$SU(N)$ chiral gauge theories on the lattice

Maarten Golterman[1,*] and Yigal Shamir[2,†]

[1]Department of Physics and Astronomy, San Francisco State University, San Francisco, California 94132, USA
[2]School of Physics and Astronomy, Raymond and Beverly Sackler Faculty of Exact Sciences, Tel-Aviv University, Ramat Aviv, 69978, Israel

(Received 12 May 2004; published 29 November 2004)

We extend the construction of lattice chiral gauge theories based on non-perturbative gauge fixing to the non-Abelian case. A key ingredient is that fermion doublers can be avoided at a novel type of critical point which is only accessible through gauge fixing, as we have shown before in the Abelian case. The new ingredient allowing us to deal with the non-Abelian case as well is the use of equivariant gauge fixing, which handles Gribov copies correctly, and avoids Neuberger's no-go theorem. We use this method in order to gauge fix the non-Abelian group (which we will take to be $SU(N)$) down to its maximal Abelian subgroup. Obtaining an undoubled, chiral fermion content requires us to gauge-fix also the remaining Abelian gauge symmetry. This modifies the equivariant Becchi-Rouet-Stora-Tyutin (BRST) identities, but their use in proving unitarity remains intact, as we show in perturbation theory. On the lattice, equivariant BRST symmetry as well as the Abelian gauge invariance are broken, and a judiciously chosen irrelevant term must be added to the lattice gauge-fixing action in order to have access to the desired critical point in the phase diagram. We argue that gauge invariance is restored in the continuum limit by adjusting a finite number of counter terms. We emphasize that weak-coupling perturbation theory applies at the critical point which defines the continuum limit of our lattice chiral gauge theory.

DOI: 10.1103/PhysRevD.70.094506 PACS numbers: 11.15.Ha, 11.30.Rd

I. INTRODUCTION

Attempts to develop a non-perturbative, lattice definition of chiral gauge theories have a long history. To date, no lattice definition of a non-Abelian chiral gauge theory which maintains exact gauge invariance is known. (For Abelian chiral gauge theories, see Ref. [1].) The fundamental difficulty is that, even if the whole collection of fermion fields is anomaly free, each lattice fermion field needs to contribute its "share" of the anomaly [2], and the regulated theory therefore tends to break the gauge invariance by irrelevant terms.

Because of those—classically, but not quantum-mechanically—irrelevant couplings, the longitudinal gauge degrees of freedom are *not* decoupled from the fermions. Extensive studies that go back to the eighties and early nineties have taught an important lesson: The uncontrolled non-perturbative dynamics of these unphysical degrees of freedom tend to spoil the desired continuum limit through the re-generation of doublers. For reviews, see Refs. [3,4].

A remedy is to regain control over the longitudinal dynamics by non-perturbative gauge fixing. This idea was first proposed in Ref. [5]. While the insight of Ref. [5] is an important one, the proposal itself was incomplete. In order that, indeed, a critical point will exist which describes the gauge-fixed target continuum theory non-perturbatively, non-trivial additional elements are needed, as first introduced in Refs. [6,7]. In subsequent work, convincing evidence was provided that fermion doublers are avoided, and that this program can be carried out successfully in Abelian lattice chiral gauge theories [8–14].[1]

In this paper, we describe in detail a proposal for the construction of $SU(N)$ chiral gauge theories on the lattice. To make it clear at the outset what our proposal does and does not accomplish, we begin with a summary of the main features and open questions of our construction.

Given an asymptotically free $SU(N)$ chiral gauge theory,[2] our method gives a prescription for how to discretize this theory in a way that satisfies the following properties:

(i) The lattice theory is local.
(ii) A straightforward, systematic weak-coupling expansion is valid near the critical point at which the target continuum theory is defined [6,7]. Near this critical point the lattice theory is manifestly renormalizable by power counting.
(iii) The fermions of the lattice theory are undoubled. In other words, the chiral fermions of the formal target continuum theory remain chiral on the lattice [8–10].
(iv) The fermion content has to be anomaly free in the usual sense. The theory accounts correctly for fermion-number violating processes [11].
(v) In order to construct the lattice theory, the target continuum theory is gauge-fixed first, before it is transcribed to the lattice, in such a way as to have

*Electronic address: maarten@stars.sfsu.edu
†Electronic address: shamir@post.tau.ac.il

[1]This statement ignores triviality, the latter being a property of Abelian gauge theories which is unrelated to the chirality of the fermion spectrum.

[2]We believe that the extension to other groups is a purely technical matter.

1550-7998/2004/70(9)/094506(22)$22.50

access to a complete set of Slavnov-Taylor identities. This is a central element of our construction, since our theory is not gauge invariant on the lattice, and both Slavnov-Taylor identities as well as power counting are needed in order to obtain a gauge-invariant continuum limit with the adjustment of a finite number of counter terms. In this sense, our proposal follows the philosophy of Ref. [5].

(vi) In order to gauge fix the non-Abelian "part" of the gauge symmetry, we employ a gauge-fixing method which may be regarded as a variant of the maximal-Abelian gauge, and is known as equivariant gauge fixing [15]. This allows us, as explained in detail in Sec. IV, to circumvent the Gribov problem in a rigorous manner. In particular, it allows us to put the well-known "Faddeev-Popov trick" on a rigorous footing, without running into the impasse of Neuberger's theorem [16].

In a non-Abelian theory without chiral fermions, equivariant gauge fixing can be implemented non-perturbatively, while maintaining an exact BRST-type invariance. With chiral fermions, gauge invariance is lost, since in order to avoid species doubling we add a Wilson term [17], which is not invariant under chiral symmetry. The gauge fixing is a key ingredient of our method: it maintains power-counting renormalizability in the absence of gauge invariance; it makes it possible to avoid doublers being generated dynamically; and it allows us to systematize the counter terms which need to be added to obtain a gauge-invariant continuum limit where the unphysical degrees of freedom decouple.

Of course, it is well known that for an arbitrary chiral gauge theory obstructions exist to this program. In the context of global (classical) chiral symmetry, the triangle anomaly appears to be a fundamental reason for the species-doubling phenomenon [2]. In the context of local chiral invariance, if the fermion spectrum is not gauge-anomaly free, it will be impossible to recover gauge invariance (and, hence, unitarity) in the continuum limit by tuning counter terms. Conversely, if the fermion spectrum does satisfy the usual anomaly-cancellation condition, one can recover gauge invariance and unitarity to all orders in perturbation theory in the continuum limit [18,19].

Other obstructions may exist which prevent us from constructing the desired continuum limit. A known example is the Witten anomaly [20,21]. Our construction does not answer the question of whether additional, as yet unknown, non-perturbative obstructions exist, and if so, how these depend on the gauge group and the fermion content.[3] If a certain non-Abelian chiral gauge theory *does* exist, our construction provides a lattice formulation of this theory in which there is strong evidence supporting the existence of a novel type of critical point yielding the correct continuum limit. All the known pitfalls are avoided in our construction.

At finite, non-zero lattice spacing our lattice definition of a chiral gauge theory is not unitary, because of the presence of unphysical states in the extended Hilbert space of a gauge-fixed gauge theory and the lack of exact BRST invariance on the lattice.[4] The existence of a systematic weak-coupling expansion in the gauge coupling makes it possible to establish unitarity to all orders in perturbation theory in the continuum limit. Non-perturbative unitarity is an issue which will need to be investigated using non-perturbative techniques.

To summarize, our construction does not prove the (non-perturbative) existence of any unitary chiral gauge theories. However, if a certain chiral gauge theory does exist, our discretization of it should be a valid discretization. Therefore our construction makes it possible to systematically investigate fundamental questions pertaining to asymptotically-free chiral gauge theories, using non-perturbative techniques.

The underlying strategy is that, keeping in mind that gauge invariance is broken explicitly on the lattice, *one constructs a lattice theory containing a suitable gauge-fixing action that admits a systematic perturbative expansion in the coupling constant near the critical point where the continuum limit is taken.* The existence of a systematic perturbative expansion has the following implication: The elementary degrees of freedom of the formal target (chiral) gauge theory, which may, by construction, be identified from the classical continuum limit of the lattice action, are indeed the elementary degrees of freedom obtained in the continuum limit of the quantum theory. In particular, no doublers are generated, and a chiral fermion spectrum can be maintained non-perturbatively. The renormalized interactions also agree with those of the target continuum theory, and all unphysical excitations decouple (at least) to all orders in perturbation theory, after adding the appropriate counter terms (as determined by the Slavnov-Taylor identities). Standard power counting can be used in order to organize the counter terms, which implies that there is a finite number of them.

The construction of a theory with a critical point as alluded to above is non-trivial. The desired critical point exists thanks to the fact that the gauge-fixing action on the lattice can be chosen such that 1) its unique absolute minimum is the configuration with all link variables equal to the identity matrix [7]; 2) the lattice theory is manifestly renormalizable by power counting (despite the fact that the regulated theory is not gauge invariant),

[3]We return to this issue in the conclusion section.

[4]For earlier work on unitarity in lattice chiral gauge theories, see Ref. [22].

because it contains kinetic terms for all four polarizations [5]. As a result, the functional integral is dominated by a single saddle point when the bare coupling g_0 is sufficiently small. This saddle point is controlled by a straightforward weak-coupling expansion. As usual, the validity of lattice perturbation theory implies the existence of a scaling region. In the case of an asymptotically free theory, moreover, a physical infra-red scale will be dynamically generated in the continuum limit $g_0 \to 0$. The critical point itself involves this limit together with appropriate adjustment of the counter terms.

There are many ingredients to the construction of a theory with the desired critical point, and most of these have already been formulated and investigated in the past in the context of Abelian chiral gauge theory. The absence of doublers has been established in rather much detail [8–10,12]. The global structure of the phase diagram was studied in Refs. [13,14],[5] which also contain further tests of the validity of perturbation theory near the critical point, and of the agreement between the light degrees of freedom of the lattice theory and of the target continuum theory. In this paper we will therefore only give a brief account of this program (see Sec. VI), referring to earlier work for details. In fact, for Abelian chiral gauge theories, no ghosts are needed if a linear gauge is used, and, using an appropriate lattice transcription of the Lorenz gauge, the construction of Abelian lattice chiral gauge theories was essentially completed before.

The remaining hurdle for non-Abelian theories has little to do with the fermions, and can first be addressed in the setting of pure Yang-Mills theories. The issue is the existence of Gribov copies [24], and the problems which arise if one tries to construct a path integral which sums over copies with a correct weight. There have been varying suggestions on how to tackle this problem. One idea is to sum over copies with a measure that includes the Faddeev-Popov determinant [25]. Unfortunately, it was shown in a rigorous setting that this does not work: Neuberger's theorem [16] asserts that the partition function of such a theory vanishes identically. Another proposal with a manifestly positive measure averages over gauge orbits through a "quenched" scalar field [26], but in this case we lack a symmetry principle such as BRST in order to recover the target continuum theory from the lattice [27].

Here, we develop the idea of Ref. [15], which can be used to fix the gauge symmetry down to the maximal Abelian subgroup, evading Neuberger's theorem while maintaining an "equivariant" BRST symmetry. The remaining Abelian gauge symmetry can then be fixed without the need to introduce ghosts, much in the same way as in our earlier work on Abelian gauge fixing. (This again avoids Neuberger's theorem.) The main content of this paper then consists of two parts: first we explain how to construct the equivariantly gauge-fixed lattice Yang-Mills theory; and second, we explain how this construction may be adapted to accommodate chiral fermions on the lattice. The main goal in the first part (Secs. II, III, IV, and V) is to show that the equivariantly gauge-fixed lattice theory is the same as the non-gauge-fixed theory. The central issue in the second part (Sec. VI) is how to gauge fix the remaining Abelian gauge symmetry on the lattice, and how to obtain the target continuum theory after adding chiral fermions, when the BRST symmetries of the target theory are broken in the regulated theory.

This paper is organized as follows. In Sec. II, we discuss the construction of an equivariantly gauge-fixed Yang-Mills theory in the continuum, extending the results of Ref. [15], and establishing an extended BRST—anti-BRST (equivariant) algebra following Ref. [28]; we do the same on the lattice in Sec. III. We concentrate on the case of interest for $G = SU(N)$ lattice chiral gauge theories, namely, when the equivariant gauge fixing leaves behind the local invariance under the maximal Abelian subgroup $H = U(1)^{N-1}$. We construct a complete path integral with a local Boltzmann weight which defines the equivariantly gauge-fixed lattice theory. In addition to the gauge field, it contains ghost fields taking values in the coset space G/H. In Sec. IV, we review Neuberger's theorem, and prove rigorously that the equivariantly gauge-fixed lattice theory is the same as the lattice theory without any gauge fixing, thus evading the theorem. By "the same" we mean that, at finite lattice spacing, correlation functions of gauge-invariant operators are the same in both theories.

Of course, in order to develop (continuum or lattice) perturbation theory for an equivariantly gauge-fixed Yang-Mills theory with gauge group G, a "second-stage" gauge fixing of the remaining subgroup $H \subset G$ will be required. Since a complete gauge fixing in a renormalizable gauge is needed for our goal, our fully gauge-fixed lattice theory also contains a Lorenz gauge-fixing term for the remaining Abelian subgroup. Here several new issues arise. In Sec. V, as a preparatory step, we address them in the context of continuum perturbation theory, again restricting ourselves to the case $G = SU(N)$, $H = U(1)^{N-1}$. We introduce a yet larger BRST-type algebra involving a new, Abelian H-ghost sector. Since we are now (linearly) gauge fixing an Abelian symmetry, the new ghosts are *free* fields. The equivariant BRST identities of the "first-stage" gauge fixing are modified, but we show that the complete algebraic setup remains sufficiently potent to guarantee unitarity (at least to all orders in perturbation theory). We develop the relevant generalized BRST identities, and, employing these, we work out a detailed example of how unitarity is maintained in perturbation theory.

In Sec. VI we finally turn to the construction of non-Abelian lattice chiral gauge theories. The complete set of

[5]See also Ref. [23].

Slavnov-Taylor identities needed to establish gauge invariance and unitarity of renormalized perturbation theory can evidently be re-derived while omitting the (free!) Abelian ghost terms from the continuum action. The target chiral gauge theory that we latticize is the *fully* gauge-fixed continuum theory without the free Abelian-ghost terms. The definition of the lattice theory includes a lattice version of the pure gauge action (for instance the plaquette action), a chiral-fermion action (for instance that of Sec. VI), the equivariant gauge-fixing action of Sec. III, a Lorenz gauge-fixing term for the remaining Abelian subgroup, an irrelevant term needed for the uniqueness of the classical vacuum, and a counter-term action. We review the mechanism that guarantees the existence of the appropriate critical point, which remains essentially the same as in our previous work on Abelian lattice chiral gauge theories. Since the lattice theory is not gauge invariant, we construct a complete set of counter terms.

We use the concluding section for additional remarks and comments. In particular, we compare our results with those of Refs. [29,30] (which describe an attempt at an exactly gauge-invariant lattice formulation of non-Abelian chiral gauge theories), and discuss future prospects. A number of technical points are relegated to the four Appendices.

II. EQUIVARIANTLY GAUGE-FIXED YANG-MILLS THEORIES—CONTINUUM

In this section we will describe the equivariant gauge fixing of a Yang-Mills theory.[6] Equivariant gauge fixing fixes only part of the gauge group G, leaving a subgroup $H \subset G$ unfixed. The main result of this section is a continuum action invariant under a set of BRST-type transformations satisfying an equivariant, extended BRST—anti-BRST algebra. (This action is also invariant under a few related symmetries.) Unless otherwise stated, the results of this section are valid for any simple, compact G and any (in general, not simple) subgroup $H \subset G$. We work in euclidean space. The Yang-Mills Lagrangian with gauge coupling g is

$$\mathcal{L}_{YM} = \frac{1}{2g^2}\mathrm{tr}(F_{\mu\nu}^2), \qquad iF_{\mu\nu} = [D_\mu(V), D_\nu(V)],$$
$$D_\mu(V) = \partial_\mu + iV_\mu, \qquad V_\mu = V_\mu^a T^a, \tag{2.1}$$

with T^a the hermitian generators of G normalized such that $\mathrm{tr}(T^a T^b) = \frac{1}{2}\delta_{ab}$, and structure constants f_{abc} defined by $[T^a, T^b] = if_{abc}T^c$. The structure constants are fully anti-symmetric in all three indices.

We will now divide the generators into a subalgebra T^i generating the subgroup H, and the rest, T^α, spanning the coset space G/H. Correspondingly, we write the vector field V as

$$V_\mu = V_\mu^a T^a = A_\mu^i T^i + W_\mu^\alpha T^\alpha. \tag{2.2}$$

We will use indices $i, j, k, \ldots$ to indicate H generators, and $\alpha, \beta, \gamma, \ldots$ for generators in G/H. We note that

$$f_{i\alpha j} = -f_{\alpha ij} = -f_{ij\alpha} = 0, \tag{2.3}$$

because the product of two elements of H should again be in H. We choose a gauge-fixing condition which is covariant under H,

$$\mathcal{F}(V) = \mathcal{D}_\mu(A)W_\mu \equiv \partial_\mu W_\mu + i[A_\mu, W_\mu], \tag{2.4}$$

where $\mathcal{D}_\mu(A)$ is a covariant derivative with respect to H. Denoting the algebras of the groups G and H by $\mathcal{G}$ and $\mathcal{H}$, we introduce $\mathcal{G}/\mathcal{H}$-valued ghost fields

$$C = C^\alpha T^\alpha, \qquad \overline{C} = \overline{C}^\alpha T^\alpha, \tag{2.5}$$

along with a coset-valued auxiliary field $b = b^\alpha T^\alpha$, and demand invariance of the gauge-fixed theory under equivariant BRST (eBRST) transformations

$$\begin{aligned} sA_\mu &= i[W_\mu, C]_{\mathcal{H}}, \\ sW_\mu &= \mathcal{D}_\mu(A)C + i[W_\mu, C]_{\mathcal{G}/\mathcal{H}}, \\ sC &= (-iC^2)_{\mathcal{G}/\mathcal{H}} = -iC^2 + X, \qquad s\overline{C} = -ib, \\ sb &= [X, \overline{C}], \end{aligned} \tag{2.6}$$

in which

$$X \equiv (iC^2)_{\mathcal{H}} = 2iT^j\mathrm{tr}(C^2T^j). \tag{2.7}$$

The transformation rule for C is similar to the standard BRST case, but for the extra term X which projects sC back onto the coset space. This modification affects the nilpotency of eBRST transformations. In fact, using that $sX = 0$,

$$s^2C = -i[X, C] = \delta_X C. \tag{2.8}$$

This does not vanish, but equals a gauge transformation (denoted by δ_ω) in H with parameter $\omega = X \in \mathcal{H}$. Requiring $s^2\overline{C} = \delta_X\overline{C}$ determines the eBRST transformation rule for b, after which one verifies that $s^2b = \delta_X b$ as well. The second eBRST variation of any physical field follows from the fact that the standard BRST transformation is nilpotent, so that only the X part in sC leads to a non-vanishing result,[7]

$$\begin{aligned} s^2A_\mu &= \mathcal{D}_\mu(A)X = \delta_X A_\mu, \\ s^2W_\mu &= -i[X, W_\mu] = \delta_X W_\mu. \end{aligned} \tag{2.9}$$

These are again precisely gauge transformations in H, proving that $s^2 = \delta_X$ is equivariantly nilpotent.[8]

Following the standard approach, we would choose as a gauge-fixing Lagrangian

[6] For the case $G = SU(2)$, $H = U(1)$, see Ref. [15].

[7] For Abelian H, $s^2A_\mu = \partial_\mu X$.

[8] It is easy to see that, for the product of any two fields Φ_1 and Φ_2, $s^2(\Phi_1\Phi_2) = (s^2\Phi_1)\Phi_2 + \Phi_1 s^2\Phi_2$.

$$\mathcal{L}'_{gf} = s\ \mathrm{tr}(2\overline{C}\mathcal{F} + i\xi g^2\overline{C}b), \tag{2.10}$$

in which ξ is a parameter. This Lagrangian is invariant under eBRST transformations, as follows from the fact that it is invariant under H transformations and from equivariant nilpotency of s. However, as we will see next, this is not the most general possible gauge-fixing Lagrangian.

It is useful to introduce the concept of anti-eBRST transformations, following Ref. [28]. Denoting anti-eBRST by $\overline{s}$, we have for the gauge fields

$$\begin{aligned}\overline{s}A_\mu &= i[W_\mu, \overline{C}]_{\mathcal{H}},\\ \overline{s}W_\mu &= \mathcal{D}_\mu(A)\overline{C} + i[W_\mu, \overline{C}]_{\mathcal{G}/\mathcal{H}}.\end{aligned} \tag{2.11}$$

Using this and a partial integration, the first term in Eq. (2.10) can be written as

$$s\ \mathrm{tr}(2\overline{C}\mathcal{F}) = -s\overline{s}\ \mathrm{tr}(W^2). \tag{2.12}$$

It is clear that the "pre-potential" $\mathrm{tr}(W^2)$ can be generalized to any H-invariant, rotationally invariant dimension-2 operator with ghost number zero. This allows us to add a term proportional to $\mathrm{tr}(\overline{C}C)$ and we will choose

$$\mathcal{L}_{gf} = -s\overline{s}\ \mathrm{tr}(W^2 + \xi g^2\overline{C}C). \tag{2.13}$$

Obviously, in order to complete the definition of $\mathcal{L}_{gf}$, we have to specify $\overline{s}C$ and $\overline{s}\,\overline{C}$. We will turn to this next.

The pre-potential in Eq. (2.13) is invariant under a discrete flip symmetry[9] on the ghost fields, $FC = \overline{C}$, $F\overline{C} = -C$. (We define $F\Phi = \Phi$ for all physical fields.) This symmetry will be useful later on, and we will define $\overline{s}$ in the ghost sector so as to have $\mathcal{L}_{gf}$ be invariant under this symmetry as well. We thus define $\overline{s}$ in the ghost sector by applying a flip transformation to Eq. (2.6), i.e., by demanding that $\overline{s}F(\text{field}) = Fs(\text{field})$. In addition to Eq. (2.11) this gives

$$\begin{aligned}\overline{s}\,\overline{C} = (-i\overline{C}^2)_{\mathcal{G}/\mathcal{H}} = -i\overline{C}^2 + \overline{X}, \qquad \overline{s}C = i\overline{b},\\ \overline{s}\,\overline{b} = -[\overline{X}, C],\end{aligned} \tag{2.14}$$

where we (temporarily) introduced a new field $\overline{b} \equiv Fb$, and in which

$$\overline{X} = FX = (i\overline{C}^2)_{\mathcal{H}} = 2iT^j\mathrm{tr}(\overline{C}^2T^j). \tag{2.15}$$

The field $\overline{b}$ is not independent if we require that the s, $\overline{s}$ algebra closes on H. While this can be worked out on the gauge fields, it is easier to do it on a matter field Φ in the fundamental representation of G, for which

$$s\Phi = -iC\Phi, \qquad \overline{s}\Phi = -i\overline{C}\Phi. \tag{2.16}$$

One finds that

$$\{s, \overline{s}\}\Phi = (-b + \overline{b} + \{\overline{C}, C\})\Phi. \tag{2.17}$$

[9]This symmetry is related to "ghost hermiticity" of Ref. [28].

Now setting

$$\overline{b} = b - \{\overline{C}, C\}_{\mathcal{G}/\mathcal{H}}, \tag{2.18}$$

Eq. (2.17) becomes an H transformation $\delta_{\tilde{X}}\Phi$ with parameter

$$\tilde{X} = i\{\overline{C}, C\}_{\mathcal{H}} = 2iT^j\mathrm{tr}(\{\overline{C}, C\}T^j). \tag{2.19}$$

We thus end up with the extended eBRST algebra

$$s^2 = \delta_X, \qquad \overline{s}^2 = \delta_{\overline{X}}, \qquad \{s, \overline{s}\} = \delta_{\tilde{X}}. \tag{2.20}$$

Note that $F\{\overline{C}, C\} = -\{\overline{C}, C\}$, so that $F\overline{b} = b$. Equation (2.18) can also be used to work out $\overline{s}b$, using Eq. (2.14). Finally, we have for all fields that

$$Fs = \overline{s}F, \qquad F\overline{s} = -sF. \tag{2.21}$$

The most general eBRST-invariant gauge-fixing Lagrangian would be any linear combination of $\mathcal{L}'_{gf}$ and $\mathcal{L}_{gf}$. However, if we insist on flip symmetry, the only possible choice is $\mathcal{L}_{gf}$, which, in addition to flip symmetry, eBRST symmetry, and H gauge invariance, also has anti-eBRST symmetry. We note that if the coset structure constants $f_{\alpha\beta\gamma}$ are all equal to zero, there is no difference between the two cases, because in that case $\overline{s}\mathrm{tr}(\overline{C}C) = -i\mathrm{tr}(\overline{C}b)$.[10] Our gauge-fixing Lagrangian is thus

$$\mathcal{L}_{gf} = -s\overline{s}\ \mathrm{tr}(W^2 + \xi g^2\overline{C}C) = \overline{s}s\ \mathrm{tr}(W^2 + \xi g^2\overline{C}C), \tag{2.22}$$

where

$$\begin{aligned}\overline{s}s\ \mathrm{tr}(W^2) = &-2\mathrm{tr}(\overline{C}\mathcal{D}_\mu(A)\mathcal{D}_\mu(A)C)\\ &+2\mathrm{tr}([W_\mu, \overline{C}]_{\mathcal{H}}[W_\mu, C]_{\mathcal{H}})\\ &-2i\ \mathrm{tr}(\overline{C}\mathcal{D}_\mu(A)[W_\mu, C]_{\mathcal{G}/\mathcal{H}})\\ &-2i\ \mathrm{tr}(b\mathcal{D}_\mu(A)W_\mu),\\ \overline{s}s\ \mathrm{tr}(\overline{C}C) = &\ \mathrm{tr}(b^2) - \mathrm{tr}(b\{\overline{C}, C\})\\ &+\mathrm{tr}((\overline{C}^2)_{\mathcal{G}/\mathcal{H}}(C^2)_{\mathcal{G}/\mathcal{H}})\\ &+\mathrm{tr}(\{\overline{C}, C\}_{\mathcal{G}/\mathcal{H}})^2 - \mathrm{tr}(\tilde{X}^2).\end{aligned} \tag{2.23}$$

This Lagrangian is invariant under flip symmetry. This can be verified directly from Eq. (2.23), but can also be seen as follows. From Eqs. (2.20) and (2.21), it follows that $Fs\overline{s} = \overline{s}F\overline{s} = -\overline{s}sF = (s\overline{s} - \delta_{\tilde{X}})F$, and thus $s\overline{s}$ commutes with F on any H-invariant expression. Since the pre-potential in Eq. (2.22) is H invariant, it follows that $\mathcal{L}_{gf}$ is invariant under flip symmetry.

We may integrate out the auxiliary field b to arrive at the form

[10]In general the second term in Eq. (2.10) cannot be written as $\overline{s}$(anything). An example with $f_{\alpha\beta\gamma} = 0$ is $G = SU(2)$, $H = U(1)$ [15].

$$\mathcal{L}_{gf} = \frac{1}{\xi g^2}\mathrm{tr}(\mathcal{D}_\mu(A)W_\mu)^2 - 2\,\mathrm{tr}(\overline{C}\mathcal{D}_\mu(A)\mathcal{D}_\mu(A)C) + 2\,\mathrm{tr}([W_\mu,\overline{C}]_{\mathcal{H}}[W_\mu,C]_{\mathcal{H}}) + i\,\mathrm{tr}\Big((\mathcal{D}_\mu(A)\overline{C}) \times[W_\mu,C] + [W_\mu,\overline{C}](\mathcal{D}_\mu(A)C)\Big) + \xi g^2\Big(\mathrm{tr}((\overline{C}^2)_{\mathcal{G}/\mathcal{H}}(C^2)_{\mathcal{G}/\mathcal{H}}) + \frac{3}{4}\mathrm{tr}(\{\overline{C},C\}_{\mathcal{G}/\mathcal{H}})^2 - \mathrm{tr}(\tilde{X}^2)\Big). \quad (2.24)$$

Note that the ghosts' differential operator $\mathcal{M}_{\alpha\beta}$, defined by the part bilinear in $\overline{C}_\alpha$ and C_β, is self-adjoint and real, and therefore symmetric. The on-shell eBRST and anti-eBRST transformation rules for $\overline{C}$ and C can be derived as usual from the equation of motion for b. The quartic ghost interactions are a novel feature, and will play an important role in Sec. IV below.

In this paper we are mainly interested in the application to chiral lattice gauge theories, where we will take $G = SU(N)$ and $H = U(1)^{N-1}$, the maximal Abelian subgroup of $SU(N)$. In this special case, even though the gauge-fixing terms break the original $SU(N)$ gauge symmetry (even the global group), there is a discrete subgroup which, in addition to the maximal Abelian subgroup H, remains a symmetry of the full gauge-fixed action. We first define this group in the fundamental, N-dimensional representation. Since now H is Abelian, we can choose all generators T^i of H to be diagonal,[11] while the remaining generators T^α are off-diagonal. They may be written as

$$T^\alpha \to T^k_{AB}, \qquad k = 1, 2, \qquad 1 \le A < B \le N, \quad (2.25)$$

where T^k_{AB}, $k = 1, 2, 3$, is defined by the requirement that if we keep only the A-th and B-th row and column, this matrix reduces to $\frac{1}{2}\sigma_k$ with all other entries of T^k_{AB} being zero. The "skewed" permutation group $\tilde{S}_N$ is defined as the subgroup of $SU(N)$ generated by the elements

$$\tilde{P}^{\,k}_{(AB)} = \exp(i\pi T^k_{AB}), \qquad k = 1, 2. \quad (2.26)$$

Acting on a vector of length N, this permutes the A-th and B-th entries, and multiplies them by a factor ± 1 or $\pm i$, while leaving the other entries unchanged. The discrete group $\tilde{S}_N$ contains, in particular, the elements $\tilde{P}^3_{(AB)}$ which also belong to H. On gauge fields and ghost-sector fields, the action of $\tilde{P}^k_{(AB)}$ is defined by

$$V_\mu \to \tilde{P}^k_{(AB)} V_\mu (\tilde{P}^k_{(AB)})^\dagger, \qquad C \to \tilde{P}^k_{(AB)} C (\tilde{P}^k_{(AB)})^\dagger, \quad (2.27)$$

and likewise for $\overline{C}$ and b. It is straightforward to check that $V_\mu = A_\mu + W_\mu$ does not transform in an irreducible representation of $\tilde{S}_N$, but that A_μ and W_μ each transform separately (and irreducibly). On the generators of H we have that

$$(\tilde{P}^k_{(AB)})^\dagger T^i \tilde{P}^k_{(AB)} = R^{ij}_{(AB)} T^j, \quad (2.28)$$

with $R_{(AB)}$ an $(N-1)\times(N-1)$ orthogonal matrix. Note that on the T^i, the group $\tilde{S}_N$ just permutes the A-th and B-th diagonal elements of each T^i. $R_{(AB)}$ is thus independent of k being 1 or 2. The pre-potential in Eq. (2.22) clearly is invariant under $\tilde{S}_N$, and thus our equivariantly gauge-fixed Yang-Mills theory is invariant.

Finally, it turns out that the gauge-fixing action (2.22) is invariant under an $SU(2)$ group that acts on the ghost fields C and $\overline{C}$. We will refer to this symmetry as ghost-$SU(2)$.[12] The three generators are

$$\Pi_+ = C_\alpha \frac{\delta}{\delta \overline{C}_\alpha}, \qquad \Pi_- = \overline{C}_\alpha \frac{\delta}{\delta C_\alpha}, \quad (2.29)$$

$$\Pi_3 = C_\alpha \frac{\delta}{\delta C_\alpha} - \overline{C}_\alpha \frac{\delta}{\delta \overline{C}_\alpha}. \quad (2.30)$$

Π_3 is the ghost-number charge. These generators satisfy the same commutation relations as $\sigma_\pm = \frac{1}{2}(\sigma_1 \pm i\sigma_2)$ and σ_3. Under ghost-$SU(2)$, the ghost fields transform as a doublet $(C, \overline{C})$.

The action is evidently invariant under the ghost-number symmetry. Let us establish its invariance under the extended, ghost-$SU(2)$ symmetry. The invariance of the terms bilinear in the ghost fields follows, as in the case of flip symmetry, from the fact that the operator $\mathcal{M}_{\alpha\beta}$ is symmetric. Turning to the on-shell four-ghost action, we may rewrite it as

$$(\xi g^2)^{-1}\mathcal{L}^{(4)}_{gh} = \mathrm{tr}\Big(-\overline{C}^2C^2 - \frac{1}{4}\{\overline{C},C\}^2 + \overline{X}X - \frac{1}{4}\tilde{X}^2\Big) = -\frac{1}{3}\mathrm{tr}\Big(\overline{C}^2C^2 - \frac{1}{4}\{\overline{C},C\}^2\Big) + \mathrm{tr}\Big(\overline{X}X - \frac{1}{4}\tilde{X}^2\Big). \quad (2.31)$$

Each trace on the last row is invariant under ghost-$SU(2)$. We used (anti)cyclicity of the trace of the product of four ghost fields, from which it follows that $\mathrm{tr}(\overline{C}^2C^2) = -\frac{1}{2}\mathrm{tr}(\{\overline{C},C\}^2)$. In the off-shell formalism, the auxiliary field transforms under ghost-$SU(2)$ as $\delta b_\alpha = \frac{i}{2}\delta(f_{\alpha\beta\gamma}\overline{C}_\beta C_\gamma)$.

[11] For instance, for $SU(2)$, $T^i \in \{\sigma_3/2\}$ with σ_k the Pauli matrices, and for $SU(3)$, $T^i \in \{\lambda_3/2, \lambda_8/2\}$, with λ_a the Gell-Mann matrices.

[12] This symmetry exists for any G and H. In Ref. [15] it is referred to as $SL(2,R)$ symmetry.

III. EQUIVARIANTLY GAUGE-FIXED YANG-MILLS THEORIES—LATTICE

In this section we show how the continuum theory constructed in the previous section can be transcribed to the lattice without any loss of symmetries (except rotational symmetry). We limit the discussion to $G = SU(N)$, $H = U(1)^{N-1}$ from now on.[13] First, the eBRST and anti-eBRST transformations of the gauge field $V_\mu = A_\mu + W_\mu$ can be summarized as

$$sV_\mu = D_\mu(V)C, \qquad \overline{s}V_\mu = D_\mu(V)\overline{C}, \tag{3.1}$$

where the derivative is covariant with respect to the full gauge group G, $D_\mu(V)C = \partial_\mu C + i[V_\mu, C]$ (cf. Eq. (2.1)). The lattice version of these transformation rules is, with $U_{x,\mu} = \exp[iV_\mu(x)]$,

$$\begin{aligned} sU_{x,\mu} &= i(U_{x,\mu}C_{x+\mu} - C_x U_{x,\mu}), \\ \overline{s}U_{x,\mu} &= i(U_{x,\mu}\overline{C}_{x+\mu} - \overline{C}_x U_{x,\mu}). \end{aligned} \tag{3.2}$$

For the Yang-Mills Lagrangian, Eq. (2.1), we will assume the usual plaquette action. For the gauge-fixing action, we choose the lattice version

$$\begin{aligned} \mathcal{L}^L_{gf} &= -s\overline{s}\sum_\mu\sum_i(-2\mathrm{tr}(T^iU_{x,\mu}T^iU^\dagger_{x,\mu}) + \xi g^2\mathrm{tr}(\overline{C}_xC_x)) \\ &= -s\overline{s}\left(\frac{1}{2}\sum_\mu\sum_\alpha W^\alpha_{x,\mu}W^\alpha_{x,\mu} + \xi g^2\mathrm{tr}(\overline{C}_xC_x) + O(V^4)\right), \end{aligned} \tag{3.3}$$

which is H invariant (recall that we take H to be Abelian). In the second line, we used that

$$\sum_{i\gamma} f_{i\gamma\alpha}f_{i\gamma\beta} = \delta_{\alpha\beta}. \tag{3.4}$$

For the lattice construction of chiral gauge theories, or in order to develop weak-coupling perturbation theory, the remaining Abelian gauge symmetry (H) will also need to be fixed. We will return to this in the sections to follow.

We will now work out the lattice equivalent of Eq. (2.24). First define

$$\mathcal{W}_{x,\mu} = -i\sum_i[U_{x,\mu}T^iU^\dagger_{x,\mu}, T^i] = W_{x,\mu} + O(V^2), \tag{3.5}$$

and lattice covariant derivatives $D^\pm_\mu$ by

$$\begin{aligned} D^+_\mu\Phi_x &= U_{x,\mu}\Phi_{x+\mu}U^\dagger_{x,\mu} - \Phi_x, \\ D^-_\mu\Phi_x &= \Phi_x - U^\dagger_{x-\mu,\mu}\Phi_{x-\mu}U_{x-\mu,\mu}. \end{aligned} \tag{3.6}$$

Before continuing, we note that both $\mathcal{W}_{x,\mu}$ and $D^-_\mu\mathcal{W}_{x,\mu} = \mathcal{W}_{x,\mu} - U^\dagger_{x-\mu,\mu}\mathcal{W}_{x-\mu,\mu}U_{x-\mu,\mu}$ live in the coset space $\mathcal{G}/\mathcal{H}$. For $\mathcal{W}_{x,\mu}$ this follows from

$$\mathrm{tr}([U_{x,\mu}T^iU^\dagger_{x,\mu}, T^i]T^j) = \mathrm{tr}(U_{x,\mu}T^iU^\dagger_{x,\mu}[T^i, T^j]) = 0.$$

For the second term in $D^-_\mu\mathcal{W}_{x,\mu}$, this follows from

$$\begin{aligned} &\mathrm{tr}(T^jU^\dagger_{x,\mu}[U_{x,\mu}T^iU^\dagger_{x,\mu}, T^i]U_{x,\mu}) = \\ &\mathrm{tr}([U_{x,\mu}T^jU^\dagger_{x,\mu}, U_{x,\mu}T^iU^\dagger_{x,\mu}]T^i) = 0. \end{aligned}$$

The off-shell gauge-fixing lattice action can now be expressed as

$$\begin{aligned} \mathcal{L}^L_{gf} = {}& 2\ \mathrm{tr}([T_i, D^+_\mu C_x][U_{x,\mu}T_iU^\dagger_{x,\mu}, D^+_\mu\overline{C}_x]) \\ &-2i\ \mathrm{tr}(b_xD^-_\mu\mathcal{W}_{x,\mu}) - 2i\ \mathrm{tr}(\{D^+_\mu C_x, \overline{C}_x\}\mathcal{W}_{x,\mu}) \\ &+\xi g^2\overline{s}s\ \mathrm{tr}(\overline{C}_xC_x), \end{aligned} \tag{3.7}$$

where $\overline{s}s\ \mathrm{tr}(\overline{C}_xC_x)$ is still given by Eq. (2.23).

Integrating out the auxiliary field we then find that on the lattice $\mathrm{tr}\mathcal{F}^2/(\xi g^2)$ (cf. Eqs. (2.4) and (2.24)) is replaced by

$$\mathcal{L}_{G/H} \equiv \frac{1}{\xi g^2}\mathrm{tr}(D^-_\mu\mathcal{W}_{x,\mu})^2. \tag{3.8}$$

It is simple to check that $D^-\mathcal{W}_{x,\mu}$ transforms covariantly under H. It follows that in the classical continuum limit $D^-_\mu\mathcal{W}_{x,\mu} \to D_\mu(V)W_\mu(x) = \mathcal{D}_\mu(A)W_\mu(x)$.

The ghost part of the Lagrangian derived from Eq. (3.3) is

$$\begin{aligned} \mathcal{L}_{\mathrm{ghost}} = {}& 2\ \mathrm{tr}([T^i, D^+_\mu C_x][U_{x,\mu}T^iU^\dagger_{x,\mu}, D^+_\mu\overline{C}_x]) \\ &-i\ \mathrm{tr}(2\{D^+_\mu C_x, \overline{C}_x\}\mathcal{W}_{x,\mu} + \{\overline{C}_x, C_x\}D^-_\mu\mathcal{W}_{x,\mu}) \\ &+\xi g^2\Big(\mathrm{tr}((\overline{C}^2)_{\mathcal{G}/\mathcal{H}}(C^2)_{\mathcal{G}/\mathcal{H}}) \\ &+\frac{3}{4}\mathrm{tr}(\{\overline{C}, C\}_{\mathcal{G}/\mathcal{H}})^2 - \mathrm{tr}(\tilde{X}^2)\Big). \end{aligned} \tag{3.9}$$

It is straightforward to verify that the ghost part of Eq. (2.24) is recovered in the classical continuum limit, by replacing first $U_{x,\mu}T^iU^\dagger_{x,\mu} \to T^i$, and then using the relation

$$\sum_i \mathrm{tr}([T^i, A][T^i, B]) = -\mathrm{tr}(A_{\mathcal{G}/\mathcal{H}}B_{\mathcal{G}/\mathcal{H}}), \tag{3.10}$$

which follows from relation (3.4).

An alternative lattice gauge-fixing action is given by

$$\mathcal{L}'^L_{gf} = -s\overline{s}\ \mathrm{tr}(\mathcal{W}^2 + \xi g^2\overline{C}C). \tag{3.11}$$

This action has the same classical continuum limit, and both lattice actions are invariant under flip symmetry (as can be seen most easily from their definition as the eBRST and anti-eBRST variations of a flip-invariant pre-potential).

There are several further remarks we wish to make before concluding this section. First, it is straightforward to check that, in both cases, the free kinetic term for the

[13]There exist straightforward generalizations of the lattice action(s) to other subgroups. For $N = 2$, see Ref. [15].

ghost fields contains a nearest-neighbor discretization of the laplacian. This implies that no species doubling occurs in the ghost sector on the lattice.

Second, both lattice actions are still invariant under the discrete $SU(N)$ subgroup $\tilde{S}_N$ introduced in the previous section. Under $\tilde{S}_N$, the lattice gauge field transforms as

$$U_\mu \to \tilde{P}^k_{(AB)} U_\mu (\tilde{P}^k_{(AB)})^\dagger, \tag{3.12}$$

consistent with Eq. (2.27). The invariance of Eq. (3.3) follows from Eq. (2.28). Finally, the lattice actions are invariant under ghost-$SU(2)$.

IV. EVADING NEUBERGER'S THEOREM

It has been shown that a gauge-fixed Yang-Mills theory with conventional BRST symmetry is not well-defined non-perturbatively [16]. With "non-perturbative" we refer to a lattice definition of the theory which maintains exact BRST symmetry. The theorem states that the partition function of such a theory, as well as the (un-normalized) expectation value of any gauge-invariant operator, vanish identically. What we wish to demonstrate in this section is that equivariant gauge fixing as described in the previous sections circumvents this problem [15].

It is instructive to review the proof of the theorem first, in order to see exactly what changes the conclusion in the equivariant case; the key ingredient is the presence of four-ghost terms in the gauge-fixing Lagrangian (2.22).

In the standard case, the lattice partition function can be written as $Z_{\mathrm{BRST}}(1)$ where

$$Z_{\mathrm{BRST}}(t) = \int [dU][db][dc][d\overline{c}] e^{-S_{inv}(U) - S_{gf}(t;U,c,\overline{c},b)}. \tag{4.1}$$

Here S_{inv} is gauge invariant and depends only on the physical fields. dU is the Haar measure on G, and with the notation [ ] we indicate products over all sites and links (for the gauge fields) or group indices (for ghost and auxiliary fields). S_{inv} may include source terms for gauge-invariant operators. The gauge-fixing term

$$S_{gf}(t) = \sum_x (t\hat{s}\ \mathrm{tr}(2\overline{c}\mathcal{F}(U)) + \xi g^2 \mathrm{tr}(b^2)) \tag{4.2}$$

is a function of the standard ghost fields c^a and $\overline{c}^a$, with one pair for each generator of G, and an auxiliary field b^a for each generator as well. Following Ref. [16], we introduced a parameter t in front of the first term in S_{gf}. Standard BRST transformations are

$$\hat{s}U_{x\mu} = i(U_{x,\mu}c_{x+\mu} - c_x U_{x,\mu}), \qquad \hat{s}c = -ic^2, \\ \hat{s}\,\overline{c} = -ib, \qquad \hat{s}b = 0, \tag{4.3}$$

and invariance of the action follows immediately from the fact that $\mathrm{tr}(b^2) = \hat{s}\ \mathrm{tr}(i\overline{c}b)$ and nilpotency of $\hat{s}$, $\hat{s}^2 = 0$.

The proof of Neuberger's theorem now follows very easily. First one observes that

$$dZ_{\mathrm{BRST}}/dt = -\langle \hat{s}\ \mathrm{tr}(2\overline{c}\mathcal{F}(U))\rangle_{\text{un-normalized}} = 0, \tag{4.4}$$

because of BRST invariance. For $t = 0$ the integral is well defined on the (finite-volume) lattice because of the compactness of the gauge-fields, and it follows that $Z(1) = Z(0) = 0$. The latter equality follows immediately from the Grassmann integration rules because the integrand for $Z(0)$ does not contain any ghosts.

Turning to the equivariant case, the path integral can be written as $Z_{\mathrm{eBRST}}(1)$ where

$$Z_{\mathrm{eBRST}}(t) = \int [dU][db][dC][d\overline{C}] e^{-S_{inv}(U) - S^L_{gf}(t;U,C,\overline{C},b)}. \tag{4.5}$$

Now there are only ghost, anti-ghost, and auxiliary fields C^α, $\overline{C}^\alpha$ and b^α for the coset generators T^α in $\mathcal{G}/\mathcal{H}$. The gauge-fixing part now corresponds to Eq. (3.3), and can be written as

$$S^L_{gf}(t) = \sum_x (ts\ \mathrm{tr}(2\overline{C}\mathcal{F}(U)) - \xi g^2 s\overline{s}\ \mathrm{tr}(\overline{C}C)). \tag{4.6}$$

Again, we have that $dZ_{\mathrm{eBRST}}/dt = 0$, because of eBRST invariance. But now $Z_{\mathrm{eBRST}}(0) \neq 0$, due to the presence of ghost fields in the term proportional to ξ in S^L_{gf}. Note that we *cannot* take $\xi \to 0$, because the b integrals do not converge in that limit. For $t = 0$, we find that

$$Z_{\mathrm{eBRST}}(1) = Z_{\mathrm{eBRST}}(0) = \int [dU] e^{-S_{inv}(U)} \int [db][dC][d\overline{C}] \times \exp\left[\xi g^2 \sum_x s\overline{s}\ \mathrm{tr}(\overline{C}C)\right]. \tag{4.7}$$

For $t = 0$ the ghosts are decoupled from the lattice gauge field and, moreover, the ghosts' partition function factorizes as the product of independent single-site integrals. The single-site ghost integral in this expression can be simplified further. Going back to Eq. (2.23), this integral may be written as $Z_{\mathrm{ghost}}(\xi g^2, 1)$ where

$$Z_{\mathrm{ghost}}(\xi g^2, t') = \int db\ dC\ d\overline{C}\ \exp[i\xi g^2 \overline{s}\ \mathrm{tr}(bC - t'\overline{C}C^2)], \tag{4.8}$$

where we have introduced another parameter t'. Using anti-eBRST invariance, we see that $\partial Z_{\mathrm{ghost}}/\partial t' = \partial Z_{\mathrm{ghost}}/\partial \xi = 0$, and thus that

$$Z_{\mathrm{ghost}}(\xi g^2, 1) = Z_{\mathrm{ghost}}(1, 0) = \int db\ dC\ d\overline{C}\ \exp[i\overline{s}\ \mathrm{tr}(bC)] = \int db\ dC\ d\overline{C}\ \exp[-\mathrm{tr}(b^2 - \tilde{X}^2)]. \tag{4.9}$$

Using this result, the rest of the proof that $Z_{\mathrm{ghost}}(1, 0) > 0$ is technical, and is relegated to Appendix A.

We see that in the equivariant case, the ghost-field integral does not vanish, and thus the full path integral does not vanish. The underlying reason for the difference with the standard case is that $\mathrm{tr}(b^2)$ itself cannot be written as the eBRST variation of anything, because $s\ \mathrm{tr}(b^2) \neq 0$. In order to build an eBRST invariant action, four-ghost terms are needed, and these render the ghost integral non-zero. It follows that the equivariantly gauge-fixed partition function, $Z_{\mathrm{eBRST}}(1)$, is equal—up to a non-zero multiplicative constant $(Z_{\mathrm{ghost}}(1, 0))^V$ where V is the volume in lattice units—to the partition function without gauge fixing, which gives the standard lattice definition of a Yang-Mills theory. We recall that S_{inv} may contain source terms for any gauge-invariant operator constructed out of the link variables $U_{x,\mu}$. Therefore, we see that the equivariantly gauge-fixed partition function generates gauge-invariant correlation functions which are rigorously equal to those generated by the non-gauge-fixed theory, for any finite volume and any finite lattice spacing.

It is instructive to restate this result in a somewhat different way. Performing a gauge transformation on $U_{x,\mu}$,

$$U^{\phi}_{x,\mu} = \phi_x U_{x,\mu} \phi^{\dagger}_{x+\mu}, \tag{4.10}$$

and multiplying the partition function by

$$1 = \int [d\phi], \tag{4.11}$$

(again $d\phi$ is the normalized Haar measure) the partition function may be written as

$$\begin{aligned} Z_{\mathrm{eBRST}}(t) &= \int [dU] Z_{\mathrm{orbit}}(t; U) e^{-S_{inv}}, \\ Z_{\mathrm{orbit}}(t; U) &= \int [d\phi][db][dC][d\overline{C}] e^{-S^L_{gf}(t; U^{\phi}, C, \overline{C}, b)}. \end{aligned} \tag{4.12}$$

S^L_{gf} is now invariant under the "orbit" eBRST transformations

$$\begin{aligned} s\phi = -iC\phi, \qquad sU = 0, \qquad sC = (-iC^2)_{\mathcal{G}/\mathcal{H}}, \\ s\overline{C} = -ib, \qquad sb = [X, \overline{C}]. \end{aligned} \tag{4.13}$$

Note that s is the same eBRST transformation as before, but the transformation of the gauge fields is now "carried" by the G-valued field ϕ. (The anti-eBRST rules may again be obtained by applying a flip transformation.) Equation (4.7) can now be restated by observing that Z_{orbit} is independent of the gauge field. This can again be seen by varying the parameter t (cf. Eq. (4.6)). It follows that $dZ_{\mathrm{orbit}}/dt = 0$, and thus $Z_{\mathrm{orbit}}(t; U) = Z_{\mathrm{orbit}}(0; U)$, the latter being independent of U. In other words, $Z_{\mathrm{orbit}}(t; U)$ is a topological field theory.

One of the consequences is that the equivariantly gauge-fixed theory is unitary if the non-gauge-fixed theory is. This is a rigorous result, as all manipulations in this section are valid for the lattice path integral in a finite volume.

V. PERTURBATIVE UNITARITY

While the results obtained in the previous two sections are interesting in their own right, our aim is to apply them to the construction of lattice chiral gauge theories. For this goal, the gauge group $G = SU(N)$ will need to be fixed completely and non-perturbatively: a complete gauge-fixing action will have to be included in the definition of the lattice theory.

Gauge fixing of the remaining subgroup H is, of course, also needed if one wishes to develop perturbation theory for any (continuum or lattice) equivariantly gauge-fixed theory. In the case at hand, the subgroup $H = U(1)^{N-1}$ is Abelian, and significant simplification occurs. In order to fix an Abelian invariance, only a simple gauge-fixing term like $\mathcal{L}_L = (1/2\alpha)\sum_i(\partial_\mu A^i_\mu)^2$ is necessary. There is no need to introduce any new ghost fields. The addition of $\mathcal{L}_L$ does break eBRST invariance, and this raises the issue of unitarity. Slavnov-Taylor identities, derived from (e)BRST invariance, are a key ingredient in the study of unitarity. Therefore one has to re-establish unitarity in the presence of $\mathcal{L}_L$.

Here we will address this question in (continuum) perturbation theory. The conclusion is that, as expected, H gauge fixing does not spoil the unitarity of the theory. Heuristically, this is easy to understand. The eBRST version of Yang-Mills theory is rigorously the same as the non-gauge-fixed version, if one restricts oneself to the physical sector, i.e., to gauge-invariant correlation functions of operators built only from the physical fields. Gauge-fixing either theory completely in order to develop perturbation theory (which for the eBRST version implies only fixing H) should not change this result.

In the context of perturbation theory, it is convenient to introduce an $\mathcal{H}$-ghost sector. As we will see, in the continuum the new ghosts are free fields that merely serve as a device to generate the relevant Slavnov-Taylor identities. Since they should decouple anyway in the continuum limit, no $\mathcal{H}$-ghosts will be introduced in the actual lattice construction of chiral gauge theories. (The eBRST and H-BRST identities of the target continuum theory are sufficient to determine the lattice counter terms to all orders; see also Ref. [31].) As for the exactly eBRST-invariant (and H un-gauge fixed) lattice theory defined in Sec. III, the H-gauge-fixing sector is an extraneous analytic device used to set up perturbation theory, in the same way as gauge fixing is needed to set up perturbation theory for the standard, fully gauge invariant lattice Yang-Mills theory.

Emphasizing again that this is only done for perturbative investigations, we introduce $\mathcal{H}$ ghost fields χ^i, $\overline{\chi}^i$ and auxiliary field β^i, and define $\mathcal{H}$-BRST transformation rules

$$\begin{aligned} s_H\Psi = \delta_\chi \Psi, \qquad s_H A^i_\mu = \partial_\mu \chi^i, \\ s_H\chi^i = 0, \qquad s_H\overline{\chi}^i = -i\beta^i, \qquad s_H\beta^i = 0, \end{aligned} \tag{5.1}$$

where Ψ stands for any of the fields V_μ (or U_μ), C, $\overline{C}$ and b. Under eBRST transformations, the new fields χ^i, $\overline{\chi}^i$ and β^i are invariant by definition. With

$$\mathcal{K} = \int d^4x \Big(\overline{\chi}^i (\partial_\mu A^i_\mu) + \frac{i}{2} \alpha g^2 \overline{\chi}^i \beta^i \Big), \qquad (5.2)$$

the additional (off-shell) gauge-fixing action can now be written as

$$S_L = s_H \mathcal{K} = \int d^4x \Big(-i\beta^i \partial_\mu A^i_\mu - \overline{\chi} \Box \chi + \frac{\alpha g^2}{2} \beta^2 \Big), \qquad (5.3)$$

where α is a gauge-fixing parameter not necessarily equal to ξ (cf. Eqs. (2.10) and (2.13)). A consequence of having s vanish when acting on any $\mathcal{H}$-ghost-sector field is that [15]

$$\{s, s_H\} = 0. \qquad (5.4)$$

This makes it possible to "port" eBRST identities to the fully gauge-fixed theory. In the equivariantly but not H-gauge-fixed theory, the eBRST identities of interest are of the form

$$\langle s\mathcal{O} \rangle = 0, \qquad (5.5)$$

with $\mathcal{O}$ any operator invariant under H gauge symmetry, i.e., operators for which $s_H \mathcal{O} = 0$. In the fully gauge-fixed theory, in which S_L is added, we have that

$$\langle s\mathcal{O} \rangle = -\langle \mathcal{O} s s_H \mathcal{K} \rangle = \langle \mathcal{O} s_H s \mathcal{K} \rangle = 0. \qquad (5.6)$$

The last step follows because $\mathcal{O}$ is invariant under s_H.[14] This proves that the same eBRST identities hold in the fully gauge-fixed theory as well. In what follows below we will also have use for the case that $\mathcal{O}$ is not invariant under H, in which case we obtain

$$\begin{aligned} \langle s\mathcal{O} \rangle &= \langle (s_H \mathcal{O})(s\mathcal{K}) \rangle \\ &= -\Big\langle (s_H \mathcal{O}) \int d^4x f_{i\alpha\beta} (\partial_\mu \overline{\chi}^i) W^\alpha_\mu C^\beta \Big\rangle. \end{aligned} \qquad (5.7)$$

Of course, in the fully gauge-fixed theory one also has Slavnov-Taylor identities derived from s_H, and they play a role in proving unitarity as well.

A comment on the appearance of $\overline{\chi}^i$ in the latter identity is in order. First, if $s_H \mathcal{O}$ does not contain the field χ^i, $\langle s\mathcal{O} \rangle$ vanishes identically. Otherwise, we may carry out the contractions and replace $\langle \chi^i \overline{\chi}^j \rangle$ by its (tree-level) propagator, because χ^i and $\overline{\chi}^i$ are free fields. Indeed, we need not have introduced the H ghosts into the theory; they are just a convenient vehicle for deriving the desired Slavnov-Taylor identities. In the remainder of this section we will keep χ^i and $\overline{\chi}^i$ as a "book-keeping device." They will not be part of our definition of lattice chiral gauge theories in the next section.

As a first application we will prove that all scattering amplitudes are independent of the gauge-fixing parameters ξ and α to all orders in perturbation theory. To do this, it is convenient to use the formalism in which the auxiliary fields b (cf. Eq. (2.22)) and β (cf. Eq. (5.3)) are kept. We show in Appendix B that Feynman rules can be consistently formulated in this framework. Consider the ξ dependence of the expectation value of a (commuting) operator $\mathcal{O}$,

$$\begin{aligned} \frac{1}{g^2} \frac{d}{d\xi} \langle \mathcal{O} \rangle &= \Big\langle \mathcal{O} \int d^4x \; s\overline{s}(\overline{C}C) \Big\rangle \\ &= -\Big\langle (s\mathcal{O}) \int d^4x \overline{s}(\overline{C}C) \Big\rangle \\ &\quad + \Big\langle \mathcal{O}(s s_H \mathcal{K}) \int d^4x \overline{s}(\overline{C}C) \Big\rangle \\ &= -\Big\langle (s\mathcal{O}) \int d^4x \overline{s}(\overline{C}C) \Big\rangle \\ &\quad + \Big\langle (s_H \mathcal{O})(s\mathcal{K}) \int d^4x \overline{s}(\overline{C}C) \Big\rangle, \end{aligned} \qquad (5.8)$$

where the last equality follows from Eq. (5.4) and the fact that $\overline{C}C$ is H-invariant. Now if we take $\mathcal{O}$ to be the product of matter fields, such as quark and anti-quark fields q and $\overline{q}$, we have that both sq and $s_H q$, as well as $s\overline{q}$ and $s_H \overline{q}$, are non-linear in these fields. Therefore, the right-hand side of Eq. (5.8) vanishes if we analytically continue to Minkowski space, amputate the fermion legs and put them on shell. In Appendix C we generalize the argument to scattering amplitudes containing gauge bosons on the external legs. We conclude that all scattering amplitudes are independent of ξ, as should be the case. A similar, even simpler, argument shows α independence as well.

We comment in passing that invariance under a continuous change of parameters in the gauge-fixing action has played an important role in the preceding section too (see e.g. Eq. (4.7)). The conclusions of the previous section are stronger because the non-perturbative setup allows us to set to zero many terms in the gauge-fixing action, while

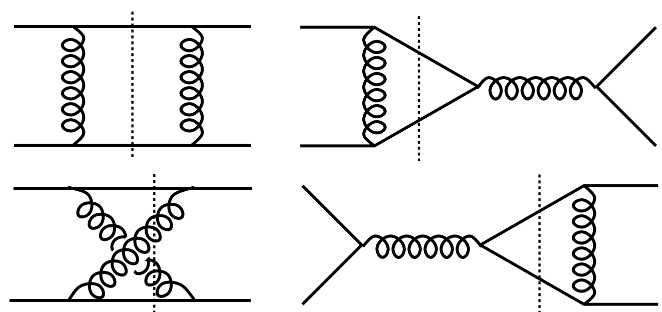

FIG. 1. Cut diagram with two fermions in the intermediate state. Solid lines denote fermions, curly lines denote gluons.

[14]We assumed that the operator $\mathcal{O}$ is anti-commuting; if it is commuting, $s\mathcal{O}$ is anti-commuting, and thus $\langle s\mathcal{O} \rangle = 0$ trivially.

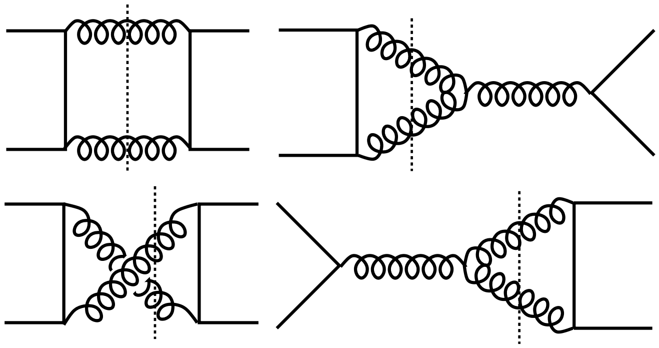

FIG. 2. Cut diagrams with two gauge bosons in the intermediate state.

maintaining the extended eBRST invariance of the theory.

In order to make the discussion less abstract, we will now work out an example that shows in more detail how eBRST and H-BRST identities play a role in proving unitarity in continuum perturbation theory.[15] With the above result in hand, we choose to work in the Feynman gauge, i.e., we now set $\xi = \alpha = 1$, thus simplifying our calculations. Our aim is to demonstrate how unitarity works *vis-à-vis* the unconventional gauge-fixing procedure introduced in this paper. Thus, we will limit ourselves to the difference between the eBRST case and the standard case.

We will consider a two-flavor vector gauge theory with $G = SU(2)$, $H = U(1) \subset G$. The gauge-sector Lagrangian will be the sum of Eqs. (2.1), (2.22), and (5.3). We add to this an $SU(2)$ doublet of massless quarks, $q = (u, d)$. For $SU(2)/U(1)$ one has $f_{\alpha\beta\gamma} = 0$, and Eq. (2.22) simplifies to

$$\mathcal{L}_{gf} = -2\,\mathrm{tr}(\bar{C}\mathcal{D}_\mu(A)\mathcal{D}_\mu(A)C) + 2\,\mathrm{tr}([W_\mu, \bar{C}][W_\mu, C]) - 2i\,\mathrm{tr}(b\mathcal{D}_\mu(A)W_\mu) + \xi g^2(\mathrm{tr}(b^2) - \mathrm{tr}(\tilde{X}^2)). \tag{5.9}$$

As usual, we will replace $A \rightarrow gA$ and $W \rightarrow gW$ in order to develop the perturbation expansion in g.

The main example we wish to discuss is that of $d\bar{u} \rightarrow d\bar{u}$ scattering at one loop, in order to demonstrate that this amplitude satisfies the optical theorem.[16] There are three types of contribution at one loop, shown in Figs. 1–3. Figure 1 contains diagrams with two-quark intermediate states, Fig. 2 contains diagrams with two-gauge-boson intermediate states, and Fig. 3 contains diagrams with one-loop vacuum polarization corrections to the one-gauge-boson intermediate state.

[15]We defer the discussion of the lattice and of chiral fermions to the next section.

[16]For a textbook discussion in standard Feynman gauge, see for instance Ref. [32].

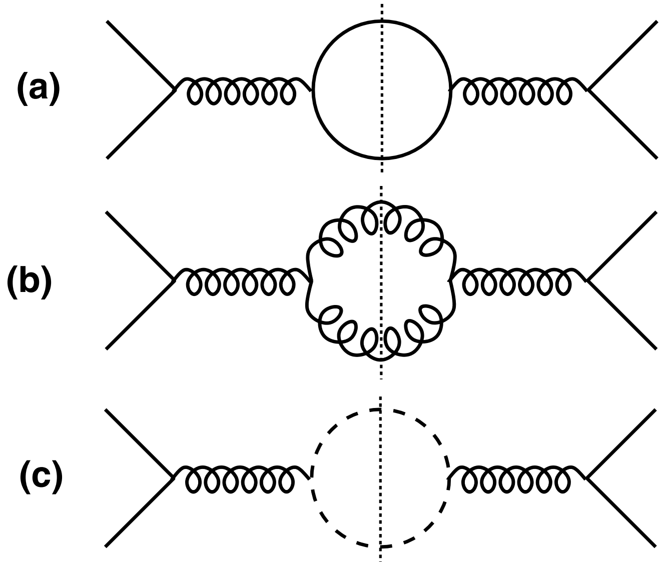

FIG. 3. Cut diagrams containing a vacuum polarization subdiagram. Dashed lines denote ghosts.

We will only be interested in contributions of the type of Figs. 2, 3(b), and 3(c), with only gauge boson and ghost contributions to the vacuum polarization. The diagrams of Fig. 1 and fermionic contributions to the vacuum polarization [Fig. 3(a)] satisfy unitarity constraints separately, and work just like in the standard case. In the case of $u\bar{d}$ scattering, the two-gauge-boson intermediate state is an A–W^- intermediate state, with $A_\mu = V_\mu^3$ and $W_\mu^\mp = (V_\mu^1 \pm iV_\mu^2)/\sqrt{2}$ in terms of the original $SU(2)$ gauge fields. Note that A is neutral under $H = U(1)$, while $W^\pm$ are charged. Since there are no (interacting) neutral ghosts, it follows immediately that there is no ghost-loop contribution to the vacuum polarization of $W^\pm$, which means that Fig. 3(c) is absent for $d\bar{u} \rightarrow d\bar{u}$. Therefore, when we cut the diagrams (as indicated in the figures), contributions from unphysical polarizations of the gauge bosons will have to cancel by themselves, without "help" from any ghost-loop diagrams. This is where our example differs from the standard case of a linear covariant gauge, in which of course ghosts corresponding to every component of the gauge field are present. In our case, the gauge fixing takes place in two stages, of which the latter is an Abelian linear gauge fixing, for which no ghosts are needed.[17]

The optical theorem relates the cut diagrams of Figs. 2 and 3(b) to the process $u\bar{d} \rightarrow AW^-$ (with both A and W^- transversely polarized). If we consider only the contributions from physical polarizations of intermediate A–W^- states to the cut diagrams, the optical theorem is evidently satisfied. Thus, our aim here is to show how

[17]Or, equivalently in perturbation theory, the corresponding ghosts are free, as manifest in Eq. (5.3).

the eBRST identities of Eq. (5.6) plus the BRST identities derived from s_H guarantee that the contributions from all unphysical polarizations to the cut diagram vanish.

The relevant eBRST identity is Eq. (5.7) with $\mathcal{O} = \overline{C}^-(x)A_\nu(y)u(v)\overline{d}(w)$. Since we are interested in the LSZ-amputated correlation function, the only terms which contribute in Eq. (5.7) are those from the linear terms in the eBRST and H-BRST transformation rules. Analytically continuing to Minkowski space, we thus obtain the identity (in Feynman gauge)

$$\begin{aligned}\langle \partial^\mu W^-_\mu(x)A_\nu(y)u(v)\overline{d}(w)\rangle_{os} &= \int d^4z\langle \overline{C}^-(x)\partial_\nu\chi(y)u(v)\overline{d}(w)\partial^\mu\overline{\chi}(z)W^-_\mu(z)C^+(z)\rangle_{os}\\ &= \int d^4z\langle\partial_\nu\chi(y)\partial^\mu\overline{\chi}(z)\rangle\langle\overline{C}^-(x)u(v)\overline{d}(w)W^-_\mu(z)C^+(z)\rangle_{os},\end{aligned} \tag{5.10}$$

where the subscript os means that all external lines have been put on shell. In the last step we made use of the fact that χ and $\overline{\chi}$ are free. Fourier transforming on x and y, this can be rewritten as

$$k^\mu\langle W^-_\mu(k)A_\nu(\ell)u(v)\overline{d}(w)\rangle_{os} = \ell_\nu\int d^4z\langle\chi(\ell)\partial^\mu\overline{\chi}(z)\rangle\langle\overline{C}^-(k)u(v)\overline{d}(w)W^-_\mu(z)C^+(z)\rangle_{os}. \tag{5.11}$$

We will also need an s_H identity for the longitudinal part of A. For this we take $\mathcal{O}' = W^-_\mu(x)\overline{\chi}(y)u(v)\overline{d}(w)$. From $\langle s_H\mathcal{O}'\rangle = 0$ we obtain

$$\ell^\nu\langle W^-_\mu(k)A_\nu(\ell)u(v)\overline{d}(w)\rangle = 0. \tag{5.12}$$

Cutting the internal W line replaces the pole $1/(k^2 + i\epsilon)\to\delta(k^2)$ and leaves a tensor $g_{\mu\rho}$ which may be expressed as a polarization sum

$$g_{\mu\rho} = -\sum_{n=1,2}\epsilon^{(n)}_\mu\epsilon^{(n)}_\rho + \epsilon^+_\mu\epsilon^-_\rho + \epsilon^-_\mu\epsilon^+_\rho \tag{5.13}$$

(similar statements apply to the cut A line). For a physical momentum $k_\mu = (\vec{k}, |\vec{k}|)$,[18] the forward (or longitudinal) and backward polarization vectors are defined as

$$\epsilon^+_\mu(\vec{k}) = \frac{1}{\sqrt{2}}(\hat{k}, 1) = \frac{1}{\sqrt{2}|\vec{k}|}k_\mu, \qquad \epsilon^-_\mu(\vec{k}) = \frac{1}{\sqrt{2}}(-\hat{k}, 1), \tag{5.14}$$

where $\hat{k}$ is the unit vector in the direction of $\vec{k}$. The normalized transverse polarization vectors $\epsilon^{(n)}$ have $\epsilon^{(n)}_4 = 0$, and are orthogonal to $\epsilon^\pm$.

We are now ready to prove unitarity of our example. Cutting the W and A lines leaves us with a sum over products of amplitudes for $d\overline{u}\to W^-A$ scattering (in this case we will refer to the W^- and A as "out-going") and $W^-A\to d\overline{u}$ scattering ("in-going" W^- and A). First, Eq. (5.12) implies that any contribution involving a forward polarized A vanishes. According to the polarization sum (5.13), this leaves us only with contributions for which the A is transverse. In this case, the W^- can still be unphysical. Again from Eq. (5.13), either the in-going or the out-going W^- has to be forward polarized, and we may apply Eq. (5.11). This equation tells us that the only non-vanishing contribution with a forward-polarized W^- has a backward polarized A on the same side of the cut (because $\epsilon^\nu\ell_\nu = 0$ for the other three A polarizations). But this means that the A on the other side of the cut has a forward polarization, and we already showed that all contributions involving a forward polarized A vanish. We conclude that none of the unphysical polarizations contribute to the imaginary part of the one-loop $d\overline{u}\to d\overline{u}$ amplitude. We verified this by explicit calculation. In an explicit calculation, one makes of course use of the fact that the d and $\overline{u}$ external legs, as well as the cut lines, are on shell.

A similar example can be worked out for the case that the in-going quark and anti-quark have the same flavor. In this case, the two-gauge-boson intermediate state of interest is a W^-W^+ state. One finds that now there is a $\mathcal{G}/\mathcal{H}$-ghost contribution to the vacuum polarization, which is needed to cancel all contributions to the cut diagram from unphysical W polarizations. The relevant identities ensuring this cancellation are (again, in Feynman gauge)

$$\langle\partial^\mu W^\pm_\mu(x)W^\mp_\nu(y)u(v)\overline{u}(w)\rangle_{os} = \langle\overline{C}^\pm(x)\partial_\nu C^\mp(y)\times u(v)\overline{u}(w)\rangle_{os}. \tag{5.15}$$

This is as one would expect, since the W's belong to the "non-Abelian part" of G. We see that the way unitarity is enforced through the (e)BRST identities is a "combination" of how it works in the standard Abelian and non-Abelian cases. Again, we verified this example by explicit calculation.

We close this section with a few comments on the differences of our gauge-fixing and that of the standard (non-Abelian) Lorenz gauge. The fact that our G/H gauge-fixing condition $\mathcal{F}(V)$ (cf. Eq. (2.4)) had to be H-covariant leads to extra vertices in our gauge-fixed theory. And, one finds indeed that these additional vertices play a role in the explicit calculations verifying unitarity in the examples discussed above.

The four-ghost vertices are another new type of vertices that appear in our theory (cf. Eqs. (2.24) and (5.9)). They do not play a role in our examples of one-loop unitarity—one would have to go to higher loops to encounter them. However, there is a simple one-loop calcu-

[18]In Minkowski space we use the metric $g_{\mu\nu} = \mathrm{diag}(-1, -1, -1, 1)$.

lation in which these new ghost vertices do play a role. First note that the ghosts have to be massless for (perturbative) unitarity to work out. In standard Lorenz gauge it is very easy to see that indeed the ghosts have to be massless, because there is a shift symmetry on the antighost field in that case. Clearly, there is no shift symmetry in our case. Still, the algebraic structure does not allow for a ghost mass term, since it cannot be obtained from the eBRST transformation of any operator. But eBRST does imply the occurrence of the four-ghost vertices, and one may indeed verify that these vertices are needed so that no mass term is generated from the one-loop ghost self-energy. This can be explicitly checked in the lattice version of our theory.[19]

VI. LATTICE CHIRAL GAUGE THEORIES

We now come to the construction of chiral gauge theories on the lattice in the framework of a fully gauge-fixed Yang-Mills theory as developed in the preceding sections. There are three steps to this task. First, we will specify the fermionic part of the lattice action. Then, after adding the chiral fermions to the theory, we choose a lattice gauge-fixing action for the H gauge fixing, and we revisit the G/H gauge fixing on the lattice, for reasons to be discussed below. The resulting action for a chiral gauge theory is not exactly invariant under (e)BRST on the lattice, and we will need to add counter terms; this constitutes the final step.

As already explained in the introduction, the existence of a systematic perturbative expansion in the coupling constant is crucial in order to recover the target chiral gauge theory in the continuum limit. We explain below how this is achieved.

We take our fermions to be left-handed and in some anomaly-free (in general, reducible) representation of the group $SU(N)$.[20] In order to avoid doublers, we introduce right-handed spectators [17] and add a Wilson term, leading to

$$\begin{aligned}\mathcal{L}_{\text{fermion}} &= \frac{1}{2}\sum_\mu(\overline{\psi}_{Lx}\gamma_\mu U_{x,\mu}\psi_{Lx+\mu} - \overline{\psi}_{Lx+\mu}\gamma_\mu U^\dagger_{x,\mu}\psi_{Lx}) \\ &\quad + \frac{1}{2}\sum_\mu(\overline{\psi}_{Rx}\gamma_\mu\psi_{Rx+\mu} - \overline{\psi}_{Rx+\mu}\gamma_\mu\psi_{Rx}) \\ &\quad - \frac{r}{2}\sum_\mu(\overline{\psi}_x\psi_{x+\mu} + \overline{\psi}_{x+\mu}\psi_x - 2\overline{\psi}_x\psi_x). \qquad (6.1)\end{aligned}$$

The right-handed fermions do not transform under the gauge group, and for $r = 0$ the fermion sector of the theory has a $G_L \times G_R$ symmetry, where only G_L is local.[21] For $r > 0$, the Wilson term breaks this symmetry to the diagonal subgroup, and thus breaks gauge, or (e)BRST, invariance explicitly. The theory is also invariant under a shift symmetry on the right-handed spectator,

$$\psi_R \rightarrow \psi_R + \epsilon_R, \qquad (6.2)$$

which protects the theory against an induced fermion mass term and any other relevant or marginal operator involving the right-handed fermion [33]. Note that the spectators become free fields in the classical continuum limit.

The chiral Wilson action, Eq. (6.1), is not the only acceptable one. In fact, within the present framework, the familiar concept of universality applies. This means that one can use any fermion action with the correct classical continuum limit. For a chiral fermion action based on domain-wall fermions (which does not work without gauge fixing [34]), see Ref. [10].

For $U_{x,\mu} = 1$, obviously the fermions are not doubled, due to the presence of the Wilson term. However, one may worry that doublers are generated dynamically, for instance, if degrees of freedom exist in the interacting theory which can form bound states with the right-handed spectator fermions.[22] Indeed, without gauge fixing, the gauge degrees of freedom are not controlled by any small parameter, and all wavelengths of these modes are equally important. In an exactly gauge invariant lattice theory (such as QCD in the commonly used lattice formulations) this does not matter, because the gauge degrees of freedom decouple from the physical degrees of freedom. In chiral gauge theories, however, each fermion field needs to contribute its share of the anomaly (even if the whole collection of fermion fields is anomaly free) [2,12], and the regulated theory therefore tends to break the gauge invariance by irrelevant terms. This is certainly the case for the chiral Wilson action used here, as well as for the chiral domain-wall fermion action of Ref. [10]. It is therefore necessary to control the gauge degrees of freedom in such a way as to decouple them in the continuum limit.

In our proposal, the lattice action contains kinetic terms for all four polarizations of the gauge-field vector bosons. Thus, all polarizations of the gauge field are controlled by the gauge coupling g, including the longitudinal modes, which are controlled by the gauge fixing. Moreover, as we will describe below, the gauge fixing on the lattice is done in such a way as to ensure the existence of a unique classical vacuum. As a consequence, lattice

[19]In dimensional regularization, the massless one-loop tadpole is zero by definition.

[20]For $N = 2$ the fermion representation should be free of the Witten anomaly as well. Simple examples are a theory with two Weyl doublets (which is not really chiral), or a theory with one Weyl fermion in the $3/2$ representation.

[21]Obviously, for $r = 0$ doublers are present; as in the standard QCD case [2], they are absent for any fixed $r > 0$. Similar statements apply to the chiral domain-wall fermion action of Ref. [10].

[22]Or alternatively the left-handed fermions may form screened bound states, decoupling from the gauge fields [35].

perturbation theory can be systematically developed. Even though our theory is not gauge invariant (hence not unitary) on the lattice, it is renormalizable. As in ordinary lattice QCD with Wilson fermions, the Wilson term remains a relevant operator for large momenta (of order π/a), and doublers do not re-appear dynamically. This was investigated in more detail in perturbation theory in Refs. [10,12]. The interactions between the fermions and the longitudinal gauge fields were also investigated numerically in the Abelian case, finding again that no doublers appear, and in fact, that there is good quantitative agreement between these non-perturbative investigations and perturbation theory [8–10,36].[23] Gauge fixing is essential for this conclusion, and it has indeed been shown that without gauge fixing fermion doublers are generated dynamically [3,4,37]. The formulation of non-Abelian chiral gauge theories developed here makes it possible to extend these numerical tests to the non-Abelian case as well.

A finite number of counter terms is added in order to adjust the theory such that (at least to any order in perturbation theory) we recover the target continuum theory, in which gauge degrees of freedom as well as spectator fermions decouple, and unitarity is restored. This is done by requiring the BRST identities of the target theory to be satisfied in the continuum limit, as already observed in Ref. [5]. Standard power counting is used in order to organize the counter terms. In particular, there is a finite number of them, and only three have mass dimension less than four.

In a gauge-fixed theory with exact BRST invariance, the Gribov problem makes it non-trivial to define the gauge-fixed theory non-perturbatively, as we have seen in the preceding sections. However, in an exactly gauge- (or BRST-) invariant lattice theory, it does not matter around which of the copies of the classical vacuum one develops perturbation theory. The same is not necessarily true when the regularized theory is not exactly gauge invariant. In that case, different Gribov copies may lead to different perturbative expansions around them; the counter terms needed to regain gauge invariance in perturbation theory around one copy may not be appropriate for some other copy, and summing or averaging over all copies may thus not yield the desired continuum limit.[24] This problem is particularly acute because of the fact that, for a simple lattice transcription of the continuum Lorenz gauge, most Gribov copies will correspond to "rough" configurations, i.e., most of them are lattice artifacts.

We deal with this problem by choosing a discretization of the gauge-fixing action such that the classical vacuum configuration $U_{x,\mu} = 1$ is the unique minimum of the action, which implies that lattice perturbation theory around the classical vacuum is valid by construction [6,7]. Thus, we will choose our gauge-fixing Lagrangian *not* just equal to the sum of $\mathcal{L}^L_{gf}$ in Eq. (3.3), and some simple lattice discretization of $\sum_i(\partial_\mu A^i_\mu)^2$. In fact, our gauge-fixing action on the lattice will break eBRST symmetry explicitly. However, this is not a "new price" to pay, since the fermion sector of the theory already breaks this symmetry explicitly, and counter terms are needed anyway.

Before we review the construction of the gauge-fixing action on the lattice, let us address a possibly confusing point. We do choose to use a lattice gauge-fixing action with a unique absolute minimum. This, however, does not mean that "continuum" Gribov copies are removed from the theory. In particular, for the classical vacuum $U_{x,\mu} = 1$, the lattice action of a continuum Gribov copy will be $O(a^2p^2)$, with p some physical scale. In the continuum limit, such copies do contribute to the functional integral. To the extent that this class of copies plays a role in the physics of non-Abelian theories, they will do so in the continuum limit of our lattice regularization.

We start from the equivariantly gauge-fixed lattice Yang-Mills theory constructed in Sec. III. In particular, we will choose the equivariant gauge-fixing action to be $\mathcal{L}_{G/H}$, defined in Eq. (3.8). Our H gauge-fixing term will be constructed as follows. Defining

$$\mathcal{A}_{x,\mu} = \frac{1}{i}\sum_i T^i \mathrm{tr}(T^i(U_{x,\mu} - U^\dagger_{x,\mu})), \tag{6.3}$$

our discretization of the continuum Lorenz gauge $\mathrm{tr}(\partial_\mu A_\mu)^2/(\alpha g^2)$ will be

$$\mathcal{L}_H = \sum_i \frac{1}{\alpha g^2}\mathrm{tr}(\partial^-_\mu \mathcal{A}_{x,\mu})^2, \tag{6.4}$$

where ∂^-_μ is the backward difference operator,

$$\partial^-_\mu \phi_x = \phi_x - \phi_{x-\mu}. \tag{6.5}$$

Note that we do not introduce ghosts for H. As a consequence, on the lattice there is no BRST symmetry corresponding to this part of the gauge fixing. This is because, contrary to the continuum, on the lattice the Faddeev-Popov operator corresponding to $\mathcal{L}_H$ is not a free lattice laplacian, but instead depends on $U_{x,\mu}$ through irrelevant terms. Therefore the identities (5.6) and (5.7), do not hold on the lattice even in the lattice theory without fermions, and certainly not in the theory with fermions (see also Ref. [31]).

Next, we wish to add an irrelevant term $\mathcal{L}_{irr}$ such that the total gauge-fixing action $\sum_x(\mathcal{L}_H + \mathcal{L}_{G/H} + \mathcal{L}_{irr})$ has an absolute minimum at the perturbative vacuum configuration $U_{x,\mu} = 1$. Lattice perturbation theory will then

[23]The numerical investigation was for $G = U(1)$. However, there is little doubt that similar results would be obtained for $G = SU(N)$.

[24]For a detailed investigation of this issue, see Ref. [14].

correspond to a systematic saddle-point approximation around the perturbative vacuum.

Following Ref. [7], first define

$$\begin{aligned}\mathcal{V}_{x,\mu} &= \frac{1}{2i}(U_{x,\mu} - U^\dagger_{x,\mu}),\\ C_x &= \sum_\mu (2 - U_{x,\mu} - U^\dagger_{x-\mu,\mu}),\\ \mathcal{B}_x &= \frac{1}{4}\sum_\mu (\mathcal{V}_{x,\mu} + \mathcal{V}_{x-\mu,\mu})^2.\end{aligned} \tag{6.6}$$

Then $\mathcal{L}_{irr}$ is defined as

$$\mathcal{L}_{irr} = \frac{\tilde{r}}{g^2}\mathrm{tr}\left(\frac{1}{2}(C^\dagger_x + C_x) + \mathcal{B}_x\right)\left(\frac{1}{2}(C^\dagger_x + C_x) - \mathcal{B}_x\right), \tag{6.7}$$

where $\tilde{r} > 0$ is a new parameter. Like the Wilson parameter r, its precise value is not important, and we will take it to be of order one. It was proved in Ref. [7] that $\mathcal{L}_{irr}$ is non-negative for all $U_{x,\mu}$ and that it vanishes only for $U_{x,\mu} = 1$.[25] Since both $\mathcal{L}_H$ and $\mathcal{L}_{G/H}$ are manifestly non-negative, and also vanish for $U_{x,\mu} = 1$, our claim follows. Putting everything together, on the lattice we take

$$\mathcal{L}_{gf,\mathrm{lattice}} = \mathcal{L}_H + \mathcal{L}_{G/H} + \mathcal{L}_{irr} + \mathcal{L}_{\mathrm{ghost}} \tag{6.8}$$

as our gauge-fixing Lagrangian. Here $\mathcal{L}_{\mathrm{ghost}}$ refers to the ghost part of $\mathcal{L}^L_{gf}$, defined in Eq. (3.9), and contains only the $\mathcal{G}/\mathcal{H}$ coset ghosts.

To gain insight into the role of $\mathcal{L}_{irr}$, it is instructive to consider the classical potential V_{class} for constant, commuting gauge fields, following Refs. [6,7]. Because of the lack of gauge invariance, mass counter terms will be needed for all gauge bosons, as we will discuss below in more detail. Including mass (counter) terms, one has

$$\begin{aligned}V_{\mathrm{class}} = \frac{\tilde{r}}{2g^2}\left[\mathrm{tr}\left(\sum_\mu V^2_\mu\right)\left(\sum_\nu V^4_\nu\right) + \ldots\right]\\ + \kappa[\mathrm{tr}(V^2_\mu) + \ldots],\end{aligned} \tag{6.9}$$

where at the classical level, we may choose $\kappa_W = \kappa_A = \kappa$ (cf. Eq. (6.11)). This potential exhibits a continuous phase transition between two phases. For $\kappa > 0$ we have a ferromagnetic (FM) phase with $\langle V_\mu \rangle = 0$. For (small) $\kappa < 0$ we have a directional ferro-magnetic phase (FMD) in which the gauge field picks up an expectation value

$$\langle V_\mu \rangle = \pm\left(\frac{|\kappa| g^2}{6\tilde{r}}\right)^{1/4}, \qquad \text{all } \mu. \tag{6.10}$$

The continuum limit will be defined by approaching the critical line from the FM ($\langle V_\mu \rangle = 0$) side. Note that the lattice expectation value (6.10) only breaks a discrete symmetry (hyper-cubic rotations), and thus no Goldstone bosons occur inside the FMD phase. This phase transition defines a novel universality class, with a greatly enlarged symmetry at the critical point, at which both gauge invariance and rotational invariance are recovered.[26] Because of the existence of a unique classical vacuum and the fact that we have a consistent power counting, lattice perturbation theory applies near this critical point.[27] The existence of this novel critical point is therefore the key ingredient making it possible to formulate a lattice gauge theory with undoubled chiral fermions. The presence of the gauge-fixing sector introduces a new direction to the phase diagram of the theory (through the coupling $\tilde{r}/g^2$), thus giving access to this critical point. (Previous attempts with Wilson fermions without gauge fixing failed because of the impossibility of reaching this novel universality class.)

The irrelevant term, $\mathcal{L}_{irr}$, represented by the V^6 term in Eq. (6.9), stabilizes the classical potential. The critical line separating the two phases with $\langle V_\mu \rangle = 0$ and with $\langle V_\mu \rangle \neq 0$ corresponds to a vanishing curvature at the origin, i.e., to a vanishing gauge-field mass-squared.[28] Classically, the transition is at $\kappa = \kappa_c = 0$. In the full quantum theory, the transition is expected to be continuous up to, possibly, effects which are non-perturbatively small in the renormalized coupling constant. Without $\mathcal{L}_{irr}$ the transition would be strongly first order (i.e., with a discontinuity in $\langle V_\mu \rangle$ of order $1/a$). $U_\mu = 1$ would not be the unique classical vacuum, and perturbation theory around this vacuum would most likely not correspond to a systematic expansion of the lattice theory.[29]

Once the existence of the critical point is secured by the irrelevant term, $\mathcal{L}_{irr}$ is indeed "irrelevant" in that it does not affect the long-distance physics near this critical point.[30] In summary, the fact that stability of the potential at the critical point is obtained through the help of an irrelevant operator implies that that irrelevant operator does not occur in the renormalized (continuum)

[25] We proved it for $G = SU(N)$ or SO(N), but expect it to be true for any simple G.

[26] The names "FM" and "FMD" reflect the phase diagram obtained in the so-called "reduced model limit," which constrains the gauge field to pure-gauge configurations $U_{x,\mu} = \phi_x \phi^\dagger_{x+\mu}$. The FM phase is analytically connected to the Higgs phase in an Abelian theory, or to the Higgs-confinement phase in a non-Abelian theory. The FM-FMD phase transition is very different from the usual Higgs transition, and separated from it by a multi-critical point [13,14,23].

[27] On the FMD side, the classical vacua exhibit the (discrete) degeneracy dictated by the spontaneous breaking of hyper-cubic rotations. As usual, perturbation theory around one of these vacua is valid in the infinite-volume limit.

[28] For more details on the nature of this critical point, see Ref. [7].

[29] For some non-perturbative investigations of this issue in an Abelian theory, see Ref. [14].

[30] The Wilson term is irrelevant in precisely the same sense: *once* the doublers have been removed, it does not affect the long-distance physics of the one remaining relativistic fermion.

Lagrangian that governs the critical point, which is the gauge-fixed Yang-Mills theory described in Secs. II and V, coupled to left-handed chiral fermions.

We now come to the final step of our construction, the addition of counter terms. As mentioned already, since (e)BRST is explicitly broken on the lattice, counter terms will need to be added in order to be able to recover the target chiral gauge theory in the continuum limit. Since our theory admits an expansion in the lattice coupling constant, and satisfies the usual power counting rules, only counter terms of engineering dimension less than or equal to four need to be included. While earlier work on counter terms in this context was carried out [5,14], we will need to redo the job, because of the different gauge condition, and the lack of shift symmetry on the anti-ghost field. The symmetries restricting the number of counter terms are hyper-cubic symmetry, CP invariance, global H invariance, the discrete subgroup $\tilde{S}_N$ of $SU(N)$ introduced in Sec. II, flip symmetry, ghost-$SU(2)$ (which includes ghost-number) symmetry, and the shift symmetry (6.2) on the right-handed spectator fermions.

We begin with counter terms of dimension two. There are three of these: mass terms for the A and W fields as well as the ghost field. We can take

$$\mathcal{L}_{ct,d=2} = \kappa_A \mathrm{tr}(\mathcal{A}_{x,\mu})^2 - 2\kappa_W \sum_i \mathrm{tr}(T^i U_{x,\mu} T^i U^\dagger_{x,\mu}) + \kappa_C \mathrm{tr}(\overline{C}_x C_x). \qquad (6.11)$$

There are no dimension-three counter terms, since they are all forbidden by the lattice symmetries. In particular, a fermion mass is excluded by shift symmetry, Eq. (6.2). (When using the chiral domain-wall fermion action [10] a similar conclusion applies in the limit of an infinite fifth dimension where (global) chiral symmetry is recovered.) In contrast, there are many marginal (dimension-four) counter terms. Below, we will use continuum notation for all dimension-four counter terms; they can easily be transcribed to the lattice, for instance by using Eqs. (6.3) and (3.5). (It does not matter how exactly the latticization is carried out, and more economical lattice versions may exist.) In the fermion sector, there are only two:

$$\mathcal{L}_{ct,\mathrm{fermion}} = \delta g_A \overline{\psi}_L \gamma_\mu (iA_\mu)\psi_L + \delta g_W \overline{\psi}_L \gamma_\mu (iW_\mu)\psi_L. \qquad (6.12)$$

Note that our fermionic counter terms only involve the left-handed fermion, since the right-handed spectator is again protected by shift symmetry. As was shown in Ref. [33], the right-handed spectators decouple automatically if the theory has a continuum limit. Because there is no global $SU(N)$ symmetry, the bare couplings g_A and g_W will have to be adjusted separately in order to maintain universality of the renormalized gauge coupling.

There is a large number of dimension-four counter terms involving only the gauge fields or ghost fields, totaling 75, including wave-function renormalizations for A, W and the ghosts. For $N = 2$ these are not all independent, but for generic N they are. We list them in Appendix D.

It would be prohibitively expensive to simultaneously tune all these counter terms numerically in order to satisfy the desired Slavnov-Taylor identities of the target theory in the continuum limit. However, for small enough gauge coupling, this is not needed. The three relevant, mass counter terms will have to be tuned non-perturbatively in order to move the lattice theory to the critical point described by the target continuum theory. In other words, the three mass terms of Eq. (6.11) need to be adjusted non-perturbatively toward infinite correlation lengths for the degrees of freedom associated with the fields A, W and the ghosts. While the ghost fields are not physical, they will have to remain (perturbatively) massless if the continuum theory is to be unitary.

In contrast, all dimension-four counter terms can be estimated with the help of lattice perturbation theory, because our lattice theory has been constructed such that perturbation theory is a systematic approximation. If tree-level precision is sufficient, this implies that they can all be omitted. If not, a one-loop calculation of these counter terms should improve the situation. While this is not a simple calculation, it can be done analytically, and parametrically as a function of the free parameters g, ξ and α. Alternatively, one may imagine a numerical determination, where first $\kappa_{A,W,C}$ are determined without any other counter terms present. After that, one may determine each of the dimension-four counter terms by considering the appropriate correlations functions, i.e., the ones to which they would contribute to lowest non-trivial order. Obviously, this still relies on the validity of perturbation theory, but may help in side-stepping questions as to which value of the coupling constant should be used in the one-loop expressions for these counter terms in a particular numerical computation.

VII. CONCLUSION

We believe that this work represents major progress in the non-perturbative construction of non-Abelian chiral gauge theories. While we already described in some detail what our construction does and does not accomplish in the Introduction, let us summarize what has been gained.

The key ingredients of the lattice chiral gauge theories constructed in this paper are the use of a renormalizable gauge, as well as the existence of a unique classical vacuum. This leads to the existence of a novel type of critical point, with a systematic weak-coupling expansion, and the target continuum chiral gauge theory is recovered to all orders in this expansion at this critical point.

It is instructive to compare our lattice construction with the standard perturbative treatment of chiral gauge theories in the continuum, for instance through the use of dimensional regularization. The continuum-regularized chiral gauge theory shares with our lattice theory the following crucial features: 1) Both regularized theories do not preserve the chiral gauge invariance. 2) As a result, counter terms have to be added to recover gauge invariance of the renormalized theory. 3) The regularized theory is in fact renormalizable *without* relying on gauge invariance, thanks to the existence of kinetic terms for all four polarization, where the longitudinal kinetic term is provided by the gauge-fixing action.

The main difference is that, in the continuum, it is simply *assumed* that some non-perturbative theory exists for which the perturbative expansion is valid. The question as to whether a non-perturbative formulation actually exists with a critical point which is indeed described by renormalized perturbation theory is not even asked. On the lattice, a theory with the appropriate critical point will have to be constructed explicitly, and this is what we did in the present paper. In the lattice construction, the use of a renormalizable gauge and the uniqueness of the classical vacuum are equally important, and independent, ingredients.

The existence of a valid perturbative expansion, or, in other words, the fact that the universality class is known, is a feature common to standard Lattice QCD and to our construction. The difference is that standard Lattice QCD is exactly gauge invariant, and gauge fixing is an extraneous device needed only to set up weak-coupling perturbation theory; in contrast, in our construction the regularized theory is not gauge invariant, and renormalizability is maintained because the gauge-fixed lattice action explicitly contains longitudinal kinetic terms for all gauge bosons. *The specific gauge fixing we adopted is thus part of the very definition of the theory.*

Of course, this does not mean that we now know that unitary, Lorentz-invariant chiral gauge theories with gauge group $G = SU(N)$ exist for anomaly-free fermion representations. For example, the $SU(2)$ theory with one fundamental-representation Weyl fermion does not exist because of a non-perturbative obstruction [20,21]. However, whether any other non-perturbative obstruction exists for a certain gauge group and fermion content is a dynamical question in the context of our construction, and can in principle be studied by non-perturbative analytical or numerical techniques.

As we explained in more detail in Sec. VI (see, in particular, the discussion around Eq. (6.9)), the single most important dynamical feature of the critical point is that the chiral nature of the fermion spectrum is maintained non-perturbatively [8].[31] Since the familiar notion of universality applies, this should be true not only for the chiral Wilson action used by us, but in fact for any lattice fermion action with the correct classical continuum limit. For the chiral domain-wall fermion action, this was demonstrated in Ref. [10]. Thus, fermion doublers are *not* generated dynamically, and no fermion masses can occur without a dynamical breaking of the gauge group. Scenarios for the dynamical symmetry breaking of a chiral gauge theory, or for light composite fermions [39], can in principle be studied within our construction.

It is interesting to contrast this state of affairs with the approach of Ref. [29]. In that approach, the goal is to construct non-Abelian lattice chiral gauge theories with exact gauge invariance. Like in Lattice QCD with exact gauge invariance, if this goal would be reached, there would be no need to worry about the back reaction of the gauge degrees of freedom on the fermion spectrum (or on any other physical degree of freedom); the unphysical degrees of freedom would decouple in the regulated theory, due to the exact gauge invariance. Obviously, this approach necessitates a complete classification of all possible non-perturbative obstructions. This problem was formulated in terms of suitable integrability conditions in Ref. [29], and was solved perturbatively in Ref. [30]. A non-perturbative solution of the integrability conditions constitutes a much harder problem, and indeed, to date no solution is known.[32] In other words, it has not been established that the necessary critical point exists within this approach for the non-Abelian case.

In our construction, we do not insist on exact gauge invariance on the lattice. Instead, because of the gauge fixing, *all* degrees of freedom are under dynamical control. The continuum limit of an asymptotically-free theory corresponds to a vanishing bare coupling constant, and the validity of the weak-coupling expansion means that the elementary degrees of freedom are those, and only those, that occur at tree level in perturbation theory. In other words, the target (chiral) gauge theory, whose particle content and interactions may be read off from the *classical* continuum limit of the lattice action, is indeed realized in the *quantum* continuum limit of the lattice theory. The lattice dynamics does not generate any new light degrees of freedom not already contained in this target continuum theory [8,9,14]. Also, the unphysical degrees of freedom of the target gauge-fixed theory decouple in the continuum limit after the adjustment of a finite number of counter terms (at least) to all orders in perturbation theory. Because perturbation theory is reliable at the lattice scale, any non-perturbative obstruction to this conclusion can only originate in some infra-red mechanism contained in the theory. This is precisely what makes it interesting to apply the construction proposed in

[31] For an explanation on how the construction by-passes the Nielsen-Ninomiya theorem [38] see Ref. [9].

[32] In the Abelian case a complete classification exists, and exact gauge invariance can be established provided the fermion spectrum satisfies one new condition apart from the usual anomaly-cancellation condition [1]. See also Ref. [4].

this paper to the study of non-Abelian chiral gauge theories.

The approach of Ref. [29] has yielded a "by-product" which is a gauge-invariant lattice weak-coupling expansion for anomaly-free chiral gauge theories [30] (see also Refs. [18,19]). It is, as already mentioned above, an open question whether a critical point exists which is controlled by this expansion, simply because the underlying non-perturbative lattice theory is not (yet) known. The perturbative solution of Ref. [30] involves the adjustment of an infinite number of irrelevant operators, in order to enforce exact gauge invariance on the lattice to any given order. It is unlikely that universality holds in that case. In the gauge-fixing approach, universality applies, and the correct continuum limit is obtained to all orders after the adjustment of a finite number of (relevant and marginal) counter terms. The gauge-fixing approach also goes beyond this in providing a fully non-perturbative lattice theory. The actual number of counter terms is undeniably large. However, in Sec. VI, we have argued that this is unlikely to be a very severe obstacle, because most can be reliably calculated in low-order perturbation theory.

We believe that our results are interesting for the case of pure Yang-Mills (or vector-like) theories as well. We demonstrated in Sec. IV that the equivariantly gauge-fixed lattice Yang-Mills theory is rigorously equivalent to the non-gauge-fixed and thus gauge-invariant theory. Among other things, it follows that the equivariantly gauge-fixed theory is unitary, because the gauge-invariant theory is. It would be interesting to see whether this conclusion can be extended non-perturbatively to the remaining maximal Abelian group H, as we did to all orders in perturbation theory in Sec. V. It is possible that the idea proposed in Ref. [40] can be extended to our case. The reason that this is not trivial, however, is the "entanglement" of the Abelian and non-Abelian degrees of freedom.

The non-perturbative study of chiral gauge theories following the approach outlined in this paper is not an easy task. With an assortment of fermion and ghost determinants, numerical investigations will certainly be very demanding. The theory contains, apart from a complex fermion determinant, also a ghost determinant (the four-ghost interaction terms can be transformed into bilinear terms with the help of bosonic auxiliary fields). We envisage to begin with a study of the non-perturbatively gauge-fixed Yang-Mills theory, with no fermions. This should be interesting by itself. Moreover, it is likely that much can be learned about the phase diagram of lattice theories as constructed here by a combination of weak- and strong-coupling analytic methods; work in this direction is in progress. Concerning the fermions, it should prove useful to begin with a study of the phase of the fermion determinant on an ensemble of quenched configurations. To the extent that chiral gauge theories are qualitatively different from vector-like gauge theories, this difference should originate in the phase of the determinant. As argued in Ref. [8], it is also sensible to check the absence of fermion doublers for non-Abelian gauge groups numerically in the quenched theory, as was done there for $G = U(1)$.

We should emphasize however, that the construction outlined here now makes such investigations at least in principle possible. We have developed a first complete non-perturbative formulation of non-Abelian chiral gauge theories, and we have provided what we believe to be compelling evidence that, if a certain asymptotically-free chiral gauge theory exists, it can be studied non-perturbatively using this formulation.

ACKNOWLEDGMENTS

We thank Aharon Casher, Jeff Greensite and Pierre van Baal for useful discussions. We would also like to thank the Institute for Nuclear Theory at the University of Washington, Seattle, for hospitality. Y. S. thanks the Physics Department at San Francisco State University for hospitality. Y. S. is supported by the Israel Science Foundation under Grant No. 222/02-1. M. G. is supported in part by the U.S. Department of Energy.

APPENDIX A

Here we show that the single-site partition function $Z_{\rm ghost}(1,0)$ defined by the last row of Eq. (4.9) is non-zero. Introducing auxiliary fields ρ_i we write

$$\int dC\, d\overline{C}\, \exp[\mathrm{tr}(\tilde{X}^2)] = \int dC\, d\overline{C}\, d\rho\, \exp\Big[-\sum_i\Big(\frac{1}{2}\rho_i^2 + \rho_i f_{i\alpha\beta}\overline{C}_\alpha C_\beta\Big)\Big] = \int d\rho\, \exp\Big(-\frac{1}{2}\sum_i \rho_i^2\Big) \times \det(\rho_i f_{i\alpha\beta}). \tag{A1}$$

The matrix $\rho_{\alpha\beta} \equiv \rho_i f_{i\alpha\beta}$ is real, anti-symmetric, and even dimensional. Hence its determinant never changes sign. This implies that the single-site ghost determinant is non-negative.[33]

It remains to prove that the ghost determinant is non-zero for *some* choice of ρ_i. This is where we will use that H is the maximal Abelian subgroup of $G = SU(N)$.[34] For definiteness, we choose $\rho_i \neq 0$ to be proportional to that

[33]We assume a suitable sign convention for the Grassmann integration measure.

[34]The argument generalizes trivially to larger subgroups H' such that $H \subset H'$. More generally, it was suggested that $Z_{\rm ghost}$ will be non-zero provided that the Euler characteristic of the coset manifold $\mathcal{G}/\mathcal{H}$ is non-zero [15].

linear combination which couples to the following linear combination of diagonal generators (in arbitrary normalization)

$$T' = \mathrm{diag}(1, 2, \ldots, N-1, -N(N-1)/2). \tag{A2}$$

It is easy to see that none of the off-diagonal $SU(N)$ generators commute with T'. Moreover the resulting matrix $\rho_{\alpha\beta}$ is skew-diagonal in the basis of off-diagonal generators introduced in Eq. (2.25). For each pair T^k_{AB}, $k = 1, 2$, we obtain a non-zero anti-symmetric two-by-two block, which implies that $\det(\rho_{\alpha\beta})$ is non-zero in this case.

APPENDIX B

In this Appendix, we comment on the Feynman rules in the formalism in which the auxiliary fields b and β are kept. The only part which is not completely straightforward is that of the tree-level two-point functions for W_μ and A_μ. The quadratic part of the action for W_μ and b (the argument for A_μ and β is similar) is, after rescaling $W_\mu \to gW_\mu$, $b \to b/g$,

$$S_{\mathrm{quad}} = \frac{1}{2}\int d^4x((\partial_\mu W_\nu)^2 - (\partial_\mu W_\mu)^2 - 2ib\partial_\mu W_\mu + \xi b^2), \tag{B1}$$

from Eqs. (2.1) and (2.22). In momentum space, this can be written as

$$S_{\mathrm{quad}} = \frac{1}{2}\int \frac{d^4p}{(2\pi)^4}(W_\mu(-p) \quad b(-p)) \times \begin{pmatrix} p^2\delta_{\mu\nu} - p_\mu p_\nu & -p_\mu \\ p_\nu & \xi \end{pmatrix}\begin{pmatrix} W_\nu(p) \\ b(p) \end{pmatrix}, \tag{B2}$$

from which it follows that

$$\begin{aligned} \langle W_\mu(p)W_\nu(q)\rangle &= \frac{1}{p^2}\left(\delta_{\mu\nu} + (\xi - 1)\frac{p_\mu p_\nu}{p^2}\right)\delta(p+q), \\ \langle W_\mu(p)b(q)\rangle &= -\langle b(p)W_\mu(q)\rangle = \frac{p_\mu}{p^2}\delta(p+q), \\ \langle b(p)b(q)\rangle &= 0. \end{aligned} \tag{B3}$$

As expected, the $\langle W_\mu W_\nu\rangle$ propagator is the same as in the more familiar formulation where the auxiliary field is integrated out.

APPENDIX C

Here we generalize the proof that the scattering matrix is independent of the gauge parameter(s) to include gauge bosons on the external legs. This requires an additional argument since the (e)BRST transformation of a gauge boson contains a term linear in a ghost field. Explicitly, $s(gW^\alpha_\mu) = \partial_\mu C^\alpha + O(g)$, $s_H(gA^i_\mu) = \partial_\mu \chi^i$. For definiteness, consider $\mathcal{O} = W^\alpha_\nu(k)\mathcal{O}_1$ where k is the W's momentum and $\mathcal{O}_1$ is a product of fermion fields (the argument generalizes trivially to scattering amplitudes involving several gauge bosons). The right-hand side of Eq. (5.8) then contains a term linear in the C ghost:

$$k_\nu\left\langle C^\alpha(k)\mathcal{O}_1 \int d^4x s\bar{s}(\overline{C}C)\right\rangle. \tag{C1}$$

In order to go on-shell we first multiply both sides of Eq. (5.8) by the W's equation-of-motion operator, contracted with a (normalized) transverse polarization vector

$$\epsilon^{(n)}_\mu(k^2 g^{\mu\nu} + (1/\xi - 1)k^\mu k^\nu) = \epsilon^{(n)\nu}k^2. \tag{C2}$$

The transverse polarizations are defined by the conditions $\epsilon^{(n)}_4 = 0$ and $\vec{\epsilon}^{(n)} \cdot \vec{k} = 0$. (The transverse polarizations are well defined already before we take the momentum on-shell. Also notice that the contraction in Eq. (C2) is independent of ξ, and thus commutes with $d/d\xi$.) Since $\epsilon^{(n)}_\mu k^\mu = 0$, the product of expressions (C1) and (C2) vanishes (even before we take the W's momentum on shell). A similar reasoning applies to scattering amplitudes involving an H-subgroup gauge boson A^i_μ. This completes the proof that all scattering amplitudes are independent of the gauge parameters ξ and α.

APPENDIX D

In this Appendix we list all four-dimensional counter terms constructed from the fields A, W, C and $\overline{C}$, which we will generically label as Φ_p, with p a label running over A, W, C and $\overline{C}$ (see Eq. (6.12) for the four-dimensional counter terms involving fermions). As mentioned in the text, we may construct the counter terms in the continuum, and then use a suitable discretization which does not violate any of the symmetries present on the lattice. These symmetries are global H invariance, the discrete group $\tilde{S}_N \in SU(N)$, flip symmetry, ghost-$SU(2)$ (which includes ghost number), hyper-cubic symmetry and CP invariance. One generates all possibilities by taking traces over products over Φ_p, with or without derivatives, up to dimension four. Since all Φ_p are traceless, there are terms containing one or two traces only. One then imposes other symmetries. For instance, for each $\overline{C}$ there has to be a C, and some possibilities have to be chosen in linear combinations which are invariant under flip and/or ghost-$SU(2)$ symmetry.

In addition, since projecting on the subalgebra $\mathcal{H}$ or the coset $\mathcal{G}/\mathcal{H}$ is invariant under $\tilde{S}_N$ and H, each trace over a product of fields can generate new terms through replacements like

$$\begin{aligned} \mathrm{tr}(\Phi_1\Phi_2\Phi_3\Phi_4) \to \mathrm{tr}(\Phi_1\Phi_2\Phi_3\Phi_4) &+ \mathrm{tr}((\Phi_1\Phi_2)_{\mathcal{H}} \times (\Phi_3\Phi_4)_{\mathcal{H}}) + \mathrm{tr}((\Phi_2\Phi_3)_{\mathcal{H}} \\ &\times (\Phi_4\Phi_1)_{\mathcal{H}}), \end{aligned} \tag{D1}$$

where each term of course comes with an arbitrary coefficient. In general, the product of two fields can be written as

$$\Phi_1\Phi_2 = (\Phi_1\Phi_2)_1 + (\Phi_1\Phi_2)_{\mathcal{H}} + (\Phi_1\Phi_2)_{\mathcal{G}/\mathcal{H}}, \quad (D2)$$

where with the subscript "1" we indicate the part proportional to the identity matrix. We can thus pick any three of these products as independent, and construct counter terms out of them. Here we will choose $\Phi_1\Phi_2$, $(\Phi_1\Phi_2)_1$ and $(\Phi_1\Phi_2)_{\mathcal{H}}$. The term of the form $(\Phi_1\Phi_2)_1$ can be written in terms of $\mathrm{tr}(\Phi_1\Phi_2)$, and thus leads to a term with two traces.

The projection onto $\mathcal{H}$ can be understood in a slightly different way. Consider the H-invariant (which is also invariant under $\tilde{S}_N$, as we showed in Sec. II)

$$\sum_i \mathrm{tr}(T^i X T^i Y), \quad (D3)$$

for any $N \times N$ matrices X and Y. In the orthogonal basis defined by $\mathrm{tr}(T^i T^j) = \frac{1}{2}\delta_{ij}$, it is possible to prove that

$$\sum_i (T^i)_{AB}(T^i)_{CD} = \frac{1}{2}\delta_{AB}\delta_{AC}\delta_{AD} - \frac{1}{2N}\delta_{AB}\delta_{CD}. \quad (D4)$$

Using this relation, one obtains

$$\sum_i \mathrm{tr}(T^i X T^i Y) = \frac{1}{2}\sum_A X_{AA}Y_{AA} - \frac{1}{2N}\mathrm{tr}(XY). \quad (D5)$$

For the $\mathcal{H}$-projected matrices $X_{\mathcal{H}} = 2T^i \mathrm{tr}(T^i X)$ and $Y_{\mathcal{H}}$ one has

$$\mathrm{tr}(X_{\mathcal{H}} Y_{\mathcal{H}}) = \sum_A X_{AA}Y_{AA} - \frac{1}{N}\mathrm{tr}(X)\mathrm{tr}(Y), \quad (D6)$$

hence

$$\mathrm{tr}(X_{\mathcal{H}} Y_{\mathcal{H}}) = 2\sum_i \mathrm{tr}(T^i X T^i Y) + \frac{1}{N}\mathrm{tr}(XY) - \frac{1}{N}\mathrm{tr}(X)\mathrm{tr}(Y). \quad (D7)$$

Applying this, for example, to the pre-potential in Eq. (3.3) gives

$$2\sum_i \mathrm{tr}(T^i U_{x,\mu} T^i U^\dagger_{x,\mu}) = \mathrm{tr}((U_{x,\mu})_{\mathcal{H}}(U^\dagger_{x,\mu})_{\mathcal{H}}) + \frac{1}{N}\mathrm{tr}(U_{x,\mu})\mathrm{tr}(U^\dagger_{x,\mu}) - 1. \quad (D8)$$

We list all counter terms below with no explicit couplings, but arbitrary real coupling constants are implied in front of each operator. If two operators are related by flip (or ghost-$SU(2)$) symmetry, we will include them together between parentheses.

Counter terms with two fields (and thus two derivatives) are

$$\begin{aligned}\mathcal{L}_{c.t.2} = {} & \mathrm{tr}(\partial_\mu W_\mu \partial_\nu W_\nu) + \mathrm{tr}(\partial_\mu W_\nu \partial_\mu W_\nu) \\ & + \mathrm{tr}(\partial_\mu W_\mu \partial_\mu W_\mu) + \mathrm{tr}(\partial_\mu A_\mu \partial_\nu A_\nu) \\ & + \mathrm{tr}(\partial_\mu A_\nu \partial_\mu A_\nu) + \mathrm{tr}(\partial_\mu A_\mu \partial_\mu A_\mu) \\ & + \mathrm{tr}(\partial_\mu \overline{C} \partial_\mu C). \end{aligned} \quad (D9)$$

Summation over *all* repeated space-time indices is assumed, so, e.g. $\mathrm{tr}(\partial_\mu W_\mu \partial_\mu W_\mu)$ is a shorthand for $\sum_\mu \mathrm{tr}(\partial_\mu W_\mu \partial_\mu W_\mu)$. Counter terms with three fields (and thus one derivative) are

$$\begin{aligned}\mathcal{L}_{c.t.3} = {} & \mathrm{tr}(\partial_\mu A_\mu A_\nu A_\nu) + \mathrm{tr}(\partial_\mu A_\nu A_\mu A_\nu) + \mathrm{tr}(\partial_\mu A_\mu A_\mu A_\mu) + \mathrm{tr}(\partial_\mu W_\mu W_\nu W_\nu) + \mathrm{tr}(\partial_\mu W_\nu \{W_\mu, W_\nu\}) \\ & + i\,\mathrm{tr}(\partial_\mu W_\nu [W_\mu, W_\nu]) + \mathrm{tr}(\partial_\mu W_\mu W_\mu W_\mu) + \mathrm{tr}(A_\mu \{\partial_\mu W_\nu, W_\nu\}) + i\,\mathrm{tr}(A_\mu [\partial_\mu W_\nu, W_\nu]) + \mathrm{tr}(A_\mu \{\partial_\nu W_\mu, W_\nu\}) \\ & + i\,\mathrm{tr}(A_\mu [\partial_\nu W_\mu, W_\nu]) + \mathrm{tr}(A_\mu \{\partial_\nu W_\nu, W_\mu\}) + i\,\mathrm{tr}(A_\mu [\partial_\nu W_\nu, W_\mu]) + \mathrm{tr}(A_\mu \{\partial_\mu W_\mu, W_\mu\}) + i\,\mathrm{tr}(A_\mu [\partial_\mu W_\mu, W_\mu]) \\ & + i\,\mathrm{tr}(A_\mu (\{\partial_\mu \overline{C}, C\} - \{\partial_\mu C, \overline{C}\})) + \mathrm{tr}(A_\mu ([\partial_\mu \overline{C}, C] - [\partial_\mu C, \overline{C}])) + i\,\mathrm{tr}(W_\mu (\{\partial_\mu \overline{C}, C\} - \{\partial_\mu C, \overline{C}\})) \\ & + \mathrm{tr}(W_\mu ([\partial_\mu \overline{C}, C] - [\partial_\mu C, \overline{C}])). \end{aligned} \quad (D10)$$

The last four terms are examples of linear combinations invariant under flip and ghost-$SU(2)$ symmetry. For the terms with four fields, we get terms with two traces,

$$\begin{aligned}\mathcal{L}_{c.t.22} = {} & \mathrm{tr}(A_\mu A_\mu)\mathrm{tr}(A_\nu A_\nu) + \mathrm{tr}(A_\mu A_\nu)\mathrm{tr}(A_\mu A_\nu) + \mathrm{tr}(A_\mu A_\mu)\mathrm{tr}(A_\mu A_\mu) + \mathrm{tr}(W_\mu W_\mu)\mathrm{tr}(W_\nu W_\nu) + \mathrm{tr}(W_\mu W_\nu)\mathrm{tr}(W_\mu W_\nu) \\ & + \mathrm{tr}(W_\mu W_\mu)\mathrm{tr}(W_\mu W_\mu) + \mathrm{tr}(A_\mu A_\mu)\mathrm{tr}(W_\nu W_\nu) + \mathrm{tr}(A_\mu A_\nu)\mathrm{tr}(W_\mu W_\nu) + \mathrm{tr}(A_\mu A_\mu)\mathrm{tr}(W_\mu W_\mu) + \mathrm{tr}(\overline{C}C)\mathrm{tr}(A_\mu A_\mu) \\ & + \mathrm{tr}(\overline{C}C)\mathrm{tr}(W_\mu W_\mu) + \mathrm{tr}(\overline{C}W_\mu)\mathrm{tr}(CW_\mu), \end{aligned} \quad (D11)$$

and, finally, terms with only one trace,

$$\begin{aligned}\mathcal{L}_{c.t.4} = {} & \mathrm{tr}(A_\mu A_\mu A_\nu A_\nu) + \mathrm{tr}(A_\mu A_\mu A_\mu A_\mu) + \mathrm{tr}(\{A_\mu, W_\nu\}\{A_\mu, W_\nu\}) + \mathrm{tr}([A_\mu, W_\nu][A_\mu, W_\nu]) + \mathrm{tr}(\{A_\mu, W_\mu\}\{A_\nu, W_\nu\}) \\ & + \mathrm{tr}([A_\mu, W_\mu][A_\nu, W_\nu]) + i\,\mathrm{tr}([A_\mu, W_\mu]\{A_\nu, W_\nu\}) + \mathrm{tr}(\{A_\mu, W_\mu\}\{A_\mu, W_\mu\}) + \mathrm{tr}([A_\mu, W_\mu][A_\mu, W_\mu]) \\ & + \mathrm{tr}(\{A_\mu, W_\nu\}\{W_\mu, W_\nu\}) + \mathrm{tr}([A_\mu, W_\nu][W_\mu, W_\nu]) + i\,\mathrm{tr}([A_\mu, W_\nu]\{W_\mu, W_\nu\}) + \mathrm{tr}(A_\mu W_\mu W_\mu W_\mu) \\ & + \mathrm{tr}(W_\mu W_\mu W_\nu W_\nu) + \mathrm{tr}(W_\mu W_\nu W_\mu W_\nu) + \mathrm{tr}(W_\mu W_\mu W_\mu W_\mu) + \mathrm{tr}((W_\mu W_\mu)_{\mathcal{H}}(W_\nu W_\nu)_{\mathcal{H}}) \\ & + \mathrm{tr}((W_\mu W_\mu)_{\mathcal{H}}(W_\mu W_\mu)_{\mathcal{H}}) + \mathrm{tr}([W_\mu, W_\nu]_{\mathcal{H}}[W_\mu, W_\nu]_{\mathcal{H}}) + \mathrm{tr}(\{W_\mu, W_\nu\}_{\mathcal{H}}\{W_\mu, W_\nu\}_{\mathcal{H}}) + \mathrm{tr}(\{\overline{C}, W_\mu\}\{C, W_\mu\}) \\ & + \mathrm{tr}([\overline{C}, W_\mu][C, W_\mu]) + \mathrm{tr}(\{\overline{C}, A_\mu\}\{C, A_\mu\}) + \mathrm{tr}([\overline{C}, A_\mu][C, A_\mu]) + \mathrm{tr}(\{\overline{C}, W_\mu\}\{C, A_\mu\} + \{\overline{C}, A_\mu\}\{C, W_\mu\}) \\ & + i\,\mathrm{tr}(\{\overline{C}, W_\mu\}[C, A_\mu] + [\overline{C}, A_\mu]\{C, W_\mu\}) + \mathrm{tr}([\overline{C}, W_\mu][C, A_\mu] + [\overline{C}, A_\mu][C, W_\mu]) + \mathrm{tr}(\{W_\mu, \overline{C}\}_{\mathcal{H}}\{W_\mu, C\}_{\mathcal{H}}) \\ & + \mathrm{tr}([W_\mu, \overline{C}]_{\mathcal{H}}[W_\mu, C]_{\mathcal{H}}) + \mathrm{tr}(\{W_\mu, \overline{C}\}_{\mathcal{H}}[W_\mu, C]_{\mathcal{H}} + [W_\mu, \overline{C}]_{\mathcal{H}}\{W_\mu, C\}_{\mathcal{H}}) + \mathrm{tr}(\overline{C}^2 C^2) + \mathrm{tr}([\overline{C}, C]_{\mathcal{H}}[\overline{C}, C]_{\mathcal{H}}) \\ & + \mathrm{tr}((\overline{C}^2)_{\mathcal{H}}(C^2)_{\mathcal{H}} - (1/4)\{\overline{C}, C\}_{\mathcal{H}}\{\overline{C}, C\}_{\mathcal{H}}). \end{aligned} \quad \text{(D12)}$$

The operator $\mathrm{tr}(\overline{C}C\overline{C}C)$ is odd under flip symmetry, and thus excluded. Note that

$$\mathrm{tr}([X, Z]\{Y, Z\}) = -\mathrm{tr}(\{X, Z\}[Y, Z]) = -\mathrm{tr}([X, Y]Z^2), \quad \text{(D13)}$$

for any X, Y and Z (if both X and Y are anti-commuting, $[X, Y]$ in the last expression should be replaced by $\{X, Y\}$). In particular, if $X = Y$, this trace vanishes. We used this relation to eliminate a number of terms in $\mathcal{L}_{c.t.4}$. The only new constraint imposed by extending ghost-number symmetry to ghost-$SU(2)$ arises from the fact that the latter mixes the two terms inside the last trace on the last row of Eq. (D12) (compare Eq. (2.31)). Note that there are no counter terms involving the Levi-Civita tensor $\epsilon_{\mu\nu\rho\sigma}$ because of CP invariance.

[1] M. Lüscher, Nucl. Phys. **B549**, 295 (1999).
[2] L. H. Karsten and J. Smit, Nucl. Phys. **B183**, 103 (1981).
[3] J. Smit, Nucl. Phys. B, Proc. Suppl. **17**, 3 (1990); D. N. Petcher, hep-lat/9301015; Y. Shamir, Nucl. Phys. B, Proc. Suppl. **47**, 212 (1996).
[4] M. Golterman, Nucl. Phys. B, Proc. Suppl. **94**, 189 (2001).
[5] A. Borrelli, L. Maiani, G.C. Rossi, R. Sisto, and M. Testa, Nucl. Phys. **B333**, 335 (1990); G.C. Rossi, R. Sarno, and R. Sisto, Nucl. Phys. **B398**, 101 (1993).
[6] Y. Shamir, Phys. Rev. D **57**, 132 (1998).
[7] M. Golterman and Y. Shamir, Phys. Lett. **B399**, 148 (1997).
[8] W. Bock, M. Golterman, and Y. Shamir, Phys. Rev. Lett. **80**, 3444 (1998).
[9] W. Bock, M. Golterman, and Y. Shamir, Nucl. Phys. B, Proc. Suppl. **63**, 147 (1998); Nucl. Phys. B, Proc. Suppl. **63**, 581 (1998).
[10] S. Basak and A. K. De, Phys. Rev. D **64**, 014504 (2001); Phys. Lett. B **522**, 350 (2001).
[11] M. Golterman and Y. Shamir, Phys. Rev. D **67**, 014501 (2003).
[12] W. Bock, M. Golterman, and Y. Shamir, Phys. Rev. D **58**, 034501 (1998).
[13] W. Bock, M. Golterman, and Y. Shamir, Phys. Rev. D **58**, 054506 (1998).
[14] W. Bock, K.C. Leung, M. Golterman, and Y. Shamir, Phys. Rev. D **62**, 034507 (2000).
[15] M. Schaden, Phys. Rev. D **59**, 014508 (1999).
[16] H. Neuberger, Phys. Lett. B **183**, 337 (1987).
[17] J. Smit, Acta Phys. Pol. B **17**, 531 (1986); P.V.D. Swift, Phys. Lett. **B145**, 256 (1984).
[18] H. Suzuki, Nucl. Phys. **B585**, 471 (2000).
[19] G. Bandelloni, C. Becchi, A. Blasi, and R. Collina, Commun. Math. Phys. **72**, 239 (1980).
[20] E. Witten, Phys. Lett. **B117**, 324 (1982).
[21] O. Bär and I. Campos, Nucl. Phys. **B581**, 499 (2000).
[22] J. Smit, Nucl. Phys. B, Proc. Suppl. **20**, 542 (1991).
[23] S. Basak, A. K. De, and T. Sinha, Phys. Lett. **B580**, 209 (2004).
[24] V. N. Gribov, Nucl. Phys. **B139**, 1 (1978).
[25] P. Hirschfeld, Nucl. Phys. **B157**, 37 (1979).
[26] C. Parrinello and G. Jona-Lasinio, Phys. Lett. **B251**, 175 (1990); D. Zwanziger, Nucl. Phys. **B345**, 461 (1990).
[27] W. Bock, M. Golterman, M. Ogilvie, and Y. Shamir, Phys. Rev. D **63**, 034504 (2001).
[28] L. Baulieu and J. Thierry-Mieg, Nucl. Phys. **B197**, 477 (1982).
[29] M. Lüscher, Nucl. Phys. **B568**, 162 (2000).
[30] M. Lüscher, J. High Energy Phys. 06 (2000) 028.
[31] W. Bock, M. Golterman, and Y. Shamir, Phys. Rev. D **58**, 097504 (1998).
[32] T.-P. Cheng and L.-F. Li, *Gauge Theory of Elementary Particle Physics* (Oxford University, Oxford, 1984); M. Peskin and D. Schroeder, *An Introduction to Quantum Field Theory* (Addison-Wesley, Reading, MA, 1995).
[33] M. Golterman and D. N. Petcher, Phys. Lett. **B225**, 159 (1989).
[34] M. Golterman, K. Jansen, D. N. Petcher, and J. C. Vink, Phys. Rev. D **49**, 1606 (1994); M. Golterman and Y. Shamir, Phys. Rev. D **51**, 3026 (1995).

[35] M. Golterman, D. N. Petcher and J. Smit, Nucl. Phys. **B370**, 51 (1992).
[36] W. Bock, M. Golterman, and Y. Shamir, hep-lat/9804015.
[37] Y. Shamir, Phys. Rev. Lett. **71**, 2691 (1993); hep-lat/9307002.
[38] H. B. Nielsen and M. Ninomiya, Nucl. Phys. **B193**, 173 (1981); Nucl. Phys. **B185**, 20 (1981); *ibid.***B195**, 541E (1982).
[39] S. Dimopoulos, S. Raby, and L. Susskind, Nucl. Phys. **B173**, 208 (1980); Nucl. Phys. **B169**, 373 (1980).
[40] M. Testa, Phys. Lett. **B429**, 349 (1998).