Why do naked singularities form in gravitational collapse? II

Pankaj S. Joshi,[1] Rituparno Goswami,[1] and Naresh Dadhich[2]
[1]*Tata Institute for Fundamental Research, Mumbai, India*
[2]*Inter-University Center for Astronomy & Astrophysics, Pune, India*
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We examine physical features that could lead to formation of a naked singularity rather than black hole, as end state of spherical collapse. Generalizing earlier results on dust collapse to general *type I* matter fields, it is shown that collapse always creates black hole if shear vanishes or density is homogeneous. It follows that nonzero shear is a necessary condition for singularity to be visible to external observers, when trapped surface formation is delayed by shearing forces or inhomogeneity within the collapsing cloud.

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It was investigated recently [1], within the framework of dust collapse models, what are the key physical features that cause the formation of a naked singularity (NS) rather than a black hole (BH) as end state of a continual gravitational collapse. It was shown that sufficiently strong shearing forces within the collapsing cloud could delay the formation of apparent horizon, thus making the singularity visible to an external observer. This is in contrast to the black hole scenario where singularity is always hidden within horizon due to early enough formation of trapped surfaces.

While dust models have provided valuable insights into dynamics of collapse, in realistic situations one would like to incorporate pressures which would be important in later stages of collapse. Also, as we know little on the form of matter or equation of state in later stages of collapse, it will be preferable to consider as general form of matter as may be possible. In fact, there have been some considerations recently of collapse with general type I matter [2] (which includes practically all known physical forms of matter [3]), and it turns out that depending on the nature of the initial data in terms of the matter distribution and metric functions, either a black hole or a naked singularity results as the final outcome of an endless collapse. The occurrence of both BH and NS end states could be generic because the initial data sets evolving to each of these outcomes form an open set [4]. Such a scenario may be of physical interest because a visible extreme strong gravity region could possibly provide with an opportunity to observe the effects of quantum gravity, as generated by the strong curvature regions [5].

In the present note, our purpose is to generalize the conclusions of [1] to include more general forms of matter, thus incorporating pressures and more generic matter fields. We intend to isolate the possible dynamic features that distinguish one outcome from the other, that is, what could cause NS to develop rather then a black hole as collapse end state. It is seen that spacetime shear again plays an important role in the case of general matter fields also towards determining the black hole or naked singularity end states.

Specifically, we show that a nonvanishing spacetime shear must be present within the collapsing cloud whenever the collapse ends in NS. Thus, nonvanishing shear (and the associated inhomogeneity) is a necessary condition for NS formation, and a black hole must always result whenever there are no shearing forces present in the collapsing cloud. A possible interpretation of this could be, whenever a naked singularity has developed, it is possibly due to the distortion of the trapped surface geometry which is caused by the presence of shear and inhomogeneity within the cloud. As the matter considered here is generic, with nonvanishing radial as well as tangential pressures, it then appears that the physical agencies such as inhomogeneities and spacetime shear present within the collapsing cloud could cause a naked singularity to develop, as distinguished from black hole outcome. In a way, this may provide a somewhat natural dynamical explanation as to why BH/NS phases develop in collapse.

Consider a spherical collapsing cloud which can be described by the general metric in the comoving coordinates (t, r, θ, ϕ) as given by,

$$ds^2 = -e^{2\nu(t,r)}dt^2 + e^{2\psi(t,r)}dr^2 + R^2(t,r)d\Omega^2 \quad (1)$$

The energy-momentum tensor for any matter field which is type I is then given in a diagonal form [3],

$$T^t_t = -\rho; \qquad T^r_r = p_r; \qquad T\theta\theta = T^\phi_\phi = p\theta \quad (2)$$

The quantities ρ, p_r and $p\theta$ are the energy density, radial and tangential pressures, respectively. We take the matter field to satisfy the *weak energy condition*, i.e., the energy density measured by any local observer be non-negative, and so for any timelike vector V^i, we must have,

$$T_{ik}V^iV^k \geq 0 \quad (3)$$

which amounts to $\rho \geq 0; \rho + p_r \geq 0; \rho + p\theta \geq 0$. We have not confined ourselves here to any special type of matter such as, e.g., dust or a perfect fluid form, but all

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forms of matter are included where the stress-energy tensor admits one timelike and three spacelike eigenvectors. In that sense, our conclusions are generic and apply to a large variety of collapse models.

Now for the metric (1) the Einstein equations take the form, in the units ($8\pi G = c = 1$)

$$\rho = \frac{F'}{R^2 R'}; \qquad p_r = -\frac{\dot{F}}{R^2 \dot{R}} \tag{4}$$

$$\nu' = \frac{2(p_\theta - p_r)}{\rho + p_r}\frac{R'}{R} - \frac{p_r'}{\rho + p_r} \tag{5}$$

$$-2\dot{R}' + R'\frac{\dot{G}}{G} + \dot{R}\frac{H'}{H} = 0 \tag{6}$$

$$G - H = 1 - \frac{F}{R} \tag{7}$$

where,

$$G(t, r) = e^{-2\psi}(R')^2; \qquad H(t, r) = e^{-2\nu}(\dot{R})^2 \tag{8}$$

In the above, the arbitrary function $F = F(t, r)$ has an interpretation of the mass function for the cloud, giving the total mass in a shell of comoving radius r. We have $F \geq 0$ from the energy conditions.

The shear tensor for the collapsing matter is given by [6],

$$\sigma^\phi_\phi = \sigma^\theta_\theta = -\frac{1}{2}\sigma^r_r = \frac{1}{3}e^{-\nu}\left(\frac{\dot{R}}{R} - \dot{\psi}\right) \tag{9}$$

Let us consider now the situation when the matter shear vanishes identically. Our purpose is to investigate to what extent this constraints the outcome of collapse. This means,

$$\frac{\dot{R}}{R} = \dot{\psi} \tag{10}$$

or, $R = q(r)e^{\psi}$. We can use the scaling freedom available in rescaling the radial coordinate r, and with a suitable rescaling we can always choose $q(r) = r$. We then have,

$$R = re^{\psi} \tag{11}$$

It then follows that the spacetime geometry (1) becomes,

$$ds^2 = -e^{2\nu(t,r)}dt^2 + e^{2\psi(t,r)}[dr^2 + r^2 d\Omega^2] \tag{12}$$

Hence we see that there are now five total field equations with six unknowns as given by, ρ, p_r, p_θ, ψ, ν and F, thus giving us the freedom of choice of one free function if we are to complete the solution. Also we require the regularity of the initial data, and, in particular, that of the density distribution, at the initial surface $t = t_i$ from which the collapse develops. The collapse condition is given as $\dot{R} < 0$, which amounts to $\dot{\psi} < 0$. It is now possible to integrate the Eq. (6), using Eq. (11), to obtain

$$e^{\nu(t,r)} = a(t)\dot{\psi} \tag{13}$$

where $a(t)$ is an arbitrary function of integration. Since the left hand side of the above equation is positive by definition, it follows that $\dot{\psi} < 0$ implies $a(t) < 0$.

The Einstein Eq. (5) implies that there is a density singularity developing at $R = 0$ and $R' = 0$. The later are the "shell-crossings" which are generally believed to be weak singularities which are removable. Hence our main interest here is to study the curvature singularity developing at $R = 0$, where physical radius of all collapsing shells go to a vanishing value. So we require that there are no shell-crossings, that is, we have $R' > 0$ in the spacetime. This is equivalent to the condition that $r\psi' > -1$ as we see from the expression for R above. Then ensuring that there are no shell-crossings ensures that the coordinate system is valid and does not breakdown till the curvature singularity at $R = 0$. In the case of the singularity at $R[r, t_s(r)] = 0$, we note that in the limit as $t \to t_s$,

$$\psi[r, t_s(r)] \to -\infty \tag{14}$$

We note that in the case of dust, which is a special case of general *Type I* matter fields considered here, the congruence of curves of collapsing matter consists of timelike *geodesics* and hence in that case no shell-crossings would imply there are no conjugate points developing in the congruence of these timelike geodesics which represent the collapsing dust particles. In fact, in this case, we have,

$$\dot{\psi} = \dot{R}'/R' \tag{15}$$

so $R' \to 0$ implies $\Theta \to -\infty$, which shows that the shell-crosses are equivalent to occurrence of conjugate points in the congruence of geodesics. Again the condition $R' > 0$ then ensures that the coordinate system is valid till the curvature singularity at $R = 0$. In the general case considered here, the collapsing matter need not move along geodesics, however, imposing the condition as above that there are no shell-crossings ensures that the coordinate system does not break down till the curvature singularity.

The expansion parameter Θ, for the infalling matter congruence for the metric (1), is given as below,

$$\Theta(t, r) = \frac{1}{e^\nu}\left[\dot{\psi} + 2\frac{\dot{R}}{R}\right] \tag{16}$$

From Eq. (10) and (13) it is then seen that, in the case of a shear-free collapse,

$$\Theta(t, r) = \Theta(t) = \frac{3}{a(t)} \tag{17}$$

The expansion has thus no space dependence in this case and is negative, as the matter is going through a process of continual collapse.

For the curvature singularity at $R = 0$, we have $\Theta \to -\infty$ as the singularity acts as a *sink* for all the curves of the collapsing congruence, and the volume elements shrink to zero along all the collapsing trajectories [3]. So the curvature singularity occurs at $a(t) = 0$. Let the time for the central shell at $r = 0$ to reach the singularity be t_{s_0}. The singularity curve $t_s(r)$ corresponds to the value $R = 0$, denoting the times at which different shells arrive at the vanishing value of the physical radius R. Suppose now $t_s(r)$ is an increasing function of r. Consider then the spacelike surface $t = t_{s_0}$. Any event on this surface with $r > 0$ then lies within the spacetime, because the singularity at $R = 0$ is reached at a later epoch for this collapsing shell. So this is a regular spacetime event at which Θ must be finite. This follows from Eq. (11) which implies that $\dot{\psi}$ is finite at all regular points because so are R and $\dot{R}$. Then, from Eq. (15), since the metric function $\nu(r, t)$ has to be regular in the spacetime, it follows that at all regular events the function $a(t)$ is finite and nonzero. (This makes physical sense as well because Θ is the parameter characterizing the volume expansion (or shrinkage) of the collapsing cloud which is finite at all regular events.) However, this is not possible because $\Theta(t_{s_0}) = -\infty$ for all values of r, as seen above. Similar argument applies if $t_s(r)$ were a decreasing function of r. It follows that $t_s(r) = t_{s_0}$, which is a constant function, and we have,

$$\Theta[t_s(r)] = \Theta(t_{s_0}) = -\infty \tag{18}$$

In other words, the singularity $t_s(r)$ is *simultaneous*.

To determine the collapse outcome in terms of either BH or NS, we need to find if nonspacelike trajectories escape away from the singularity, thus making it visible. The singularity will be naked if there are future directed nonspacelike curves that reach faraway observers, and in the past which terminate at the singularity. But the singularity curve being constant, the collapse is simultaneous. This necessarily gives rise to a covered singularity at $R = 0$, and there cannot be any outgoing future directed nonspacelike geodesics coming out from the same. Because, if there were any such curves, given by say $t = t(r)$ in the (t, r) plane, which came out from $t = t_s, r = 0$, then the time coordinate must increase along these paths. This is, however, impossible as there is complete collapse at $t = t_{s_0}$, and there is no spacetime beyond that. Hence no values $t > t_{s_0}$ are allowed within the spacetime which does not extend beyond the singularity. Hence no nonspacelike trajectories come out of the singularity and the collapse gives rise necessarily to a black hole in the spacetime (see also [7]). Similar argument applies to the points on the singularity curve at $r = 0$ or $r > 0$.

We have shown that if a naked singularity is to result as collapse outcome, the presence of nonvanishing shear within the cloud is a necessary condition. Hence in general the vanishing of shear implies black hole, and absence of NS formation, for a generic matter field. We note that presence of shear is essential for NS to occur, but presence of shear will not in general imply presence of NS necessarily.

We considered here a sufficiently general form of matter, and so conclusions are generic to that extent, though limited to spherical symmetry. To get an insight into how shear operates in a dynamically evolving scenario, we consider below a collapse evolution where we assume the matter density to be homogeneous throughout. We construct an explicit class of collapse models to understand how shear works to affect the collapse. The choice of a homogeneous density profile, implies matter is general but we have $\rho(r, t) = \rho(t)$ Now, choose the class of velocity profiles for the collapsing shells as given by the choice,

$$\nu(t, r) = A(R) \tag{19}$$

Here the function $A(R)$ is any arbitrary, suitably differentiable function of the physical radius R of the cloud, with the initial constraint $A(R)|_{t=t_i} = \nu_0(r)$ Again, from the Einstein Eq. (5) we get,

$$\nu_0(r) = \int_0^r \left(\frac{2(p_{\theta_0} - p_{r_0})}{(\rho_0 + p_{r_0})} - \frac{p'_{r_0}}{\rho_0 + p_{r_0}} \right) dr \tag{20}$$

where p_{θ_0}, p_{r_0} and ρ_0 denote the pressure and density profiles at the initial epoch. Let us now assume that the initial pressures have physically reasonable behavior at the center $r = 0$, in that the pressure gradients vanish, i.e. $p'_{r_0}(0) = p'_{\theta_0}(0) = 0$, and also that the difference between radial and tangential pressures vanishes at the center, i.e. $p_{r_0}(0) - p_{\theta_0}(0) = 0$, which ensures the regularity of the initial data at the center of the cloud. Then, from Eq. (20), it is evident that $\nu_0(r)$ has the form $\nu_0(r) = r^2 g(r)$, where $g(r)$ is at least a C^1 function of r for $r = 0$, and at least a C^2 function for $r > 0$. From this we can now generalize the form of $A(R)$ as $A(R) = R^2 g_1(R)$, where $g_1(R)$ is a suitably differentiable function and $g_1(R)|_{t=t_i} = g(r)$. Now $\rho = \rho(t)$ and (4) gives,

$$F = \frac{1}{3}\rho(t)R^3; \qquad p_r = -\rho(t) - \frac{1}{3}\dot{\rho(t)}\frac{R}{\dot{R}} \tag{21}$$

Also, using Eq. (19) in Eq. (6), we have,

$$G(t, r) = b(r)e^{2A} \tag{22}$$

Here $b(r)$ is another arbitrary function of the radial coordinate r. (A comparison with dust collapse models interprets $b(r)$ as the velocity function for the shells). We can write $b(r) = 1 + r^2 b_0(r)$. Thus we see that for an ϵ ball around the central shell, the function G behaves as $G \approx e^{2A}$ Now using Eq. (7) we get

$$\frac{\dot{R}}{R} = -e^A \sqrt{2g_1(R) + \frac{1}{3}\rho(t)} \tag{23}$$

Hence, it is evident that in the vicinity of the singularity, that is in the limit $R \to 0$ and $\rho \to \infty$, and close to the central shell,

$$\frac{\dot{R}}{R} = f(t) \tag{24}$$

Here $f(t)$ is another function of time. Thus from Eqs. (24) and (21) we see that in the limit of approach to the singularity and near the central shell, the radial pressure behaves as, $p_r = p_r(t)$ Again, in the same limit we can write the tangential pressure as,

$$2p_\theta = RA_{,R}(\rho + p_r) + 2p_r \approx 2p_r(t) \tag{25}$$

It is clear from the above that in the case of a homogeneous collapse, a large class of solutions as above, and characterized by the functions $A(R)$ exists, for which both the radial and tangential pressures homogenize close to the center and in the vicinity of the singularity. Hence collapse becomes necessarily shear-free at this limit (note that in spherical symmetry homogeneity, through the no energy flux condition, implies vanishing shear). The final outcome of such a homogeneous collapse is then necessarily black hole.

It follows that collapse outcome is a black hole whenever the collapsing matter is shear-free or homogeneous. The end product of collapse could be different from black hole (i.e. naked singularity) only if the collapsing matter is both shearing as well as inhomogeneous. This is a generic feature to the extent that the matter field we have considered is general enough. So for a number of classes of general *Type I* matter fields, the homogeneous collapsing configurations are subclasses of shear-free collapse, because a collapse which is homogeneous in density tends to be shear-free in the limit of approach to the singularity, as seen above. This shows there is an interesting tying up of shear and inhomogeneity for a collapsing matter cloud to end up in a naked singularity. Shear could be a physical process that distorts the shape of apparent horizon surface to expose the singularity. This matches with why shear-free collapse always ends in black hole. Interestingly, a similar tying up of inhomogeneity and shear (anisotropy) appears to exist for nonsingular cosmological models as was argued by one of us [8]. We thus see that the collapse of a general matter field could generically tend to an outcome which is observed in the dust case [1]. This raises an interesting possibility whether general collapse configurations could also tend to a dustlike model in the vicinity of the singularity.

Finally, consider the equation for outgoing radial null geodesics $dt/dr = e^{\psi-\nu}$. One could now write the above in terms of the variables ($u = r^\alpha, R$). Choosing $\alpha = \frac{5}{3}$, and using Eq. (7) we get,

$$\frac{dR}{du} = \frac{3}{5}\left(\frac{R}{u} + \frac{\sqrt{v}v'}{\sqrt{\frac{R}{u}}}\right)\left(\frac{1-\frac{F}{R}}{\sqrt{G}[\sqrt{G}+\sqrt{H}]}\right) \tag{26}$$

where we have written $R = rv$. Now if the null geodesics do terminate at the singularity in the past with a definite tangent, then at the singularity we have $\frac{dR}{du} > 0$, in the (u, R) plane with a finite value. It follows that all points $r > 0$ on the singularity curve are then covered because $F/R \to \infty$ (when weak energy condition is satisfied and pressures are positive then $F(r)$ tends to a finite positive value for any $r > 0$ on the singularity curve) with $\frac{dR}{du} \to -\infty$. So noncentral singularities will always be covered for a general *type I* matter distribution. The central singularity could however be visible.

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