

# Corrections to the Cardy-Verlinde formula from the generalized uncertainty principle

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In this Letter, we compute the corrections to the Cardy-Verlinde formula of the  $d$ -dimensional Schwarzschild black hole. These corrections stem from the generalized uncertainty principle. Then we show one can take into account the generalized uncertainty principle corrections of the Cardy-Verlinde entropy formula by just redefining the Virasoro operator  $L_0$  and the central charge  $c$ .

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## I. INTRODUCTION

It is commonly believed that any valid theory of quantum gravity must necessarily incorporate the Bekenstein-Hawking definition of black hole entropy [1,2] into its conceptual framework. However, the microscopic origin of this entropy remains an enigma for two reasons. First of all although the various counting methods have pointed to the expected semiclassical result, there is still a lack of recognition as to what degrees of freedom are truly being counted. This ambiguity can be attributed to most of these methods being based on dualities with simpler theories, thus obscuring the physical interpretation from the perspective of the black hole in question. Secondly, the vast and varied number of successful counting techniques only serve to cloud up an already fuzzy picture.

The Cardy-Verlinde formula proposed by Verlinde [3] relates the entropy of a certain conformal field theory (CFT) with its total energy and its Casimir energy in arbitrary dimensions. Using the AdS<sub>d</sub>/CFT<sub>d-1</sub> (anti-de Sitter) [4] and dS<sub>d</sub>/CFT<sub>d-1</sub> correspondences [5], this formula has been shown to hold exactly for different black holes (see, for example, [6–15]).

Black hole thermodynamic quantities depend on the Hawking temperature via the usual thermodynamic relations. The Hawking temperature undergoes corrections from many sources: the quantum corrections [16], the self-gravitational corrections [17], and the corrections due to the generalized uncertainty principle.

The generalized uncertainty principle corrections are not tied down to any specific model of quantum gravity; these corrections can be derived using arguments from string theory [18] as well as other approaches to quantum gravity [19].

In this Letter we concentrate on the corrections due to the generalized uncertainty principle. In Sec. II we review the connection between uncertainty principle and thermodynamic quantities, and then we drive the corrections to these quantities due to the generalized uncertainty principle [20]. In Sec. III we consider the

generalized Cardy-Verlinde formula of a  $d$ -dimensional Schwarzschild black hole [21,22]; then we obtain the generalized uncertainty principle corrections to this entropy formula.

## II. THE GENERALIZED UNCERTAINTY PRINCIPLE

A  $d$ -dimensional Schwarzschild black hole of mass  $M$  is described by the metric

$$ds^2 = -\left(1 - \frac{16\pi G_d M}{(d-2)\Omega_{d-2} c^2 r^{d-3}}\right) c^2 dt^2 + \left(1 - \frac{16\pi G_d M}{(d-2)\Omega_{d-2} c^2 r^{d-3}}\right)^{-1} dr^2 + r^2 d\Omega_{d-2}^2, \quad (1)$$

where  $\Omega_{d-2}$  is the metric of the unit  $S^{d-2}$  and  $G_d$  is the  $d$ -dimensional Newton's constant. Since the Hawking radiation is a quantum process, the emitted quanta must satisfy the Heisenberg uncertainty principle

$$\Delta x_i \Delta p_j \geq \hbar \delta_{ij}, \quad (2)$$

where  $x_i$  and  $p_j$ ,  $i, j = 1, \dots, d-1$ , are the spatial coordinates and momenta, respectively. By modeling a black hole as a  $d$ -dimensional cube of size equal to twice its Schwarzschild radius  $r_s$ , the uncertainty in the position of a Hawking particle at the emission is

$$\Delta x \approx 2r_s = 2\lambda_d \left(\frac{G_d M}{c^2}\right)^{1/(d-3)}, \quad (3)$$

where

$$\lambda_d = \left(\frac{16\pi}{(d-2)\Omega_{d-2}}\right)^{1/(d-3)}. \quad (4)$$

Using Eq. (2), the uncertainty in the energy of the emitted particle is

$$\Delta E \approx c \Delta p \approx \frac{M_{pl} c^2}{2\lambda_d} m^{-1/(d-3)}, \quad (5)$$

where  $m = M/M_{pl}$  is the mass in Planck units and  $M_{pl} = (\hbar^{d-3}/c^{d-5} G_d)^{1/(d-2)}$  is the  $d$ -dimensional Planck mass.  $\Delta E$  can be identified with the characteristic temperature of the black hole emission, i.e., the Hawking temperature.

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Setting the constant of proportionality to  $(d-3)/2\pi$  we get

$$T = \frac{(d-3)}{4\pi\lambda_d} M_{pl} c^2 m^{-1/(d-3)}. \quad (6)$$

The entropy is

$$S = \frac{4\pi\lambda_d}{d-2} m^{(d-2)/(d-3)} = \frac{(d-3)}{(d-2)} \frac{Mc^2}{T}. \quad (7)$$

We now determine the corrections to the above results due to the generalized uncertainty principle. The general form of the generalized uncertainty principle is

$$\Delta x_i \geq \frac{\hbar}{\Delta p_i} + \alpha^2 l_p^2 \frac{\Delta p_i}{\hbar}, \quad (8)$$

where  $l_{pl} = (\hbar G_d/c^3)^{1/(d-2)}$  is the Planck length and  $\alpha$  is a dimensionless constant of order 1. There are many derivations of the generalized uncertainty principle, some heuristic and some more rigorous. Equation (8) can be derived in the context of string theory [18], non-commutative quantum mechanics [19], and from minimum length consideration [23]. The exact value of  $\alpha$  depends on the specific model. The second term on the right-hand side of Eq. (8) becomes effective when momentum and length scales are of the order of Planck mass and of the Planck length, respectively. This limit is usually called quantum regime. Inverting Eq. (8), we obtain

$$\begin{aligned} \frac{\Delta x_i}{2\alpha^2 l_{pl}^2} \left[ 1 - \sqrt{1 - \frac{4\alpha^2 l_{pl}^2}{\Delta x_i^2}} \right] &\leq \frac{\Delta p_i}{\hbar} \\ &\leq \frac{\Delta x_i}{2\alpha^2 l_{pl}^2} \left[ 1 + \sqrt{1 - \frac{4\alpha^2 l_{pl}^2}{\Delta x_i^2}} \right]. \end{aligned} \quad (9)$$

The corrections to the black hole thermodynamic quantities can be calculated by repeating the above argument. Setting  $\Delta x = 2r_s$ , the generalized uncertainty principle-corrected Hawking temperature is

$$T' = \frac{(d-3)\lambda_d}{2\pi\alpha^2} m^{1/(d-3)} \left[ 1 - \sqrt{1 - \frac{\alpha^2}{\lambda_d^2 m^{2/(d-3)}}} \right] M_{pl} c^2. \quad (10)$$

Equation (10) may be Taylor expanded around  $\alpha = 0$ :

$$T' = \frac{(d-3)}{4\pi\lambda_d} m^{-1/(d-3)} \left[ 1 + \frac{\alpha^2}{4\lambda_d^2 m^{2/(d-3)} + \dots} \right] M_{pl} c^2. \quad (11)$$

### III. GENERALIZED UNCERTAINTY PRINCIPLE CORRECTIONS TO THE CARDY-VERLINDE FORMULA

The entropy of a  $(1+1)$ -dimensional CFT is given by the well-known Cardy formula [24]

$$S = 2\pi \sqrt{\frac{c}{6} \left( L_0 - \frac{c}{24} \right)}, \quad (12)$$

where  $L_0$  represent the product  $ER$  of the energy and radius, and the shift of  $\frac{c}{24}$  is caused by the Casimir effect. After making the appropriate identifications for  $L_0$  and  $c$ , the same Cardy formula is also valid for CFT in arbitrary spacetime dimensions  $(d-1)$  in the form [3]

$$S_{\text{CFT}} = \frac{2\pi R}{d-2} \sqrt{E_c(2E - E_c)}, \quad (13)$$

the so-called Cardy-Verlinde formula, where  $R$  is the radius of the system,  $E$  is the total energy, and  $E_c$  is the Casimir energy, defined as

$$E_c = (d-1)E - (d-2)TS. \quad (14)$$

So far, mostly asymptotically AdS and dS black hole solutions have been considered [4–15]. In [21], it is shown that even the Schwarzschild and Kerr black hole solutions, which are asymptotically flat, satisfy the modification of the Cardy-Verlinde formula

$$S_{\text{CFT}} = \frac{2\pi R}{d-2} \sqrt{2EE_c}. \quad (15)$$

This result holds also for the various charged black hole solution with asymptotically flat spacetime [22].

In this section we compute the generalized uncertainty principle corrections to the entropy of a  $d$ -dimensional Schwarzschild black hole described by the Cardy-Verlinde formula Eq. (15). The Casimir energy Eq. (14) now will be modified due to the uncertainty principle corrections as

$$E'_c = (d-1)E' - (d-2)T'S'. \quad (16)$$

It is easily seen that

$$\begin{aligned} 2E'E'_c &= 2(d-1)E'^2 - 2(d-2)E'T'S' \\ &= \frac{8(d-1)\pi^2 T'^2}{(d-3)^2} - \frac{4\pi(d-2)T'^2 S'}{d-3}. \end{aligned} \quad (17)$$

We substitute expressions (16) and (17) which were computed to first order in  $\alpha^2$  in the Cardy-Verlinde formula in order that generalized uncertainty principle corrections to be considered:

$$\begin{aligned} S'_{\text{CFT}} &= S_{\text{CFT}} \left\{ 1 + \frac{\pi T}{(d-3)E_c} \right. \\ &\quad \left. \times \left[ \frac{4\pi\Delta T}{d-3} - (d-2)T\Delta S - 2(d-2)\Delta TS \right] \right\}, \end{aligned} \quad (18)$$

where

$$\Delta T = \frac{(d-3)\alpha^2 m^{-3/(d-3)}}{16\pi\lambda_d^3} M_{pl} c^2, \quad (19)$$

and

$$\Delta S = \frac{-\pi\alpha^2 m^{(d-4)/(d-3)}}{(d-4)\lambda_d}, \quad d > 4. \quad (20)$$

We would like to express the modified Cardy-Verlinde entropy formula in terms of the energy and Casimir energy, therefore rewriting  $T$ ,  $S$ ,  $\Delta T$ , and  $\Delta S$  in terms of energy as following:

$$T = \frac{(d-3)E}{2\pi}, \quad (21)$$

$$S = \frac{2\pi(d-3)}{(d-2)} (2\lambda_d)^{3-d} (M_{pl} c^2)^{d-2} E^{2-d}, \quad (22)$$

$$\Delta T = \frac{(d-3)}{2\pi(M_{pl} c^2)^2} \alpha^2 E^3, \quad (23)$$

$$\Delta S = \frac{-\pi\alpha^2}{(d-4)\lambda_d} \left( \frac{2\lambda_d E}{M_{pl} c^2} \right)^{4-d}. \quad (24)$$

To obtain the last equation we have used the Eqs. (5) and (20). Then, Eq. (18) can be rewritten as

$$S'_{\text{CFT}} = S_{\text{CFT}} \left\{ 1 + \frac{\alpha^2}{2E_c} \left[ \frac{2E^3}{(M_{pl} c^2)^2} + (d-3)E^{5-d} (M_{pl} c^2)^{d-4} \times (2\lambda_d)^{3-d} \left( \frac{d-2}{d-4} - 2(d-3) \right) \right] \right\}. \quad (25)$$

As we saw in the above discussion these corrections are caused by the generalized uncertainty principle.

For the Schwarzschild black holes, the dual CFT lives on a flat space, and thus the energy has no subextensive part. Since the Casimir energy vanishes, the Cardy-Verlinde formula (13) makes no sense in this case. In the two-dimensional conformal field theory, when the conformal weight of the ground state is zero, we have

$$S = 2\pi\sqrt{\frac{cL_0}{6}}. \quad (26)$$

If we use  $E_c R = (d-2)S_c/2\pi$  in (13), where  $S_c$  is the Casimir entropy, and drop the subtraction of  $E_c$  in analogy with Eq. (26), we obtain the generalization to  $(d-1)$  dimensions,

$$S = \frac{2\pi}{d-2} \sqrt{\frac{cL_0}{6}}, \quad (27)$$

where  $L_0 = ER$  and  $c/6 = (d-2)S_c/\pi = 2E_c R$ . Then, we can take into account the generalized uncertainty principle corrections of the Cardy-Verlinde entropy formula by just redefining the Virasoro operator and the central charge as follows:

$$L'_0 = E'R = \frac{R\lambda_d}{\alpha^2} m^{1/(d-3)} \left[ 1 - \sqrt{1 - \frac{\alpha^2}{\lambda_d^2 m^{2/(d-3)}}} \right] M_{pl} c^2, \quad (28)$$

$$\begin{aligned} c' &= 12E'_c R \\ &= \frac{12R\lambda_d}{\alpha^2} m^{1/(d-3)} \left\{ \left[ 1 - \sqrt{1 - \frac{\alpha^2}{\lambda_d^2 m^{2/(d-3)}}} \right] M_{pl} c^2 \right. \\ &\quad \times \left[ (d-1) - (d-2)(d-3) \left( \frac{\alpha}{\lambda_d} \right)^{d-2} \right. \\ &\quad \left. \left. \times I \left( 1, d-4, \frac{\lambda_d m^{1/(d-3)}}{\alpha} \right) \right] \right\}. \end{aligned} \quad (29)$$

Also the first order corrections to the  $L_0$  and  $c$  are given by

$$\begin{aligned} \Delta L_0 &= L'_0 - L_0 = (E' - E)R \\ &= \frac{R\alpha^2}{8\lambda_d^3 m^{3/(d-3)}} M_{pl} c^2 = \frac{\alpha^2}{4\lambda_d^2 m^{2/(d-3)}} L_0, \end{aligned} \quad (30)$$

$$\begin{aligned} \Delta c &= c' - c = 12R(E'_c - E_c) \\ &= 12R \left[ \frac{(d-1)\alpha^2}{\lambda_d^3 m^{3/(d-3)}} M_{pl} c^2 + (2\lambda_d)^{3-d} \right. \\ &\quad \left. \times (M_{pl} c^2)^{d-4} \alpha^2 E^{5-d} \left( \frac{d-2}{d-4} - (d-3) \right) \right]. \end{aligned} \quad (31)$$

Thus, this redefinition can be considered as a renormalization of the quantities entering in the Cardy formula.

#### IV. CONCLUSION

In this paper we have examined the effects of the generalized uncertainty principle in the generalized Cardy-Verlinde formula. The general form of the generalized uncertainty principle is given by Eq. (8). Black hole thermodynamic quantities depend on the Hawking temperature via the usual thermodynamic relation. The Hawking temperature undergoes corrections from the generalized uncertainty principle as Eq. (10). Then we have obtained the corrections to the entropy of a dual

conformal field theory live on flat space as Eqs. (18) and (25). We have considered this point that the Cardy-Verlinde (generalized Cardy-Verlinde) formula is the outcome of a striking resemblance between the thermodynamics of CFTs with asymptotically Ads (flat) duals and CFTs in two dimensions. After that we have obtained the corrections to the quantities entering the Cardy-Verlinde formula: Virasoro operator and the central charge. The

corresponding problem for the Schwarzschild-AdS metric is in progress by the author.

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