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# Timelike naked singularity

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We construct a class of spherically symmetric collapse models in which a naked singularity may develop as the end state of collapse. The matter distribution considered has negative radial and tangential pressures, but the weak energy condition is obeyed throughout. The singularity forms at the center of the collapsing cloud and continues to be visible for a finite time. The duration of visibility depends on the nature of energy distribution. Hence the causal structure of the resulting singularity depends on the nature of the mass function chosen for the cloud. We present a general model in which the naked singularity formed is timelike, neither pointlike nor null. Our work represents a step toward clarifying the necessary conditions for the validity of the Cosmic Censorship Conjecture.

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The cosmic censorship conjecture (CCC) has been widely recognized as one of the most important open problems in gravitational physics today. This is because several important areas in the theory and applications of black hole physics crucially depend on CCC. Nevertheless, the CCC remains unproved and there exists no mathematically precise and definite statement for the CCC which one could try to prove (see, e.g., [1–6] for some recent reviews, and references therein).

For this reason, a detailed study of dynamically developing gravitational collapse models within the framework of general relativity becomes rather essential. The hope is that such a study may allow us to formulate a provable statement of the CCC, if it is correct in some form. Such investigations also help us to discard certain statements of the CCC which might sound plausible but for which there exist counter-examples which show that the CCC cannot be valid in such a form. They may even illustrate the physical conditions that give rise to naked singularities (NS) or black holes (BH) as end states of a realistic gravitational collapse. So far, such dynamical collapse studies have focused largely on collapse models that create either BH or NS, depending on the nature of the initial profiles of density, pressure, and velocity from which the collapse develops. In many of these cases, when an NS develops, it is located at the center of the spherically symmetric cloud (a central singularity, see, e.g., [5–7]). In that case, there will exist families of nonspacelike future directed geodesics, which will be accessible to distant observers in the future, and which will terminate at the singularity in the past, thus making it visible in principle. This is opposed to the BH case where the apparent horizon forms early enough to cover all of the singularity, with no portion of it remaining visible to outside observers.

If we require the pressure to be positive then the "central" singularity, if it is naked, corresponds to a singularity along a visible null line. The remainder of the singularity is spacelike and covered by a horizon. In this paper, however, we permit the pressure to be negative and examine the structure of the singularity. We construct an explicit solution in which the singularity may be timelike. It may even change its character, being timelike along a certain region and, after being visible for a finite time, turning spacelike and being covered. The collapsing matter is described by a particularly chosen matter field that satisfies the weak energy condition although the radial and tangential pressures are negative and unequal. While what is presented here is a specific construction of a class of collapse models, involving somewhat special choices, we make sure that physical reasonability conditions such as the energy conditions and the regularity of the initial data at the initial surface are respected.

The spherically symmetric metric in a general form can be written as

$$ds^2 = -e^{2\nu(t,r)}dt^2 + e^{2\psi(t,r)}dr^2 + R^2(t,r)d\Omega^2, \quad (1)$$

where $d\Omega^2$ is the line element on a two-sphere. Choosing the co-moving frame, the stress-energy tensor for a general (type I) matter field is given in a diagonal form as

$$T^t_t = -\rho; \qquad T^r_r = p_r; \qquad T^\theta_\theta = T^\phi_\phi = p_\theta, \quad (2)$$

where $\rho$, $p_r$ and $p_\theta$ are the energy density, and the radial and tangential pressures, respectively. We assume that the matter field satisfies the *weak energy condition*, that is, the energy density as measured by any local observer is non-negative and, for any timelike vector $V^i$,

$$T_{ik}V^iV^k \geq 0, \quad (3)$$

which amounts to,

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$$\rho \geq 0; \qquad \rho + p_r \geq 0; \qquad \rho + p_\theta \geq 0. \tag{4}$$

The initial data consists of three metric functions, the energy density, and the radial and tangential pressures at the initial time $t = t_i$. This is given in terms of six arbitrary functions of the radial coordinate, *viz.*, $\nu(t_i, r) = \nu_0(r)$, $\psi(t_i, r) = \psi_0(r)$, $R(t_i, r) = r$, $\rho(t_i, r) = \rho_0(r)$, $p_r(t_i, r) = p_{r0}(r)$, $p_\theta(t_i, r) = p_{\theta 0}(r)$, where, using the scaling freedom for the radial coordinate $r$ we have chosen $R(t_i, r) = r$ at the initial epoch. The dynamic evolution of the initial data is then determined by the Einstein equations, which for the metric (1) become ($8\pi G = c = 1$),

$$\rho = \frac{F'}{R^2 R'}, \qquad p_r = -\frac{\dot{F}}{R^2 \dot{R}}, \tag{5}$$

$$\nu' = \frac{2(p_\theta - p_r)}{\rho + p_r}\frac{R'}{R} - \frac{p_r'}{\rho + p_r}, \tag{6}$$

$$-2\dot{R}' + R'\frac{\dot{G}}{G} + \dot{R}\frac{H'}{H} = 0, \tag{7}$$

$$G - H = 1 - \frac{F}{R}, \tag{8}$$

where $F = F(t, r)$ is an arbitrary function. In spherically symmetric spacetimes $F(t, r)$ is interpreted as the mass function, with $F \geq 0$. In order to preserve the regularity of the initial data we must also require $F(t_i, 0) = 0$, *i.e.*, the mass function should vanish at the center of the cloud. The functions $G$ and $H$ are defined as $G(t, r) = e^{-2\psi}(R')^2$ and $H(t, r) = e^{-2\nu}(\dot{R})^2$.

All the initial data above are not mutually independent: from Eq. (6) we find that the function $\nu_0(r)$ is determined in terms of rest of the initial data. Also, by rescaling of the radial coordinate $r$, the number of independent initial data functions reduces to four. We then have a total of five field equations with seven unknowns, $\rho$, $p_r$, $p_\theta$, $\psi$, $\nu$, $R$, and $F$, giving us the freedom of choice of two free functions. Selection of these functions, subject to the given initial data and weak energy condition, determines the matter distribution and metric of the space-time and thus leads to a particular collapse evolution of the initial data. At this point it is convenient to introduce a scaling variable $v(t, r)$, defined as

$$R(t, r) = rv(t, r), \tag{9}$$

where,

$$v(t_i, r) = 1, \;\; v[t_s(r), r] = 0, \;\text{ and } \dot{v} < 0, \tag{10}$$

the last condition being necessary for a collapse. Let us consider the following choice of the allowed free functions, $F(t, r)$ and $\nu(t, r)$,

$$F(t, r) = r^3 \mathcal{M}(r) v, \tag{11}$$

where $r^3\mathcal{M}(r)$ is a suitably differentiable and monotonically nondecreasing function, and

$$\nu(t, r) = \nu_0(R). \tag{12}$$

The function $\mathcal{M}$ may be expanded in a Taylor series about $r = 0$,

$$\mathcal{M}(r) = \mathcal{M}_0 + \mathcal{M}_2 r^2 + \mathcal{M}_3 r^3 + \ldots. \tag{13}$$

Then, from Eq. (5), we have

$$\rho = \frac{3\mathcal{M}v + r[\mathcal{M}_{,r}v + \mathcal{M}v']}{v^2[v + rv']} \tag{14}$$

and

$$p_r = -\frac{\mathcal{M}(r)}{v^2}. \tag{15}$$

The above choice of mass function therefore implies that the radial pressure is negative (see also, [8]). The *weak energy condition*, however, does hold. If $R' = v + rv'$ and $F'$ are both positive in the Eq. (14), then clearly $\rho \geq 0$. Again, for $\rho + p_r \geq 0$ at all epochs, it must be true that

$$(2\mathcal{M}_0 + 4\mathcal{M}_2 r^2 + 5\mathcal{M}_3 r^3 + \ldots)v \geq 0. \tag{16}$$

But, because $v \geq 0$, it follows that if the condition $\rho + p_r \geq 0$ is satisfied at the initial epoch, it is satisfied throughout the evolution. Finally, from Eq. (6),

$$\rho + p_\theta = \frac{1}{2}(\rho + p_r)[1 + R\nu_0(R)_{,R}] + \frac{r^2\mathcal{M}}{R^2} \geq 0 \tag{17}$$

if $[1 + R\nu_0(R)_{,R}] \geq 0$ for all epochs. This provides a necessary condition for the weak energy condition to be satisfied.

At the initial epoch we then have

$$\rho_0(r) = 3\mathcal{M}_0 + 5\mathcal{M}_2 r^2 + 6\mathcal{M}_3 r^3 + \ldots \tag{18}$$

and

$$p_{r_0}(r) = -[\mathcal{M}_0 + \mathcal{M}_2 r^2 + \mathcal{M}_3 r^3 + \ldots] \tag{19}$$

At the initial epoch, the radial and the tangential pressures must be equal at the center and all the pressure gradients must vanish. It follows that the initial tangential pressure must have the form

$$p_{\theta_0}(r) = -[\mathcal{M}_0 + p_{\theta_2} r^2 + p_{\theta_3} r^3 + \ldots]. \tag{20}$$

Hence, from Eq. (6) we see that $\nu_0(r)$ becomes,

$$\nu_0(r) = a_2 r^2 + a_3 r^3 + \ldots, \tag{21}$$

where

$$a_2 = \frac{p_{\theta_2} + \mathcal{M}_2}{2\mathcal{M}_0}, \qquad a_3 = \frac{p_{\theta_3} + \mathcal{M}_3}{2\mathcal{M}_0},$$

and, from Eq. (12),

$$\nu(t, r) = \nu_0(R) = a_2 R^2 + a_3 R^3 + \ldots \tag{22}$$

The dynamic evolution of $p_\theta(t, r)$ is obtained by inserting Eq. (12) in Eq. (6) and simplifying to get,

$$p_\theta(r, v) = p_r + \frac{Rp_r'}{2R'} + \frac{1}{2}\nu_0(R)_{,R}R(\rho + p_r). \quad (23)$$

There exists, therefore, an $\epsilon$ ball around the central shell for which $p_\theta = p_r$ and the perfect fluid equation of state is valid.

Using Eq. (12) in Eq. (7), we get,

$$G(t, r) = b(r)e^{2\nu_0(R)}, \quad (24)$$

where $b(r)$ is another arbitrary function of $r$. In corresponding dust models, we can write $b(r) = 1 + r^2b_0(r)$, where $b_0(r)$ is the energy distribution function of the collapsing shells. Thus, the metric (1) becomes,

$$ds^2 = -e^{2(a_2R^2+\ldots)}dt^2 + \frac{R'^2e^{-2(a_2R^2+\ldots)}dr^2}{1 + r^2b_0(r)} + R^2d\Omega^2 \quad (25)$$

and is valid for small values of $r$, for all epochs, i.e., for all values of $v(r, t)$, till the singularity.

Solving the equation of motion (8) we find that

$$\dot{v} = -e^{2\nu_0(rv)}\sqrt{v^2(2a_2 + 2a_3rv\ldots) + b_0(r)e^{2\nu_0(rv)} + \mathcal{M}(r)}, \quad (26)$$

which may be integrated to obtain

$$t(v, r) = \int_v^1 \frac{e^{-2\nu_0(rv)}dv}{\sqrt{v^2(2a_2 + 2a_3rv\ldots) + b_0(r)e^{2\nu_0(rv)} + \mathcal{M}(r)}}. \quad (27)$$

We note that the radial coordinate $r$ is treated as a constant in the above equation, which gives the time taken for a shell labeled $r$, to reach a later epoch $v$ in collapse from the initial epoch $v = 1$.

It is clear that an explicit solution of the above integral will give a closed form solution of the form $t = f(v, r)$ or, inversely, $v = g(t, r)$, which will then determine the metric function $R = rg(t, r)$ thereby giving an exact solution for the metric (25). Unfortunately, the integral cannot be expressed in closed form and we make a Taylor expansion of the integral about the center of the cloud.

$$t(v, r) = t(v, 0) + r\mathcal{X}(v) + O(r^2) \quad (28)$$

where the function $\mathcal{X}(v)$ is given by,

$$\mathcal{X}(v) = -\frac{1}{2}\int_v^1 dv \frac{2v^4a_3 + b_1}{[\sqrt{b_0(0) + 2v^2a_2 + \mathcal{M}(0)}]^3} \quad (29)$$

If a closed form solution of $R$ exists up to the first approximation, it will be of the form $R = r\mathcal{X}^{-1}\{[t(v, r) - t(v, 0)]/r\}$. Therefore, by expanding as above we are actually solving for $R$ and so for the metric (25) to the first approximation, although we do not write it in closed form. This is because it is only the sign of $\mathcal{X}(0)$ that determines the final end state of the collapse, which is the issue of interest here.

The time taken for the central shell at $r = 0$ to reach the singularity, $t_s(0)$, is given by

$$t_s(0) = \int_0^1 \frac{dv}{\sqrt{v^2(2a_2) + b_0(0) + \mathcal{M}(0)}}. \quad (30)$$

The time taken for the other shells ($r \neq 0$) to reach the singularity, $t_s(r)$, can be given as,

$$t_s(r) = t_s(0) + r\mathcal{X}(0) + O(r^2), \quad (31)$$

where the function $\mathcal{X}(0)$ is given by,

$$\mathcal{X}(0) = -\frac{1}{2}\int_0^1 dv \frac{2v^4a_3 + b_1}{[\sqrt{b_0(0) + 2v^2a_2 + \mathcal{M}(0)}]^3}. \quad (32)$$

We see that by suitable choice of the coefficients of initial density, pressure and energy profiles we can make $\mathcal{X}(0)$ positive or negative. Furthermore, at the singularity, for a constant $v$ surface we have

$$\lim_{v\to 0} v' = \sqrt{b_0(r) + \mathcal{M}(r)}\mathcal{X}(0) + \mathcal{O}(r^2) \quad (33)$$

and, because we have expressions for $v'$ and $\dot{v}$, near the central singularity, we can in principle calculate $v(r, t)$ in the neighborhood of the central singularity. This solves the system of Einstein equations.

The apparent horizon is the boundary of the trapped region of the space-time and is given by $R/F = 1$. If the neighborhood of the center gets trapped earlier than the singularity, then it will be covered and a black hole will be the final state of the collapse. Otherwise, the singularity can be naked with nonspacelike future directed trajectories escaping from it to outside observers. Using (11), we find that the apparent horizon is just the surface

$$r^2\mathcal{M}(r) = 1. \quad (34)$$

Therefore, if $r_b^2\mathcal{M}(r_b) < 1$ there will be no trapped surfaces in space-time, where $r_b$ is the radial coordinate of the boundary of the cloud.

It is simplest to examine the nature of the singularity by noting that it occurs at $R = 0$. This implies that at the singularity,

$$ds^2 = \left[\exp(2\psi) - \exp(2\nu)\frac{R'^2}{\dot{R}^2}\right]dr^2. \quad (35)$$

If the right hand side is negative, the singularity is timelike. Therefore, for a timelike singularity, $G - H > 0$, or

$$1 - r^2\mathcal{M}(r) > 0. \quad (36)$$

But, because the function $r^2\mathcal{M}(r)$ is monotonically nondecreasing, it follows that the singularity is timelike near $r = 0$, becomes null at $r^2\mathcal{M}(r) = 1$ and finally spacelike when $r^2\mathcal{M}(r) > 1$.

This result may also be obtained explicitly near the center by examining the outgoing null geodesics. To see this specifically at $r = 0$, the outgoing radial null lines are given by

$$\frac{dt}{dr} = e^{\psi - \nu}, \tag{37}$$

which, at the singularity, corresponds to

$$\left(\frac{dt}{dr}\right)_{null} = \lim_{v \to 0} \frac{r v'}{\sqrt{1 + r^2 b_0(r)}}. \tag{38}$$

In order to find the existence or otherwise of an outgoing null geodesic from the singularity we substitute the value of $v'$ at the singularity in the above equation to obtain

$$\left(\frac{dt}{dr}\right)_{null} = \left[\frac{r\sqrt{b_0(r) + \mathcal{M}(r)}}{\sqrt{1 + r^2 b_0(r)}}\right]\left(\frac{dt}{dr}\right)_s. \tag{39}$$

If

$$\left(\frac{dt}{dr}\right)_s = \mathcal{X}(0) \geq 0 \tag{40}$$

then, for all values of $r$ for which

$$\left[\frac{r\sqrt{b_0(r) + \mathcal{M}(r)}}{\sqrt{1 + r^2 b_0(r)}}\right] \leq 1 \tag{41}$$

or $1 - r^2 \mathcal{M}(r) > 0$ , there will be a visible outgoing null geodesic leaving the singularity.

The model discussed here is based on a special choice of the mass function. The fluids described by this choice are not intended to describe an actual physical system: both the radial and the tangential pressures are negative. Nevertheless the initial data satisfy appropriate regularity conditions and the weak energy condition is maintained throughout. Therefore, the system can serve as a guide to further clarifying and precisely formulating the CCC.

Understanding what is possible in dynamically developing collapse models is necessary to arrive at a plausible concrete statement of the CCC. We have an example that shows that even in spherically symmetric collapse, the naked singularity need not always be either pointlike or null, but can have an interesting causal structure, including being timelike.

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