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Analytical approximation for $\langle\varphi^2\rangle$ of a quantized scalar field in ultrastatic asymptotically flat spacetimes

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Analytical approximations for $\langle\varphi^2\rangle$ of a quantized scalar field in ultrastatic asymptotically flat spacetimes are obtained. The field is assumed to be both massive and massless, with an arbitrary coupling ξ to the scalar curvature, and in a zero or nonzero temperature vacuum state. The expression for $\langle\varphi^2\rangle$ is divided into low- and high-frequency parts. The expansion for the high-frequency contribution to this quantity is obtained. This expansion is analogous to the DeWitt-Schwinger one. As an example, the low-frequency contribution to $\langle\varphi^2\rangle$ is calculated on the background of the small perturbed flat spacetime in a quantum state corresponding to the Minkowski vacuum at the asymptotic. The limits of the applicability of these approximations are discussed.

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I. INTRODUCTION

The investigations of black hole evaporation and particle production by an expanding universe have acted as a stimulus for a detailed and systematic investigation of the theory of quantum fields propagating on curved spacetimes. The main objects to calculate from quantum field theory in curved spacetime are the quantities $\langle\varphi^2\rangle$ and $\langle T^\mu_\nu\rangle$ where φ is the quantum field and T^μ_ν is the stress-energy tensor operator for φ. The renormalized stress-energy tensor $\langle T^\mu_\nu\rangle$ is an important quantity for the construction of a self-consistent model of an evaporating black hole, while the mean-square field $\langle\varphi^2\rangle$ plays a role in the study of theories with spontaneous symmetry breaking. The functional dependence $\langle T^\mu_\nu\rangle$ on the metric $g_{\mu\nu}$ allows us to study the evolution of the background geometry driven by the quantum fluctuation of the matter fields propagating on it. This is the so-called backreaction, governed by the semiclassical Einstein equations

$$G^\mu_\nu = 8\pi\langle T^\mu_\nu\rangle. \tag{1}$$

However, the exact results for $\langle\varphi^2\rangle$ and $\langle T^\mu_\nu\rangle$ in four dimensions are not numerous (see, for example, [1]). Numerical computations of these quantities are as a rule extremely intensive [2–4].

One of the most widely used techniques to obtain information about these quantities is the DeWitt-Schwinger (DS) expansion [5]. It may be used to give the expansions for $\langle\varphi^2\rangle$ and $\langle T^\mu_\nu\rangle$ in terms of powers of the small parameter

$$\frac{1}{mL} \ll 1, \tag{2}$$

where m is the mass of the quantized field and L is the characteristic scale of change of the background gravitational field [6].

The analytical approximations to $\langle\varphi^2\rangle$ and $\langle T^\mu_\nu\rangle$ for the conformally coupled massless fields [3,4,7–12] give good results. Nevertheless, there still remains the problem of the extension of this type of approximations' applicability limits. If the quantum field is massive but the mass of the field does not satisfy the condition (2) the analytical approximations to $\langle\varphi^2\rangle$ and $\langle T^\mu_\nu\rangle$ are even less numerous [3,4,11–14].

In this paper, approximate expressions for $\langle\varphi^2\rangle_{\rm ren}$ of a quantized scalar field in ultrastatic asymptotically flat spacetimes are derived. The field is assumed to be either massless or massive with an arbitrary coupling ξ to the scalar curvature R, and in a zero or nonzero temperature vacuum state. The expression for $\langle\varphi^2\rangle_{\rm ren}$ is divided into low- and high-frequency parts. The Bunch-Parker approach [15] is used for the derivation of high-frequency contributions (HFC) to these quantities. As in the case of a massless field the quantum state of the field with mass $m \ll 1/L$ is essentially determined by the topology of spacetime and the boundary conditions. In this paper such dependence is determined by the low-frequency contribution (LFC). As an example, this contribution is calculated on the background of the small perturbed flat spacetime in a quantum state corresponding to the Minkowski vacuum at the asymptotic.

The paper is organized as follows. In Sec. II the expressions for the Euclidean Green's function of a scalar field with arbitrary mass and curvature coupling in a ultrastatic spacetime is divided into low- and high-frequency parts. In Sec. III the WKB approximation of the high-frequency contribution to $\langle\varphi^2\rangle$ is derived. The low-frequency contribution is derived and the renormalization procedure is described in Sec. IV. The results are summarized in Sec. V. In the Appendix the expanding of the renormalization counterterm for $\langle\varphi^2\rangle$ in the powers of the coordinate difference of the separated points is described. The units $\hbar = c = G = k_B = 1$ are used throughout the paper. The sign conventions are those of Misner, Thorne, and Wheeler [16].

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II. GREEN'S FUNCTION

The metric in the Euclidean section for an ultrastatic spacetime is given by

$$ds^2 = d\tau^2 + g_{ab}dx^a dx^b, \tag{3}$$

where $\tau = it$ is the Euclidean time, and g_{ab} are arbitrary functions of spatial coordinates x^1, x^2, and x^3. (Latin indices run from 1 to 3; Greek indices run from 0 to 3.)

In this paper, the point-splitting method is employed for the regularization of ultraviolet divergences. When the points are separated one can show that

$$\langle\varphi^2\rangle_{\text{unren}} = G_{\text{E}}(x^\mu, \tilde{x}^\nu). \tag{4}$$

The Euclidean Green's function G_{E} is a solution of the equation

$$[\Box_x - m^2 - \xi R(x^a)]G_{\text{E}}(x^\mu, \tilde{x}^\nu) = -\frac{\delta(\tau, \tilde{\tau})\delta^{(3)}(x^a, \tilde{x}^b)}{\sqrt{g^{(3)}(x^a)}}, \tag{5}$$

where m is the mass of the scalar field, ξ is its coupling to the scalar curvature $R(x^a)$, and $g^{(3)}(x^a) = \det g_{ab}(x^1, x^2, x^3)$.

If the scalar field is at zero temperature then $\delta(\tau, \tilde{\tau})$ and $G_{\text{E}}(x^\mu, \tilde{x}^\nu)$ can be expanded as

$$\delta(\tau, \tilde{\tau}) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} d\omega e^{i\omega(\tau-\tilde{\tau})}, \tag{6}$$

$$G_{\text{E}}(x^\mu, \tilde{x}^\nu) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} d\omega e^{i\omega(\tau-\tilde{\tau})}\tilde{G}_{\text{E}}(\omega; x^a, \tilde{x}^b). \tag{7}$$

If the field is at temperature T, then the Green's function is periodic in $\tau - \tilde{\tau}$ with period $1/T$. In this case $\delta(\tau, \tilde{\tau})$ and $G_{\text{E}}(x^\mu, \tilde{x}^\nu)$ have the expansions

$$\delta(\tau, \tilde{\tau}) = T\sum_{n=-\infty}^{\infty} \exp[i\omega_n(\tau - \tilde{\tau})], \tag{8}$$

$$G_{\text{E}}(x^\mu, \tilde{x}^\nu) = T\sum_{n=-\infty}^{\infty} \exp[i\omega_n(\tau - \tilde{\tau})]\tilde{G}_{\text{E}}(\omega_n; x^a, \tilde{x}^b), \tag{9}$$

where $\omega_n = 2\pi Tn$. In both cases $\tilde{G}_{\text{E}}(\omega; x^a, \tilde{x}^b)$ satisfies the equation

$$\frac{1}{\sqrt{g^{(3)}(x^c)}}\frac{\partial}{\partial x^a}\left[\sqrt{g^{(3)}(x^c)}g^{ab}(x^c)\frac{\partial}{\partial x^b}\tilde{G}_{\text{E}}(\omega; x^a, \tilde{x}^b)\right] - (\omega^2 + m^2 + \xi R)\tilde{G}_{\text{E}}(\omega; x^a, \tilde{x}^b) = -\frac{\delta^{(3)}(x^a, \tilde{x}^b)}{\sqrt{g^{(3)}(x^a)}}. \tag{10}$$

In the case $m \gg 1/L$, where L is a characteristic scale of the variation of the background gravitational field, it is possible to construct the iterative procedure of the solution of this equation with the small parameter $1/mL$ [5,15]. This procedure gives the standard expansion of $\langle\varphi^2\rangle_{\text{unren}}$ in terms of the powers of mL.

In the case

$$m \lesssim \frac{1}{L} \tag{11}$$

a small parameter of the iterative procedure does not exist. Nevertheless, let us divide G_{E} into low- and high-frequency parts, as was done in [12],

$$G_{\text{E}}(x^\mu, \tilde{x}^\nu) = G_{\text{E}}^{\text{LFC}}(x^\mu, \tilde{x}^\nu) + G_{\text{E}}^{\text{HFC}}(x^\mu, \tilde{x}^\nu), \tag{12}$$

where

$$G_{\text{E}}^{\text{LFC}}(x^\mu, \tilde{x}^\nu) = \frac{1}{\pi}\int_0^{1/\lambda_0} d\omega \cos[\omega(\tau - \tilde{\tau})]\tilde{G}_{\text{E}}(\omega; x^a, \tilde{x}^b), \tag{13}$$

$$G_{\text{E}}^{\text{HFC}}(x^\mu, \tilde{x}^\nu) = \frac{1}{\pi}\int_{1/\lambda_0}^{\infty} d\omega \cos[\omega(\tau - \tilde{\tau})]\tilde{G}_{\text{E}}(\omega; x^a, \tilde{x}^b), \tag{14}$$

if the scalar field is at zero temperature and

$$G_{\text{E}}^{\text{LFC}}(x^\mu, \tilde{x}^\nu) = T\tilde{G}_{\text{E}}(0; x^a, \tilde{x}^b) + 2T\sum_{n=1}^{n_0-1}\cos[\omega_n(\tau - \tilde{\tau})] \times\tilde{G}_{\text{E}}(\omega_n; x^a, \tilde{x}^b), \tag{15}$$

$$G_{\text{E}}^{\text{HFC}}(x^\mu, \tilde{x}^\nu) = 2T\sum_{n=n_0}^{\infty}\cos[\omega_n(\tau - \tilde{\tau})]\tilde{G}_{\text{E}}(\omega_n; x^a, \tilde{x}^b) \tag{16}$$

if the field is at temperature T. Then the expansion of $G_{\text{E}}^{\text{HFC}}(x^\mu, \tilde{x}^\nu)$ in terms of the powers of a small parameter

$$\varepsilon_{\text{WKB}} = \frac{\lambda_0}{L} \ll 1 \tag{17}$$

can be obtained by the analogy with the methods of evaluation of the DeWitt-Schwinger expansion. If the field is at temperature T then

$$\lambda_0 = \frac{1}{2\pi Tn_0} \tag{18}$$

and

$$\varepsilon_{\text{WKB}} = \frac{1}{2\pi Tn_0 L} \ll 1. \tag{19}$$

Below the main points of the Bunch and Parker approach [15] for obtaining $G_{\rm E}^{\rm HFC}(x^\mu, \tilde{x}^\nu)$ are outlined.

III. HIGH-FREQUENCY CONTRIBUTION TO $\langle\varphi^2\rangle$

Let us introduce Riemann normal coordinates y^a in 3D space with the origin at the point $\tilde{x}^a$ [17]. In these coordinates one has

$$g_{ab}(y^a) = \eta_{ab} - \tfrac{1}{3}R_{acbd}y^c y^d + O(y^3), \qquad (20)$$

$$g^{ab}(y^a) = \eta^{ab} + \tfrac{1}{3}R^{a\ b}_{\ c\ d}y^c y^d + O(y^3), \qquad (21)$$

$$g^{(3)}(y^a) = 1 - \tfrac{1}{3}R_{ab}y^a y^b + O(y^3), \qquad (22)$$

where the coefficients here and below are evaluated at $y^a = 0$ (i.e., at the point $\tilde{x}^a$), and η_{ab} denotes the three-dimensional Euclidean metric. All indices are raised and lowered with the metric η_{ab}. Defining $\overline{G}(\omega; y^a)$ by

$$\overline{G}(\omega; y^a) = \sqrt{g^{(3)}(y)}\tilde{G}_{\rm E}(\omega; y^a) \qquad (23)$$

and retaining in (10) only terms with coefficients involving two derivatives of the metric or fewer one finds that $\overline{G}(\omega; y^a)$ satisfies the equation

$$\eta^{ab}\frac{\partial^2\overline{G}}{\partial y^a\partial y^b} - \omega^2\overline{G} + \frac{1}{3}R^{ab}_{cd}y^c y^d\frac{\partial^2\overline{G}}{\partial y^a\partial y^b} - \left[m^2 + \left(\xi - \frac{1}{3}\right)R\right]\overline{G} = -\delta^{(3)}(y). \qquad (24)$$

Note that quantities R_{abcd}, R_{ab}, and R evaluated in metric (3) and those evaluated in 3D metric g_{ab} coincide. Let us present

$$\overline{G}(\omega; y^a) = \overline{G}_0(\omega; y^a) + \overline{G}_1(\omega; y^a) + \overline{G}_2(\omega; y^a) + \cdots, \qquad (25)$$

where $\overline{G}_i(\omega; y^a)$ has a geometrical coefficient involving i derivatives of the metric at point $y^a = 0$. Then these functions satisfy the equations

$$\eta^{ab}\frac{\partial^2\overline{G}_0(\omega; y^a)}{\partial y^a\partial y^b} - \omega^2\overline{G}_0(\omega; y^a) = -\delta^{(3)}(y), \qquad (26)$$

$$\eta^{ab}\frac{\partial^2\overline{G}_1(\omega; y^a)}{\partial y^a\partial y^b} - \omega^2\overline{G}_1(\omega; y^a) = 0, \qquad (27)$$

$$\eta^{ab}\frac{\partial^2\overline{G}_2(\omega; y^a)}{\partial y^a\partial y^b} - \omega^2\overline{G}_2(\omega; y^a) + \frac{1}{3}R^{a\ b}_{\ c\ d}y^c y^d\frac{\partial^2\overline{G}_0(\omega; y^a)}{\partial y^a\partial y^b} - \left[m^2 + \left(\xi - \frac{1}{3}\right)R\right]\overline{G}_0(\omega; y^a) = 0. \qquad (28)$$

The function $G_0(\omega; y^a)$ satisfies the condition

$$R^{a\ b}_{\ c\ d}y^c y^d\frac{\partial^2\overline{G}_0(\omega; y^a)}{\partial y^a\partial y^b} - R^a_b y^b\frac{\partial\overline{G}_0(\omega; y^a)}{\partial y^a} = 0, \qquad (29)$$

since $G_0(\omega; y^a)$ is the function only of $\eta_{ab}y^a y^b$ [15]. Therefore Eq. (28) may be rewritten

$$\eta^{ab}\frac{\partial^2\overline{G}_2(\omega; y^a)}{\partial y^a\partial y^b} - \omega^2\overline{G}_2(\omega; y^a) + \frac{1}{3}R^a_b y^b\frac{\partial\overline{G}_0(\omega; y^a)}{\partial y^a} - \left[m^2 + \left(\xi - \frac{1}{3}\right)R\right]\times \overline{G}_0(\omega; y^a) = 0. \qquad (30)$$

Let us introduce the local momentum space associated with the point $y^a = 0$ by making the 3-dimensional Fourier transformation

$$\overline{G}_i(\omega; y^a) = \iiint_{-\infty}^{+\infty}\frac{dk_1 dk_2 dk_3}{(2\pi)^3}\exp(ik_a y^a)\overline{G}_i(\omega; k^a). \qquad (31)$$

It is not difficult to see that

$$\overline{G}_0(\omega; k^a) = \frac{1}{k^2 + \omega^2}, \qquad (32)$$

$$\overline{G}_1(\omega; k^a) = 0, \qquad (33)$$

where $k^2 = \eta^{ab}k_a k_b$. In momentum space Eq. (30) gives

$$-(k^2 + \omega^2)\overline{G}_2(\omega; k^a) - \frac{1}{3}R^a_b k_a\frac{\partial\overline{G}_0(\omega; k^a)}{\partial k_b} - (m^2 + \xi R)\overline{G}_0(\omega; k^a) = 0. \qquad (34)$$

Hence

$$\overline{G}_2(\omega; k) = \frac{-m^2 - \xi R}{(k^2 + \omega^2)^2} + \frac{2}{3}\frac{R^{ab}k_a k_b}{(k^2 + \omega^2)^3}. \qquad (35)$$

Substituting (31)–(33) and (35) in (25) and integrating leads to

$$\overline{G}(\omega; y^a) = \iiint_{\infty}^{+\infty}\frac{dk_1 dk_2 dk_3}{(2\pi)^3}\exp(ik_a y^a)\left[\frac{1}{k^2 + \omega^2} - \frac{(m^2 + \xi R)}{(k^2 + \omega^2)^2} + \frac{2R^{ab}k_a k_b}{3(k^2 + \omega^2)^3}\right] = \frac{1}{8\pi\exp(|\omega|y)}\left[\frac{-m^2 - (\xi - 1/6)R}{|\omega|} + \frac{2}{y} - \frac{R_{ab}y^a y^b}{6y}\right], \qquad (36)$$

where $y = \sqrt{\eta_{ab}y^a y^b}$. Using the definition of $\overline{G}(\omega; y^a)$ (23) and expansion (22) one finds

$$\tilde{G}_{\rm E}(\omega; y^a) = \left(1 + \frac{1}{6}R_{ab}y^a y^b\right)\overline{G}(\omega; y^a) = \frac{1}{8\pi \exp(|\omega|y)}\left[\frac{-m^2 - (\xi - 1/6)R}{|\omega|} + \frac{2}{y} + \frac{R_{ab}y^a y^b}{6y}\right]. \tag{37}$$

The necessary condition for the validity of this approximation is

$$\omega > \frac{1}{\lambda_0} \gg \frac{1}{L}. \tag{38}$$

Hence one can evaluate $G_{\rm E}^{\rm HFC}$ if the field is at zero temperature

$$G_{\rm E}^{\rm HFC}(\Delta\tau, y^a) = \frac{1}{\pi}\int_{1/\lambda_0}^{\infty} d\omega \cos[\omega\Delta\tau]\tilde{G}_{\rm E}(\omega; y^a) = \frac{1}{8\pi^2}\left\{\left[m^2 + \left(\xi - \frac{1}{6}\right)R\right]\left[C + \frac{1}{2}\times \ln\left|\frac{(y^2 + \Delta\tau^2)}{\lambda_0^2}\right|\right] - \frac{2}{y\lambda_0} + \frac{1}{\lambda_0^2} + \frac{2}{(y^2 + \Delta\tau^2)} + \frac{R_{ab}y^a y^b}{6(y^2 + \Delta\tau^2)}\right\}, \tag{39}$$

where C is Euler's constant, and $\Delta\tau = \tau - \tilde{\tau}$.

If the field is at temperature T then the necessary condition for the validity of expression (37) is

$$n \geq n_0 \gg \frac{1}{2\pi TL} \tag{40}$$

and

$$G_{\rm E}^{\rm HFC}(\Delta\tau, y^a) = 2T\sum_{n=n_0}^{\infty}\cos[\omega_n\Delta\tau]\tilde{G}_{\rm E}(\omega_n; x^a, \tilde{x}^b) = \frac{1}{8\pi^2}\left\{\left[m^2 + \left(\xi - \frac{1}{6}\right)R\right]\left[C + \frac{1}{2}\ln|(y^2 + \Delta\tau^2)(2\pi T)^2| + \psi(n_0)\right] - 2\pi T\left(n_0 - \frac{1}{2}\right)\frac{2}{y} + (2\pi T)^2\left(n_0^2 - n_0 + \frac{1}{6}\right) + \frac{2}{(y^2 + \Delta\tau^2)} + \frac{R_{ab}y^a y^b}{6(y^2 + \Delta\tau^2)}\right\}, \tag{41}$$

where $\psi(z)$ is the logarithmic derivative of the gamma function (i.e., the digamma function). The Plana sum formula [18] has been used to compute the sums over n in the last expression.

IV. LOW-FREQUENCY CONTRIBUTION TO $\langle\varphi^2\rangle$ AND RENORMALIZATION PROCEDURE

The behavior of low-frequency modes is determined by the boundary conditions and the topological structure of the spacetime. As an example, let us consider the evaluation of the low-frequency contribution to $\langle\varphi^2\rangle$ on the background of the small perturbed flat spacetime in a quantum state corresponding to Minkowski vacuum at the asymptotic. If the characteristic scale of the gravitational field inhomogeneity λ satisfies the condition

$$\frac{\lambda}{\lambda_0} \ll 1 \qquad (\text{or } \lambda T n_0 \ll 1), \tag{42}$$

the low-frequency contributions to $\langle\varphi^2\rangle$ can be expanded in terms of the powers of this small parameter. Below the zeroth-order term of this expansion will be used for the approximation of the low-frequency contributions to $\langle\varphi^2\rangle$. This means that we choose the long-wave modes approximately coincident with long-wave modes of Minkowski vacuum. For these modes in a zero temperature quantum state

$$G_{\rm E}^{\rm LFC}(\Delta\tau, y^a) = \frac{1}{(2\pi)^4}\int_{-1/\lambda_0}^{1/\lambda_0} d\omega \iiint_{-\infty}^{\infty} d^3p \frac{\exp(i\omega\Delta\tau + ip_\alpha y^\alpha)}{(\omega^2 + p_1^2 + p_2^2 + p_3^2 + m^2)} = \frac{1}{4\pi^3}\int_{-1/\lambda_0}^{1/\lambda_0} d\omega e^{i\omega\Delta\tau}\int_0^\infty dp \frac{p\sin(py)}{y(\omega^2 + p^2 + m^2)} = \frac{1}{8\pi^2}\int_{-1/\lambda_0}^{1/\lambda_0} d\omega e^{i\omega\Delta\tau}\frac{\exp(-y\sqrt{\omega^2 + m^2})}{y}. \tag{43}$$

In the limit $\Delta\tau \to 0$, $y \to 0$

$$G_{\rm E}^{\rm LFC}(\Delta\tau, y^a) = \frac{1}{8\pi^2}\int_{-1/\lambda_0}^{1/\lambda_0} d\omega e^{i\omega\Delta\tau}\left[\frac{1}{y} - \sqrt{\omega^2 + m^2} + O(y)\right] = \frac{1}{8\pi^2}\left[2\frac{\sin(\Delta\tau/\lambda_0)}{y\Delta\tau} - \frac{1}{\lambda_0}\sqrt{\frac{1}{\lambda_0^2} + m^2} - m^2\ln\left|\frac{1/\lambda_0 + \sqrt{1/\lambda_0^2 + m^2}}{m}\right| + O(y)\right]. \tag{44}$$

If we take into account conditions (11) and (17), i.e.,

$$m \ll \frac{1}{\lambda_0}, \tag{45}$$

then

$$G_{\rm E}^{\rm LFC}(\Delta\tau, y^a) = \frac{1}{8\pi^2}\left\{\frac{2}{y\lambda_0}\left[1 - O\left(\frac{\Delta\tau^2}{\lambda_0^2}\right)\right] - \frac{1}{\lambda_0^2} - \frac{m^2}{2}\left[1 + \ln\left|\frac{4}{\lambda_0^2 m^2}\right| + O(\lambda_0^2 m^2)\right]\right\} + O\left(\frac{\lambda^2}{\lambda_0^4}\right). \tag{46}$$

The analogous calculations for the temperature quantum state give

$$\begin{aligned} G_{\rm E}^{\rm LFC}(\Delta\tau, y^a) &= T\tilde{G}_{\rm E}(0; y^a) + 2T\sum_{n=1}^{n_0-1}\cos[\omega_n\Delta\tau]\tilde{G}_{\rm E}(\omega_n; y^a) \\ &= \frac{1}{8\pi^2}\left\{2\pi T\left(n_0 - \frac{1}{2}\right)\frac{2}{y}[1 - O(T^2 n_0^2\Delta\tau^2)] - (2\pi T)^2(n_0^2 - n_0) - \frac{m^2}{2}\left[1 - \frac{1}{n_0}\right.\right. \\ &\quad \left.\left. + \ln\left|\frac{4(2\pi T)^2 n_0^2}{m^2}\right| + O(m^2 T^2 n_0^2)\right]\right\} + O(\lambda^2 T^4 n_0^4). \end{aligned} \tag{47}$$

The renormalization of $\langle\varphi^2\rangle_{\rm unren}$ is achieved by subtracting the renormalization counterterm [19] and then letting $\tilde{x}^\mu \to x^\mu$

$$\langle\varphi^2\rangle_{\rm ren} = \lim_{\tilde{x}^\mu\to x^\mu}[\langle\varphi^2\rangle_{\rm unren} - \langle\varphi^2\rangle_{\rm DS}], \tag{48}$$

where

$$\langle\phi^2\rangle_{\rm DS} = \frac{1}{8\pi^2\sigma} + \frac{1}{8\pi^2}\left[m^2 + \left(\xi - \frac{1}{6}\right)R\right]\left[C + \frac{1}{2}\ln\left(\frac{m_{\rm DS}^2|\sigma|}{2}\right)\right] - \frac{m^2}{16\pi^2} + \frac{1}{96\pi^2}R_{\mu\nu}\frac{\sigma^\mu\sigma^\nu}{\sigma}, \tag{49}$$

σ is one-half the square of the distance between the points x and $\tilde{x}$ along the shortest geodesic connecting them, σ^μ is the covariant derivative of σ, and the constant $m_{\rm DS}$ is equal to the mass m of the field for a massive scalar field. For a massless field $m_{\rm DS}$ is an arbitrary parameter due to the infrared cutoff in $\langle\varphi^2\rangle_{\rm DS}$. A particular choice of the value of $m_{\rm DS}$ corresponds to a finite renormalization of the coefficients of terms in the gravitational Lagrangian and must be fixed by experiment or observation. The details of the calculations of $\langle\phi^2\rangle_{\rm DS}$ are discussed in the Appendix:

$$\langle\phi^2\rangle_{\rm DS} = \frac{1}{8\pi^2}\left\{\frac{2}{y^2 + \Delta\tau^2} + \left[m^2 + \left(\xi - \frac{1}{6}\right)R\right]\left[C + \frac{1}{2}\ln\left(\frac{m_{\rm DS}^2}{4}|y^2 + \Delta\tau^2|\right)\right] - \frac{m^2}{2} + \frac{R_{ab}y^a y^b}{6(y^2 + \triangle\tau^2)}\right\}. \tag{50}$$

V. RESULTS

Using Eqs. (4), (12), (39), (46), (48), and (50) one finds

$$\langle\varphi^2\rangle_{\rm ren} = \lim_{\substack{\Delta\tau\to 0\\ y\to 0}}[G_{\rm E}^{\rm LFC}(\Delta\tau, y^a) + G_{\rm E}^{\rm HFC}(\Delta\tau, y^a) - \langle\varphi^2\rangle_{\rm DS}] = \frac{R}{16\pi^2}\left(\xi - \frac{1}{6}\right)\ln\left|\frac{4}{\lambda_0^2 m_{\rm DS}^2}\right| + O\left(\frac{\varepsilon_{\rm WKB}^2}{L^2}\right) + O\left(\frac{\lambda^2}{\lambda_0^4}\right). \tag{51}$$

If the field is at temperature T then

$$\begin{aligned} \langle\varphi^2\rangle_{\rm ren} = &\frac{R}{16\pi^2}\left(\xi - \frac{1}{6}\right)\left[\ln\left|\frac{4(2\pi T)^2}{m_{\rm DS}^2}\right| + 2\psi(n_0)\right] + \frac{(2\pi T)^2}{48\pi^2} + \frac{m^2}{8\pi^2}\left[\psi(n_0) - \ln(n_0) + \frac{1}{2n_0}\right] + O\left(\frac{\varepsilon_{\rm WKB}^2}{L^2}\right) \\ &+ O(\lambda^2 T^4 n_0^4). \end{aligned} \tag{52}$$

Let us cite the conditions of the validity of expressions (51) and (52) once more:

$$m \lesssim \frac{1}{L}, \tag{53}$$

$$\lambda \ll \lambda_0 \ll L \quad \left(\text{or } \lambda \ll \frac{1}{2\pi T n_0} \ll L\right). \tag{54}$$

If also

$$n_0 \gg 1, \tag{55}$$

the digamma function $\psi(n_0)$ is given by

$$\psi(n_0) = \ln(n_0) - \frac{1}{2n_0} + O\left(\frac{1}{n_0^2}\right). \tag{56}$$

Hence the expression (52) can be rewritten as

$$\langle\varphi^2\rangle_{\rm ren} = \frac{R}{16\pi^2}\left(\xi - \frac{1}{6}\right)\ln\left|\frac{4(2\pi T)^2 n_0^2}{m_{\rm DS}^2}\right| + \frac{(2\pi T)^2}{48\pi^2} + O\left(\frac{1}{n_0 L^2}\right) + O\left(\frac{\varepsilon_{\rm WKB}^2}{L^2}\right) + O(\lambda^2 T^4 n_0^4). \quad (57)$$

The presence of the arbitrary parameter λ_0 in the expression (51) is a generic feature of local approximation schemes [3,4,10–12,14,20]. For a conformally invariant field this parameter can be absorbed into the definition of constant $m_{\rm DS}$.

Note that the approximation which corresponds to the analytical approximation of Anderson, Hiscock, and Samuel [3] for $\langle\varphi^2\rangle$ in the case of ultrastatic asymptotically flat spacetime can be obtained by one use of the high-frequency approximation of $\tilde{G}_{\rm E}(\omega; y^a)$ [see Eq. (37)] for all the values of ω. However the use of the high-frequency approximation of $\tilde{G}_{\rm E}(\omega; y^a)$ for $\omega \ll 1/\lambda_0$ does not seem obvious. Nevertheless, in the case of a conformally coupled massless field such a procedure gives good results in the asymptotically flat region [3,4,10,14].

APPENDIX

In this Appendix the expanding of $\langle\phi^2\rangle_{\rm DS}$ in powers of $x^\alpha - \tilde{x}^\alpha$ is described.

Let points $P(x)$ and $\tilde{P}(\tilde{x})$ be connected by the shortest geodesic $x^\alpha = x^\alpha(s)$, where s is the canonical parameter. The functions $x^\alpha(s)$ can be expanded in the Taylor series about point $\tilde{P}(\tilde{x})$:

$$x^\alpha = \tilde{x}^\alpha + \frac{1}{1!}\frac{dx^\alpha}{ds}\Delta s + \frac{1}{2!}\frac{d^2x^\alpha}{ds^2}(\Delta s)^2 + \frac{1}{3!}\frac{d^3x^\alpha}{ds^3}(\Delta s)^3 + O((\Delta s)^4), \quad (A1)$$

where the coefficients are evaluated at $\tilde{P}(\tilde{x})$. Using the geodesic equation

$$\frac{d^2x^\alpha}{ds^2} + \Gamma^\alpha_{\beta\gamma}\frac{dx^\beta}{ds}\frac{dx^\gamma}{ds} = 0, \quad (A2)$$

one finds

$$\frac{d^3x^\alpha}{ds^3} = (-\partial_\gamma\Gamma^\alpha_{\sigma\beta} + 2\Gamma^\alpha_{\sigma\delta}\Gamma^\delta_{\beta\gamma})\frac{dx^\sigma}{ds}\frac{dx^\beta}{ds}\frac{dx^\gamma}{ds}, \quad (A3)$$

where ∂_γ denotes the partial derivative with respect to x^γ. Hence Eq. (A1) can be rewritten

$$x^\alpha = \tilde{x}^\alpha + \frac{1}{1!}u^\alpha\Delta s + \frac{1}{2!}(-\Gamma^\alpha_{\beta\gamma}u^\beta u^\gamma)(\Delta s)^2 + \frac{1}{3!}\times(-\partial_\gamma\Gamma^\alpha_{\sigma\beta} + 2\Gamma^\alpha_{\sigma\delta}\Gamma^\delta_{\beta\gamma})u^\sigma u^\beta u^\gamma(\Delta s)^3 + O((\Delta s)^4), \quad (A4)$$

where $u^\alpha = dx^\alpha/ds$. This equation can be inverted

$$u^\alpha\Delta s = \epsilon^\alpha + \tfrac{1}{2}\Gamma^\alpha_{\gamma\beta}\epsilon^\gamma\epsilon^\beta + (\tfrac{1}{6}\Gamma^\alpha_{\sigma\delta}\Gamma^\delta_{\beta\gamma} + \tfrac{1}{6}\partial_\gamma\Gamma^\alpha_{\sigma\beta})\epsilon^\sigma\epsilon^\beta\epsilon^\gamma + O(\epsilon^4), \quad (A5)$$

where $\epsilon^\alpha = x^\alpha - \tilde{x}^\alpha$. If we use determinations σ^μ and σ [21], we can write

$$\begin{aligned}\sigma(x,\tilde{x}) &= \tfrac{1}{2}g_{\alpha\beta}u^\alpha u^\beta(\Delta s)^2\\ &= \tfrac{1}{2}g_{\alpha\beta}\epsilon^\alpha\epsilon^\beta + \tfrac{1}{2}g_{\alpha\beta}\Gamma^\alpha_{\gamma\delta}\epsilon^\beta\epsilon^\gamma\epsilon^\delta\\ &\quad+\tfrac{1}{6}g_{\alpha\beta}\Gamma^\alpha_{\gamma\delta}\Gamma^\delta_{\varepsilon\upsilon}\epsilon^\beta\epsilon^\gamma\epsilon^\varepsilon\epsilon^\upsilon\\ &\quad+\tfrac{1}{6}g_{\alpha\beta}(\partial_\varepsilon\Gamma^\alpha_{\gamma\delta})\epsilon^\beta\epsilon^\gamma\epsilon^\delta\epsilon^\varepsilon\\ &\quad+\tfrac{1}{8}g_{\alpha\beta}\Gamma^\alpha_{\gamma\delta}\Gamma^\beta_{\varepsilon\upsilon}\epsilon^\gamma\epsilon^\delta\epsilon^\varepsilon\epsilon^\upsilon + O(\epsilon^5).\end{aligned} \quad (A6)$$

Hence the resulting expression for $\langle\varphi^2\rangle_{\rm DS}$ is

$$\langle\varphi^2\rangle_{\rm DS} = \frac{1}{4\pi^2}\left[\frac{1}{g_{\alpha\beta}\epsilon^\alpha\epsilon^\beta} - \frac{g_{\alpha\beta}\Gamma^\alpha_{\gamma\delta}\epsilon^\beta\epsilon^\gamma\epsilon^\delta}{(g_{\alpha\beta}\epsilon^\alpha\epsilon^\beta)^2} - \left(\frac{1}{3}g_{\alpha\beta}\Gamma^\alpha_{\gamma\delta}\Gamma^\delta_{\varepsilon\upsilon} + \frac{1}{3}g_{\alpha\beta}\partial_\upsilon\Gamma^\alpha_{\gamma\varepsilon} + \frac{1}{4}g_{\alpha\delta}\Gamma^\alpha_{\beta\gamma}\Gamma^\delta_{\varepsilon\upsilon}\right)\frac{\epsilon^\beta\epsilon^\gamma\epsilon^\varepsilon\epsilon^\upsilon}{(g_{\alpha\beta}\epsilon^\alpha\epsilon^\beta)^2} + \frac{(g_{\alpha\beta}\Gamma^\alpha_{\gamma\delta}\epsilon^\beta\epsilon^\gamma\epsilon^\delta)^2}{(g_{\alpha\beta}\epsilon^\alpha\epsilon^\beta)^3}\right] + \frac{m^2 + (\xi - \frac{1}{6})R}{8\pi^2}\left[C + \frac{1}{2}\ln\left(\frac{m_{\rm DS}^2}{4}|g_{\alpha\beta}\epsilon^\alpha\epsilon^\beta|\right)\right] - \frac{m^2}{16\pi^2} + \frac{1}{48\pi^2}\frac{R_{\alpha\beta}\epsilon^\alpha\epsilon^\beta}{g_{\alpha\beta}\epsilon^\alpha\epsilon^\beta} + O(\epsilon). \quad (A7)$$

In coordinates τ, y^a, y^a are the Riemann normal coordinates with their origin at the point $\tilde{x}^a$ in 3D space.

$$\epsilon^0 = \Delta\tau, \qquad \epsilon^a = y^a. \quad (A8)$$

At point $\tilde{P}$ in these coordinates also [17]

$$\Gamma^\alpha_{\beta\gamma} = 0 \quad (A9)$$

and

$$g_{\alpha\beta}(\partial_\upsilon\Gamma^\alpha_{\gamma\delta})\epsilon^\beta\epsilon^\gamma\epsilon^\delta\epsilon^\upsilon = \frac{\eta_{ab}}{6}(R^a{}_{cde} + R^a{}_{dce})y^b y^c y^d y^e = 0. \quad (A10)$$

After the substitution of these expressions into (A7) one finds that $\langle\varphi^2\rangle_{\rm DS}$ has the form (50).

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[1] J.S. Dowker and R. Critchley, Phys. Rev. D **13**, 3224 (1976); L.S. Brown and J.P. Cassidy, *ibid.* **15**, 2810 (1977); T.S. Bunch and P.C.W. Davies, Proc. R. Soc. London A **360**, 117 (1978); T.S. Bunch, J. Phys. A **12**, 517 (1979); B. Allen and A. Folacci, Phys. Rev. D **35**, 3771 (1987); A. Folacci, J. Math. Phys. (N.Y.) **32**, 2813 (1991); **32**, 2828 (1991); K. Kirsten and J. Garriga, Phys. Rev. D **48**, 567 (1993).

[2] P. Candelas, Phys. Rev. D **21**, 2185 (1980); M.S. Fawcett, Commun. Math. Phys. **89**, 103 (1983); K.W. Howard and P. Candelas, Phys. Rev. Lett. **53**, 403 (1984); B.P. Jensen and A. Ottewill, Phys. Rev. D **39**, 1130 (1989); B.P. Jensen, J.G. McLaughlin, and A.C. Ottewill, *ibid.* **45**, 3002 (1992); P. R. Anderson, W.A. Hiscock, and D.J. Loranz, Phys. Rev. Lett. **74**, 4365 (1995); E.R. Bezerra de Mello, V.B. Bezerra, and N.R. Khusnutdinov, Phys. Rev. D **60**, 063506 (1999).

[3] P.R. Anderson, W.A. Hiscock, and D.A. Samuel, Phys. Rev. D **51**, 4337 (1995).

[4] P.B. Groves, P.R. Anderson, and E.D. Carlson, Phys. Rev. D **66**, 124017 (2002).

[5] B.S. DeWitt, *Dynamical Theory of Groups and Field* (Gordon and Breach, New York, 1965).

[6] V.P. Frolov and A.I. Zel'nikov, Phys. Lett. **115B**, 372 (1982); **123B**, 123 (1983); Phys. Rev. D **29**, 1057 (1984); R. Herman, *ibid.* **58**, 084028 (1998); J. Matyjasek, *ibid.* **61**, 124019 (2000); H. Koyama, Y. Nambu, and A. Tomimatsu, Mod. Phys. Lett. A **15**, 815 (2000); J. Matyjasek, *ibid.* **63**, 084004 (2001).

[7] D.N. Page, Phys. Rev. D **25**, 1499 (1982); M.R. Brown and A.C. Ottewill, *ibid.* **31**, 2514 (1985); M.R. Brown, A.C. Ottewill, and D.N. Page, *ibid.* **33**, 2840 (1986).

[8] T. Zannias, Phys. Rev. D **30**, 1161 (1984).

[9] V.P. Frolov and A.I. Zel'nikov, Phys. Rev. D **35**, 3031 (1987).

[10] R. Balbinot, A. Fabbri, P. Nicolini, V. Frolov, P. Sutton, and A. Zelnikov, Phys. Rev. D **63**, 084029 (2001).

[11] A.A. Popov and S.V. Sushkov, Phys. Rev. D **63**, 044017 (2001); A.A. Popov, *ibid.* **64**, 104005 (2001).

[12] A.A. Popov, Phys. Rev. D **67**, 044021 (2003).

[13] A. Tomimatsu and H. Koyama, Phys. Rev. D **61**, 124010 (2000); S.V. Sushkov, Gravitation Cosmol. **6**, 45 (2000); V. Frolov and A. Zelnikov, Phys. Rev. D **63**, 125026 (2001); V. Frolov, S. Sushkov, and A. Zelnikov, Phys. Rev. D **67**, 104003 (2003).

[14] R. Balbinot, A. Fabbri, P. Nicolini, and P. Sutton, Phys. Rev. D **66**, 024014 (2002).

[15] T.S. Bunch and L. Parker, Phys. Rev. D **20**, 2499 (1979).

[16] C.W. Misner, K.S. Thorne, and J.A. Wheeler, *Gravitation* (Freeman, San Francisco, 1973).

[17] A.Z. Petrov, *Einstein Spaces* (Pergamon, Oxford, 1969).

[18] B.T. Wittaker and G.N. Watson, *A Course of Modern Analysis* (Cambridge University Press, Cambridge, UK, 1927).

[19] S.M. Christensen, Phys. Rev. D **14**, 2490 (1976).

[20] V. Frolov, P. Sutton, and A. Zelnikov, Phys. Rev. D **61**, 024021 (2000).

[21] J.L. Synge, *Relativity: The General Theory* (North-Holland Publishing, Amsterdam, 1964).