanics in principle imposes no fixed relation between the number of bosonic and fermionic degrees of freedom. Moreover, we did not need to assume the existence of a well-defined fermion parity $(-1)^F$ on the states, and thus representations could exist which are not Bose-Fermi paired. As simple illustrations of the minimal one-to-one matching consider first a putative BPS vortex in a theory with minimal $\mathcal{N} = 1$ SUSY (i.e. two supercharges) in $2 + 1$D. Such a configuration would require a worldvolume description with two bosonic zero modes, but only one fermionic mode. This is not permitted by the argument above, and indeed no configurations of this type are known. Vortices always exhibit at least two fermionic zero modes and are thus BPS only in $\mathcal{N} = 2$ theories in $2 + 1$D. As a second example, consider an SU(2) Yang-Mills instanton. Within the $\mathcal{N} = 1$ superconformal algebra they possess eight bosonic, but only four fermionic, zero modes, and are indeed BPS. However, to exist as solitons we must lift them to $4 + 1$D where the minimal superconformal algebra possesses 16 generators and BPS instantons then exhibit eight fermionic zero modes restoring the minimal one-to-one matching.

Proceeding further, one notes that a one-to-one matching between bosonic and fermionic modes (in practice a two-to-one matching of phase space variables) is possible only in the absence of nontrivial constraints from Lorentz invariance, namely, when the worldvolume is $0 + 1$ or $1 + 1$-dimensional. These cases still cover the majority of

[1]Exceptions to this correspondence are known within supergravity [22], where the condition that Killing spinors be normalizable at infinity becomes nontrivial.

[2]This includes the cases $D = 2, 3$ and 4 which we will focus on here, but the argument should generalize appropriately to dimensions without Majorana spinors.

solitons present within theories in 3 + 1D, and this minimal matching is known to occur in many cases. The first example where Lorentz invariance does impose a constraint arises for BPS walls in 3 + 1D, where the mode matching must be one-to-two. It is this case that will be of interest here.

The discussion above focused on the translational or, more generally, the ''super-Goldstone'' sector of the moduli space $\mathcal{M}_{\rm SUSY}$. The constraints imposed by Lorentz invariance on $\tilde{\mathcal{M}}$ are the same, but in general the realization of supersymmetry may be somewhat different. This is the issue to which we turn next.

B. Supersymmetry enhancement for $\tilde{\mathcal{M}}$: the $\mathcal{N} = 1$ S^3 sigma model

The realization of SUSY in the translational sector of the moduli space is highly constrained by kinematics. In contrast, the reduced moduli space may contain moduli which are unrelated to translational zero modes and the structure of the superalgebra. Of course, this is not necessarily the case if we consider a multisoliton configuration where $\tilde{\mathcal{M}}$ will include moduli corresponding to relative translations, but we have in mind a situation where $\tilde{\mathcal{M}}$ is instead associated with other broken global symmetries. In this case, we will argue that the reduced moduli space may exhibit an apparent enhancement of supersymmetry at the two-derivative level relative to the full dynamics on $\mathcal{M}$.

To motivate why supersymmetry enhancement for the low energy dynamics on $\tilde{\mathcal{M}}$ can be rather natural, we will first present an explicit example. Consider an $\mathcal{N} = 1$ sigma model in 1 + 1D with target space S^3 accompanied by its round metric [23], for which we introduce spherical polar coordinates $\phi^a = \{\theta, \xi, \phi\}$,

$$ds^2 = r[d\theta^2 + \sin^2\theta(d\xi^2 + \sin^2\xi d\phi^2)]. \tag{7}$$

We also turn on a (real) superpotential,

$$\mathcal{W}(\phi) = m\cos\theta, \tag{8}$$

which depends on only one of the angular coordinates parametrizing the S^3. The theory then has two vacua at $\theta = 0, \pi$.

Classical BPS kinks exist which interpolate between the two vacua, having mass

$$M_{\rm sol} = \mathcal{Z} = 2m, \tag{9}$$

and satisfying the Bogomol'nyi equation,

$$\partial_z\phi^a = g^{ab}\partial_b\mathcal{W}(\phi). \tag{10}$$

The solutions have the simple sine-Gordon form

$$\theta_{\rm sol}(z) = 2\arctan\left\{\exp\left[-\frac{m}{r}(z - z_0)\right]\right\},$$
$$\xi_{\rm sol} = \xi_0, \qquad \phi_{\rm sol} = \phi_0, \tag{11}$$

exhibiting three bosonic moduli $\{z_0, \xi_0, \phi_0\}$.

These bosonic moduli are Goldstone modes for the symmetries broken by the wall: z_0 is associated with the breaking of translation invariance; ξ_0 and ϕ_0 arise from the SO(3) global symmetry of the target space which is preserved in the vacua but broken to SO(2) by the kink solution. We thus anticipate that ξ_0 and ϕ_0 coordinatize the coset $SO(3)/SO(2) \simeq S^2$. This may be verified by inserting the solution into the bosonic Lagrangian and computing the induced metric for the bosonic zero modes [2], on allowing for weak time-dependence,

$$\begin{aligned} ds^2_{\mathcal{M}} &= 2mdz_0^2 + h_{ij}dx^i dx^j \\ &= 2mdz_0^2 + \frac{2r^2}{m}[d\xi_0^2 + \sin^2\xi d\phi_0^2], \\ i, j &= 1, 2, \end{aligned} \tag{12}$$

where h_{ij} is the metric of the reduced moduli space $\tilde{\mathcal{M}}$. The bosonic moduli space is thus

$$\mathcal{M} = \mathbb{R}\times\tilde{\mathcal{M}} = \mathbb{R}\times S^2, \tag{13}$$

with the natural metric on each factor.

Let us now consider the fermionic sector. The S^3 coordinates ϕ^a are partnered under $\mathcal{N} = 1$ SUSY by a set of two-component Majorana spinors, ψ^a_α, $\alpha = 1, 2$. For each bosonic zero mode x^i, one finds a corresponding (one-component) fermionic partner η^i in the lower component of ψ^a_α,

$$\psi^a_{\rm sol} = \eta^i\frac{\partial\phi^a_{\rm sol}}{\partial x^i}\begin{pmatrix}0\\1\end{pmatrix} + \text{ nonzero modes.} \tag{14}$$

Only one of these modes is guaranteed to exist by virtue of the fact that the solution is classically 1/2-BPS and thus breaks one of the two supercharges. The broken supercharge is realized as

$$Q_1 = 2\mathcal{Z}\eta^z, \tag{15}$$

in terms of this ''Goldstino'' mode. Here η^z is the superpartner of z_0.

We now come to a rather surprising feature of this system. The reduced moduli space $\tilde{\mathcal{M}}$ is a Kähler manifold and, since the bosonic and fermionic zero modes are paired, exhibits $\mathcal{N} = 2$ supersymmetry. One of these supercharges is Q_2, the unbroken charge present in the bulk theory, while the second which we will call $\tilde{Q}_2$ exists only due to the complex structure J associated with $\tilde{\mathcal{M}}$. In the coordinate system (12),

$$J = \begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}. \tag{16}$$

We can represent the supercharges as[3]

[3]Implying that the algebra is restricted to the rest-frame, we take $\dot{z}_0 = 0$.

$$\mathcal{Q}^I \equiv \begin{cases} Q_2 = h_{ij}\dot{x}^i\eta^j \\ \tilde{Q}_2 = h_{ij}J^j_k\dot{x}^i\eta^k \end{cases} \tag{17}$$

and, noting that $\{\eta^i, \eta^j\} = h^{ij}$, one can verify that they satisfy the algebra of $\mathcal{N} = 2$ supersymmetric quantum mechanics (SQM),

$$\{\mathcal{Q}^I, \mathcal{Q}^J\} = \mathcal{H}_{\rm SQM}\delta^{IJ}, \tag{18}$$

where $\mathcal{H}_{\rm SQM} = (M - Z)$ is the worldline Hamiltonian. Introducing the complex coordinate

$$w = e^{i\phi_0}\tan\frac{\xi_0}{2} \tag{19}$$

on $\tilde{\mathcal{M}}$, and its fermionic partner

$$\psi = \frac{1}{2}\sec^2\frac{\xi_0}{2}e^{i\phi_0}(\eta^\xi + i\sin\xi_0\eta^\phi), \tag{20}$$

we can rewrite the algebra in the form

$$\{\mathcal{Q}, \mathcal{Q}^*\} = \mathcal{H}_{\rm SQM}, \qquad (\mathcal{Q})^2 = (\mathcal{Q}^*)^2 = 0, \tag{21}$$

where

$$\begin{aligned} \mathcal{Q} &= \frac{1}{2}(\mathcal{Q}^1 + i\mathcal{Q}^2) = h_{w\bar{w}}\dot{\bar{w}}\psi, \\ \mathcal{Q}^* &= \frac{1}{2}(\mathcal{Q}^1 - i\mathcal{Q}^2) = h_{w\bar{w}}\dot{w}\bar{\psi}. \end{aligned} \tag{22}$$

At this point we should emphasize that the arguments for enhanced supersymmetry presented above refer to the low energy or two-derivative sector of the worldvolume theory. Since supersymmetry does not enforce this enhancement, nor indeed the Kähler structure of the reduced moduli space, it seems inevitable that higher derivative terms on the worldvolume will not respect $\mathcal{N} = 2$ supersymmetry. We will not attempt to verify this in detail,[4] as we will focus on the worldvolume vacuum structure for which the two-derivative sector of the theory is sufficient.

In this specific example, one can show that on quantization there are no supersymmetric vacua, and thus no quantum BPS kinks, since $(Q_2)^2$ is bounded from below by the scalar curvature $\mathcal{R}$ of $\tilde{\mathcal{M}}$ which is clearly positive. More precisely [23], Q_2 can be realized as the Dirac operator on $\tilde{\mathcal{M}}$,

$$Q_2 = \frac{1}{\sqrt{2}}\sigma^j(-i\nabla_j), \tag{23}$$

and thus one finds

$$(Q_2)^2 = \mathcal{H}_{\rm SQM} = -\nabla^2 + \frac{1}{8}\mathcal{R}, \tag{24}$$

where ∇^2 is the Laplacian on $\tilde{\mathcal{M}}$.

[4] An example of this kind in the translational sector was noted by Townsend [24].

Although we focused on one particular example, the mechanism for SUSY enhancement exhibited above clearly generalizes readily to, for example, sigma models with target spaces which are (nontrivial) U(1) bundles R over Kähler manifolds K (e.g. regular Sasakian manifolds), where the soliton profile in the transverse coordinate z lies entirely within the S^1 fibre,

$$\begin{array}{ccc} S^1(z) & \longrightarrow & R \\ & & \downarrow \\ & & K \end{array} \tag{25}$$

and it would clearly be interesting to explore other generalizations. It is worth emphasizing here that a global feature of this kind is ultimately what is responsible for enhancing the supersymmetry on $\tilde{\mathcal{M}}$. In other words, the existence of a globally defined Kähler form on $\tilde{\mathcal{M}}$, while not strictly proven through our local considerations, is ensured by an underlying geometric structure. Note also that the nontriviality of the fibration is a necessary condition ensuring that the $\{\xi_0, \phi_0\}$ zero modes are normalizable, i.e., that they are localized to the kink.

Here we will note only one natural extension of the example above, which is directly relevant to our subsequent discussion of BPS walls in SQCD. We can embed the $\mathcal{N} = 1$ S^3 model in a Kähler $\mathcal{N} = (2, 2)$ sigma model with target space $T^*(S^3)$. The bosonic soliton solutions persist, and depend only on the base S^3 coordinates. Consequently, the bosonic moduli space is unchanged. However, the cotangent directions supply an additional set of fermionic zero modes, so that the bosonic and fermionic moduli are now paired one-to-two, and the reduced moduli space preserves the action of four supercharges constructed as above with η reinterpreted as a two-component Majorana spinor. This in fact is crucial as the system can then be lifted to a nonchiral theory in 1 + 1D with $\mathcal{N} = (2, 2)$ supersymmetry and, more importantly, the natural $\mathcal{N} = 2$ theory in 2 + 1D on the worldvolume of a domain wall.

In the next section, we will review the origin of this geometric structure within the context of BPS domain walls in $\mathcal{N} = 1$ SQCD, and describe in some detail the worldvolume dynamics on the reduced moduli space.

III. DOMAIN WALL MODULI IN $\mathcal{N} = 1$ SQCD

In the first part of this section we briefly review the arguments which determine the topology of the reduced k-wall moduli space in SU(N) SQCD with $N_f = N$ flavors [10]. We then present a more explicit construction for SU(2), exhibiting the enhancement of supersymmetry on the reduced moduli space, and describing how the imposition of a hierarchical structure for the quark mass matrix leads to a potential which lifts the flavor moduli. This potential, at least for linear deformations, is geometrically the norm squared of a U(1) Killing vector.

A. Counting k-walls and the moduli space for $N_f = N$

$\mathcal{N} = 1$ SQCD with $N_f = N$ flavors is obtained by adding N chiral superfields, Q_f and $\tilde{Q}^{\bar{g}}$ ($f, \bar{g} = 1, \ldots, N$), transforming, respectively, in the fundamental and antifundamental representations of the gauge group, to the fields of $\mathcal{N} = 1$ SYM with gauge group SU(N). This matter content will ensure that the gauge symmetry is completely broken in any vacuum in which the matter fields have a nonzero vacuum expectation value. Provided the mass gap is sufficiently large, the gauge fields may then be integrated out, obtaining a low energy effective description in terms of the meson moduli $M_f^{\bar{g}} = Q_f \tilde{Q}^{\bar{g}}$.

The superpotential describing the resulting low energy dynamics is given by

$$\mathcal{W} = \mathrm{Tr}(\hat{m}M) + \lambda(\det M - \Lambda_N^{2N}), \tag{26}$$

in terms of the meson matrix M, the dynamical scale Λ_N, and a Lagrange multiplier λ. The Lagrange multiplier is to be understood as a heavy classical field, for consistency with the nonrenormalization theorem, which enforces a reduced form of the quantum constraint [3],

$$\det M - B\tilde{B} = \Lambda_N^{2N}, \tag{27}$$

containing in addition the baryon fields B and $\tilde{B}$. These fields have been set to zero (their vacuum values) in (26) as they do not play a role in the wall configurations we will consider here.

An important constraint on the accessible parameter space is the requirement that the vacua of the theory, and generic domain wall trajectories, lie at weak coupling where the gauge modes, which have been integrated out, are indeed heavy. This condition is satisfied if the quark mass matrix $\hat{m}$ is chosen in a specific hierarchical form, and the choice which retains the maximal global symmetry is given by

$$\hat{m} = \mathrm{diag}\{m, m, \ldots, m, m_N\}, \qquad \Lambda_N \gg m_N \gg m. \tag{28}$$

The vacua are then given by diagonal meson vacuum expectation values (VEVs) with components (no summation over i),

$$\langle M_i^i \rangle_k = \left(\frac{m_N}{m}\right)^{1/N} \Lambda_N^2 \omega_N^k, \qquad \omega_N^k = \mathrm{e}^{2\pi k/N}, \\ i = 1, \ldots, N-1, \qquad k = 0, \ldots, N-1. \tag{29}$$

The vacua are weakly coupled if the hierarchy is sufficiently large, i.e., we require $m_N/m \gg e^N$. If we restrict our attention to energy scales below m_N, the effective dynamical scale is $\Lambda_{N-1}^{2N+1} = m_N \Lambda_N^{2N}$.

For the specific problem of deducing the multiplicity of BPS walls the need for a hierarchical mass matrix can be circumvented [10]. This counting problem amounts to computing the Cecotti-Fendley-Intriligator-Vafa (CFIV) index [25], which is formally defined as the following trace, suitably regularized, over the Hilbert space with boundary conditions appropriate to a k-wall [25,26],

$$\nu_k \equiv \mathrm{Tr} F(-1)^F, \tag{30}$$

where F is the fermion number operator. Note that only shortened multiplets contribute. It will be useful to briefly recall two approaches to the computation of this index in the present context (see [10] for further details).

1. Hierarchical regime: Counting permutations

It is convenient to define dimensionless fields $X = \hat{m}M(\mu\Lambda_N^2)^{-1}$, with $\mu \equiv (\det\hat{m})^{1/N}$, in terms of which the superpotential exhibits the maximal SU(N) flavor symmetry,

$$\mathcal{W} = \mu\Lambda_N^2[\mathrm{Tr}X + \lambda(\det X - 1)], \tag{31}$$

while the hierarchical structure of the mass matrix is now visible only in the rescaled Kähler potential. The superpotential depends only on the eigenvalues $\{\eta_i\}$ of X,

$$\mathcal{W} = \mu\Lambda_N^2\left[\sum_{i=1}^N \eta_i + \lambda\left(\prod_{i=1}^N \eta_i - 1\right)\right], \tag{32}$$

which exhibits the vacua at the roots of unity, $\langle \eta_i \rangle_k = \omega_N^k$. Specifying boundary conditions relevant for a k-wall, the trajectory of each eigenvalue is characterized by its winding number $w(\eta)$ which can take one of two possible values: $w_1 = k/N$ and $w_2 = k/N - 1$ (see also [27]). The Bogomol'nyi equations then ensure that $N - k$ of the eigenvalues carry winding number w_1 and k carry winding number w_2. It follows immediately that the wall multiplicity is given by the number of permutations of the eigenvalues subject to these conditions, i.e.

$$\nu_k = \binom{N}{k} = \frac{N!}{k!(N-k)!}. \tag{33}$$

One observes that, since this construction depends only on constraints on the N eigenvalues, it can be applied consistently in the decoupling limit of the N^{th} flavor.

2. Symmetric regime: Quantizing moduli

An alternative approach, developed in [10], involves noting that the CFIV index can also be deduced from the Witten index $\mathrm{Tr}(-1)^F$ [4] of the worldvolume theory on $\tilde{\mathcal{M}}_k$. Thus it depends only on the topology of the reduced moduli space of BPS walls. These moduli are determined by the flavor symmetries broken by the wall and parametrize a Kähler manifold.[5] In particular, it is only the induced metric on this space which is sensitive to the precise specification of quark masses; the topology is invariant. One then recalls that the CFIV index is inde-

[5]The Kähler structure of the reduced moduli space, which is not *a priori* imposed by SUSY, will be discussed in more detail below.

pendent of smooth diffeomorphisms of the Kähler potential [25], and so we can restore its symmetry by such a diffeomorphism if so desired.

The result (33) can then be understood via quantization of the classical moduli space Lagrangian. In particular, it follows from the constraints on the eigenvalues that the maximal flavor symmetry that the k-wall can preserve is

$$\mathrm{S\,U}(k) \times \mathrm{SU}(N-k) \times \mathrm{U}(1), \tag{34}$$

which is a subgroup of the full flavor symmetry $\mathrm{SU}(N)$. Consequently, taking care with discrete factors, there must be localized Goldstone modes on the wall parametrizing the Grassmannian coset [10],

$$\tilde{\mathcal{M}}_k = G(k,N) \equiv \frac{\mathrm{U}(N)}{\mathrm{U}(k)\times \mathrm{U}(N-k)}. \tag{35}$$

The CFIV index then reduces to the worldvolume Witten index of the supersymmetric Grassmannian sigma model, given by the Euler characteristic, with the result

$$\nu_k = \chi[G(k,N)] = \frac{N!}{k!(N-k)!} \tag{36}$$

for the multiplicity of k-walls, in the presence of a suitable infrared regulator, consistent with the result above.

This latter computation relies heavily on the invariance of the index under D-term deformations, in order to deform the theory to a symmetric mass regime. We now wish to study this worldvolume theory in more detail and consequently will need to consider more carefully the transition back to the weakly coupled hierarchical mass regime.

B. The SU(2) case and enhanced supersymmetry

In order to provide a more explicit discussion of the resulting worldvolume dynamics on the moduli space of BPS walls, we will limit our attention in what follows to the simplest example with gauge group SU(2) and $N_f = 2$ flavors.

In addressing the full worldvolume dynamics, we are no longer at liberty to perform diffeomorphisms of the Kähler metric, and so it will be useful to introduce another dimensionless meson field $Z = M\Lambda_N^{-2}$ in terms of which the symmetry breaking induced by the hierarchical mass matrix is visible within the superpotential. A convenient basis is then provided by the following decomposition,

$$\begin{aligned} Z &= U_{\alpha_2-\alpha_1}(Z_0 1 + iZ_i\sigma^i)U_{\alpha_2-\alpha_1}, \\ U_\alpha &= \exp\left(\frac{i}{4}\alpha\sigma^3\right), \end{aligned} \tag{37}$$

where the (axial) rotation angle is the relative phase of the two quark masses; $m_k = |m_k|e^{i\alpha_k}$ for $k = 1, 2$. In this basis, the moduli space constraint takes the form,

$$\sum_{a=0}^{3} Z_a^2 = 1, \tag{38}$$

and it describes a smooth complex submanifold of $\mathbb{C}^4$, known as the deformed conifold [28]. This manifold is symplectically equivalent to $T^*(S^3)$.

In studying the BPS wall spectrum, it will be convenient to first consider the decoupling regime with hierarchical quark masses.

1. The decoupling regime

We first consider the regime where

$$\left|\frac{m_2}{m_1}\right| \gg 1, \tag{39}$$

so that the second flavor can be integrated out. The superpotential can be written as follows

$$\begin{aligned} \mathcal{W} = e^{i\gamma}|m_1|\Lambda_2^2&\left[Z_+ + \left|\frac{m_2}{m_1}\right|Z_-\right] \\ &+ \lambda(Z_+Z_- + Z_1^2 + Z_2^2 - 1), \end{aligned} \tag{40}$$

where $Z_\pm = Z_0 \pm iZ_3$ and $\gamma = (\alpha_1 + \alpha_2)/2$ is an overall phase. In the decoupling limit Z_1 and Z_2, since they are sensitive to the heavy quark VEV, are set to zero $\langle Z_1\rangle = \langle Z_2\rangle = 0$, and thus the moduli space contracts to

$$Z_+Z_- = 1, \tag{41}$$

a submanifold which is locally $\mathbb{R} \times S^1$. Solving this constraint directly, one recovers the Affleck-Dine-Seiberg superpotential for the 1-flavor theory [6]. We will instead proceed by restricting the fields Z_+ and Z_- to lie on the S^1 real section of (41), since this contains the two vacua at $\langle Z_+\rangle = \langle Z_-\rangle^{-1} = \pm\sqrt{|m_2/m_1|}$. Introducing an angular coordinate $\theta \in [0, \pi]$, we define

$$Z_+ = Z_-^{-1} = \sqrt{\left|\frac{m_2}{m_1}\right|}e^{\pm i\theta}, \tag{42}$$

which ensures that the physical meson field M_{11} scales as $(\Lambda_1^5/m_1)^{1/2}$ and thus remains finite in the decoupling limit. The classical Kähler potential for M_{11}, which is reliable in this hierarchical regime, also scales as $(\Lambda_1^5/m_1)^{1/2}$.

The superpotential reduces to

$$\mathcal{W} = 2e^{i\gamma}\sqrt{|m_1|\Lambda_1^5}\cos\theta, \tag{43}$$

which we recognize as equivalent, up to normalization, to the (real) superpotential of the S^3 model analyzed in Sec. II. The Bogomol'nyi equation takes the sine-Gordon form,

$$\partial_z\theta = -2|m_1|\sin\theta, \tag{44}$$

and thus the solution,

$$\theta_{\rm sol}(z) = 2\arctan(e^{-2|m_1|(z-z_0)}), \qquad (45)$$

exhibits a single bosonic modulus z_0 corresponding to the soliton position. We also observe from the $\mathbb{Z}_2$ ambiguity in (42) that there are two solutions [11], consistent with the value of the index $\nu_1^{N=2} = 2$.

Since there is no reduced moduli space for domain walls in this regime, we will not discuss the realization of supersymmetry explicitly. We note only that the translational sector is described by a single free $\mathcal{N} = 1$ scalar multiplet in 2 + 1D. The results above will nonetheless provide a useful comparison to those we will derive in the symmetric mass regime below.

2. The symmetric regime

We would now like to consider this system outside the decoupling regime. Although we will ultimately return to the controllable hierarchical mass regime (albeit with m_2 finite), we will first abstract slightly and consider what happens when we set the quark masses equal $m_1 = m_2 = \mu$. Although this puts the wall trajectory at strong-coupling, it turns out that the enhanced symmetry will still provide important constraints, and essentially the only assumption we need to make is that the effective description in terms of meson moduli is still valid. In practice, we do this simply to study the kinematic structure of the resulting worldvolume superalgebra, and we will partially remove the need for this assumption in the next subsection where we consider how the resulting picture is modified on detuning the two quark masses.

To proceed, it is now convenient to write the superpotential in the following form

$$\mathcal{W} = e^{i\gamma}\Lambda_2^2[\bar{m}Z_0 + i\Delta m Z_3] + \lambda\left(\sum_{a=0}^{3} Z_a^2 - 1\right), \qquad (46)$$

where the (real) mass parameters are

$$\bar{m} = |m_1| + |m_2|, \qquad \Delta m = |m_2| - |m_1|. \qquad (47)$$

Setting $\Delta m = 0$, we observe that the two vacua, $Z_0 = \pm 1$, now lie at the poles of the S^3 which forms the real section of the surface $\sum_{a=0}^{3} Z_a^2 = 1$. Supersymmetry demands that the metric on this latter space be Kähler. However its precise form is subject to quantum corrections and is not known except in the asymptotic regime where $M_f^{\bar{g}} \gg \Lambda_2^2$. Fortunately, one can show that not only the vacua but also the wall solutions lie entirely within the S^3 section [10] and we can ignore the metric structure of the cotangent directions. Moreover, when both mass terms are set to zero, the theory preserves an enhanced SU(2) $\times$ SU(2) symmetry which demands that the induced metric on the base S^3 be the round one. We can introduce a suitable set of coordinates $\{\phi'^a\}$ for the surface $\sum_a Z_a^2 = 1$, or a submanifold thereof, which makes the symmetry of this embedding manifest, and we denote the induced line element $d\Omega_3'(\phi'^a)$.

Let us also introduce a second coordinate system for the S^3, $\{\phi^a\}$, given by the embedding into flat space, with induced line element $d\Omega_3(\phi^a)$. Note that one obtains the same result for the embedding within the classical Kähler geometry $\mathrm{Tr}\sqrt{(\bar{Z}Z)}$. The relation between the two induced metrics $d\Omega_3'(\phi'^a)$ and $d\Omega_3(\phi^a)$ is nontrivial, and determined by the renormalization of the Kähler potential. However, symmetry demands that we have

$$d\Omega_3'(\phi'^a) = f(\phi^a)d\Omega_3(\phi^a), \qquad (48)$$

with a conformal factor $f(\phi^a)$, consistent with the isometries, which must be nonsingular to preserve the known vacuum structure. Note that this is a stronger constraint than would apply to the entire Kähler metric. We now see that, although $f(\phi^a)$ is unknown in general, it will enter the Bogomol'nyi equation for BPS walls in such a form that it can be "removed" by a field-dependent rescaling of the transverse spacetime coordinate to the wall. Such a rescaling will affect the wall profile, but will not affect the symmetries of the system and will allow us to proceed with an analysis of the kinematics. Thus, for this subsection, we will perform this rescaling and set $\phi'^a = \phi^a$. The induced metric on the S^3, in spherical polar coordinates $\{\theta, \xi, \phi\}$, then takes the form

$$ds^2_{\rm base} = \Lambda_2^2[d\theta^2 + \sin^2\theta(d\xi^2 + \sin^2\xi d\phi^2)], \qquad (49)$$

where the normalization is fixed by the only dimensionful scale available, the dynamical scale Λ_2 of $N_f = 2$ SQCD. When we turn the equal mass perturbation back on, the corrections will be of order μ/Λ_2 which are subleading in the light quark mass regime we consider here. In the next subsection, we will consider unequal mass perturbations which will move the wall trajectory back toward the weakly coupled region.

We can now utilize the same coordinate system, $\{\theta, \xi, \phi\}$, to rewrite the superpotential, restricted to the S^3 real section, in the form,

$$\mathcal{W} = e^{i\gamma}\mu\Lambda_2^2 \mathrm{Tr}Z \longrightarrow 2e^{i\gamma}\mu\Lambda_2^2\cos\theta, \qquad (50)$$

which is once again equivalent, up to normalization, to the superpotential of the S^3 model analyzed in Sec. II, and the superpotential in the hierarchical regime deduced above. In the latter context, the use of the same notation for the angle θ entering the superpotential is not accidental and will be justified later in this section. The vacua lie at the poles $\theta = 0, \pi$, and the Bogomol'nyi equations reduce to

$$\partial_z\theta = -2\mu\sin\theta, \qquad \partial_z\xi = \partial_z\phi = 0, \qquad (51)$$

which are naturally equivalent to Eq. (10), and are solved once again by the sine-Gordon soliton (11),

$$\theta_{\rm sol}(z) = 2\arctan(e^{-2\mu(z-z_0)}),\qquad \xi_{\rm sol} = \xi_0, \qquad \phi_{\rm sol} = \phi_0. \tag{52}$$

We conclude that the bosonic moduli space is the same as that obtained within the S^3 model, namely $\mathcal{M}^{\rm N=2} = \mathbb{R} \times \mathbb{C}\mathrm{P}^1$, which is consistent with the general discussion above. Integrating over the wall profile, and reconstructing the spatial dependence using Lorentz invariance, leads to the corresponding bosonic moduli space Lagrangian,

$$\mathcal{L}_{\rm bose} = \int d^3x\left[-T_1 + \frac{1}{2}T_1\partial_\mu z_0 \partial^\mu z_0 + \frac{1}{2}h_{ij}\partial_\mu x^i \partial^\mu x^j\right], \tag{53}$$

where $T_1 = 4\mu\Lambda_2^2$ is the 1-wall tension, and h_{ij} is the metric on the moduli space, given by

$$\begin{aligned} ds^2_{\mathcal{M}} &= T_1 dz_0^2 + h_{ij}dx^i dx^j \\ &= T_1 dz_0^2 + R_{\tilde{\mathcal{M}}}(d\xi_0^2 + \sin^2\xi_0 d\phi_0^2), \end{aligned} \tag{54}$$

with

$$R_{\tilde{\mathcal{M}}} = \frac{\Lambda_2^2}{\mu} \tag{55}$$

the scale of the reduced moduli space.

We are now in a position to explore the realization of supersymmetry on the reduced moduli space. The first point to note, following the comments at the end of Sec. II, is that the present system has twice as many fermions as the S^3 model considered earlier. The second set of fermions arise from the cotangent directions of $T^*(S^3)$. We can choose a basis where the complex fermions lying in the chiral multiplet Z decompose into two (real) sets, one $\psi_{1\alpha}$ the $\mathcal{N} = 1$partner of the S^3 coordinates of the base, and the other $\psi_{2\alpha}$ the $\mathcal{N} = 1$partner of the cotangent directions. One then finds that a second set of fermionic zero modes arise from ψ_{21}. The fermionic mode decomposition takes the form

$$\begin{aligned} \psi^a_{1\alpha} &= \left[\eta_1^z \frac{\partial\phi^a_{\rm sol}}{\partial z_0} + \eta_1^i \frac{\partial\phi^a_{\rm sol}}{\partial x^i}\right]\begin{pmatrix}0\\1\end{pmatrix}_\alpha + \text{nonzero modes}, \\ \psi^a_{2\alpha} &= \left[\eta_2^z \frac{\partial\phi^a_{\rm sol}}{\partial z_0} + \eta_2^i \frac{\partial\phi^a_{\rm sol}}{\partial x^i}\right]\begin{pmatrix}1\\0\end{pmatrix}_\alpha + \text{nonzero modes}, \end{aligned} \tag{56}$$

where $\{\eta^i_A\}$ are two sets of fermionic operators satisfying

$$\{\eta^z_A, \eta^z_B\} = \frac{1}{T_1}\delta_{AB}, \qquad \{\eta^i_A, \eta^j_B\} = h^{ij}\delta_{AB}, \tag{57}$$

where h_{ij} is the reduced moduli space metric. Thus we now find in full a one-to-two matching between the number of bosonic versus fermionic zero modes. It is important that since the worldvolume is now 2 + 1-dimensional, this matching condition is a requirement of Lorentz invariance—a constraint that was not present in our earlier discussion of 1 + 1D kinks.

With this constraint in mind, it is convenient to combine the fermionic moduli η_A into a two-component spinor $\eta = (\eta_1, \eta_2)$. The center-of-mass sector now comprises a real scalar z_0 and a Majorana spinor η^z, which is sufficient to compose a scalar multiplet of $\mathcal{N} = 1$ SUSY in 2 + 1D. This is the sector of the theory generated by spontaneous breaking of translational invariance and the two broken supercharges, since the state is 1/2-BPS.

The reduced moduli space $\mathbb{C}\mathrm{P}^1$ is Kähler, and so from the discussion of Sec. II, we would anticipate some enhancement of supersymmetry in this sector. Indeed, it is clear that essentially the same construction as before, now augmented with two-component spinors η^i, will lead to the dynamics admitting $\mathcal{N} = 2$supersymmetry in 2 + 1D, or four supercharges, only two of which can be identified with the unbroken generators of the bulk superalgebra. This conclusion has important consequences, as this theory can be shown to have two supersymmetric vacua (at least in 1 + 1D or less), in contrast to the "chiral" theory which was realized in the S^3 model.

In preparation for the following subsection, it will be useful to describe explicitly the construction of the supercharges. To this end, we will compactify the theory on a 2-torus of radius R, and consider the $\mathcal{N} = (2, 2)$ superalgebra in 1 + 1D:

$$\begin{aligned} \{Q_\alpha, Q^\dagger_\beta\} &= 2(\gamma^\mu\gamma^0)_{\alpha\beta}P_\mu, \\ \{Q_\alpha, Q_\beta\} &= 2i(\gamma^5\gamma^0)_{\alpha\beta}\bar{Z}, \\ \{Q^\dagger_\alpha, Q^\dagger_\beta\} &= 2i(\gamma^5\gamma^0)_{\alpha\beta}Z, \end{aligned} \tag{58}$$

choosing the γ-matrices as follows

$$\gamma^0 = \sigma_2, \qquad \gamma^1 = i\sigma_3, \qquad \gamma^5 = \gamma^0\gamma^1 = -\sigma_1. \tag{59}$$

We can now rotate to a Majorana basis $Q_\alpha = e^{-i\gamma/2}(Q^1_\alpha + iQ^2_\alpha)/\sqrt{2}$ within which

$$\{Q^i_\alpha, Q^j_\beta\} = 2\delta^{ij}(\gamma^\mu\gamma^0)_{\alpha\beta}P_\mu + 2i(\gamma^5\gamma^0)_{\alpha\beta}|Z|\sigma_3^{ij}, \tag{60}$$

where $\gamma \equiv \arg(Z)$, and takes the form $\gamma = (\alpha_1 + \alpha_2)/2$ in the present case. In the rest-frame,

$$\begin{aligned} (Q^1_1)^2 = (Q^2_2)^2 &= M + |Z|, \\ (Q^2_1)^2 = (Q^1_2)^2 &= M - |Z|, \end{aligned} \tag{61}$$

where $M = T_1R^2$ in terms of the wall tension. Thus, we see that for a BPS wall configuration Q^2_1 and Q^1_2 are the unbroken supercharges which will be realized within the worldvolume theory.

To compute these supercharges in terms of the moduli we recall that for a Wess-Zumino model, as we have here, the complex supercharge is given by

$$Q = \int dz(g_{ab}\gamma^\mu\partial_\mu\phi^a\gamma^0\psi^b + i\partial_{\bar{b}}\bar{W}\gamma^0\psi^{*b}). \tag{62}$$

To move to the Majorana basis, we decompose $\psi = e^{-i\gamma/2}(\psi^1 + i\psi^2)/\sqrt{2}$, and obtain

$$Q^1 = \int dz \begin{pmatrix} g_{ab}\dot{\phi}^a & g_{ab}\partial_z\phi^a + e^{i\gamma}\partial_{\bar{b}}\bar{\mathcal{W}}_0 \\ g_{ab}\partial_z\phi^a - e^{i\gamma}\partial_{\bar{b}}\bar{\mathcal{W}}_0 & g_{ab}\dot{\phi}^a \end{pmatrix}\psi^{1b}, \tag{63}$$

with Q^2_α given by a similar expression in terms of ψ^2. Inserting the solutions for the fermionic zero modes one obtains,

$$Q_\alpha \equiv (Q^1_2, Q^2_1) = T_1\dot{z}_0\eta^z_\alpha + h_{ij}\dot{x}^i\eta^i_\alpha, \tag{64}$$

where we have combined the two unbroken charges into a spinor, using the corresponding fermionic zero modes η_α. This is recognizable as a spinor analogue of the unbroken supercharge within the S^3 model. If we now drop the translational zero modes, and restrict Q_α to the reduced moduli space, with $x^i = \{\xi_0, \phi_0\}$, then we discover that there is a second unbroken spinor supercharge, existing by virtue of the complex structure J associated with $\tilde{\mathcal{M}} = S^2$, introduced earlier in (16). We can then form a complex spinor charge $\mathcal{Q}^I_\alpha$

$$\mathcal{Q}^I_\alpha \equiv \begin{cases} Q_\alpha = h_{ij}\dot{x}^i\eta^j_\alpha, \\ \tilde{Q}_\alpha = h_{ij}J^j_k\dot{x}^i\eta^k_\alpha, \end{cases} \tag{65}$$

and these charges satisfy the algebra of $\mathcal{N} = 4$ SQM or more importantly, when lifted back to 2 + 1D, the $\mathcal{N} = 2$ superalgebra.

The worldvolume theory is then an $\mathcal{N} = 2$ $\mathbb{C}$P^1 sigma model and, as noted above, the Witten index for this theory is equal to two, consistent with our counting of domain walls. Therefore, within this system, at least when compactified to 1 + 1D or below, there are indeed two quantum vacua, and thus two BPS walls. This worldvolume structure also has important consequences for worldvolume BPS solitons, a subject that we turn to in the next section.

The crucial distinction to be made here with the $\mathcal{N} = 1$ algebra arising for kinks in the S^3 model is that with two-component fermions the model has an additional potential term associated with the Riemann tensor,

$$\Delta V = -\frac{1}{12}R_{ijkl}\bar{\eta}^i\eta^j\bar{\eta}^k\eta^l, \tag{66}$$

with $\bar{\eta} = \eta^T\gamma_0$, which precisely cancels the zero-point curvature term in (24) in the quantum action of the unbroken supercharges on the ground states. One way to understand this is to recall[6] that, while the one-component worldvolume supercharges which arise in the $\mathcal{N} = 1$ S^3 model are realized quantum mechanically in terms of the Dirac operator, or alternatively (anti-) holomorphic (or Dolbeault) exterior derivatives [20], on the reduced moduli space,

$$(\mathcal{Q}, \mathcal{Q}^*) \leftrightarrow (\bar{\partial}^\dagger, \bar{\partial}), \tag{67}$$

the spinor supercharges arising in the $\mathcal{N} = 2$ $T^*(S^3)$ model are realized in terms of (de Rham) exterior derivatives [4],

$$(Q, Q^\dagger) \leftrightarrow (d, d^*). \tag{68}$$

The supersymmetric vacua in the latter case correspond to normalizable harmonic forms, of which there are two for S^2 corresponding to the Betti numbers $b_0 = b_2 = 1$. However, supersymmetric vacua of the $\mathcal{N} = 1$ theory would be normalizable *holomorphic* harmonic forms on the same manifold. The presence of such forms on a Kähler manifold, which would necessarily have antiholomorphic partners, is forbidden by the uniqueness of harmonic forms in each de Rham cohomology class.

3. Quark mass splitting and a potential on the moduli space

In the preceding discussion, we abstracted slightly in ignoring the deformation imposed by considering a hierarchical mass matrix for the quarks. As noted above, this choice is enforced if we wish to retain a weakly coupled description of the vacua between which the wall interpolates. In this subsection, we rectify this by turning on this deformation and demonstrating that the effect on the reduced moduli space is, at linear order, to introduce a new potential given by the norm squared of a U(1) Killing vector. Such a potential is naturally associated with the fact that turning on the quark mass difference, $\Delta m \equiv |m_1| - |m_2|$, breaks the non-Abelian part of the global symmetry from SU(2) $\rightarrow$ U(1). An important feature of this particular deformation on the worldvolume is that it preserves the enhanced $\mathcal{N} = 2$ SUSY [21].

Using the same coordinate system as above, and restricting once again to the real section, we can write the superpotential in the form

$$\mathcal{W} = \mathcal{W}_0 + i\Delta\mathcal{W} = e^{i\gamma}\bar{m}\Lambda^2_2[\cos\theta + i\epsilon\sin\theta\cos\xi], \tag{69}$$

where the (real) deformation parameter is

$$\epsilon \equiv \frac{\Delta m}{\bar{m}}. \tag{70}$$

Rather than study the exact wall solutions within this system, we will consider the impact at leading order in ϵ

[6]Similar issues arise in comparing the spectrum of dyons of magnetic charge two in gauge theories with $\mathcal{N} = 2$ [29] and $\mathcal{N} = 4$ supersymmetry [30].

on the moduli space dynamics valid at $\Delta m = 0$. Working to linear order in ϵ, it is consistent to make use of the unperturbed wall solution in constructing the worldvolume supercharges. The deformation is then apparent in the presence of a correction term,

$$\Delta Q^1 = \int dz \begin{pmatrix} 0 & e^{i\gamma}\partial_b \Delta \bar{W} \\ -e^{i\gamma}\partial_b \Delta \bar{W} & 0 \end{pmatrix} \psi^{2b}, \tag{71}$$

with the correction to Q^2_α given by a similar expression.

It is now clear that at linear order in ϵ the broken supercharges Q^1_1 and Q^2_2 are not corrected on setting the nonzero modes to zero. This is consistent with the fact that corrections to the central charge start at $\mathcal{O}(\epsilon^2)$,

$$Z = 2e^{i\gamma}\bar{m}\Lambda_2^2\left(1 - \frac{1}{2}\epsilon^2 + \cdots\right). \tag{72}$$

In contrast, the unbroken supercharges are corrected, and evaluating them using the zeroth order Bogomol'nyi equations, we find

$$\begin{aligned} Q^1_2 &= T_1 \dot{z}_0 \eta^z_1 + h_{ij}\dot{x}^i \eta^j_1 + \pi\epsilon\Lambda_2^2 \sin\xi_0 \eta^\xi_2, \\ Q^2_1 &= T_1 \dot{z}_0 \eta^z_2 + h_{ij}\dot{x}^i \eta^j_2 - \pi\epsilon\Lambda_2^2 \sin\xi_0 \eta^\xi_1. \end{aligned} \tag{73}$$

The relative sign for the perturbations to Q^1_2 and Q^2_1 ensures that $\{Q^1_2, Q^2_1\} = 0$ in the rest-frame as required.

We would now like to determine whether or not this linearized deformation has preserved the additional supersymmetry, associated with the complex structure on the reduced moduli space. In fact we can verify this explicitly. To proceed, let us drop the decoupled translational mode as above and relabel the supercharges acting on the reduced moduli space as follows

$$\begin{aligned} Q^1_2 &\longrightarrow Q^1_L \\ &= 2\frac{\Lambda_2^2}{\bar{m}}\left[\dot{\xi}_0\eta^\xi_1 + \sin^2\xi_0\dot{\phi}_0\eta^\phi_1 + \frac{1}{2}\pi\epsilon\bar{m}\sin\xi_0\eta^\xi_2\right], \\ Q^2_1 &\longrightarrow Q^2_R \\ &= 2\frac{\Lambda_2^2}{\bar{m}}\left[\dot{\xi}_0\eta^\xi_2 + \sin^2\xi_0\dot{\phi}_0\eta^\phi_2 - \frac{1}{2}\pi\bar{m}\epsilon\sin\xi_0\eta^\xi_1\right]. \end{aligned} \tag{74}$$

Remarkably enough one can write down a second set of supercharges leading to the same Hamiltonian,

$$\begin{aligned} Q^2_L &= 2\frac{\Lambda_2^2}{\bar{m}}\sin\xi_0\left[\dot{\xi}_0\eta^\phi_1 - \dot{\phi}_0\eta^\xi_1 + \frac{1}{2}\pi\epsilon\bar{m}\sin\xi_0\eta^\phi_2\right], \\ Q^1_R &= 2\frac{\Lambda_2^2}{\bar{m}}\sin\xi_0\left[\dot{\xi}_0\eta^\phi_2 - \dot{\phi}_0\eta^\xi_2 - \frac{1}{2}\pi\epsilon\bar{m}\sin\xi_0\eta^\phi_1\right], \end{aligned} \tag{75}$$

and one can verify that $\{Q^1_L, Q^2_L\} = \{Q^1_R, Q^2_R\} = 0$, and $(Q^2_L)^2 = (Q^1_R)^2 = \mathcal{H}$. It follows that we can build complex combinations of the form, $Q_L = (Q^1_L + iQ^2_L)/2$ and $Q_R = (Q^1_R + iQ^2_R)/2$, i.e.

$$\begin{aligned} Q_L = \frac{\Lambda_2^2}{\bar{m}}\Big[&(\dot{\xi}_0\eta^\xi_1 + \sin^2\xi_0\dot{\phi}_0\eta^\phi_1) + i\sin\xi_0(\dot{\xi}_0\eta^\phi_1 - \dot{\phi}_0\eta^\xi_1) \\ &+ \frac{1}{2}i\pi\epsilon\bar{m}\sin^2\xi_0\eta^\phi_2 + \frac{1}{2}\pi\epsilon\bar{m}\sin\xi_0\eta^\xi_2\Big], \end{aligned} \tag{76}$$

such that

$$\{Q_L, \bar{Q}_L\} = \{Q_R, \bar{Q}_R\} = \mathcal{H}, \tag{77}$$

with the other anticommutators vanishing in the absence of central charges.

This structure is of course not accidental. We can make the underlying complex structure manifest, by introducing complex coordinates associated with the stereographic projection. If, as in (19), we define:

$$w = e^{i\phi_0}\tan\frac{\xi_0}{2}, \tag{78}$$

the corresponding map for the fermions is given by

$$\psi_L = \frac{1}{2}\sec^2\frac{\xi_0}{2}e^{i\phi_0}(\eta^\xi_1 + i\sin\xi_0\eta^\phi_1), \tag{79}$$

with a similar relation for ψ_R in terms of η^ξ_2 and η^ϕ_2. With these redefinitions, the somewhat lengthy expressions above for Q_L and Q_R take the simple form

$$\begin{aligned} Q_L &= h_{w\bar{w}}[\dot{\bar{w}}\psi_L + \pi\epsilon\bar{m}\,\bar{w}\,\psi_R], \\ Q_R &= h_{w\bar{w}}[\dot{\bar{w}}\psi_R - \pi\epsilon\bar{m}\,\bar{w}\,\psi_L], \end{aligned} \tag{80}$$

with the remaining supercharges given by $\bar{Q}_L$ and $\bar{Q}_R$. The Fubini-Study metric is

$$h_{\bar{w}w} = 4\frac{\Lambda_2^2}{\bar{m}}\frac{1}{(1+|w|^2)^2} = \frac{2R_{\tilde{\mathcal{M}}}}{(1+|w|^2)^2}. \tag{81}$$

This is precisely the structure expected for a deformation by a Killing vector proportional to a "twisted" [31–33] or "real" mass term in 1 + 1D or 2 + 1D respectively, thus preserving $\mathcal{N} = 2$ SUSY. In fact, since ϵ is a real parameter, we see that this deformation is most directly interpreted as a real mass term in 2 + 1D, as one would expect for the worldvolume theory of a wall in 3 + 1D. In this context ψ_L and ψ_R are then the upper and lower components, respectively, of a complex spinor.

We have focused on the impact of this deformation on the supercharges, since we were working to linear order and making use of the undeformed soliton solution. This deformation is visible at the bosonic level as a potential given by the norm squared of a U(1) Killing vector $G = G^i\partial_i$ for rotations in ϕ_0,

$$G^i = \frac{1}{2}\pi\Delta m\delta^{i\phi_0}. \tag{82}$$

However, this contribution is of second order in the perturbation. Formally, we obtain

$$(Q_2^1)^2 = (Q_2^1)^2 = \mathcal{H} = \frac{1}{2} h_{ij} \dot{x}^i \dot{x}^j + \frac{1}{4} \bar{m} \Lambda_2^2 (\pi \epsilon)^2 \sin^2 \xi_0, \tag{83}$$

and thus the induced potential is of the form

$$\Delta V = \frac{1}{2} h_{ij} G^i G^j = \frac{1}{4} \bar{m} \Lambda_2^2 (\pi \epsilon)^2 \sin^2 \xi_0. \tag{84}$$

Strictly speaking we have not verified that this structure indeed persists at second order in ϵ. The difficulty is that in perturbing away from the symmetric point, we lose any semblance of control over the induced metric on the moduli space, and one cannot rule out singularities arising in the truncation to the real section—these would most likely take the form of cusps appearing at the vacua. This hinders a purely bosonic construction via completing the square in the Hamiltonian á la Bogomol'nyi. Nevertheless, we would like to emphasize here that the picture one obtains from (84) is entirely consistent with the results we obtained earlier in the opposite (hierarchical) limit in which $|m_2/m_1| \to \infty$. In particular, the potential implies that the vacua lie at $\xi_0 = 0, \pi$. From the polar coordinatization of the real section, we see that this contracts the moduli space as follows:

$$Z_\pm = Z_0 \pm i Z_3 = e^{\pm i\theta}, \qquad Z_1 = Z_2 = 0. \tag{85}$$

This is entirely consistent with the behavior of the wall solutions we observed in the hierarchical limit, accounting for the fact that here $|m_1/m_2| = 1 + \mathcal{O}(\epsilon)$. This consistency suggests that although we have only considered the perturbation at linear order, the resulting physical picture is valid more generally.

In concluding this section, we will comment briefly on some subtleties that arise in extending these arguments to higher N. Firstly, since the reduced moduli space for 1-walls, $\mathbb{C}P^{N-1}$, can always be embedded within a suitably oriented real section of the meson moduli space $\det M = \Lambda_N^{2N}$, it seems clear that the one-to-two pairing between bosonic and fermionic zero modes will hold more generally. This ensures that multiplets when realized in terms of $\mathcal{N} = 1$ SUSY are necessarily reducible, although this structure may of course be lifted once one goes beyond the two-derivative level. With this matching, $\mathcal{N} = 2$ SUSY would follow immediately given a Kähler metric on the reduced moduli space. It is this latter property, namely, that the induced geometry is in fact *globally* Kähler, which appears difficult to prove in generality. In the symmetric mass regime, it of course follows directly from the construction of the moduli space as a Kähler quotient. However, this regime is not weakly coupled and in the tractable hierarchical mass regime one loses the isometry constraints on the induced metric. Nonetheless, the explicit construction in the SU(2) case is certainly suggestive that supersymmetry enhancement also arises for generic N and $\tilde{\mathcal{M}}_k$.

It is worth noting that this conclusion is rather novel when the Kähler structure is not imposed by the residual supersymmetry of the BPS state. For pointlike or stringlike solitons one has additional freedom through the possibility of realizing SUSY using one-component fermions. Indeed, this is the conventional manner in which worldline theories for, e.g., lumps in Kähler sigma models, and monopoles in $\mathcal{N} = 2$ SYM, get around the apparent contradiction of being 1/2-BPS states while at the same time having a Kähler, or, respectively, hyper-Kähler, moduli space [20,29]. The worldvolume theories in question can be thought of as reductions of (0,2) and (0,4) sigma models in $1 + 1$D, and this structure can be understood from the fact that the same bosonic moduli space arises in theories with twice as much supersymmetry, namely, hyper-Kähler sigma models and $\mathcal{N} = 4$ SYM, where the additional fermionic zero modes restore the "nonchiral" structure to the worldvolume superalgebra.

IV. ON 1/4-BPS WALL INTERSECTIONS

In this section, we will turn our attention to a second set of BPS configurations present in $\mathcal{N} = 1$ SQCD. An inspection of the $\mathcal{N} = 1$ supertranslation algebra in $3 + 1$D shows that it admits central charges supported by domain walls and also stringlike sources (see Appendix). The corresponding charges transform in the (0,1) and (1/2, 1/2) Lorentz representations, respectively. The SQCD theories considered here are not expected to exhibit BPS string solutions, but one has the possibility of forming (1/4-BPS) intersections or junctions of domain walls supported by both wall and string charges. One class of 1/4-BPS junctions arises from a multispoke configuration of N domain walls in theories with N degenerate vacua. The 1/4-BPS criterion amounts to the statement that the superpotential evaluated on a path through each wall surrounding the junction traces out a closed polygon [34–37]. However, for SQCD, the existence of a degenerate spectrum of k-walls [10,11], presents the possibility of forming a novel class of domain wall-junction configurations consisting of only *two* walls.[7] It is these configurations that we will study in this section, first from the bulk perspective, and then from the worldvolume point of view of the constituent walls. We will generally restrict our attention to gauge group SU(2) with $N_f = 2$, and make use of the worldvolume theory constructed in the previous section.

Before describing the explicit construction, we recall some well-known (and some less well-known) features of

[7]A similar class of string junctions consisting of just two strings was studied recently [38] in the context of gauge theories with eight supercharges. From the standpoint of the bulk theory such string junctions turn out to represent "confined monopoles" in the Higgs phase.

the kinematics. To this end, it is convenient to represent the superalgebra in 2 + 1D, which we can do by lifting the corresponding discussion of Sec. III, phrased in a 1 + 1D language appropriate to domain walls, to 2 + 1D compactified on a circle of radius L. Using the same notation, with the identification $\gamma^2 = i\gamma^5$, we can extend (60) as follows [36,39]:

$$\{Q^i_\alpha, Q^j_\beta\} = 2(\gamma^\mu\gamma^0)_{\alpha\beta}\delta^{ij}P_\mu + 2i(\gamma^0)_{\alpha\beta}\epsilon^{ij}Z_{\rm S} + 2(\gamma^2\gamma^0)_{\alpha\beta}(\sigma_3)^{ij}(|Z_{\rm W}|L), \tag{86}$$

which includes, in addition to the lift of the kink (or wall) charge, denoted $Z_{\rm W}L$, a new (real) charge, $Z_{\rm S}$, associated with localized objects in 2 + 1D—which we have taken to be positive to simplify the discussion. In the rest-frame, it is sufficient to focus on the sector of the two unbroken supercharges in the background of a BPS domain wall, namely Q^1_2 and Q^2_1, which we relabel as Q'_i for $i = 1, 2$ respectively. The rest-frame algebra in this subsector takes the form,

$$\{Q'_i, Q'_j\} = 2\delta_{ij}(M - |Z_{\rm W}|L) - 2(\sigma^1)_{ij}Z_{\rm S}, \tag{87}$$

from which we observe that, in a background with both central charges nonzero, only one of these supercharges can annihilate the state, and the Bogomol'nyi bound takes the form

$$M > |Z_{\rm W}|L + Z_{\rm S}. \tag{88}$$

1/4-BPS junction configurations are required to saturate this bound.

When we lift this picture one further dimension to 3 + 1D, an additional subtlety arises from the fact that the charge $Z_{\rm S}$, now associated with stringlike sources, transforms as a vector and is not algebraically independent of the momentum, i.e., in 3 + 1D,

$$\{Q_\alpha, \bar{Q}_{\dot\alpha}\} = 2(\gamma^\mu)_{\alpha\dot\alpha}(P_\mu + Z^{\rm S}_\mu). \tag{89}$$

Ignoring the wall charges for now, if we orient the stringlike source in the x_3 direction, we see that the BPS bound takes a somewhat unusual form

$$T^{(S)} \geq P_3 + Z^{\rm S}_3 \geq Z^{\rm S}_3, \tag{90}$$

where the second relation follows on noting that for configurations which saturate the bound (88) in 2 + 1D, the allowed boost in the x_3 direction is chiral, namely, in the current basis (with positive $Z_{\rm S}$) P_3 is required to be strictly positive [34]. The crucial point here is that although the central charge is not algebraically independent of the momentum, it is *dynamically* distinguished by the existence of an alternate means of identifying P_μ via the conserved, and symmetric, energy-momentum tensor. One can of course pick the "rest" frame $P_3 = 0$ to recover a more standard form of the Bogomol'nyi bound as discussed in [34–36], but one can alternatively "boost" the BPS soliton (see Fig. 2). The additional

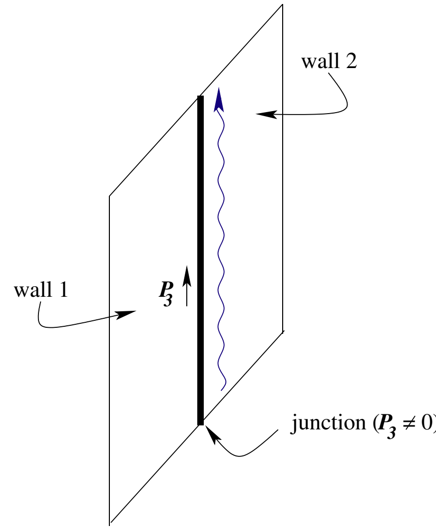

FIG. 2 (color online). A schematic representation of a "boosted" BPS junction, i.e., a junction superposed with a wave of momentum P_3.

invariant which accounts for this is $P_\mu Z^\mu$, which in the present coordinate system reduces to P_3.

This example illustrates the general point that the full spectrum of central charges is not always obtained by lifting the algebra to the maximal allowed dimension, and then matching the full number of components in the anticommutator of supercharges, minus the momenta, with the allowed set of tensor central charges. The reason is that not all vectorial charges can be absorbed into the momenta since, although they are not algebraically independent, they are dynamically distinguished. We provide a discussion of the central charge structure along these lines in the Appendix.

Returning to the bound (90) in the present context, the fact that the only configurations currently known which saturate this bound in $\mathcal{N} = 1$ theories are wall-junction configurations may partially be explained by considering the number of localized zero modes, as discussed in Section II. The one broken supercharge furnishes the junction with a single fermionic zero mode. This is paired with a single bosonic zero mode whose origin is best understood by viewing the junction as a kink-soliton on the wall worldvolume. The bosonic zero mode then arises from the breaking of translational invariance along the wall. A second translational zero mode, associated with the position of the junction in the orthogonal direction is not localized as it corresponds to a shift of wall itself. In this sense the worldvolume structure of the junction is quite distinct from a localized source such as a vortex.

In 3 + 1D this structure has a natural interpretation in terms of the extension of the zero modes to worldvolume fields in 1 + 1D, i.e. we can now complete the single fermionic zero mode to a chiral fermion, which we can choose to be left moving. In the bosonic sector the single translational zero mode is completed to a bosonic field, which we can decompose into a left and a right-mover. Only the left-mover will be paired with the fermionic zero mode under the residual chiral (0,1) worldsheet supersymmetry [34]. Now, when we turn on P_3, we can interpret the resulting junction which remains BPS as equivalent to the "bare" $P_3 = 0$ junction superposed with a left moving wave of momentum P_3. This configuration is illustrated schematically in Fig. 2.

With these preliminaries, we now return to the specific case of interest, namely, 2-wall junctions in $\mathcal{N} = 1$ SQCD, and consider these solutions first in the hierarchical quark mass limit with only one light flavor.

A. Junction tension for $N_f = 1$

We first consider the hierarchical regime for gauge group SU(2), and integrate out the second flavor as in the corresponding discussion of Sec. III. Since we take the decoupling limit directly, and thus solve the constraint $Z_+Z_- = 1$ explicitly for Z_+, it is useful to introduce another dimensionless field Y in the form $Y = \sqrt{M_{11}}(\Lambda_1^5 m_1^{-1})^{-1/4} = \sqrt{Z_+}(m_2 m_1^{-1})^{-1/4}$, such that after decoupling

$$\mathcal{W} = \sqrt{m_1\Lambda_1^5}(Y^2 + Y^{-2}), \quad \text{and} \quad \mathcal{K} = \sqrt{\Lambda_1^5 m_1^{-1}}\bar{Y}Y. \tag{91}$$

Provided we take $m_1 \ll \Lambda_1$, the vacua $\langle Y^2\rangle = \pm 1$ lie at weak coupling, and one can construct the two BPS wall configurations we exhibited in (45) (first obtained in [11]), which we reproduce here in the form (with $\tilde{\theta} \in [-\pi, \pi]$),

$$Y_{\rm wall}^2 = e^{i\tilde{\theta}(x)}, \qquad \tilde{\theta}_{\rm sol}(x) = \pm 2\arctan(e^{-2|m_1|(z-z_0)}). \tag{92}$$

The labeling of the two walls, W_+ and W_-, reflects whether the phase of Y^2 interpolates between the two (real) vacua via the upper or lower half-plane. The corresponding trajectories are illustrated in Fig. 3.

Having two degenerate walls, we can contemplate the possibility of a 2-wall-junction in the form illustrated in Fig. 3. We choose coordinates so that the walls interpolate from $k = 0$ at $x = -\infty$ to $k = 1$ at $x = +\infty$, and position the walls at $x_0 = 0$. The spatial worldvolume dimension of the walls transverse to the junction will be denoted y, with the junction located at $y_0 = 0$.

Qualitatively, we see that at large $|y|$, remote from the junction, the field profiles are essentially those for the wall trajectories (92), i.e. $W_\pm$ for y positive or negative. However, the evolution in y must interpolate smoothly between W_+ and W_-. A (presumably rapid) transition necessarily occurs near $x = 0, y = 0$ where the junction is located. In particular, such a smooth interpolation means that near $y = 0$ our x trajectory necessarily runs through the shaded domain of small Y shown on the right of Fig. 3, implying that strong dynamics must become important.

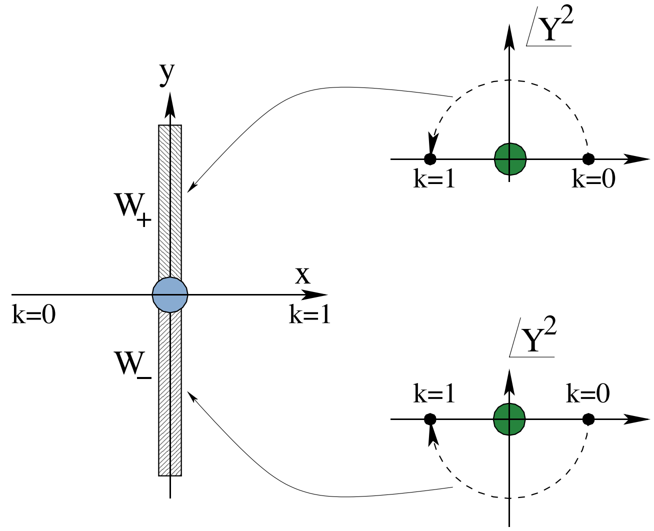

FIG. 3 (color online). The wall-junction geometry, indicating the field profiles in the W_+ and W_- components. Note that, by continuity, the fields near the junction must pass through the strong-coupling regions near $Y = 0$, where the low energy description breaks down.

We can understand this more clearly by studying the Bogomol'nyi equation which follows either by minimizing the energy, or equivalently requiring a configuration preserving one of the four supercharges in the $\mathcal{N} = 1$ algebra. Introducing the complex spatial coordinate $z = x + iy$, the equation can be written as [12,37]

$$g_{\bar{Y}Y}\partial_z Y = -\frac{1}{2}\frac{\partial\bar{\mathcal{W}}}{\partial\bar{Y}}, \tag{93}$$

which in the present case reduces to

$$\partial_{\hat{z}} Y = -\bar{Y} + \bar{Y}^{-3}, \tag{94}$$

on introducing a dimensionless coordinate $\hat{z} = |m_1|z$.

For configurations satisfying the Bogomol'nyi equation, the junction tension T_j saturates the BPS bound for the (1/2, 1/2) central charge [36],

$$T_j = -\frac{1}{2}\sqrt{\Lambda_1^5 m_1^{-1}}\oint a_k dx_k, \; a_k = i\bar{Y}\overleftrightarrow{\partial}_k Y, \qquad k = 1, 2, \tag{95}$$

where the integral runs over a large contour in the xy plane. In the problem at hand it is convenient to choose a rectangular contour which must lie in the plane of Fig. 3. Then, on the vertical sides of the contour (i.e. those parallel to the wall) the field Y is essentially constant; therefore, $a_k = 0$. Moreover, on the horizontal sides of the contour (i.e. those perpendicular to the wall) only the phase of Y changes, and so

$$a_k = -\partial_k \tilde{\theta}, \tag{96}$$

where the phase $\tilde{\theta}$ was defined in Eq. (92). We do not need to know precisely how $\tilde{\theta}$ depends on x since the contour integral can be done directly,

$$\oint a_k dx_k = -\Delta\tilde{\theta}, \quad \text{with} \quad \Delta\tilde{\theta} = 2\pi, \tag{97}$$

where the numerical result holds for the field configuration depicted in Fig. 3. Thus,

$$T_j = \pi\sqrt{\left|\frac{\Lambda_1^5}{m_1}\right|}. \tag{98}$$

This tension is determined [36] by the (real) central charge in the anticommutator $\{\bar{Q}, Q\}$ as in (89). However, in contrast to the wall tension, it is not holomorphic in parameters, and thus we cannot extrapolate this result to large m where one recovers pure SYM.

Note that we can interpret (98) as implying that the thickness of the junction in the y direction is of the same order as the thickness of the wall itself in the x direction and is large, $\sim m^{-1}$. Because of this fact the presence of an unknown core in the wall-junction (which is inevitable since the Y trajectory runs through the strong-coupling domain) is unimportant numerically since the relevant scale is Λ_1. However, this point necessarily means that the junction cannot fully be described within this effective theory, and we can ask whether the worldvolume perspective may help in this regard.

B. Resolving the singularity on the worldvolume

The preceding analysis indicates that the $N_f = 1$ system is rather inadequate for describing the detailed structure of the junction solution. In particular, the boundary conditions at infinity in the plane transverse to the axis of symmetry ensure the following symmetry of the solution,

$$Y(z) \to \bar{Y}(\bar{z}). \tag{99}$$

Along with continuity, this implies that the field Y^2 must pass through zero at the core of the junction. Since the potential diverges at this point, we see that the description breaks down within the shaded domain sketched at the right in Fig. 3, and we cannot expect to find a solution (even numerically) in this region. A similar singularity is seen to arise, for similar reasons, for analogous 2-wall junctions for higher N.

It is interesting then to address this question directly from the worldvolume point of view, by adding an additional light flavor so that the two component walls arise from the dynamics of a $\mathbb{C}P^1$ sigma model, or more precisely a massive sigma model where the mass term is identified with $|\pi\Delta m|/2$ as discussed in the previous section. This theory possesses 1/2-BPS kink solitons, and it is natural to identify these kinks as the worldvolume description of 1/4-BPS 2-wall junctions. We will now provide evidence for this identification by verifying that the kink tension reproduces the tension of the junction, given in (98), in the appropriate limit.

Using complex coordinates for S^2, as introduced in (78), the bosonic sector of the massive $\mathbb{C}P^1$ sigma model becomes

$$\mathcal{L} = \frac{2R_{\tilde{\mathcal{M}}}}{(1+|w|^2)^2}\left[|\partial_\mu w|^2 + \frac{1}{4}|\pi\Delta m|^2|w|^2\right], \tag{100}$$

where $R_{\tilde{\mathcal{M}}}$ is the Kähler parameter denoting the size of the reduced moduli space. When $|\Delta m|$ is large relative to any dynamically generated scale, the theory has classical vacua at $w = 0, \infty$. Using the coordinate relation from (78), $w = \tan\frac{\xi_0}{2} e^{i\phi_0}$, one finds that classical BPS kink solutions exist which satisfy (yet) another sine-Gordon equation [33]

$$\partial_y \xi_0 = \pm\frac{1}{2}\pi|\Delta m|\sin\xi_0, \qquad \partial_y\phi_0 = 0. \tag{101}$$

The corresponding tension of the junction is given by

$$T_j = \frac{1}{2}\pi|\Delta m|(2R_{\tilde{\mathcal{M}}}) + \mathcal{O}(\Lambda_{\rm wv}), \tag{102}$$

where for the moment we assume Δm is large and so provides the dominant mass scale.

In the present case $R_{\tilde{\mathcal{M}}}$ was computed in (55), and we obtain

$$T_j = \pi\left|\frac{|m_1| - |m_2|}{\sqrt{|m_1 m_2|}}\right|\Lambda_2^2. \tag{103}$$

A simple check on this result follows on integrating out one of the flavors. On sending $m_2 \to \infty$, we must keep $\Lambda_1^5 = m_2\Lambda_2^4$ fixed, so that

$$T_j = \pi\left|\frac{|m_1| - |m_2|}{\sqrt{|m_1 m_2|}}\right|\sqrt{\frac{\Lambda_1^5}{|m_2|}} \xrightarrow{m_2\to\infty} \pi\sqrt{\frac{\Lambda_1^5}{|m_1|}}, \tag{104}$$

which agrees precisely with the result obtained earlier in Eq. (98) from a direct analysis of the 1-flavor model [despite being derived in the small Δm regime]. It is worthy of note that the earlier determination that the worldvolume real mass perturbation was indeed a real parameter also finds a nice consistency check in this expression. The resulting junction tension depends nonholomorphically on m_1 as one expects from the bulk point of view.

Examining (103) we observe that as the mass splitting is reduced we become sensitive to quantum effects on the worldvolume, and indeed this is to be expected as the bulk theory is also strongly coupled for $m_1 \sim m_2$. In this regime, the junction configuration is still described by a $\mathbb{C}P^1$ kink, and we conclude that the solution will be nonsingular whenever the worldvolume IR dynamics is sufficient to generate a mass gap. For example, it is sufficient to compactify one of the spatial dimensions on an S^1 of circumference L. The effective $1 + 1$D dynamics then generates a dynamical scale of the form

$$\Lambda_{\rm wv} = \mu \exp\left(-\frac{2\pi}{g^2(\mu)}\right), \qquad g^2(\mu) = \frac{1}{R_{\tilde{\mathcal{M}}} L}, \qquad (105)$$

and it is this parameter which enters the (1/2, 1/2) central charge and sets the tension, or "effective mass", of the junction, reduced now to a localized soliton. This determines a contribution to T_j which is necessarily independent of the contribution from the walls. Unfortunately, since the relevance of such a worldvolume scale only becomes apparent on entering the strong-coupling regime when $m_1 \sim m_2$ it is difficult to make any concrete identification with bulk $3 + 1$D parameters.[8] Nonetheless, this scale does have a direct physical interpretation in $3 + 1$D as the intrinsic junction tension.

The enhancement of worldvolume supersymmetry for the reduced wall moduli space also has important consequences for this identification of 1/4-BPS bulk junctions with 1/2-BPS worldvolume kinks. In particular, while the junction preserves only one of the bulk supercharges, the kink preserves in addition one of the supernumerary charges present on the worldvolume. Moreover, since the junction not only preserves two worldvolume supercharges, but also breaks two, it necessarily exhibits two fermionic zero modes. Recalling the discussion at the start of Sec. IV, we see that this is not the minimal chiral content that one would anticipate based on the breaking of bulk supersymmetry. In actual fact, the kink solutions also have two bosonic moduli, the center-of mass position and the phase $\phi = \phi_0$ as is apparent from (101). Thus the moduli space is two-dimensional

$$\mathcal{M}_{\rm kink} = \mathbb{R} \times \tilde{\mathcal{M}}_1, \qquad (106)$$

where the reduced moduli space is $\tilde{\mathcal{M}}_1 = S^1$. Note that only one of these bosonic moduli—the translational mode—would have been anticipated from a consideration of the bulk kinematics. We see that the bosonic and fermionic moduli form two (0,1) chiral multiplets and two bosonic singlets. On the worldvolume, this structure is enforced by the broken supersymmetry. However, from the bulk point of view the second (0,1) chiral multiplet and singlet are not required by supersymmetry considerations, but presumably correspond to a Goldstone multiplet arising through the breaking of flavor symmetry, as is the case for the wall itself.

In this context, the $\mathcal{N} = 2$ worldvolume SUSY resolves an apparent paradox that arises when one tries to verify that these junction solutions are BPS saturated at the quantum level. In particular, were the worldvolume to possess only $\mathcal{N} = 1$ SUSY, putative BPS junctions would have to be realized as one-component multiplets. An index to count such multiplets was introduced in [23], which is formally expressed as

$$\nu_{\rm LSV} = \frac{1}{2Z}\{{\rm Tr} Q_{\rm broken}\}^2, \qquad (107)$$

in terms of the broken supercharge. An observation of [23] which is particularly relevant here is that the index necessarily vanishes when the fermion parity $(-1)^F$ is well-defined. This indicates that the multiplet is generally reducible (containing two states) and can lift from the BPS bound. In the present case, as noted above, the kinks have an even number of bosonic, and consequently fermionic, moduli. On quantization, the latter furnish a representation of the Clifford algebra, in this case $\gamma^i = \sigma_1, \sigma_2$, from which we can construct $\gamma_5 = \sigma_3$ which represents $(-1)^F$. Thus one would necessarily conclude that no short $\mathcal{N} = 1$ multiplets are allowed and there would be no reason to expect that these junctions should saturate the 1/4-BPS bound in $3 + 1$D. This would be rather puzzling, and indeed as we have discussed this problem is resolved due to the enhanced SUSY on the reduced moduli space so that the junctions lie in BPS multiplets of $\mathcal{N} = 2$ SUSY, and are instead counted by the CFIV index on the worldvolume.

C. Extensions for SU(N)

An immediate technical advantage of the realization of junctions as BPS kinks on the wall worldvolume, is that we can utilize our knowledge of these configurations for arbitrary N to infer analogous results for junctions, which are in fact rather difficult to obtain directly. Thus we now identify 1/4-BPS 2-wall junctions in the SU(N) theory with $N_f = N$ with kinks in the worldvolume $\mathbb{C}P^{N-1}$ sigma model deformed by the relevant real mass terms. We will limit our remarks here to two issues, namely, the multiplicity of 2-wall junctions, and their tension.

In order to make this discussion concrete we must again resort to compactifying the theory on a circle to ensure that the low energy effective theory on the wall worldvolume is $1 + 1$-dimensional and develops a mass gap. We can then vary the quark masses across the range where the dynamical scale $\Lambda_{\rm wv}$ becomes important and, for example, sit in the strong-coupling region where $\Delta m \ll \Lambda_{\rm wv}$. Note that whether or not this restriction changes the physical conclusion is tied to the question of whether entering the strong-coupling domain in the bulk effectively induces a mass gap within the (decompactified) worldvolume theory.

Because of $\mathcal{N} = 2$ SUSY, the junction multiplicity is formally given by the CFIV index as noted above. Focusing just on minimal walls for arbitrary N, a generic intersection between two of the N possible walls, will connect walls differing by p units of phase—we will refer to this as a p-junction. The number of p-junctions is

[8] In contrast, an identification of the quantum scale is possible within the analogous $\mathbb{C}P^{N-1}$ worldsheet dynamics of "non-Abelian" vortices in the $\mathcal{N} = 2$ Higgs phase [38].

formally (on compactification on S^1, and taking the limit $|\Delta m| \to 0$)

$$\nu_{\mathrm{CFIV}} = \binom{N}{p}. \tag{108}$$

In practice, as noted above, the result can be somewhat different in the limit $|\Delta m| \gg \Lambda_{\mathrm{wv}}$, which may in fact be the only accessible regime in 2 + 1D. In particular, in this regime in 1 + 1D one can turn on an arbitrary integer "dyonic" charge [33], due to a coupling to the corresponding U(1) current in the superalgebra [40], although this is also reflected in a change in the mass. Only a certain number of these states survive (as above) in the limit $|\Delta m| \to 0$, due to the presence of marginal stability curves [33].

Turning to the tension, in the hierarchical regime $|\Delta m| \gg \Lambda_{\mathrm{wv}}$, the result is a natural generalization of (102) determined by the various real mass terms. More interesting perhaps is that, within the compactified regime with $|\Delta m| \ll \Lambda_{\mathrm{wv}}$, the result translated directly from that for $\mathbb{C}\mathrm{P}^{N-1}$ kinks is

$$T_p = \Lambda_{\mathrm{wv}} \sin\frac{2\pi p}{N}. \tag{109}$$

Although this is a rather familiar formula in the context of wall-like solitons, it takes on an interesting new interpretation here as the junction is a stringlike source, albeit wrapped on a small circle in the present construction. Since this result is naively protected by the enhanced $\mathcal{N} = 2$ worldvolume SUSY,[9] we see that the wall junctions actually realize the "sine-formula" for the ratio of "string" tensions for differing values of p first observed for strings in softly broken $\mathcal{N} = 2$ SYM by Douglas and Shenker [41].[10]

This structure is not expected to apply to generic p-strings in confining vacua of $\mathcal{N} = 1$ SYM, since these states are non-BPS, but here we find a situation where the sine-formula appears to be exact, due to the enhanced SUSY on the worldvolume. However, we should reiterate that this discussion has been framed within a specific scenario. If we decompactify the extra spatial dimension, then to retain control over the vacuum structure, one needs to reintroduce a hierarchy for the quark masses. The kink spectrum, and also the tension, then changes considerably on moving outside a "curve of marginal stability", and many more states are present classified by U(1) charges associated with the residual Abelian flavor symmetries in the hierarchical case. After decompactification, if we try to remove the hierarchy the system reenters a strong-coupling regime that at present appears intractable.

[9]This statement requires some caution as reference to Eq. (104) indicates that, due to the embedding, the dependence of the tension on the mass scale in the hierarchical limit is not holomorphic. This is in accord with expectations for the junction charge in the bulk.

[10]A heuristic model relating the wall tension and the string tension (109) was discussed recently in [42].

V. CONCLUDING REMARKS

In this paper we have presented a detailed exploration of the worldvolume moduli space dynamics of 1/2-BPS domain walls in $\mathcal{N} = 1$ SQCD with gauge group SU(2) and $N_f = 2$ flavors. We have also discussed how novel 1/4-BPS 2-wall junctions may be realized as kinks within the worldvolume theory. We concentrated on the SU(2) example where much of the analysis could be performed explicitly, but we anticipate that most of the conclusions should extend to the generic SU(N) case with $N_f = N$ flavors. In particular, the appearance of an enhanced $\mathcal{N} = 2$ worldvolume supersymmetry on the reduced moduli space is essentially guaranteed by the corresponding construction as a Kähler quotient. In this concluding section, we will make a couple of more speculative remarks on localized worldvolume solitons which may (or may not) find a bulk interpretation.

The $\mathcal{N} = 2$ algebra in 2 + 1D includes, in addition to a tensorial central charge for the 1 + 1D kink which we have interpreted as a 2-wall-junction, a Poincaré invariant charge supported by localized lump solitons. When the worldvolume theory is naturally embedded in the relevant linear sigma model, lumps are realized as semilocal vortices. Consequently, with reference to the interpretation of such walls as D-branes for SYM strings [43], it is tantalizing to speculate that these configurations may have a relation to the endpoints of SQCD strings.[11] Note, in particular, that at energy scales below the UV cutoff on the wall, of order $1/\mu$, such strings are stable to quark pair production. The result ν_k for the wall multiplicity is also consistent with the interpretation that 1-walls lie in the fundamental representation of SU(N), or more generally admit an action of the corresponding Weyl group, and form antisymmetric bound states. Moreover, although these configurations are BPS on the worldvolume, they would indeed be non-BPS within the bulk.

In spite of these intriguing hints it seems difficult, for several reasons, to make a precise identification of this type. For example, lumps carry integer charges $\pi_2(\mathbb{C}P^1) = \mathbb{Z}$, rather than charges under $\mathbb{Z}_N$ that one might associate with the center-of the gauge group. A contraction of the charge lattice, $\mathbb{Z} \to \mathbb{Z}_N$, might occur due to physics occurring above the worldvolume UV cutoff, but there is another more significant roadblock in the way of a quantitative study of this question. This is the fact that in the hierarchical mass regime where the theory is tractable there is a potential on the moduli space. In the

[11]Note that worldvolume vortices on BPS walls were shown to represent string endpoints in gauge theories with eight supercharges [44] (see also [45]).

presence of such a deformation, lumps are no longer stable, via Derrick's theorem, unless one turns on additional U(1) charges. Such time-dependent Q-lumps are known, for $N > 2$, to have a ringlike structure and locally carry the junction charge [46]. Thus these configurations appear as domain wall bubbles on the worldvolume. It is far from clear what may happen to these configurations as one sends $\Delta m \to 0$ and returns to strong-coupling, and this hinders a direct bulk interpretation.

In a similar regard, we can also speculate about configurations which one might dub "junctions of junctions". In particular, the counting argument for junctions described above suggests that, even in the minimal SU(2) case, there are two inequivalent junctions. One may then anticipate that a further intersection, now within the worldvolume, would be possible—a 1/4-BPS state on the worldvolume (since it sources both the kink and lump central charges), but again non-BPS in the bulk.

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APPENDIX A: ON CENTRAL CHARGES IN $D = 2, 3$ AND 4

In this Appendix we will briefly discuss the central charge (CC) content of the superalgebras in $D = 2, 3$ and 4 relevant to this paper, and their inter-relations.

One may recall that some time after the minimal four dimensional superalgebra was first written down by Golfand and Likhtman [47], central charges were introduced algebraically as Poincaré invariant, and thus scalar, elements of the superalgebra commuting with all the other generators [48,49] (see also [50]). Their dynamical role was subsequently made apparent by Witten and Olive [1], who showed that such charges are supported by the topological charges of solitons. While it was appreciated for some time that not all central charges are Lorentz scalars (see e.g. [51]), the dynamical role of these additional tensorial charges was not fully understood until somewhat later, when they were shown to be nonzero in the presence of extended objects (p-branes) within supergravity [52] (see also [53]). Their occurrence in $\mathcal{N} = 1$ SYM in $D = 4$ (via a quantum anomaly) was first observed in [9].

We will concentrate on the algebras in $D = 2, 3$ and 4, for which the analyses in [54] for $D = 2$, and [39] (see also [36]) for $D = 3$ and 4, are particularly relevant. While most of what follows comprises review material collected here for completeness, we will extend the discussion in [39] of vectorial central charges, namely, those with the Lorentz structure of P_μ.

1. Minimal SUSY

Limiting ourselves to two, three and four dimensions we observe that the minimal number of supercharges is 2, 3, and 4, respectively. Two-dimensional theories with a single supercharge, although algebraically possible, require the loss of F and $(-1)^F$. Therefore, if one wishes to keep the distinction between "bosons" and "fermions", the minimal number of supercharges in $D = 2$ is two.

Working in a real representation with ν_Q supercharges, it is clear that, generally speaking, the maximal possible number of CC's is determined by the dimension of the symmetric matrix $\{Q_i, Q_j\}$ of size $\nu_Q \times \nu_Q$, namely,

$$\nu_{\rm CC} = \frac{\nu_Q(\nu_Q + 1)}{2}. \qquad \text{(A1)}$$

In fact, D anticommutators have the Lorentz structure of the energy-momentum operator P_μ. Therefore, up to D central charges could be absorbed in P_μ. However, in particular situations this number can be smaller, since although algebraically the corresponding CC's have the same structure as P_μ, they are dynamically distinguishable. The point is that P_μ is uniquely defined through the conserved and symmetric energy-momentum tensor of the theory.

The total set of CC's can be arranged by classification with respect to their Lorentz structure. Below we will present this classification for minimal supersymmetry in $D = 2, 3$ and 4. We then consider the extended $\mathcal{N} = 2$ supersymmetry algebras in $D = 2$ and $D = 3$ obtained via dimensional reduction from $D = 4$, and consider the analogous decomposition in terms of Lorentz and R-symmetry representations.

A. D = 2

Consider two-dimensional theories with two supercharges. From the discussion above, on purely algebraic grounds, three CC's are possible:

$$\{Q_\alpha, Q_\beta\} = 2(\gamma^\mu\gamma^0)_{\alpha\beta}(P_\mu + Z_\mu) + 2i(\gamma^5)_{\alpha\beta}Z, \qquad \text{(A2)}$$

one Lorentz-scalar Z and a two-component vector Z_μ. The latter case would require the existence of a vector order parameter taking distinct values in different vacua. This will break Lorentz invariance and supersymmetry of the vacuum state. Limiting ourselves to supersymmetric vacua we conclude that only one (real) Lorentz-scalar central charge is possible. This central charge is relevant to kinks in $\mathcal{N} = 1$ theories.

B. D = 3

The central charge allowed in this case is a Lorentz-vector Z_μ, i.e.

$$\{Q_\alpha, Q_\beta\} = 2(\gamma^\mu\gamma^0)_{\alpha\beta}(P_\mu + Z_\mu), \qquad \text{(A3)}$$

which we should arrange to be orthogonal to P_μ. By an appropriate choice of reference frame it can always be cast in the form (0,0,1). In fact, this is the central charge of the previous section elevated by one dimension. It is associated with a domain wall (or string) oriented along the second axis.

C. $D = 4$

Maximally one can have ten CC's which are decomposed into Lorentz representations as $(0, 1) + (1, 0) + (1/2, 1/2)$:

$$\begin{aligned}\{Q_\alpha, \bar{Q}_{\dot\alpha}\} &= 2(\gamma^\mu)_{\alpha\dot\alpha}(P_\mu + Z_\mu),\\ \{Q_\alpha, Q_\beta\} &= (\Sigma^{\mu\nu})_{\alpha\beta}\bar{Z}_{[\mu\nu]},\\ \{\bar{Q}_{\dot\alpha}, \bar{Q}_{\dot\beta}\} &= (\bar\Sigma^{\mu\nu})_{\dot\alpha\dot\beta}Z_{[\mu\nu]},\end{aligned} \qquad \text{(A4)}$$

where $(\Sigma^{\mu\nu})_{\alpha\beta} = (\sigma^\mu)_{\alpha\dot\alpha}(\bar\sigma^\nu)^{\dot\alpha}_\beta$ is a chiral version of $\sigma^{\mu\nu}$ (see e.g. [55]). The antisymmetric tensors $Z_{[\mu\nu]}$ and $\bar{Z}_{[\mu\nu]}$ are associated with domain walls, and reduce to a complex number and a spatial vector orthogonal to the domain wall. The $(1/2, 1/2)$ CC Z_μ is a Lorentz-vector orthogonal to P_μ. It is associated with strings (flux tubes), and reduces to one real number and a three-dimensional unit spatial vector parallel to the string.

2. Extended SUSY

We will limit our attention here to exploring the reduction of the minimal SUSY algebra in $D = 4$ to $D = 2$ and 3, namely, the $\mathcal{N} = 2$ SUSY algebra in those dimensions. As should be clear from the discussion above, the maximal number of CC's is of course the same, and the only distinction we must make is to provide a decomposition into both Lorentz and R-symmetry irreps.

A. $\mathcal{N} = 2$ in $D = 3$

The superalgebra can be decomposed into Lorentz and R-symmetry tensorial structures as follows:

$$\begin{aligned}\{Q^i_\alpha, Q^j_\beta\} = {}& 2(\gamma^\mu\gamma^0)_{\alpha\beta}[(P_\mu + Z_\mu)\delta^{ij} + Z^{(ij)}_\mu]\\ &+ 2(\gamma^0)_{\alpha\beta}Z^{[ij]},\end{aligned} \qquad \text{(A5)}$$

where γ^0 is the charge conjugation matrix. The maximal set of ten CC's enter as a triplet of spacetime vectors Z^{ij}_μ—which we decompose into an R-symmetry singlet trace term, denoted Z_μ, and a trace-free symmetric combination $Z^{(ij)}_\mu$—and a singlet $Z^{[ij]}$. The singlet CC is associated with vortices (or lumps), and corresponds to the reduction of the $(1/2, 1/2)$ charge or the 4^{th} component of the momentum vector in $D = 4$. The R-symmetry singlet Z_μ is algebraically indistinguishable from the momentum and is equivalent to the vectorial charge in the $\mathcal{N} = 1$ algebra. The traceless symmetric combination $Z^{(ij)}_\mu$ can be reduced to a complex number and vectors specifying the orientation of a codimension one source. We see that these are the direct reduction of the (0,1) and (1,0) wall charges in $D = 4$.

B. $\mathcal{N} = 2$ in $D = 2$

Lorentz invariance now provides a much weaker constraint, and one can in principle consider different (p, q) superalgebras with $p \neq q$. We will focus here only on the nonchiral $\mathcal{N} = (2, 2)$ case corresponding to dimensional reduction of the $\mathcal{N} = 1 D = 4$ algebra. The tensorial decomposition is as in (A5), but with the decomposition of $D = 3$ spacetime vectors into $D = 2$ vectors and a singlet,

$$\begin{aligned}\{Q^i_\alpha, Q^j_\beta\} = {}& 2(\gamma^\mu\gamma^0)_{\alpha\beta}[(P_\mu + Z_\mu)\delta^{ij} + Z^{(ij)}_\mu]\\ &+ 2i(\gamma^5\gamma^0)_{\alpha\beta}(\delta^{ij}Z + Z^{(ij)}) + 2(\gamma^0)_{\alpha\beta}Z^{[ij]},\end{aligned} \qquad \text{(A6)}$$

We discard all vectorial charges Z^{ij}_μ in this case for the same reasons as noted above in the $\mathcal{N} = 1$ case, namely, they would imply SUSY breaking in the vacuum. This leaves two singlets $Z^{(ij)}$, which are the reduction of the domain wall charges in $D = 4$ and correspond to topological kink charges, and two further singlets Z and $Z^{[ij]}$, arising via reduction from P_2 and the vortex charge in $D = 3$. The latter charges also arise for kinks in the presence of twisted mass terms [54].

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