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Gauge five-brane moduli in four-dimensional heterotic models

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We present a Kähler potential for four-dimensional heterotic M-theory which includes moduli describing a gauge five-brane living on one of the orbifold fixed planes. This result can also be thought of as describing compactifications of either of the weakly coupled heterotic strings in the presence of a gauge five-brane. This is the first example of a Kähler potential in these theories which includes moduli describing background gauge field configurations. Our results are valid when the solitons width is much smaller than the size scale of the Calabi-Yau threefold and can be used to provide a more complete description of some moving brane scenarios. We point out that, in general, it is not consistent to truncate away the gauge five-brane moduli in a simple manner.

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I. INTRODUCTION

Heterotic M-theory [1,2], the compactification of the Hořava-Witten strongly coupled limit of the $E_8 \times E_8$ heterotic string [3,4] on a manifold of SU(3) holonomy, is one of the most promising corners of the M-theory moduli space studied to date from a phenomenological point of view. The theory combines phenomenological successes of its weekly coupled counterpart [5] with a natural mechanism for obtaining the correct strength of gravitational interactions, through a kind of "large extra dimensions" mechanism [1,6].

The vacua associated with heterotic M-theory, which were presented in [1,2,7], are nontrivial domain wall solutions with a warping in the direction of the bulk and, more importantly from the point of view of this paper, gauge field expectation values on at least one of the fixed planes (other papers written on the vacua of this theory include [8,9]). This gauge field background is taken to live entirely within the Calabi-Yau threefold (i.e., it is not allowed to depend on the four external directions and is taken to be zero when its index is external) in order to maintain four-dimensional Poincaré invariance. It is this vacuum which has been used in the study of four-dimensional phenomenology and modulus evolution. Because of the model's many successes from a particle physics standpoint extensive studies have been made of the moduli evolution about this vacuum, the basic solutions being provided in [10]. However, to the authors' knowledge, no one has included any of the moduli describing the background gauge field configuration in obtaining such cosmological solutions. The reason for this is quite simple—the relevant kinetic terms are not known (although some other information about these moduli has been obtained in [11–15]). The reason for this lack of a four-dimensional theory is quite simple to understand. The most straightforward way to calculate such kinetic terms would be to start with a background solution which described a Calabi-Yau compactification of the theory, including the sections of the gauge bundles living on the fixed planes. One would then take the integration constants in this solution, promote them to be four-dimensional fields, and plug the resulting configuration into the higher dimensional action. Integrating out the internal dimensions would then naively result in the desired terms in the four-dimensional effective action. However, such exact solutions on a compact Calabi-Yau threefold are not known rendering this calculation impossible.

The advent of some recent scenarios based upon moving and colliding M5 branes [16,17] has made the need to include some of the gauge bundle moduli in our cosmological analysis even more pressing. The M5 branes concerned can be included in the vacuum solution without breaking $N = 1$ supersymmetry if they are oriented parallel to the fixed planes in the bulk with two of their world volume directions wrapping a holomorphic curve within the Calabi-Yau [1,18]. The scenarios mentioned above are based upon the position modulus of an M5 brane evolving in such a way that the object collides with an orbifold fixed plane [19–21]. However it is not the case that during such a collision the M5 brane disappears with nothing else in the situation changing—this would result in an inconsistency in the cohomology condition, for example, which essentially says that the charges on the fixed planes and M5 branes should sum to zero. Various considerations, including study of the cohomology condition and examination of extra light states which appear at collision, lead us to believe that one thing that might happen on collision is a so-called small instanton transition [22,23]. Here, during collision, the M5 brane disappears and is replaced with a gauge five-brane living on the relevant orbifold fixed plane. A gauge five-brane is a solitonic object made completely out of low-energy fields—including gauge fields [24]. The object appears

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with fundamental length scale width just after the collision and then could spread out with time to become more diffuse. The moduli which describe the gauge five-brane are examples of gauge bundle moduli. The easiest way to understand this is to observe that the soliton is essentially a Yang-Mills instanton with various gravitational field dressings. Once the gauge field core of the object is known the dressing can be determined completely in terms of this (up to certain discrete choices which are available) and so the moduli describing the gauge five-brane are simply the moduli of the Yang-Mills instanton—i.e., moduli associated with gauge bundle on that fixed plane.

As we shall see, bundle moduli cannot in general be consistently truncated off and, therefore, represent an essential part of heterotic low-energy effective theories which have been widely neglected so far. In particular, they must be included in cosmological scenarios, such as those mentioned above, where the gauge bundle may evolve in time. For example, one would like to understand whether or not the gauge five-brane *does* spread out after a small instanton transition. A prerequisite for such an investigation would be knowledge of the kinetic terms for the appropriate moduli.

Given this situation in this paper we present a calculation of the effective four-dimensional theory which describes the centered moduli space of the gauge five-brane (neglecting nonperturbative potentials). This theory contains, for instance, the size modulus for the soliton mentioned above. To our knowledge this is the first example of a Kähler potential which includes gauge bundle moduli describing the background configuration of gauge fields in Heterotic M-theory.

In obtaining this four-dimensional action we circumvent the problem of obtaining an explicit background solution with which to work by realizing that the gauge five-brane, at least when its width is small with respect to the curvature scale of the Calabi-Yau, is in some sense a very localized object. In such a regime the five-brane does not, outside of its core, probe the directions transverse to it to any significant degree [22]. In particular, in some senses, it does not know if the transverse space is compact or asymptotically flat. The idea then is to construct an approximate solution for the gauge and gravitational fields which is only valid close to the gauge five-brane (in a manner to be made explicit later). One then has to see if the effective action can be reliably calculated with only this limited information—i.e., can we calculate the effective theory describing the object without knowing what happens far from its world volume in the transverse space. We find that the answer is, as one might expect physically, in the affirmative.

Although we will be working in the heterotic M-theory set up here it should be stressed that similar configurations to the gauge five-brane appear in many other phenomenologically viable compactifications of string theory. Many of the comments made above would equally well apply to these cases and one would expect the method we present to be viable there as well. For example one could consider a situation in type I where we have an instanton based configuration living on a stack of D_p branes. Such a configuration could have been created by the collision of a D_{p-4} brane [25]. We would like to stress that these solitonic objects, as well as the ones we consider directly in this paper, do not have to be created by brane collisions. They can exist in these vacua independently of such considerations.

The outline of this paper is as follows. In Section II we shall introduce the higher dimensional action upon which our analysis is based. We will review Strominger's solution [24] describing a gauge five-brane in flat space and shall then proceed to generalize this solution to give an approximate configuration upon which we can carry out our dimensional reduction. In Section III we proceed to the calculation of the effective action in four dimensions, first outlining a subtlety associated with promoting the integration constants of the background configuration to be four-dimensional moduli fields, and then performing the dimensional reduction necessary to obtain the four-dimensional theory. In Section IV we present our results, in particular, couching our findings in terms of a Kähler potential. In Section V we comment on possible directions of future work.

Our index conventions are as follows. Indices $\mu, \nu, \ldots$ and $M, N, \ldots$ label world volume directions of the gauge five-brane with $\mu, \nu = 0, 1, 2, 3$, and $M, N = 4, 5$. The μ directions will eventually correspond to four-dimensional uncompactified space while the M directions will be associated with a holomorphic 2 cycle in a Calabi-Yau threefold. Indices $A, B, \ldots$ label the transverse dimensions with $A, B = 6, 7, 8, 9$. These directions will eventually correspond to directions in the compactified space transverse to the gauge five-brane. We shall use indices $a, b = 4, .., 9$ to denote a general direction in the internal space.

II. HIGHER DIMENSIONAL ACTION AND BACKGROUND SOLUTION

Our starting point is the low-energy effective action of the $E8 \times E8$ heterotic string. This action also provides an effective description of ten-dimensional heterotic M-theory [26] and, with a suitable change of gauge group, the weakly coupled SO(32) heterotic string at low energies. Thus the discussion and results presented in this paper are equally valid in these corners of the M-theory moduli space. Working with this description of these theories is valid to the approximations we shall be making and results in considerable simplification as compared to carrying out the analysis, for example, in the 11-dimensional picture of heterotic M-theory. One obvious simplification as compared to that case is that we no longer have to worry about warping in the Hořava-Witten orbifold direction as this is already included in the effective theory we are using to the approximation we require. The action is given by

$$S_{10} = \frac{1}{2\kappa_{10D}^2} \int d^{10}x\sqrt{-g}e^{2\phi}\Big[-R - 4(\partial\phi)^2 + \frac{1}{3}H^2 + \frac{\alpha'}{30}\mathrm{Tr}F^2 + \ldots\Big], \tag{1}$$

where the field strength H takes the usual form

$$H = dB - \frac{\alpha'}{30}\omega_{3YM} + \ldots. \tag{2}$$

The ...'s here express the fact that we have dropped some terms which we will not need, for the particular calculation we are interested in, to our approximations. The traces in this expression are in the adjoint of $E_8 \times E_8$. The Chern-Simons three-form associated with the $E_8 \times E_8$ gauge fields is denoted by ω_{3YM} and B is a two-form potential. The field strength of the $E_8 \times E_8$ gauge fields is F and ϕ is the ten-dimensional dilaton. The action is valid to first order in α' which is the order we will be working to throughout this paper.

A. Gauge five-brane solution in asymptotically flat space

There is a solution of this theory due to Strominger [24] which describes a gauge five-brane in ten-dimensional asymptotically flat space (the 11-dimensional counterpart of this solution in the heterotic M-theory case was given in [27]). While as it stands this is obviously not a suitable background solution for our purposes it will be very important in the following analysis and so we take the time to describe it in some detail. The solution assigns values to the ten dimensional fields as follows

$$A_B = \sigma_\gamma\theta^\gamma \frac{2i\rho^2 r^C\bar{\sigma}_{CB}}{R^2(R^2+\rho^2)}\bar{\sigma}_\delta\theta^\delta, \tag{3}$$

$$e^{-2\phi} = e^{-2\phi_0}\Big[1 + 8\alpha'\frac{R^2+2\rho^2}{(R^2+\rho^2)^2}\Big], \tag{4}$$

$$ds_{10}^2 = \eta_{\mu\nu}dx^\mu dx^\nu + \delta_{MN}dx^M dx^N + e^{2\phi_0-2\phi}(\delta_{AB}dx^A dx^B), \tag{5}$$

$$H_{ABC} = \epsilon_{ABC}{}^D\partial_D\phi. \tag{6}$$

In these expressions ϕ_0, ρ, and θ^γ are constants and we make use of the following definitions

$$(r^C) = (x^6, x^7, x^8, x^9), \tag{7}$$

$$R^2 = \delta_{AB}r^A r^B. \tag{8}$$

We also have the constraint

$$\sum_{\gamma=1}^{4}(\theta^\gamma)^2 = 1 \tag{9}$$

on the quantities θ^γ.

Indices μ, ν, ... and M, N, ... label world volume directions with μ, $\nu = 0, 1, 2, 3$ and M, $N = 4, 5$. Indices A, B, ... label the transverse dimensions with A, $B = 6, 7, 8, 9$. We have split the world volume indices into two groups like this and have introduced r and R to make our notation compatible with the discussion in later sections when we will wrap two of the world volume directions of the gauge five-brane up on a holomorphic cycle in a Calabi-Yau threefold. We define $\sigma_A = (1_{[2]\times[2]}, i\vec{\tau})$ where τ^i, $i = 1, 2, 3$ are the Pauli matrices. The Hermitian conjugate matrices are $\bar{\sigma}_A = (1_{[2]\times[2]}, -i\vec{\tau})$ and we define the self dual and anti self dual two index objects, $\sigma_{AB} = \frac{1}{4}(\sigma_A\bar{\sigma}_B - \sigma_B\bar{\sigma}_A)$ and $\bar{\sigma}_{AB} = \frac{1}{4}(\bar{\sigma}_A\sigma_B - \bar{\sigma}_B\sigma_A)$. The completely antisymmetric symbol in four dimensions is denoted ϵ^s_{ABCD}, and $\epsilon^s_{6789} = 1$. The associated tensor is denoted ϵ_{ABCD}.

It should be noted that while this solution is in a singular gauge [28] (the gauge field given above is divergent at $r^C = 0$) any physical (i.e., gauge invariant) quantity associated with it is everywhere finite.

The solution is accurate, as is the action we have presented, to first order in α'. It describes a soliton with six world volume dimensions and four transverse ones. The object is, at its core, a Yang-Mills instanton embedded within some SU(2) subgroup of the E_8 associated with the fixed plane on which it exists in the higher dimensional picture. This Yang-Mills configuration has, as is well known, several collective coordinates. These are integration constants of the solution which describe flat directions in the object's moduli space. For example the object has some finite width in the four transverse directions which is determined by the constant ρ in the solutions given above. The instanton has an orientation within the SU(2) which is determined by the parameters θ^γ. Although there are four θ's they are subject to one constraint (24), and so the size and SU(2) orientation together makes a total of four parameters which span the so-called centered moduli space of the instanton. In addition to the centered moduli space the object has four collective coordinates which describe its motion in the transverse space and 112 zero modes associated with the embedding of SU(2) within E_8. We shall concentrate on the centered moduli space in this paper and leave the analysis of these extra degrees of freedom for future work. It is fairly easy to see why this is a consistent thing to do, at least in the case of the translation moduli. The first nontrivial test that it is possible to consistently truncate off the position moduli is given by the observation [28] that the moduli space of a pure Yang-Mills instanton factors into a product of the centered moduli space and the space of the position moduli. Since this result must be regained in the limit where we "freeze" the other moduli this is a necessary condition for our truncation to be consistent. The real test for consistent truncation however comes from checking that the position moduli enter the effective four-dimensional theory bilinearly. We have indeed checked that this is the case and so we are justified in truncating to the centered moduli space.

In any case one may wonder whether these gauge bundle moduli span the moduli space of the gauge five-brane or whether there are other moduli associated with the "Neveu-Schwarz" (NS) dressing. As was demonstrated in [24] however, given the gauge field configuration presented in (3) we can determine the gravitational dressing, also given in Eqs. (3), and no more integration constants appear (although different discrete choices are possible [29–31]). The fact that this object is based upon a self-dual solution to the Yang-Mills equations will be of central importance when we come to talk of generalizing the work presented in this paper. There exists a powerful tool for obtaining such configurations, in the form of the Atiyah Drinfeld Hitchin Manin (ADHM) construction [28], which we can use as a starting point for the analysis of more complicated situations.

B. Wrapping up the gauge five-brane

As we have already mentioned, the solution given in the previous section is, as it stands, of no use for our current purposes as the gauge five-brane described therein lives in asymptotically flat ten-dimensional space. We wish to describe a situation where the manifold we are working on is not $\mathcal{M}^{10}$ but $\mathcal{M}^4 \times X$ where X is a compact six-dimensional manifold. To preserve $\mathcal{N} = 1$ supersymmetry in four dimensions X has to have SU(3) holonomy with respect to the generalized connection including the three-form field strength [32]. In addition, for $\mathcal{N} = 1$ supersymmetry and four-dimensional Poincaré invariance we require that four of the world volume directions of our gauge five-brane span $\mathcal{M}^4$ with the remaining two wrapping a holomorphic 2 cycle within X [1].

Now obtaining an exact solution describing a compact Calabi-Yau manifold with an associated gauge bundle which includes a piece of the background configuration which can be identified as a wrapped gauge five-brane is, as was mentioned in the introduction, beyond the capabilities of current technology. However we can obtain an *approximate* solution which describes such a situation and which is valid only near the cycle which the five-brane wraps. We shall find in the next section that this approximate solution is all that we require, as might have been expected on physical grounds, to calculate the terms due to the presence of the five-brane in the desired effective action.

So how do we construct this approximate solution? We shall really only require one property of our compactification manifold in order to construct such an approximation as a generalization of the solution we have already encountered. That property is,

(i) Near the 2 cycle which the gauge five-brane wraps the compact space may be written as $X = C_2 \times C_4$, where C_2 is a Riemann surface and C_4 is a complex four-dimensional space.

However, in order to demonstrate the existence of Calabi-Yau threefolds with this property and to give us a concrete context in which to carry out our calculation we shall concentrate on the class of compact metric configurations which can be obtained by blowing up six-dimensional orbifolds of SU(3) holonomy. Note "SU(3) holonomy" here denotes the holonomy of the metric neglecting the back reaction due to the presence of the gauge five-brane. Our analysis, of course, includes this back reaction and so our full metric is not of SU(3) holonomy.

We shall therefore start with some six-dimensional orbifold which in addition to having SU(3) holonomy has the factorization property mentioned above near the 2 cycle on which we shall wrap our gauge five-brane. In order to simplify the following calculation we shall also require that the point group of our orbifold is such as to project out off diagonal metric moduli. This is not necessary for our method to work but results in considerable simplification of the calculation, in particular, in keeping the number of moduli we have to deal with at a manageable level, while still retaining the essential ingredients we are interested in.

An example of an orbifold which has all three of these properties is a $Z_8 - I$ Coxeter orbifold with an SO(5) $\times$ SO(9) lattice [33]. The orbifold is constructed as follows. We define three complex coordinates

$$z_1 = x^4 + ix^5, \tag{10}$$

$$z_2 = x^6 + ix^7, \tag{11}$$

$$z_3 = x^8 + ix^9. \tag{12}$$

To construct the orbifold we start with flat space spanned by these complex coordinates. We then make identifications under the point group. In our case the action of the point group can be written as follows

$$z_1 \rightarrow e^{2\pi i(1/4)} z_1, \tag{13}$$

$$z_2 \rightarrow e^{2\pi i(1/8)} z_2, \tag{14}$$

$$z_3 \rightarrow e^{-2\pi i(3/8)} z_3. \tag{15}$$

As we have said, this particular choice is an example where the off diagonal metric moduli are projected out by the orbifolding. This can be seen by considering the action of the point group on an off diagonal component of the metric such as $g_{z_1\bar{z}_2}$.

Having gauged the point group we then perform identifications under a lattice to obtain a compact manifold. We use the root lattice of SO(5) $\times$ SO(9), which is compatible with our choice of point group and with our need for a suitable 2 cycle on which to wrap the brane [33]. C_2 will be identified with the space which is compactified by modding out by the SO(5) lattice and C_4 with the space which is modded out by the SO(9) lattice. The orbifold fixed loci in this compactification are then blown up using some appropriate resolutions [34] leaving us with a class of Calabi-Yau manifolds with the desired properties.

Wrapping the gauge five-brane solution up on a 2 cycle is then simple. We shall work in the "downstairs" picture where we simply consider the fundamental region of the orbifold. We choose the size of the fixed loci blow ups and the gauge five-brane width to be much smaller than the overall size of the orbifold. We then choose a holomorphic 2 cycle to wrap around which is determined by choosing a point in C_4 which is far from any of the orbifolds resolved fixed loci and the rest of the bundle which is also assumed to be localized (perhaps in the form of more gauge five-branes). The solution given in Section II A can then be generalized to wrap this cycle with the trivial modification of making the identifications that turn two of the world volume directions into the cycle (the solution has symmetries which are compatible with this).

All we then have to do is use coordinate transformations to introduce the constants which will form the metric moduli of X and introduce constants which will become the imaginary parts in their complexifications. We then have a solution which is valid near to the 2 cycle which will be appropriate to use in our dimensional reduction. Of course the solution differs substantially from the real situation we are interested in far from the gauge five-brane in the transverse space, our approximate solution being asymptotically flat in these directions. However as we shall see in the next section we can show that this approximate solution, valid in this restricted volume to an approximation to be made more concrete later, is all we require to compute the effective theory we desire.

The coordinate transformations we use to introduce the metric moduli are, in real coordinates

$$x^A \to \mathcal{V}^{1/6}{}_{(A)} x^A, \tag{16}$$

where we have introduced the notation $\mathcal{V}_{(A)}$. This is taken to mean $\mathcal{V}_{(1)} = \mathcal{V}_{(2)} = \mathcal{V}_1$, $\mathcal{V}_{(3)} = \mathcal{V}_{(4)} = \mathcal{V}_2$. We have chosen to use this notation as each volume modulus, while associated with one component of the metric in complex coordinates ($g_{z_i \bar{z}_i}$ for $i = 1 \dots 3$), is associated with two real coordinates.

The constant two-form potential contributions which will form the completion of the complex volume moduli can simply be added to the configuration and it will remain a solution. These can be seen in the equation for the two-form below (they are the χ's). It should be noted however that the off diagonal components of the two-form are projected out for our choice of point group in exactly the same way as we saw above for the metric moduli.

Combining all of this information we may now write down our approximate background solution including all of the necessary integration constants.

$$A_B = \sigma_\gamma \theta^\gamma \frac{2i\rho^2 r^C \bar{\sigma}_{CB} \mathcal{V}^{1/6}_{(B)}}{R^2(R^2 + \rho^2)} \bar{\sigma}_\delta \theta^\delta, \tag{17}$$

$$e^{-2\phi} = e^{-2\phi_0}\left[1 + 8\alpha' \frac{R^2 + 2\rho^2}{(R^2 + \rho^2)^2}\right], \tag{18}$$

$$\begin{aligned} ds^2_{10} = {} & f g_{\mu\nu} dx^\mu dx^\nu + \mathcal{V}^{1/3}_3 \delta_{MN} dx^M dx^N \\ & + e^{2\phi_0 - 2\phi}(\mathcal{V}^{1/3}_{(A)} \delta_{AB} dx^A dx^B), \end{aligned} \tag{19}$$

$$B_{AB} = B^{bg}_{AB} + \frac{1}{6}\chi_{(1)}\Pi^{(1)}_{AB} + \frac{1}{6}\chi_{(2)}\Pi^{(2)}_{AB}, \tag{20}$$

$$B_{MN} = B^{bg} + \frac{1}{6}\chi_{(3)}\Pi^{(3)}_{MN}. \tag{21}$$

We have denoted three harmonic two-forms on a flat orbifold of our type as (in the absence of the instanton) $\Pi^{(1)}_{ab} = (+1|_{a=1,b=2}, -1|_{a=2,b=1})$, $\Pi^{(2)}_{ab} = (+1|_{a=3,b=4}, -1|_{a=4,b=3})$, and $\Pi^{(3)}_{ab} = (+1|_{a=5,b=6}, -1|_{a=6,b=5})$. In addition, B^{bg} is the order α' background contribution to the two-form due to the presence of the five-brane.

We now have the following definitions for r^C and R

$$(r^C) = (\mathcal{V}^{1/6}_1 x^6, \mathcal{V}^{1/6}_1 x^7, \mathcal{V}^{1/6}_2 x^8, \mathcal{V}^{1/6}_2 x^9), \tag{22}$$

$$R^2 = \delta_{AB} r^A r^B. \tag{23}$$

We also still have the constraint on the θ^γ's

$$\sum_{\gamma=1}^{4} (\theta^\gamma)^2 = 1. \tag{24}$$

$\mathcal{V}_3$ is a metric modulus associated with the size of the 2 cycle the five-brane wraps, i.e., it is the size modulus associated with C_2, and χ_3 is its corresponding axion. $\mathcal{V}_1$ and $\mathcal{V}_2$ are metric moduli associated with the size and shape of the four-dimensional transverse space C_4 and χ_1 and χ_2 are their corresponding axions. f is a Weyl rescaling factor which will be chosen by demanding that the Einstein Hilbert term in four dimensions is canonically normalized in terms of the four-dimensional metric $g_{\mu\nu}$.

Now that we have this approximate solution one might naively think that it is easy to compute the four-dimensional effective action describing the gauge five-brane. The "usual" procedure would be to take integration constants in this approximate solution and promote them to be four-dimensional fields. One would then take the resulting configuration, substitute it into the higher dimensional action, and integrate out the six compactified dimensions. If our physical argument that the effective theory can be calculated without knowing what happens far from the gauge five-brane's world volume is true then we should be able to do all of this without picking up non-negligible contributions from the part of the transverse space on which we cannot trust our solution. We would then end up with a four-dimensional effective action which would be valid to some well controlled approximations. In fact we shall see that, while this is broadly speaking how the calculation proceeds, a few subtleties arise which we have to deal with before we can obtain our result.

III. THE FOUR-DIMENSIONAL MODULI SPACE EFFECTIVE ACTION

As just stated the dimensional reduction we will now proceed to carry out is not completely straightforward. There are essentially three different subtleties involved in the calculation. The first of these arises in the promotion of ten-dimensional integration constants to become four-dimensional moduli fields. We find in the situation at hand that it is necessary to introduce the concept of compensators in this process as explained in Subsection III A. The second subtlety involves using small amounts of information we have about the *exact* vacuum solution under consideration to show that certain terms from the higher dimensional action do not contribute to the four-dimensional effective theory to the approximations at hand. This is explained in detail in the first half of Subsection III A. The final subtlety, which we discuss in the second half of Subsection III B, has to do with making sure that we make sensible definitions of our moduli fields.

Throughout this section the reader should bear in mind that our goal is to obtain the four-dimensional effective action which is presented in Eq. (65).

A. Promoting constants in the background solution and the inclusion of compensators

The first thing we have to do in the calculation of the effective action is to take the constants which will become the moduli we wish to describe and replace them with four-dimensional fields. We immediately encounter a subtlety in doing this, albeit one that is well known in other contexts [28]. Consider the following contribution to the gauge field

$$A_\mu = \Omega_m^{(A)} \partial_\mu m. \tag{25}$$

Here m is some modulus, a sum over moduli being implied on the right hand side, and we recall that μ labels an external four-dimensional direction.

Now let us construct a configuration by taking some background solution, promoting its integration constants to be four-dimensional fields, and adding to it contributions to the gauge field of the form given in Eq. (25). When we then take the four-dimensional fields m to be constant we recover the background solution for any value of the so-called compensators $\Omega_m^{(A)}$. Similar points could be made for the metric and two-form fields. We could have

$$g^{\rm comp}_{\mu a} = \Omega^{(g)}_{(a|m|}\partial_{\mu)} m, \tag{26}$$

$$B^{\rm comp}_{\mu a} = \Omega^{(B)}_{[a|m|}\partial_{\mu]} m. \tag{27}$$

The reason for only considering one four-dimensional index in these expressions will become clear in a moment. It is clear then that we need some way of determining what values these compensators should take when we promote the integration constants of our background solution to be four-dimensional moduli fields.

The idea of a moduli space approximation such as the one we are going to make is that a solution to the resulting four-dimensional effective action can be raised, using the ansatz used for dimensional reduction, to give a solution to the higher dimensional equations of motion to some approximations. One of the approximations that is always made is the slowly changing moduli approximation. This is that the higher dimensional equations of motion will only be solved by such a configuration up to second order in four-dimensional derivatives. Expanding the higher dimensional equations of motion in powers of four-dimensional derivatives we obtain the following.

- Zeroth Order These equations are simply the background equations of motion—if we have chosen our background configuration correctly these are automatically satisfied to our approximations.
- First Order These are the equations which determine the compensators—see below.
- Second Order These are the higher dimensional manifestations of the moduli equations of motion.

Thus the first order equations determine the compensators. This is best demonstrated by an example so let us consider the gauge field compensators. The contribution to the higher dimensional equation of motion for the gauge field at first order in four-dimensional derivatives is

$$\begin{aligned}&\mathcal{V}_{(a)}^{-(1/3)}\mathcal{V}_{(b)}^{-(1/3)}\frac{1}{3}\partial_\mu \chi_{(a)}\Pi^{(a)}_{ab}F_{ab}\alpha' \\ &+\alpha'\mathcal{D}_a(\mathcal{V}_{(a)}^{-(1/3)}[-\partial_m A_a\partial_\mu m+\mathcal{D}\Omega_m^{(A)}\partial_\mu m]) \\ &=0. \end{aligned}\tag{28}$$

Here $\mathcal{D}$ is a gauge covariant derivative and summation is implied over repeated indices. We recall that the indices $a, b \ldots$ cover the entire Calabi-Yau threefold, i.e., $a, b = 4, \ldots, 9$.

Now for this equation to be satisfied the coefficient of $\partial_\mu m$ has to vanish for each m. This leads to a separate equation for each gauge compensator. For $m \neq \chi$ we have

$$\mathcal{D}_a(\mathcal{V}_{(a)}^{-(1/3)}\mathcal{D}_a\Omega_m^{(A)}) = \mathcal{D}_a(\mathcal{V}_{(a)}^{-(1/3)}\partial_m A_a). \tag{29}$$

However, for $m = \chi_{(A)}$ we have

$$\mathcal{D}_a(\mathcal{V}_{(a)}^{-(1/3)}\mathcal{D}_a\Omega_{\chi_{(a)}}^{(A)}) = -\mathcal{V}_{(a)}^{-(1/3)}\mathcal{V}_{(b)}^{-(1/3)}\frac{1}{3}\Pi_{ab}^{(a)}F_{ab}. \tag{30}$$

We see from the structure of the source terms in these equations that the general rule is that if the background gauge field depends on a certain modulus then we have to include a gauge field compensator for that degree of freedom when we promote integration constants to be moduli. The exceptions to this rule are the fields $\chi_{(i)}$ ($i = 1\ldots3$) which have an additional source term causing them to give rise to compensators even if (as is indeed the case) the background gauge field is not dependent on them. Given these equations and our approximation to the background solution given in the previous section we can calculate expressions for the gauge field compensators. Of course, as before, our results will only be valid near the world volume of the gauge five-brane. In addition we will need two boundary conditions to fix the compensators uniquely (due to them being determined by second order differential equations). The next question then is how do we choose these boundary conditions.

The first boundary condition is simply that when we compute any physical (i.e., gauge invariant) quantity we require it to be nonsingular at the core of the gauge five-brane. This condition is enough to determine one of the integration constants that is present in the general solutions to Eqs. (29) and (30). The second boundary condition is that we require that, in our approximate solution where the transverse space is asymptotically flat, the compensator does not diverge at large distances from the gauge five-brane. This second condition is clearly necessary if our approximation is to work but can also be justified on physical grounds. The need for compensators here is a direct consequence of the presence of the five-brane—they are not needed in the case where we ignore the instanton moduli. As they are sourced by the gauge five-brane we would not expect the compensators to diverge as we go away from the soliton core in the transverse space.

Using our Eqs. (29) and (30) and these boundary conditions we can then compute the approximations to the gauge field compensators associated with the configuration presented in the previous section. The result is given below for those moduli whose compensators do not obviously vanish

$$\Omega_\rho^{(A)} = 0, \tag{31}$$

$$\Omega_{\theta_\gamma}^{(A)} = \frac{-i\rho^2\sigma_\gamma\bar{\sigma}_\delta\theta^\delta}{r^2+\rho^2}, \tag{32}$$

$$\begin{aligned}\Omega_{\mathcal{V}1}^{(A)} =& \sigma_\gamma\theta^\gamma\frac{i}{3\mathcal{V}_1}\\ &\times\bigg\{\frac{(\bar{\sigma}_C r^C\sigma_{B7}r^B r^7\sigma_D r^D + \bar{\sigma}_C r^C\sigma_{B6}r^B r^6\sigma_D r^D)}{R^2(R^2+\rho^2)}\\ &-\frac{(\bar{\sigma}_6 r^6\sigma_C r^C + \bar{\sigma}_7 r^7\sigma_C r^C)}{2R^2}\\ &+\frac{[(r^6)^2+(r^7)^2]}{2R^2}\bigg\}\bar{\sigma}_\delta\theta^\delta,\end{aligned} \tag{33}$$

$$\begin{aligned}\Omega_{\mathcal{V}2}^{(A)} =& \sigma_\gamma\theta^\gamma\frac{i}{3\mathcal{V}_2}\\ &\times\bigg\{\frac{[\bar{\sigma}_C r^C\sigma_{B9}r^B r^9\sigma_D r^D + \bar{\sigma}_C r^C\sigma_{B8}r^B r^8\sigma_D r^D]}{R^2(R^2+\rho^2)}\\ &-\frac{[\bar{\sigma}_3 r^3\sigma_C r^C + \bar{\sigma}_9 r^9\sigma_C r^C]}{2R^2}\\ &+\frac{[(r^8)^2+(r^9)^2]}{2R^2}\bigg\}\bar{\sigma}_\delta\theta^\delta,\end{aligned} \tag{34}$$

$$\Omega_{\chi_1}^{(A)} = \sigma_\gamma\theta^\gamma\frac{i\rho^2}{3}\frac{\bar{\sigma}_C r^C\bar{\sigma}_{67}\sigma_D r^D}{\mathcal{V}_1^{1/3}R^2(R^2+\rho^2)}\bar{\sigma}_\delta\theta^\delta, \tag{35}$$

$$\Omega_{\chi_2}^{(A)} = \sigma_\gamma\theta^\gamma\frac{i\rho^2}{3}\frac{\bar{\sigma}_C r^C\bar{\sigma}_{89}\sigma_D r^D}{\mathcal{V}_2^{1/3}R^2(R^2+\rho^2)}\bar{\sigma}_\delta\theta^\delta. \tag{36}$$

Thus when we promote integration constants to moduli fields in our background solution, prior to dimensional reduction, we must include these compensators when we write down the gauge field using the form shown in (25). It should be noted that, while our boundary conditions are enough to give isolated values to the integration constants, in one case at least a different discrete choice for the compensator is possible. The expression $\Omega_{\theta^\gamma}^{(A)} = -i\sigma_\gamma\bar{\sigma}_\delta\theta^\delta$ also fits all of the above criteria for our compensators. However, this choice turns the θ's into the parameters of a local gauge transformation and so they drop out of the four-dimensional effective action altogether. In short this choice means we are not including enough true integration constants in our ansatz—the ansatz is incomplete.

We should say a few words about gauge invariance at this point. Despite the compensators being gauge dependent quantities it is easy to show that the result they contribute to, i.e., the moduli space metric, is a gauge invariant quantity. This is essentially because the compensators are parts of gauge fields and so transform appropriately. The action of course is made out of gauge invariant quantities, a point we will return to briefly later, and thus leads to an invariant moduli space metric for our result.

We mentioned that we will also obtain compensators for the metric and two-form field in very similar ways. Fortunately, for the calculation we are interested in in this paper, the explicit forms of these compensators are not required and so we shall not give them here.

Collecting all the information we have presented in this section together we can now write down an approximation to the promoted background solution, where all of the integration constants have been replaced with four-dimensional fields, which is valid near the gauge five-brane's world volume. This is the configuration which it is appropriate to use in performing a reliable dimensional reduction to obtain the effective theory that we desire.

$$A_B = \sigma_\gamma \theta^\gamma \frac{2i\rho^2 r^C \bar{\sigma}_{CB} \mathcal{V}_{(B)}^{1/6}}{(B)R^2(R^2+\rho^2)} \bar{\sigma}_\delta \theta^\delta, \qquad A_\mu = \Omega_m^{(A)} \partial_\mu m, \tag{37}$$

$$e^{-2\phi} = e^{-2\phi_0}\left[1 + 8\alpha' \frac{R^2 + 2\rho^2}{(R^2+\rho^2)^2}\right], \tag{38}$$

$$\begin{aligned} ds_{10}^2 = {} & f g_{\mu\nu} dx^\mu dx^\nu + \mathcal{V}_3^{1/3} \delta_{MN} dx^M dx^N \\ & + e^{2\phi_0 - 2\phi}(\mathcal{V}_{(A)}^{1/3} \delta_{AB} dx^A dx^B) \\ & + 2\Omega^{(g)}_{(a|m|}\partial_{\mu)} m dx^a dx^\mu, \end{aligned} \tag{39}$$

$$B_{AB} = B_{AB}^{bg} + \frac{1}{6}\chi_{(A)} \Pi_{AB}^{(A)}, \tag{40}$$

$$B_{MN} = B^{bg} + \frac{1}{6}\chi_{(3)} \Pi_{MN}^{(3)}, \tag{41}$$

$$B_{a\mu} = \Omega^{(B)}_{[a|m|}\partial_{\mu]} m, \tag{42}$$

$$B_{\mu\nu} = \text{constant.} \tag{43}$$

B. Calculating the moduli space effective action

We are now in a position to calculate the centered moduli space metric of the gauge five-brane. Our procedure for this calculation comes in several parts.

Firstly we must make sure that we use all of the information we have at our disposal. By a careful examination of what we know about the nature of the exact background configuration being considered we can show that some of the terms that could contribute to the moduli space metric in fact do not. Although we do not have a complete solution to first order in α' describing the Calabi-Yau and its associated gauge bundle we do know a small amount of information about this "complete solution" (as opposed to the approximate one which we do have which is valid near to the five-brane's world volume).

- X is compact.

(i) Except for near to the resolved orbifold fixed points we know the solution for the NS fields everywhere on the compact manifold to zeroth order in α'. It is simply the relevant zeroth order parts of the approximate ansatz given in Eqs. (37)–(43).

(i) We know what order in α' various contributions come in at. This information can be gleaned from an examination of the equations of motion. For example gauge field compensators come in at zeroth order in α', as can be seen from Eqs. (29) and (30) (and we do not need any higher order corrections to them to our approximations). The metric compensators, however, are first order, their zeroth order contribution vanishing.

(i) We have a small amount of information about the index structure of the full solution. In particular we know that the compensators that we have to include, to our approximations, have one four-dimensional and one Calabi-Yau index.

We can use this information to eliminate terms (i.e., show that they are zero) in our effective action calculation as follows. We start with the ten-dimensional effective action which we repeat here for convenience

$$\begin{aligned} S_{10} = {} & \frac{1}{2\kappa_{10D}^2} \int d^{10}x \sqrt{-g} e^{2\phi}\left[-R - 4(\partial\phi)^2 + \frac{1}{3}H^2 \right. \\ & \left. + 2\alpha' \mathrm{tr} F^2 + \ldots\right]. \end{aligned} \tag{44}$$

We have changed our convention here so that all traces from now on will be taken in the fundamental of SU(2). This results in the changed coefficient of the Yang-Mills kinetic term, for example, as compared with Eq. (1).

We are now going to imagine that we have a full solution which describes the Calabi-Yau compactification and the full gauge bundle, including the gauge five-brane. We imagine plugging this solution into the different terms of the ten-dimensional action and performing the integration over the compact space. Using only the information given above about this solution, and working only up to first order in α', we start to eliminate terms.

Let us start with the ten-dimensional dilatonic Einstein Hilbert term. Consider the $O(\alpha')$ contributions to this term due to the presence of the $O(\alpha')$ gravitational compensators. Using the known zeroth order parts of the reduction ansatz and the fact that the gravitational compensators are of order α' we find that this term has the following form

$$\begin{aligned} & -\frac{1}{2\kappa^2} \int d^{10}x \sqrt{-g} e^{2\phi} R|_{\text{gravitational compensators}} \\ & = -\frac{1}{2\kappa^2} \int d^{10}x f^2 \mathcal{V}_1^{1/3} \mathcal{V}_2^{1/3} \mathcal{V}_3^{1/3} e^{2\phi_0} [R_{IJ}|_{\alpha'=0} g^{IJ}_{\text{comp}} \\ & + \nabla^I(\nabla^J g^{\text{comp}}_{IJ} - g^{KL}_{\alpha'=o} \nabla_I g^{\text{comp}}_{KL})]. \end{aligned} \tag{45}$$

Now we use the index structure of the compensator and the zeroth order solution to eliminate the first and last terms on the right hand side

$$-\frac{1}{2\kappa^2}\int d^{10}x\sqrt{-g}e^{2\phi}R|_{\text{gravitational compensators}} = -\frac{1}{\kappa^2}\int d^{10}xf^2\mathcal{V}_1^{1/3}\mathcal{V}_2^{1/3}\mathcal{V}_3^{1/3}e^{2\phi_0}[\nabla^I(\nabla^J g_{IJ}^{\text{comp}})]. \tag{46}$$

Using some standard identities, some information about our zeroth order solution, remembering that our internal manifold is compact and working to first order in α' we obtain

$$-\frac{1}{2\kappa^2}\int d^{10}x\sqrt{-g}e^{2\phi}R|_{\text{gravitational compensators}} = -\frac{1}{\kappa^2}\int d^{10}x\sqrt{-g}e^{2\phi}(\partial_\mu\phi_0)(\nabla_a g_{\text{comp}}^{\mu a}). \tag{47}$$

This can be written, using the same information, as the integral of a divergence over the compactification manifold and so vanishes.

Note, in particular, in the above that the gravitational compensators do not contribute to $\sqrt{-g}$ to first order in α'. Combinations such as $g_{\text{comp}}^{a\mu}\partial_\mu\phi\partial_a\phi$ are likewise higher order in α'. This means finally that, as promised in Section III A, the gravitational compensators completely drop out of the calculation and need not be considered in what follows.

Next we consider the $O(\alpha')$ contribution which is due to the two-form compensators. The relevant term in the ten-dimensional action is the following one

$$\frac{1}{2\kappa_{10D}^2}\int d^{10}x\sqrt{-g}e^{2\phi}\left(\frac{1}{3}H^2\right). \tag{48}$$

This has a contribution due to the B field compensators which becomes, when we use the fact that the two form compensators are $O(\alpha')$, work to $O(\alpha')$, use our knowledge of the zeroth order solution, and use Eq. (2),

$$\frac{1}{2\kappa_{10D}^2}\int d^{10}x\sqrt{-g}e^{2\phi}\left(\frac{1}{3}H^2\right)|_{\text{compensators}} = \frac{1}{2\kappa_{10D}^2}\int d^{10}x\mathcal{V}_{(a)}^{-2/3}\left[\frac{1}{3}\partial_\mu\chi_{(a)}\Pi_{ab}^{(a)}(dB_{ab\mu}^{\text{comp}})2\right]. \tag{49}$$

Then we use our knowledge of the compensators index structure to obtain

$$\frac{1}{2\kappa_{10D}^2}\int d^{10}x\sqrt{-g}e^{2\phi}\left(\frac{1}{3}H^2\right)|_{\text{compensators}} = \frac{1}{2\kappa_{10D}^2}\int d^{10}x\mathcal{V}_{(a)}^{-2/3}\left[\frac{1}{3}\partial_\mu\chi_{(a)}\Pi_{ab}^{(a)}(4\partial_a B_{b\mu}^{\text{comp}})2\right]. \tag{50}$$

This is again the integral of a total divergence over a compact manifold and so vanishes. It should be noted that since we are working with a gauge dependent term in singular gauge there is a risk of obtaining nonzero boundary terms when performing such integration by parts. Fortunately in our case we find that they all vanish. We shall return to the subject of gauge invariance of our results shortly. At any rate, we see that, as was the case with the gravitational compensators, the two-form compensators drop out of our calculation and as was mentioned earlier we do not need to know their precise form, even in the region near to the gauge five-brane's world volume.

Having eliminated the terms which arise due to the presence of the metric and two-form compensators we can now go on to the next stage of our procedure to obtain a reliable calculation (one that we can show only depends on the form of the reduction ansatz near the gauge five-brane) of the four-dimensional effective action. This next stage is to deal with some problematic terms which *do* seem to depend on the form of the reduction ansatz in the region where it is not a good approximation to the true configuration.

Consider, for example, the dilaton

$$\phi = \tilde{\phi}_0 + \alpha'\tilde{\Phi}. \tag{51}$$

Here, $\tilde{\phi}_0$ is some constant—the zeroth order part of the solution and $\alpha'\tilde{\Phi}$ is a correction due to the presence of the gauge five-brane which is some complicated function of moduli and Calabi-Yau coordinates. This expression should be compared to Eq. (38) which describes our approximation to this solution near to the gauge five-brane's world volume. An example of the problematic terms mentioned above would be an $O(\alpha')$ correction term to the four-dimensional effective action which is proportional to the integral over the transverse space of $\tilde{\Phi}$. If we try and perform this integral using our approximate solution we find that we get a divergent answer—a clear sign that this correction term receives significant contributions from portions of the compactification manifold which are far from the gauge five-brane's world volume where our approximate solution is valid. This might naively be thought to be an indication that our method cannot be made to work. However, the problem can be solved by making a sensible definition of the constant/modulus $\tilde{\phi}_0$. We denote the average of this correction over the compact space as follows

$$\frac{(\int d^6x\alpha'\tilde{\Phi})}{V_{CY_3}} = \alpha'\langle\tilde{\Phi}\rangle. \tag{52}$$

Here, V_{CY_3} is the coordinate volume of the Calabi-Yau. The trick is to define our dilatonic modulus so as to absorb our ignorance of the situation far from the gauge five-brane's world volume into this already arbitrary constant. We perform the following manipulations

$$\phi = \tilde{\phi}_0 + \alpha'\langle\tilde{\Phi}\rangle + [\alpha'\tilde{\Phi} - \alpha'\langle\tilde{\Phi}\rangle] \tag{53}$$

$$\equiv \phi_0 + [\alpha'\Phi]. \tag{54}$$

In other words we absorb the average of the correction over the internal space into the definition of the constant ϕ_0. Since this constant takes an arbitrary value in the background solution this is something we are perfectly entitled to do. With the dilaton in this form some of the potentially problematic terms which we could not calculate in the four-dimensional effective theory vanish. For example, the analogue of the term we mentioned above is proportional to the integral over the transverse space of Φ which is zero. We will deal with the remaining troublesome terms with a similar trick involving the axions in a short while. But first let us make a digression to work out the four-dimensional Einstein Hilbert term and so fix f, the Weyl rescaling factor in the metric.

Using all we have learned so far we find for the four-dimensional Einstein Hilbert term

$$-\frac{1}{2\kappa_4^2}\int d^4x\sqrt{-g_4}f\mathcal{V}_1^{1/3}\mathcal{V}_2^{1/3}\mathcal{V}_3^{1/3}e^{2\phi_0}R_{4D}. \qquad (55)$$

From this we see that for a canonically normalized Einstein Hilbert term in four dimensions we must choose

$$f=\mathcal{V}_1^{-(1/3)}\mathcal{V}_2^{-(1/3)}\mathcal{V}_3^{-(1/3)}e^{-2\phi_0}. \qquad (56)$$

We now return to the examination of possible problematic terms in our calculation. The only remaining terms which require knowledge of the field configuration far from the gauge five-brane's world volume come from the H^2 term in the ten-dimensional action. These terms are due to the background B field configuration as opposed to the B field compensators whose contributions to the effective action have already been shown to vanish. The terms are again of order α' and are given by

$$\frac{1}{2\kappa^2}\int d^{10}x\mathcal{V}_{(a)}^{-(2/3)}\frac{4}{3}\partial^\mu\chi_{(a)}\Pi_{ab}^{(a)}\partial_\mu B_{ab}^{\rm bg}. \qquad (57)$$

This is an $O(\alpha')$ term as the background two- form field is of that order. As with the terms we examined previously if we try and calculate this integral using our approximate solution we obtain an answer which depends on the form of the solution in the region of the transverse space where our approximation breaks down. As before we can deal with this problem by making a judicious, and perfectly legitimate, redefinition of our moduli. This time we redefine the geometrical axions to absorb the average of the background two-form over the transverse space

$$\Pi^{(c)ab}B_{ab}=\frac{1}{6}\chi_{(c)}2+\Pi^{(c)ab}B_{ab}^{\rm bg}, \qquad (58)$$

$$\Pi^{(c)ab}B_{ab}=\frac{1}{6}(\chi_{(c)}2+6\langle\Pi^{(c)ab}B_{ab}^{\rm bg}\rangle)+[\Pi^{(c)ab}B_{ab}^{\rm bg}-\langle\Pi^{(c)ab}B_{ab}^{\rm bg}\rangle]. \qquad (59)$$

We redefine $\chi_{(c)}$ to be the terms in the first set of (nonsquare) brackets in the second line of Eq. (59). As was the case in our redefinition of the dilatonic modulus this completely eliminates the problematic $O(\alpha')$ corrections by absorbing our ignorance of the "asymptotic" configuration into the arbitrary constant associated with the modulus.

We may now, finally, proceed to plug our approximate reduction ansatz into the remaining terms in the-ten dimensional effective action. We then find that in the calculation of the 4d effective action—i.e., in performing the integration over the compact dimensions of the Calabi-Yau threefold—that we do not need to know the reduction ansatz to an accuracy beyond that provided by our approximation. In other words the terms involving five-brane moduli only depend on the form of the reduction ansatz close to the gauge five-brane's world volume.

Let us be a little more precise about what we mean by this. Consider patching our instanton into the transverse space by multiplying all of the fields by some smoothing function which is one inside some radius r_1 which includes the core, 0 outside some radius $r_2>r_1$ and which smoothly interpolates between these values between these two radii. We find that our results would be unchanged by such a procedure—whatever the specific form of the smoothing function—provided that $r_1\gg\rho$. Our approximation neglects terms which are suppressed with respect to the ones which we keep by factors of $\frac{\rho^2}{r_1^2}$.

We find that all of the terms in the four-dimensional action involving instanton moduli come from just two terms in the higher dimensional action. They are

$$\frac{1}{2\kappa^2}\int d^{10}x\sqrt{-g}e^{2\phi}\left(-\frac{4}{3}\alpha' dB\omega_{3YM}+2\alpha'{\rm tr}F^2\right), \qquad (60)$$

where we have suppressed the index structure in these expressions for conciseness. So all we have to do is to plug the zeroth order ansatz into this expression and perform the appropriate integrals.

The reader may be wondering what has happened to gauge invariance in all this. The ${\rm tr}F^2$ term is clearly gauge invariant. However at first sight the ${\rm d}B\omega_{3YM}$ term is not as it usually pairs with a $({\rm d}B)^2$ term to form a gauge invariant object to first order in α'. In fact we can show that, to the degree that is needed for our calculation, this term in the action is indeed gauge invariant in its own right to first order in α', up to our approximations. The proof is as follows. Schematically the two-form and gauge field change in the following manner under an infinitesimal gauge transformation

$$\delta A=d\Lambda+[A,\Lambda], \qquad (61)$$

$$\Rightarrow\delta\omega_{3YM}={\rm dtr}(\Lambda dA), \qquad (62)$$

$$\delta B=2\alpha'{\rm tr}(\Lambda dA). \qquad (63)$$

Here Λ is an infinitesimal parameter describing the gauge transformation. This means that, to first order in α', our ${\rm d}B\omega_{3YM}$ term undergoes a change under such a transformation proportional to

$$\int d^6x\alpha' \mathrm{d}[\mathrm{d}B\mathrm{tr}(\Lambda dA)]. \tag{64}$$

This is the integral of a total derivative. If we are working at the level of our approximate solution this leads to two possibilities. Either the gauge transformation we are considering dies off sufficiently quickly away from the five-brane's world volume that we can still perform our calculation reliably in the resultant gauge or it does not. In the first case due to the total derivative structure of the integrand in Eq. (64) we find that our moduli space effective action indeed will not change if we perform this gauge transformation on our reduction ansatz. In the second case we will find that we simply cannot reliably perform the calculation in this gauge. Thus we find that in any gauge where we can actually compute the answer we desire our answer is unique and so our results are compatible with gauge invariance.

IV. THE RESULT

When we apply the procedure detailed in the previous section, taking into account all of the various subtleties, we find we can indeed reliably calculate the moduli space effective action of the gauge five-brane without knowing the exact form of the solution far from the instanton core in the transverse space. The effective theory including the geometric moduli and the gauge five-brane moduli is the following

$$\begin{aligned} S = \frac{1}{2\kappa_4^2}\int d^4x\sqrt{-g}\Big(&-R + \frac{1}{2}(\partial\varphi)^2 + \frac{1}{2}[(\partial\beta_1)^2 + (\partial\beta_2)^2 + (\partial\beta_3)^2] + e^{-2\varphi}(\partial\sigma)^2 + \frac{2}{9}[e^{-2\beta_1}(\partial\chi_1)^2 + e^{-2\beta_2}(\partial\chi_2)^2 \\ &+ e^{-2\beta_3}(\partial\chi_3)^2] + q_{G5}\{8e^{-\beta_1-\beta_2}[(\partial\rho)^2 + \rho^2(\partial\theta)^2] + \rho^2e^{-\beta_1-\beta_2}[(\partial\beta_1)^2 + (\partial\beta_2)^2] - 4\rho e^{-\beta_1-\beta_2}\partial\rho(\partial\beta_1 \\ &+ \partial\beta_2) - \frac{2}{9}\rho^2e^{-\beta_1-\beta_2}[e^{-2\beta_1}(\partial\chi_1)^2 + e^{-2\beta_2}(\partial\chi_2)^2] + \frac{4}{9}\rho^2e^{-2\beta_1-2\beta_2}\partial\chi_1\partial\chi_2 + \frac{8}{3}\rho^2e^{-\beta_1-\beta_2}(\theta^2\partial\theta^1 - \theta^1\partial\theta^2 \\ &+ \theta^3\partial\theta^4 - \theta^4\partial\theta^3)(e^{-\beta_1}\partial\chi_1 + e^{-\beta_2}\partial\chi_2)\}\Big). \end{aligned} \tag{65}$$

This expression employs the following definition

$$(\partial\theta)^2 = \sum_{\gamma=1}^{4}(\partial\theta^\gamma)^2. \tag{66}$$

We have also introduced $q_{G5} = \frac{\alpha'(2\pi)^2}{V_{\mathrm{trans}}}$ and V_{trans} is the coordinate volume of the transverse space to the five-brane. To get the result in this form, which exhibits the usual normalizations of low-energy heterotic M-theory, we have made a number of field redefinitions. We have $\mathcal{V}_a = e^{3\beta_a}$, and $\phi_0 = \frac{1}{2}\varphi - \frac{1}{2}(\beta_1 + \beta_2 + \beta_3)$. In particular these choices ensure that the zeroth order (in α') kinetic terms have the usual form and normalization.

The terms on the first line of the action then form the usual result for the four-dimensional effective action of heterotic M-theory (or indeed weakly coupled heterotic string theory) accurate to first order in α'. To recap β_3 is a volume modulus for the 2 cycle our gauge five-brane wraps and χ_3 is its associated axion. β_1 and β_2 are volume moduli associated with the four-dimensional transverse space and χ_1 and χ_2 are their associated axions. Finally φ here is the four-dimensional dilaton and σ is its axion.

The remaining terms, which are contained within the square brackets, contain the kinetic terms of, and cross couplings to, the gauge five-brane moduli. We recall that ρ is the solitons width while the θ's describe its SU(2) orientation. So, for example, if we want to describe how a gauge five-brane spinning in SU(2) space generates the axions $\chi_{(i)}$ from a four-dimensional viewpoint all we have to do is obtain an appropriate cosmological solution of this action.

Let us recap the approximations we have made and thus when our effective action is valid. We have made all of the standard approximations employed in obtaining four-dimensional actions in this context. These are the slowly changing moduli approximation, working to first order in α' (or more precisely to first order in ϵ_w in the language of [18]) and ignoring towers of massive states (which corresponds to "the other" ϵ expansion of [18]). The action also does not include contributions from nonperturbative corrections.

The new approximation that we have made here is that $\rho \ll V_{\mathrm{trans}}^{1/6}$ (there could be corrections to our action which are suppressed by powers of $\frac{\rho^2}{V_{\mathrm{trans}}^{1/3}}$). In addition, in order for our supergravity description to be valid we require $\rho^2 \gg \alpha'$.

The first thing to notice about this result is that if we artificially "turn off" gravity—i.e., if we drop all the metric moduli we obtain the Lagrangian density $(\partial\rho)^2 + \rho^2(\partial\theta)^2$. The moduli space associated with these kinetic terms is simply the usual centered moduli space of a single Yang-Mills instanton [28]. One might think, remembering the constraint (24) on the θ's, that this moduli space is simply $\mathcal{R}^4$. However it is in fact $\frac{\mathcal{R}^4}{Z_2}$. This is because the physical situation is unchanged by the transformation $\theta^\gamma \rightarrow -\theta^\gamma$. Hence we should make the identification $\theta^\gamma = -\theta^\gamma$ which results in the moding out by Z_2 in the moduli space. Because of this identification the Yang-Mills instanton moduli space has a conical singularity at the point where the size modulus vanishes. In fact it is clear that this conical singularity survives in our full result and this pathology at $\rho = 0$ is one of the indicators that we cannot trust our effective action down to arbitrarily small gauge five-brane widths. Another indication of this is that if we examine the curvature at the core of the object it diverges as we take the size to zero. It should be noted that the moduli space of a single Yang-Mills instanton is, from the above discussion, obviously hyper-Kähler as it is locally simply four-dimensional flat space.

We can see that the Yang-Mills instanton moduli space is embedded within our result in a highly nontrivial way involving many prefactors and cross terms. A very good check that we have got the calculation of all of these terms correct is that the result is compatible with $\mathcal{N} = 1$ supersymmetry. In other words we should check that the full moduli space is Kähler.

We have checked this in two different ways. First we have directly calculated the holonomy of the moduli space. This is achieved by using the Riemann tensor to obtain the generators of the holonomy group. One may then count the number of linearly independent generators to obtain the dimensionality of the group. By examining the dimensionality of all the different possible subgroups of SO(N), where N is the dimensionality of the manifold under consideration, one can then in many cases show that only one subgroup has the dimensionality that we find—giving us the holonomy.

Now the moduli space is a direct product of a manifold spanned by $(\varphi, \sigma, \beta_3, \chi_3)$ and one spanned by $(\beta_1, \chi_1, \beta_2, \chi_2, \rho, \theta^\gamma)$. We know from standard results that the first manifold is Kähler so we just need to find the holonomy of the second one. We find that its holonomy group fills out the entirety of U(4) and hence our result is indeed compatible with $\mathcal{N} = 1$ supersymmetry. We also see that the hyper-Kähler moduli space of the Yang-Mills instanton is reduced to being merely Kähler when embedded within this context.

The second way in which we can demonstrate that our result is Kähler is to write down a Kähler potential and complex structure which is associated with our component action (65). We find the following

$$K = -\ln(S + \bar{S}) - \ln(T_1 + \bar{T}_1) - \ln(T_2 + \bar{T}_2) - \ln(T_3 + \bar{T}_3) + \frac{16\alpha'(|C_1|^2 + |C_2|^2)}{\sqrt{(T_1 + \bar{T}_1)(T_2 + \bar{T}_2)}}, \tag{67}$$

$$C_1 = e^{-(\beta_1/4)-(\beta_2/4)}(Y_1 + iY_2), \tag{68}$$

$$C_2 = e^{-(\beta_1/4)-(\beta_2/4)}(Y_3 + iY_4), \tag{69}$$

$$T_1 = e^{\beta_1} + \frac{2}{3}i\chi_1 + 4\alpha' e^{[(\beta_1-\beta_2)/2]}(|C_1|^2 + |C_2|^2), \tag{70}$$

$$T_2 = e^{\beta_2} + \frac{2}{3}i\chi_2 + 4\alpha' e^{[(\beta_2-\beta_1)/2]}(|C_1|^2 + |C_2|^2), \tag{71}$$

$$T_3 = e^{\beta_3} + \frac{2}{3}i\chi_3, \tag{72}$$

$$S = e^{\varphi} + \sqrt{2}i\sigma. \tag{73}$$

Here we have defined the fields Y_γ as $Y_\gamma = \rho\theta^\gamma$. We see that we have the usual Kähler potential of heterotic -string/M-theory with an additional term, the final one, which is due to the presence of the gauge five-brane. This additional term is just the Kähler potential for a simple Yang-Mills instanton, modified by the addition of some factors of real parts of T superfields. Similarly the definition of the T superfields in terms of component fields is just the usual one with a couple of modifications at $O(\alpha')$ due to the presence of the gauge five-brane. C_1 and C_2 are again just the usual Yang-Mills instanton expressions modified by some overall factors of different e^β's.

We can make a number of comments about the physics that follow from these results purely from an examination of the component action and Kähler structure.

(i) First of all it is clearly not consistent, in the case of the compactifications considered here, to take the universal case for the metric moduli in the presence of a generic changing instanton configuration. In other words, due to the nontrivial factors of e^{β_1} and e^{β_2} in the gauge five-brane moduli kinetic terms, for example, it is inconsistent to set $\beta_1 = \beta_2 = \beta_3$ (however as we shall see shortly we can set $\beta_1 = \beta_2$ if we make some compatible truncations of the other fields).

(i) The dilaton and the size of the 2 cycle (φ and β_3) do not feel the presence of the gauge five-brane from the point of view of the four-dimensional theory. The terms in which they appear are not modified from the results where the bundle moduli are ignored.

(i) It can be seen from the last five terms in Eq. (65) that it is not consistent to truncate off the gauge five-brane moduli by setting them to be nonzero constants. In fact it is not possible to truncate them away by setting all the Y's to zero either, even though they appear bi-linearly in the above expressions. This is because setting all the Y's to zero in this manner corresponds to setting ρ to zero and as mentioned above our effective description is not valid in this region of moduli space. This result is in contrast to the case of matter fields for example [2]. In that case there is no analogue of our Z_2 identification and so the matter fields (which appear bi-linearly as well) can be truncated away by simply setting them to zero. Returning to the gauge five-brane case, more complicated forms of truncation are possible in certain special cases. We shall see such a special case where we can clearly truncate off the instanton moduli, in certain combinations with other fields, in a moment.

(i) The form of the instanton corrections to the Kähler potential and complex structure is reminiscent of the analogous corrections obtained by the inclusion of matter fields. This is perhaps not particularly surprising given that there are some similarities in the origins of these four-dimensional fields. It should be emphasized, however, that many qualitative differences exist in how these two types of moduli arise.

Now the result presented above appears fairly complicated and depends on a reasonably large number of fields (twelve real moduli). However, we can consistently truncate away many of the moduli to leave a simpler system with fewer degrees of freedom. For example the simplest nontrivial truncation which includes at least one instanton modulus is the following

$$K = -2\ln(T + \bar{T}) + \frac{32\alpha'|C|^2}{(T + \bar{T})}, \tag{74}$$

$$T = e^{\beta} + \frac{2}{3}i\chi + 8\alpha'|C|^2, \tag{75}$$

$$C = e^{-(\beta/2)}(Y_1 + iY_2). \tag{76}$$

Here we have taken the situation where $\beta_1 = \beta_2 = \beta$, $\chi_1 = \chi_2 = \chi$, ϕ, σ, β_3, and χ_3 are taken constant, $Y_1 = Y_3$ and $Y_2 = Y_4$. We can obtain a component action from this Kähler potential and consistently truncate off the axions to obtain the following 4d theory.

$$S = \frac{1}{2\kappa^2}\int d^4x\sqrt{-g}[-R + (\partial\beta)^2 + 8q_5 e^{-\beta}(\partial\hat{\rho})^2]. \tag{77}$$

We have defined $\hat{\rho} = e^{-(\beta/2)}\rho$. Since this simplification was obtained via consistent truncation a solution to this simple theory corresponds to a solution to the full higher dimensional equations of motion to our approximations. One could obtain cosmological solutions to such an action with ease. These solutions are the simplest examples of what we need to make the brane collision scenarios mentioned in the introduction more complete. We have obtained such solutions and these will be presented as part of some future work [35].

V. FURTHER WORK

There are many ways in which this work can be used as a basis for future study. Here we will list a few of the more interesting possibilities.

(i) One could generalize the results presented here to the case where we consider more than one gauge five-brane. It would not even be necessary to restrict such a study to the case where the gauge five-branes do not overlap (a trivial modification of the above results). Such complicated situations are probably tractable due to the fact that we have a very powerful mechanism for obtaining self-dual gauge field configurations in the form of the ADHM construction [28]. This construction can provide us with analytic solutions for configurations containing many instantons. These can overlap, have different positions and sizes as well as different SU(2) orientations. Following the procedure outlined in [24] we can then use these gauge field configurations as the core for a system of gauge five-branes. The NS dressing can be determined once the gauge field configuration is known. Similar calculations to the one presented here could then be performed for these more complicated situations. Indeed by using a Kummer style construction for the Calabi-Yau threefold, such as the one we have employed here, and by taking the case where the gauge field background is entirely in the form of (either overlapping or not overlapping) gauge five-branes one could write down a four-dimensional theory which includes *all* of the moduli present in the compactification.

(i) There are other parts of the gauge bundle which would yield to our approach. For example there is another object which takes a Yang-Mills instanton as its core—the so-called symmetric solution [29–31] (this is related to the discrete choices in determining the NS dressing that we mentioned earlier). We could equally well apply our method to this object and obtain the low-energy effective theory which includes its moduli. Unlike our case this object is an example of a standardly embedded configuration. Another difference to the object we have considered in this paper is that the size modulus of the symmetric solution is quantized. This would mean that there would not be analogues of the particular continuous moduli we have considered here for that case.

(i) One could try to obtain a more complete description of the gauge five-brane's four-dimensional effective theory by combining the kinetic terms described here with the work which has been done on obtaining nonperturbative potentials for gauge bundle moduli [11–14]. In particular it would be interesting to identify the moduli in these papers which correspond to those described here, for example, the instanton size.

(i) The action presented in this paper could be used to derive a number of different types of cosmological solutions. For example one could seek to describe the cosmological effects of a gauge five-brane spreading out with time [35] or spinning in SU(2) space. Such solutions could be of critical importance in certain cosmological scenarios [16,17].

(i) We have already described how our results could be used to improve the description of cosmological scenarios based upon small instanton transitions. However we would also like to stress that gauge five-branes can live on the orbifold fixed planes irrespective of whether or not the system has undergone such phase transitions. Therefore it is possible to base scenarios purely on the dynamics of such objects. For example, if we were to include the position moduli of the five-brane in our analysis we could imagine basing some kind of brane inflation scenario on gauge five-branes and anti gauge five-branes. This soliton-antisoliton inflation could potentially have some quite nice properties. For example when inflation ends with the collision of the instanton and anti-instanton they would presumably annihilate in a manner which is describable within the regime of low-energy field theory—both objects simply being made out of low-energy fields [1]. The energy from such an annihilation could reheat the universe—the fact that the colliding objects are annihilating on an orbifold fixed plane presumably means that it would be natural for a sizable proportion of the resulting energy to be dumped into matter fields.

In other words gauge five-branes can be every bit as useful in developing various scenarios as their "fundamental" counterparts—and in addition these solitonic objects have extra attractive features such as variable widths and the fact that they are entirely built out of low-energy fields.

In short it is now possible to start an analysis of the effect of certain types of gauge bundle moduli on different cosmological scenarios for the first time.

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[1]Although some caution is called for with this statement given the results presented in [36] for a situation which one would think would be subject to similar arguments. The two situations *are* different however. In particular in our case the two colliding objects would have no net five-brane charge.

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