New Hamiltonian formalism and quasilocal conservation equations of general relativity

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I describe the Einstein's gravitation of $3+1$ dimensional spacetimes using the (2,2) formalism without assuming isometries. In this formalism, quasilocal energy, linear momentum, and angular momentum are identified from the four Einstein's equations of the divergence-type, and are expressed geometrically in terms of the area of a two-surface and a pair of null vector fields on that surface. The associated quasilocal balance equations are spelled out, and the corresponding fluxes are found to assume the canonical form of energy-momentum-flux as in standard field theories. The remaining non-divergence-type Einstein's equations turn out to be the Hamilton's equations of motion, which are derivable from the *nonvanishing* Hamiltonian by the variational principle. The Hamilton's equations are the evolution equations along the out-going null geodesic whose *affine* parameter serves as the time function. In the asymptotic region of asymptotically flat spacetimes, it is shown that the quasilocal quantities reduce to the Bondi energy, linear momentum, and angular momentum, and the corresponding fluxes become the Bondi fluxes. The quasilocal angular momentum turns out to be zero for any two-surface in the flat Minkowski spacetime. I also present a candidate for quasilocal *rotational* energy which agrees with the Carter's constant in the asymptotic region of the Kerr spacetime. Finally, a simple relation between energy-flux and angular momentum-flux of a generic gravitational radiation is discussed, whose existence reflects the fact that energy-flux always accompanies angular momentum-flux unless the flux is an s-wave.

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I. INTRODUCTION

It has been known for sometime that the Plebañski equation [1], the self-dual Einstein's equation of 4-dimensional Euclidean space, can be obtained as the large n limit of the equations of motion of a certain class of $sl(n)$-valued nonlinear sigma models in two dimensions [2–4]. The equivalence of these equations defined in two different dimensions is quite unexpected, but if one realizes that large n limit of the $sl(n)$ Lie algebra is just the Lie algebra of area-preserving diffeomorphisms of an auxiliary 2-dimensional surface [5–7], and that the equations of motion of $sl(\infty)$-valued nonlinear sigma models in two dimensions are in fact partial differential equations on four dimensional space, then one might be more comfortable with the idea of describing 4-dimensional self-dual Einstein's gravity as a limit of a certain class of 2-dimensional field theories, and can show that the two theories are in fact identical. This correspondence is supported further by the observation that the Plebañski equation and the $sl(n)$-valued 2-dimensional nonlinear sigma models are both integrable.

One may be interested in extending this idea of describing $3+1$ dimensional theories from $1+1$ dimensional perspective without the self-dual restriction and in the Lorentzian regime. There are several advantages of such a description, if it is possible at all. They stem from the fact that $1+1$ dimensional field theories are usually more manageable than $3+1$ dimensional ones, both classically and quantum mechanically. For example, a number of field theories in $1+1$ dimensions are renormalizable, due to the *dimensionlessness* of field variables in a naive power counting. There would be an enormous gain if one ever succeeds in describing the Einstein's gravitation in $3+1$ dimensions as a limit of some kind of $1+1$ dimensional field theories which eventually proves to be renormalizable. This idea sounds strange but does not seem impossible, since the renormalizability is highly sensitive to the spacetime dimensions on which the theories are defined.

These reasonings led us to seek the possibility whether the Einstein's gravity in $3+1$ dimensions without the self-dual restriction is describable as a $1+1$ dimensional field theory [8]. The idea was simply to split a $3+1$ dimensional spacetime into a $1+1$ dimensional base manifold and a 2-dimensional fibre space, and write down the Einstein-Hilbert action. Then the Einstein-Hilbert action becomes $1+1$ dimensional field theory action, where the infinite dimensional group of diffeomorphisms of 2-dimensional fibre space becomes the Yang-Mills gauge symmetry. But this program was successful only in a formal sense, since the resulting $1+1$ dimensional action did not seem much useful, which made the whole idea of describing the Einstein's gravitation as $1+1$ dimensional field theory questionable. The follow-up idea was to use the gauge freedom of the $3+1$ dimensional spacetime [9–14]. If one chooses one of the spacetime coordinates as the affine parameter of the out-

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going null geodesic, then it turns out that the $1 + 1$ dimensional field theory description of the Einstein's gravitation is simplified significantly.

The purpose of this paper is to present several unexpected results that I obtained in the (2,2) fibre bundle description of the Einstein's gravitation, and discuss their physical implications. First, I will present quasilocal balance equations of energy, linear momentum, and angular momentum for an arbitrary compact two-surface, which are just two-surface integrals of the four divergence-type equations that are part of the Einstein's equations [13–15]. Quasilocal energy, linear momentum, and angular momentum are expressed in the coordinate-independent and geometric way in terms of the area of a two-surface and the in- and out-going null vector fields at each point of that surface [16,17]. They are Bondi-like, since their rates of changes are given by fluxes of the canonical form [18]

$$T_{0\alpha}\eta^{\alpha} \sim \sum_{I} \pi_I \mathcal{L}_{\eta} q^I, \qquad (1.1)$$

where η is an appropriate vector field defined at each point of a two-surface.

Second, problems of defining quasilocal angular momentum and associated rotational energy have been particularly subtle issues [19–24]. This is due to the fact that the very notion of rotation depends on the choice of the coordinates, which implies that one can always remove the effects of rotation by working in a corotating coordinate system. On the other hand, it is natural to demand that the angular momentum and the rotational energy of any compact two-surface in the flat Minkowski spacetime be zero. It will be seen that our quasilocal angular momentum and rotational energy not only become zero for any two-surface in the flat Minkowski spacetime, but also reduce to the standard values of the total angular momentum and the *Carter's constant* in the asymptotic region of the Kerr spacetime, respectively [25–28]. In this sense, our quasilocal rotational energy may be regarded as a quasilocal generalization of the Carter's constant of a generic gravitational field.

Third, using the affine parameter of the out-going null geodesic as the time coordinate, I will write down the Hamiltonian of the Einstein's theory [29]. I will obtain the Hamilton's equations of motion from this Hamiltonian using appropriate boundary conditions, which determine the time flows of the field variables. Together with the quasilocal balance equations (or the constraint equations depending on the signature of the 3-dimensional hypersurface), it will be seen that the Hamilton's equations of motion constitute the full Einstein's equations.

Finally, I will present a simple but general relation between quasilocal energy-flux and angular momentum-flux of a generic gravitational radiation that has no isometries. It is a generalization of the well-known relation of mass-loss and angular momentum-loss [30],

$$\delta U = \frac{\omega}{m_z}\delta L_z \qquad (1.2)$$

for small perturbations around a stationary and axisymmetric spacetime, where ω and m_z are the frequency and azimuthal angular momentum of the perturbations, respectively. To my knowledge, such a relation between these gravitational fluxes of the most general type has not been discussed before, but it strongly indicates that our identifications of fluxes are physically correct, since energy-flux always carries angular momentum-flux unless the radiation is an s-wave.

This paper is organized as follows. In Sec. II, I will introduce the kinematics of the (2,2) fibre bundle formalism, and write down the Einstein's equations. Then I will discuss $1 + 1$ dimensional gauge theory aspects of the Einstein's gravitation of $3 + 1$ dimensions from this fibre bundle point of view.

In Sec. III, I will study the four Einstein's equations that are first-order in the derivatives along the out-going null vector field. These equations, which are the natural analogs of the Einstein's constraint equations in the $3 + 1$ formalism, turn out to be divergence-type equations. It is from the two-surface integrals of these equations that one obtains quasilocal balance equations of gravitational energy, linear momentum, and angular momentum. I will also present quasilocal gravitational rotational energy. The Carter's constant, which is usually interpreted as a measure of intrinsic rotation of gravitational field, is known to exist for a certain class of spacetimes that have two commuting Killing symmetries, and for the Kerr spacetime, it is just the total angular momentum squared. Our quasilocal rotational energy reduces to the Carter's constant for asymptotically Kerr spacetimes, as is shown in Sec. VI, and therefore, may be regarded as a quasilocal generalization of the Carter's constant to spacetimes that have no isometries.

In Sec. IV, it will be shown that the remaining Einstein's equations, which are second-order in the derivatives along the out-going null vector field, are the Hamilton's equations of motion derivable from a non-vanishing Hamiltonian by the variational principle. The details of this derivation are given in the Appendix. Thus, together with the quasilocal balance equations (or constraint equations depending on the signature of the 3-dimensional hypersurface), the Hamilton's equations of motion make up for the full Einstein's equations in this formalism.

In Sec. V, quasilocal energy, linear momentum, and angular momentum of the previous sections will be expressed in the coordinate-independent and geometric way, using the area of a two-surface and a pair of null vector fields orthogonal to that surface. Relative to a given background spacetime against which these quasilocal quantities are measured, quasilocal energy and linear momentum are given by the rates of changes of the area of the two-surface along the in- and out-going null vector fields, respectively, and quasilocal angular momentum associated with a vector field ξ is given by two-surface integral of the projection of the *twist* of the in- and out-going null vector fields onto ξ modulo a background-dependent subtraction term.

In Sec. VI, I will study the quasilocal balance equations at the null infinity and show that they all agree with the well-known Bondi formulae of energy-loss, momentum-loss, and angular momentum-loss. In order to show these correspondences, it is necessary to find the asymptotic fall-off rates of the metric and their derivatives near the null infinity, using the affine parameter of the out-going null geodesic as the radial coordinate [13,14,31–33]. I will present the asymptotic fall-off rates in this section. It will be shown that the quasilocal rotational energy in the asymptotic region of the asymptotically Kerr spacetimes agrees with the Carter's constant of the Kerr spacetime.

In Sec. VII, a general relation between quasilocal energy-flux and angular momentum-flux will be presented for a generic gravitational radiation. When restricted to small perturbations around a stationary and axi-symmetric spacetime, it will be shown that this relation reduces to the well-known relation of mass-loss and angular momentum-loss in the perturbation theory of the Kerr black hole [30].

In the Appendix, I present in detail the derivation of the non-divergence type Einstein's equations as the Hamilton's equations of motion associated with a non-vanishing gravitational Hamiltonian.

II. KINEMATICS

In this section, I will introduce the kinematics of the (2,2) fibre bundle formalism [34,35], and write down the Einstein's equations. This section serves mainly to fix the notations. Let us consider the following line element

$$ds^2 = -2dudv - 2hdu^2 + \phi_{ab}(dy^a + A_+^a du + A_-^a dv) \times (dy^b + A_+^b du + A_-^b dv), \tag{2.1}$$

where $+, -$ stands for u, v, respectively [9–14,36–39]. In order to understand the geometry of this metric, it is convenient to introduce the following vector fields $\{\hat{\partial}_\pm\}$ defined as

$$\hat{\partial}_+ := \partial_+ - A_+^a \partial_a, \tag{2.2}$$

$$\hat{\partial}_- := \partial_- - A_-^a \partial_a, \tag{2.3}$$

where

$$\partial_+ = \frac{\partial}{\partial u}, \qquad \partial_- = \frac{\partial}{\partial v}, \qquad \partial_a = \frac{\partial}{\partial y^a} \quad (a = 2, 3). \tag{2.4}$$

The inner products of the vector fields $\{\hat{\partial}_\pm, \partial_a\}$ are given by

$$\begin{aligned} &<\hat{\partial}_+, \hat{\partial}_+> = -2h, \qquad <\hat{\partial}_+, \hat{\partial}_-> = -1, \\ &<\hat{\partial}_-, \hat{\partial}_-> = 0, \qquad <\hat{\partial}_\pm, \partial_a> = 0, \\ &<\partial_a, \partial_b> = \phi_{ab}. \end{aligned} \tag{2.5}$$

The hypersurface $u =$ constant is an out-going null hypersurface generated by $\hat{\partial}_-$ whose norm is zero. The hypersurface $v =$ constant is generated by $\hat{\partial}_+$ whose norm is $-2h$, which can be either negative, zero, or positive, depending on whether $\hat{\partial}_+$ is timelike, null, or spacelike, respectively. The vector fields $\{\hat{\partial}_\pm\}$ are called horizontal since they are orthogonal to $\{\partial_a\}$, and two dimensional section spanned by $\{\hat{\partial}_\pm\}$ has the Lorentzian signature. The intersection of two hypersurfaces $u, v =$ constant defines a spacelike two-surface N_2 labeled by $\{y^a\}$, which is assumed to be compact with a positive-definite metric ϕ_{ab} on it (see Fig. 1). The metric ϕ_{ab} is decomposed into the area element e^σ and the conformal two-metric ρ_{ab} normalized to have a unit determinant

$$\phi_{ab} = e^\sigma \rho_{ab} \quad (\det\rho_{ab} = 1). \tag{2.6}$$

For later uses, let us express the in-going null vector field n and out-going null vector field l in term of $\{\hat{\partial}_\pm\}$. They are given by

$$n = \hat{\partial}_+ - h\hat{\partial}_-, \tag{2.7}$$

$$l = \hat{\partial}_-, \tag{2.8}$$

and satisfy the normalization condition

$$<n, l> = -1. \tag{2.9}$$

Notice that $\partial/\partial v$ is either spacelike or null, since its norm is given by

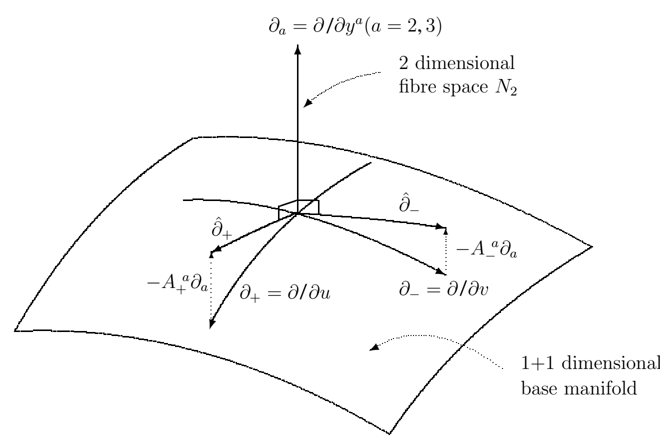


FIG. 1. This figure shows the geometry of the (2,2) fibre bundle splitting of $3 + 1$ dimensional spacetime. The $1 + 1$ dimensional base manifold is spanned by $\{\partial_\pm\}$ and the two dimensional fibre space N_2 by $\{\partial_a\}$. The horizontal vector fields $\{\hat{\partial}_\pm\}$ are orthogonal to N_2, and $A_\pm^a$ are the connections valued in the Lie algebra of the diffeomorphisms of N_2.

$$<\frac{\partial}{\partial v},\frac{\partial}{\partial v}> = e^{\sigma}\rho_{ab}A^a_-A^b_- \geq 0. \tag{2.10}$$

The coordinate v increases uniformly as l evolves, since we have

$$\mathcal{L}_l v = 1. \tag{2.11}$$

In the gauge where $A^a_- = 0$, l is given by

$$l = \frac{\partial}{\partial v}, \tag{2.12}$$

which tells us that v becomes the affine parameter of the out-going null geodesic l.

The complete set of the vacuum Einstein's equations are found to be [11]

$$(a)\quad e^{\sigma}D_+D_-\sigma + e^{\sigma}D_-D_+\sigma + 2e^{\sigma}(D_+\sigma)(D_-\sigma) - 2e^{\sigma}(D_-h)(D_-\sigma) - \frac{1}{2}e^{2\sigma}\rho_{ab}F^a_{+-}F^b_{+-} + e^{\sigma}R_2 - he^{\sigma}\left\{(D_-\sigma)^2 - \frac{1}{2}\rho^{ab}\rho^{cd}(D_-\rho_{ac})(D_-\rho_{bd})\right\} = 0, \tag{2.13}$$

$$(b)\quad -e^{\sigma}D_+^2\sigma - \frac{1}{2}e^{\sigma}(D_+\sigma)^2 - e^{\sigma}(D_-h)(D_+\sigma) + e^{\sigma}(D_+h)(D_-\sigma) + 2he^{\sigma}(D_-h)(D_-\sigma) + e^{\sigma}F^a_{+-}\partial_a h - \frac{1}{4}e^{\sigma}\rho^{ab}\rho^{cd}(D_+\rho_{ac})(D_+\rho_{bd}) + \partial_a(\rho^{ab}\partial_b h) + h\left\{-e^{\sigma}(D_+\sigma)(D_-\sigma) + \frac{1}{2}e^{\sigma}\rho^{ab}\rho^{cd}(D_+\rho_{ac})(D_-\rho_{bd}) + \frac{1}{2}e^{2\sigma}\rho_{ab}F^a_{+-}F^b_{+-} - e^{\sigma}R_2\right\} + h^2e^{\sigma}\left\{(D_-\sigma)^2 - \frac{1}{2}\rho^{ab}\rho^{cd}(D_-\rho_{ac})(D_-\rho_{bd})\right\} = 0, \tag{2.14}$$

$$(c)\quad 2e^{\sigma}(D_-^2\sigma) + e^{\sigma}(D_-\sigma)^2 + \frac{1}{2}e^{\sigma}\rho^{ab}\rho^{cd}(D_-\rho_{ac})(D_-\rho_{bd}) = 0, \tag{2.15}$$

$$(d)\quad D_-(e^{2\sigma}\rho_{ab}F^b_{+-}) - e^{\sigma}\partial_a(D_-\sigma) - \frac{1}{2}e^{\sigma}\rho^{bc}\rho^{de}(D_-\rho_{bd})(\partial_a\rho_{ce}) + \partial_b(e^{\sigma}\rho^{bc}D_-\rho_{ac}) = 0, \tag{2.16}$$

$$(e)\quad -D_+(e^{2\sigma}\rho_{ab}F^b_{+-}) - e^{\sigma}\partial_a(D_+\sigma) - \frac{1}{2}e^{\sigma}\rho^{bc}\rho^{de}(D_+\rho_{bd})(\partial_a\rho_{ce}) + \partial_b(e^{\sigma}\rho^{bc}D_+\rho_{ac}) + 2he^{\sigma}\partial_a(D_-\sigma) + he^{\sigma}\rho^{bc}\rho^{de}(D_-\rho_{bd})(\partial_a\rho_{ce}) + 2e^{\sigma}\partial_a(D_-h) - 2\partial_b(he^{\sigma}\rho^{bc}D_-\rho_{ac}) = 0, \tag{2.17}$$

$$(f)\quad -2e^{\sigma}D_-^2h - 2e^{\sigma}(D_-h)(D_-\sigma) + e^{\sigma}D_+D_-\sigma + e^{\sigma}D_-D_+\sigma + e^{\sigma}(D_+\sigma)(D_-\sigma) + \frac{1}{2}e^{\sigma}\rho^{ab}\rho^{cd}(D_+\rho_{ac})(D_-\rho_{bd}) + e^{2\sigma}\rho_{ab}F^a_{+-}F^b_{+-} - 2he^{\sigma}\left\{D_-^2\sigma + \frac{1}{2}(D_-\sigma)^2 + \frac{1}{4}\rho^{ab}\rho^{cd}(D_-\rho_{ac})(D_-\rho_{bd})\right\} = 0, \tag{2.18}$$

$$(g)\quad h\{e^{\sigma}D_-^2\rho_{ab} - e^{\sigma}\rho^{cd}(D_-\rho_{ac})(D_-\rho_{bd}) + e^{\sigma}(D_-\rho_{ab})(D_-\sigma)\} - \frac{1}{2}e^{\sigma}(D_+D_-\rho_{ab} + D_-D_+\rho_{ab}) + \frac{1}{2}e^{\sigma}\rho^{cd}\{(D_-\rho_{ac})(D_+\rho_{bd}) + (D_-\rho_{bc})(D_+\rho_{ad})\} - \frac{1}{2}e^{\sigma}\{(D_-\rho_{ab})(D_+\sigma) + (D_+\rho_{ab})(D_-\sigma)\} + e^{\sigma}(D_-\rho_{ab})(D_-h) + \frac{1}{2}e^{2\sigma}\rho_{ac}\rho_{bd}F^c_{+-}F^d_{+-} - \frac{1}{4}e^{2\sigma}\rho_{ab}\rho_{cd}F^c_{+-}F^d_{+-} = 0. \tag{2.19}$$

Here R_2 is the scalar curvature of N_2, and the diffN_2-covariant derivatives are given by [8,10],

$$F^a_{+-} = \partial_+A^a_- - \partial_-A^a_+ - [A_+, A_-]^a_{\mathrm{L}}, \tag{2.20}$$

$$D_\pm\sigma = \partial_\pm\sigma - [A_\pm, \sigma]_{\mathrm{L}}, \tag{2.21}$$

$$D_\pm h = \partial_\pm h - [A_\pm, h]_{\mathrm{L}}, \tag{2.22}$$

$$D_\pm\rho_{ab} = \partial_\pm\rho_{ab} - [A_\pm, \rho]_{\mathrm{L}ab}. \tag{2.23}$$

In general, the diffN_2-covariant derivative of a tensor density $f_{ab\cdots}$ with weight w with respect to the diffeomorphisms of N_2 is given by

$$D_\pm f_{ab\cdots} = \partial_\pm f_{ab\cdots} - [A_\pm, f]_{\mathrm{L}ab\cdots}, \tag{2.24}$$

where the bracket $[A_\pm, f]_{\mathrm{L}ab\cdots}$ is the Lie derivative of $f_{ab\cdots}$ along $A_\pm := A^a_\pm\partial_a$,

$$[A_\pm, f]_{\mathrm{L}ab\cdots} := A^c_\pm\partial_c f_{ab\cdots} + f_{cb\cdots}\partial_a A^c_\pm + f_{ac\cdots}\partial_b A^c_\pm + \cdots + w(\partial_c A^c_\pm)f_{ab\cdots}. \tag{2.25}$$

For instance, the diffN_2-covariant derivatives of the area element e^{σ} and the conformal metric ρ_{ab} which are scalar and tensor density with weight 1 and -1 with respect to the diffN_2 transformations, respectively, are given by

$$D_{\pm}e^{\sigma} = \partial_{\pm}e^{\sigma} - A_{\pm}^{c}\partial_{c}e^{\sigma} - (\partial_{a}A_{\pm}^{a})e^{\sigma}, \tag{2.26}$$

$$D_{\pm}\rho_{ab} = \partial_{\pm}\rho_{ab} - A_{\pm}^{c}\partial_{c}\rho_{ab} - \rho_{cb}\partial_{a}A_{\pm}^{c} - \rho_{ac}\partial_{b}A_{\pm}^{c} + (\partial_{c}A_{\pm}^{c})\rho_{ab}. \tag{2.27}$$

If one uses the Leibniz rule in (2.26), then one has

$$D_{\pm}\sigma = \partial_{\pm}\sigma - A_{\pm}^{c}\partial_{c}\sigma - \partial_{a}A_{\pm}^{a} = \partial_{\pm}\sigma - [A_{\pm}, \sigma]_{\mathrm{L}}, \tag{2.28}$$

which is just the Eq. (2.21).

The spacetime integral of the *scalar* curvature of the metric (2.1) is given by

$$I = \int dudvd^{2}yL + \text{surface integrals}, \tag{2.29}$$

where L is given by [9–14]

$$\begin{aligned} L = &-\frac{1}{2}e^{2\sigma}\rho_{ab}F_{+-}^{a}F_{+-}^{b} + e^{\sigma}(D_{+}\sigma)(D_{-}\sigma) \\ &-\frac{1}{2}e^{\sigma}\rho^{ab}\rho^{cd}(D_{+}\rho_{ac})(D_{-}\rho_{bd}) - e^{\sigma}R_{2} \\ &- 2e^{\sigma}(D_{-}h)(D_{-}\sigma) - he^{\sigma}(D_{-}\sigma)^{2} \\ &+\frac{1}{2}he^{\sigma}\rho^{ab}\rho^{cd}(D_{-}\rho_{ac})(D_{-}\rho_{bd}). \end{aligned} \tag{2.30}$$

Each term in (2.30) is manifestly invariant under the diffeomorphisms of N_2, since the $\{y^{a}\}$-dependence of each term is completely hidden in the diffN_2-covariant derivatives. In this sense one may regard N_2 as a kind of *internal* space as in Yang-Mills theory, with the infinite dimensional group of diffeomorphisms of N_2 as the associated gauge symmetry. Thus, the Einstein's gravitation of $3+1$ dimensional spacetimes is describable as $1+1$ dimensional Yang-Mills type gauge theory interacting with $1+1$ dimensional scalar fields σ, h, and nonlinear sigma fields ρ_{ab} whose interactions are dictated by the above Lagrangian density L. If one uses the diffN_2 gauge freedom so that $A_{-}^{a} = 0$, then the metric (2.21) becomes identical to the metric of the null hypersurface formalism studied in [38]. In this paper, however, I shall retain the A_{-}^{a} field, since its presence will make the coordinate choice less restrictive and the diffN_2-invariant Yang-Mills type gauge theory aspect more transparent.

III. A SET OF QUASILOCAL BALANCE EQUATIONS

Notice that the equations (2.13), (2.14) and (2.17) are partial differential equations that are *first-order* in D_{-} derivatives. Therefore, it is of particular interest to study these equations, since they are the analogs of the Einstein's constraint equations in the standard $3+1$ formalism. Thus, in this (2,2) formalism, the *natural* vector field that defines the evolution is D_{-}. Then the momentum variables $\pi_{I} = \{\pi_{h}, \pi_{\sigma}, \pi_{a}, \pi^{ab}\}$ conjugate to the configuration variables $q^{I} = \{h, \sigma, A_{+}^{a}, \rho_{ab}\}$ are defined as

$$\pi_{I} := \frac{\partial L}{\partial(D_{-}q^{I})}. \tag{3.1}$$

They are found to be

$$\pi_{h} = -2e^{\sigma}(D_{-}\sigma), \tag{3.2}$$

$$\pi_{\sigma} = -2e^{\sigma}(D_{-}h) - 2he^{\sigma}(D_{-}\sigma) + e^{\sigma}(D_{+}\sigma), \tag{3.3}$$

$$\pi_{a} = e^{2\sigma}\rho_{ab}F_{+-}^{b}, \tag{3.4}$$

$$\pi^{ab} = he^{\sigma}\rho^{ac}\rho^{bd}(D_{-}\rho_{cd}) - \frac{1}{2}e^{\sigma}\rho^{ac}\rho^{bd}(D_{+}\rho_{cd}). \tag{3.5}$$

Conversely, one can express D_{-} derivatives of the configuration variables in terms of the conjugate momenta as follows,

$$D_{-}h = -\frac{1}{2}e^{-\sigma}\pi_{\sigma} + \frac{1}{2}D_{+}\sigma + \frac{1}{2}he^{-\sigma}\pi_{h}, \tag{3.6}$$

$$D_{-}\sigma = -\frac{1}{2}e^{-\sigma}\pi_{h}, \tag{3.7}$$

$$F_{+-}^{a} = e^{-2\sigma}\rho^{ab}\pi_{b}, \tag{3.8}$$

$$D_{-}\rho_{ab} = \frac{1}{h}e^{-\sigma}\rho_{ac}\rho_{bd}\pi^{cd} + \frac{1}{2h}D_{+}\rho_{ab}. \tag{3.9}$$

Notice that π^{ab} is traceless

$$\rho_{ab}\pi^{ab} = 0, \tag{3.10}$$

due to the identities

$$\rho^{ab}D_{\pm}\rho_{ab} = 0 \tag{3.11}$$

which are direct consequences of the condition

$$\det \rho_{ab} = 1. \tag{3.12}$$

The Hamiltonian density H_0 is given by [13,14]

$$H_{0} := \pi_{I}D_{-}q^{I} - L = H + \text{total divergences}, \tag{3.13}$$

where H is

$$\begin{aligned} H = &-\frac{1}{2}e^{-\sigma}\pi_{\sigma}\pi_{h} + \frac{1}{4}he^{-\sigma}\pi_{h}^{2} - \frac{1}{2}e^{-2\sigma}\rho^{ab}\pi_{a}\pi_{b} \\ &+\frac{1}{2h}e^{-\sigma}\rho_{ac}\rho_{bd}\pi^{ab}\pi^{cd} + \frac{1}{2}\pi_{h}(D_{+}\sigma) \\ &+\frac{1}{2h}\pi^{ab}(D_{+}\rho_{ab}) + \frac{1}{8h}e^{\sigma}\rho^{ab}\rho^{cd}(D_{+}\rho_{ac})(D_{+}\rho_{bd}) \\ &+ e^{\sigma}R_{2}. \end{aligned} \tag{3.14}$$

Notice that H and H_0 are Hamiltonian densities that differ by total divergences only. In terms of the canonical variables, the first-order equations (2.13), (2.14), and (2.17) can be written as, after a little algebra,

$$\begin{aligned}\text{(i)}\quad & \pi^{ab}D_+\rho_{ab} + \pi_\sigma D_+\sigma - hD_+\pi_h \\ & - \partial_+(h\pi_h + 2e^\sigma D_+\sigma) + \partial_a(h\pi_h A_+^a + 2A_+^a e^\sigma D_+\sigma \\ & + 2he^{-\sigma}\rho^{ab}\pi_b + 2\rho^{ab}\partial_b h) = 0,\end{aligned} \tag{3.15}$$

$$\text{(ii)}\quad H - \partial_+\pi_h + \partial_a(A_+^a\pi_h + e^{-\sigma}\rho^{ab}\pi_b) = 0, \tag{3.16}$$

$$\begin{aligned}\text{(iii)}\quad & \partial_+\pi_a - \partial_b(A_+^b\pi_a) - \pi_b\partial_a A_+^b - \pi_\sigma\partial_a\sigma + \partial_a\pi_\sigma \\ & - \pi_h\partial_a h - \pi^{bc}\partial_a\rho_{bc} + \partial_b(\pi^{bc}\rho_{ac}) + \partial_c(\pi^{bc}\rho_{ab}) \\ & - \partial_a(\pi^{bc}\rho_{bc}) = 0.\end{aligned} \tag{3.17}$$

Notice that (3.15) and (3.16) are divergence-type equations of the following form [15]

$$A + \partial_+B + \partial_a C^a = 0. \tag{3.18}$$

One can also express (3.17) as another divergence-type equation. If we contract (3.17) by an arbitrary function ξ^a of $\{v, y^b\}$ such that

$$\partial_+\xi^a = 0, \tag{3.19}$$

then we have

$$\begin{aligned}& \pi^{ab}\mathcal{L}_\xi\rho_{ab} + \pi_\sigma\mathcal{L}_\xi\sigma + \pi_h\mathcal{L}_\xi h + \pi_a\mathcal{L}_\xi A_+^a - \partial_+(\xi^a\pi_a) \\ & + \partial_a(-\xi^a\pi_\sigma + 2\pi^{ab}\xi^c\rho_{bc} + A_+^a\xi^b\pi_b) = 0,\end{aligned} \tag{3.20}$$

which is in the same form as (3.18). Here $\mathcal{L}_\xi f_{ab\cdots}$ is the Lie derivative defined in (2.25),

$$\mathcal{L}_\xi f_{ab\cdots} = [\xi, f]_{\mathrm{L}ab\cdots} \quad (\xi := \xi^a\partial_a). \tag{3.21}$$

Integrals of the divergence-type equations (3.15), (3.16), and (3.20) over a compact two-surface N_2 become, after normalizing by $1/16\pi$,

$$\begin{aligned}\frac{\partial}{\partial u}U(u,v) = \frac{1}{16\pi}\oint d^2y(&\pi^{ab}D_+\rho_{ab} + \pi_\sigma D_+\sigma \\ & - hD_+\pi_h),\end{aligned} \tag{3.22}$$

$$\frac{\partial}{\partial u}P(u,v) = \frac{1}{16\pi}\oint d^2yH, \tag{3.23}$$

$$\begin{aligned}\frac{\partial}{\partial u}L(u,v;\xi) = \frac{1}{16\pi}\oint d^2y(&\pi^{ab}\mathcal{L}_\xi\rho_{ab} + \pi_\sigma\mathcal{L}_\xi\sigma \\ & - h\mathcal{L}_\xi\pi_h - A_+^a\mathcal{L}_\xi\pi_a),\end{aligned} \tag{3.24}$$

where we used the identities

$$\oint d^2y\pi_h\mathcal{L}_\xi h = -\oint d^2yh\mathcal{L}_\xi\pi_h, \tag{3.25}$$

$$\oint d^2y\pi_a\mathcal{L}_\xi A_+^a = -\oint d^2yA_+^a\mathcal{L}_\xi\pi_a. \tag{3.26}$$

Here $U(u,v)$, $P(u,v)$, and $L(u,v;\xi)$ are two-surface integrals defined as

$$U(u,v) := \frac{1}{16\pi}\oint d^2y(h\pi_h + 2e^\sigma D_+\sigma) + \bar{U}, \tag{3.27}$$

$$P(u,v) := \frac{1}{16\pi}\oint d^2y(\pi_h) + \bar{P}, \tag{3.28}$$

$$L(u,v;\xi) := \frac{1}{16\pi}\oint d^2y(\xi^a\pi_a) + \bar{L} \quad (\partial_+\xi^a = 0), \tag{3.29}$$

where $\bar{U}$, $\bar{P}$, and $\bar{L}$ are undetermined subtraction terms that satisfy the conditions

$$\frac{\partial\bar{U}}{\partial u} = \frac{\partial\bar{P}}{\partial u} = \frac{\partial\bar{L}}{\partial u} = 0. \tag{3.30}$$

Notice that choices of subtraction terms are not unique, since it is the subtraction terms that *define* the references against which these quasilocal quantities are measured. A natural criterion for the "right" choice of subtraction terms would be that values of quasilocal quantities reproduce "standard" values in the well-known limiting cases, but otherwise they can be chosen arbitrarily.

Let us notice that the integrand of the r.h.s. of (3.24) assumes the canonical form of angular momentum-flux,

$$T_{0a}\xi^a \sim \sum_I \pi_I\mathcal{L}_\xi q^I, \tag{3.31}$$

where ξ is tangent to the two-surface N_2. One can also put the r.h.s. of (3.22) into the canonical form of energy-flux,

$$T_{0\alpha}\eta^\alpha \sim \sum_I \pi_I\partial_+q^I, \tag{3.32}$$

where $\eta^\alpha = \delta_+^\alpha$. To show this, let us contract (3.17) with ${A_+}^a$ and integrate over N_2. Then we have

$$\begin{aligned}\oint d^2y(A_+^a\partial_+\pi_a) = \oint d^2y(&\pi^{ab}\mathcal{L}_{A_+}\rho_{ab} + \pi_\sigma\mathcal{L}_{A_+}\sigma \\ & - h\mathcal{L}_{A_+}\pi_h).\end{aligned} \tag{3.33}$$

If we use the Eq. (3.33) and the diffN_2-covariant derivative D_+ defined in (2.24), then (3.22) can be written as

$$\begin{aligned}\frac{\partial}{\partial u}U(u,v) = \frac{1}{16\pi}\oint d^2y(&\pi^{ab}\partial_+\rho_{ab} + \pi_\sigma\partial_+\sigma \\ & - h\partial_+\pi_h - A_+^a\partial_+\pi_a),\end{aligned} \tag{3.34}$$

where the r.h.s. indeed assumes the canonical form of energy-flux [18].

In the region where $\hat{\partial}_+$ is timelike ($2h > 0$), the Eqs. (3.23), (3.24), and (3.34) are quasilocal balance equations that relate the instantaneous rates of changes of two-surface integrals at a given u-time to the associated net fluxes across the timelike tube generated by $\hat{\partial}_+$ [see Fig. 2(a)]. Let us remark that, unlike the Tamburino-Winicour's quasilocal conservation equations that are "weak" conservation equations since the Ricci flat conditions (i.e., the full vacuum Einstein's equations) were assumed in their derivation [40], our quasilocal balance equations are "strong" conservation equations in that only four Einstein's equations of the divergence-type were used in the derivation.

In the region where $\hat{\partial}_+$ is spacelike ($2h < 0$), the vector field $\partial/\partial u$ is spacelike,

$$< \frac{\partial}{\partial u}, \frac{\partial}{\partial u} > = -2h + e^{\sigma}\rho_{ab}A_+^a A_+^b > 0, \tag{3.35}$$

so that u is the radial coordinate [see Fig. 2(b)]. Then the Eqs. (3.15), (3.16), and (3.17) are constraint equations rather than balance equations, and each equation splits into a divergence term and a source term. The source term either assumes the canonical form of energy-momentum "density" $\Sigma\pi_I \mathcal{L}_\xi q^I$ and $\Sigma\pi_I \partial_+ q^I$ as in Eqs. (3.31) and (3.32), or is given by the Hamiltonian density H in (3.14). Notice that the source term is not "flux" but "density," since the $v =$ constant hypersurface is now a 3-dimensional spacelike hypersurface. These constraint equations describe how the quasilocal quantities at a given u-radius change as the radius u changes on a given spacelike hypersurface. Thus, the difference of two-surface integrals evaluated on two successive two-spheres on a given spacelike hypersurface is given by the 3-dimensional spatial integral of the "density" over the region between the two-spheres of interest. This is exactly what happens in Maxwell's theory, where the Gauss law constraint is given by

$$\vec{\nabla} \cdot \vec{E} = 4\pi\rho. \tag{3.36}$$

If one integrates (3.36) over a 3-dimensional spacelike region V whose boundaries are two-spheres S_1 and S_2, then one has

$$\oint_{S_1} E_n da - \oint_{S_2} E_n da = 4\pi \int_V \rho dv. \tag{3.37}$$

Thus, in Maxwell's theory, the difference of two-surface integrals of the "momentum" E_n on two successive two-spheres is given by the spatial integral of the charge density over the volume between the two-spheres. In this paper, however, we shall be concerned with the case $2h > 0$ only, and the case $2h \leq 0$ will be discussed elsewhere [41].

If we introduce a function $W_R(u, v)$ defined as

$$W_R(u, v) := \frac{1}{16\pi} \int_0^u du \oint d^2y A_+^a \partial_+ \pi_a, \tag{3.38}$$

then we have

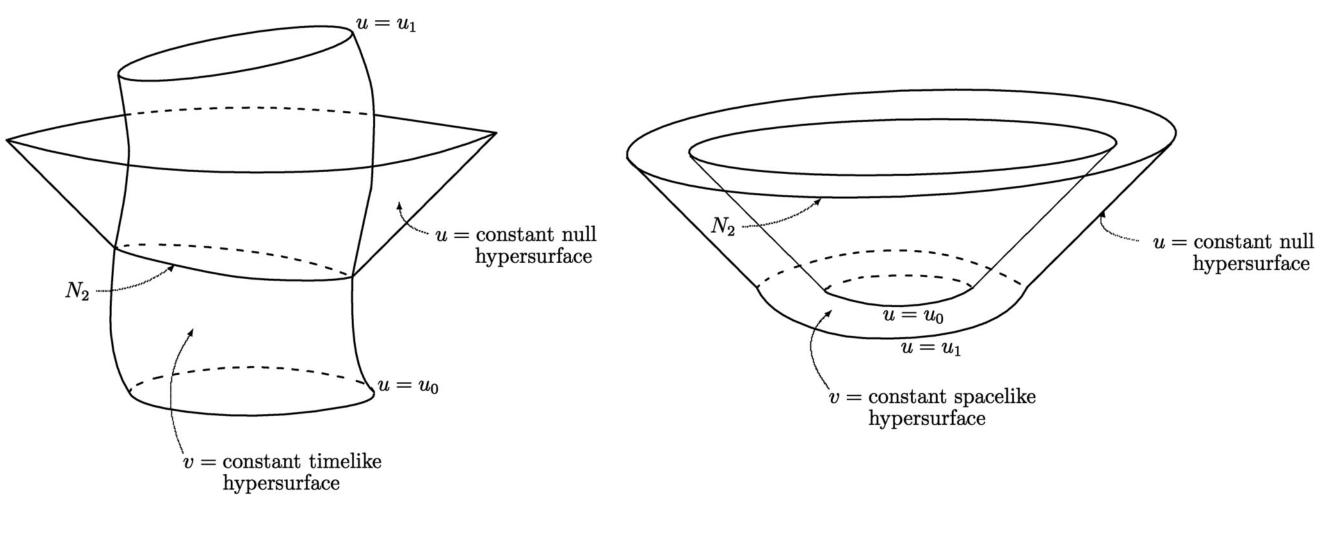
$$\frac{\partial}{\partial u} W_R(u, v) = \frac{1}{16\pi} \oint d^2y A_+^a \partial_+ \pi_a, \tag{3.39}$$

so that (3.34) can be written as

$$\frac{\partial}{\partial u}\{U(u, v) + W_R(u, v)\} = \frac{1}{16\pi} \oint d^2y(\pi^{ab}\partial_+\rho_{ab} + \pi_\sigma \partial_+ \sigma - h\partial_+ \pi_h). \tag{3.40}$$

Notice that the r.h.s. of (3.39) has the form

$$\frac{\partial}{\partial u} W_R(u, v) \sim \sum_a \Omega^a \partial_+ L_a, \tag{3.41}$$



(a) $2h > 0$ (b) $2h < 0$

FIG. 2. (a) This figure shows the spacetime geometry in the region where $2h > 0$, and $v =$ constant hypersurface is timelike. (b) This figure shows the spacetime geometry in the region where $2h < 0$, and $v =$ constant hypersurface is spacelike. In both cases N_2 is represented by a circle.

where

$$\Omega^a \sim A_+^a, \quad (3.42)$$

$$L_a \sim \frac{1}{16\pi}\pi_a, \quad (3.43)$$

which represents the work done per unit u-time by changing the angular momentum L_a of the system that has the angular velocity Ω^a. From this perspective, the Eq. (3.39) is the work done on N_2 per unit u-time by changing the angular momentum density $\pi_a/16\pi$ of gravitational field that has the angular velocity A_+^a at each point of N_2. This observation suggests that $W_R(u, v)$ be identified as the quasilocal *rotational* energy of N_2. Indeed, as is shown in Sec. VI, $W_R(u, v)$ reduces to the Carter's constant for the asymptotically Kerr spacetimes, the total angular momentum squared [25–28]. This is a strong indication that supports our identification of $W_R(u, v)$ as the rotational energy of gravitational field in the circumstances where *no* isometries are present.

In the limit where A_+^a is independent of u such that

$$\partial_+ A_+^a = 0, \quad (3.44)$$

$W_R(u, v)$ becomes

$$W_R(u, v) = \frac{1}{16\pi}\oint_{u=u} d^2y(A_+^a \pi_a) - \frac{1}{16\pi} \times \oint_{u=0} d^2y(A_+^a \pi_a). \quad (3.45)$$

IV. HAMILTON'S EQUATIONS OF MOTION

Let us define the Hamiltonian K as the integral of H in (3.14),

$$K := \int du \oint d^2y\{H + \lambda(\det\rho_{ab} - 1)\}, \quad (4.1)$$

where λ is a Lagrange multiplier that enforces the unimodular condition (3.12). In the Appendix, we have shown that the Eqs. (2.15), (2.16), (2.18), and (2.19) are the Hamilton's equations of motion

$$D_- q^I = \frac{\delta K}{\delta \pi_I}, \quad (4.2)$$

$$D_- \pi_I = -\frac{\delta K}{\delta q^I}, \quad (4.3)$$

where $\{\pi_I, q^I\}$ are

$$\pi_I = \{\pi_h, \pi_\sigma, \pi_a, \pi^{ab}\}, \qquad q^I = \{h, \sigma, A_+^a, \rho_{ab}\}, \quad (4.4)$$

assuming the boundary conditions

$$\delta\sigma = \delta\rho_{ab} = 0 \quad (4.5)$$

at the endpoints of the u-integration. Thus, together with the divergence-type Einstein's Eqs. (2.13), (2.14), and (2.17), from which follow the integral equations (3.22), (3.23), and (3.24) that may be interpreted as either the quasilocal balance equations or constraint equations depending on the signature of the 3-dimensional hypersuface, the Hamilton's equations of motion (4.2) and (4.3) make up for the complete set of the vacuum Einstein's equations. Thus, in the (2,2) fibre bundle formalism, the Einstein's equations split into 12 first-order Hamilton's equations of motion dictating the evolution along the outgoing null geodesic and the four quasilocal balance equations or the constraint equations that implement the Hamilton's evolution equations.

V. GEOMETRICAL INTERPRETATIONS

Two-surface integrals (3.27), (3.28), and (3.29) can be given geometric interpretations in terms of the area of N_2 and null vector fields orthogonal to N_2. In order to see this, it is necessary to recall the definitions of the in- and out-going null vector fields n and l given by (2.7) and (2.8), respectively.

A. Quasilocal Energy

Let us observe that, apart from the subtraction term $\bar{U}$, (3.27) can be written as the Lie derivative of the area A of N_2 along n. Notice that we have

$$\oint d^2y(h\pi_h + 2e^\sigma D_+\sigma) = 2\oint d^2y e^\sigma(D_+\sigma - hD_-\sigma) = 2\oint d^2y \mathcal{L}_n e^\sigma. \quad (5.1)$$

But one has

$$\oint d^2y \mathcal{L}_n e^\sigma = \mathcal{L}_n \mathcal{A}, \quad (5.2)$$

where $\mathcal{A}$ is given by

$$\mathcal{A} = \oint d^2y e^\sigma. \quad (5.3)$$

The identity (5.2) follows from the fact that the order of d^2y integration and the Lie derivation $\mathcal{L}_n$ is interchangeable, since the in-going null vector field n is orthogonal to N_2. Thus we have

$$U(u, v) = \frac{1}{8\pi}\mathcal{L}_n\mathcal{A} + \bar{U}. \quad (5.4)$$

In order to fix $\bar{U}$, it is necessary to introduce a reference spacetime. In principle, the reference spacetime can be chosen arbitrarily, provided that the pullback of the background metric to N_2 is the same as $e^\sigma\rho_{ab}$. If we denote the coordinates of the reference spacetime as $(\bar{u}, \bar{v}, y^a)$, then its metric can be written as

$$d\bar{s}^2 = -2d\bar{u}d\bar{v} - 2\bar{h}d\bar{u}^2 + e^\sigma\rho_{ab}(dy^a + \bar{A}_+^a d\bar{u} + \bar{A}_-^a d\bar{v})(dy^b + \bar{A}_+^b d\bar{u} + \bar{A}_-^b d\bar{v}), \quad (5.5)$$

where $\{\bar{h}, \bar{A}^a_{\pm}\}$ are the embedding degrees of freedom of N_2 into the reference spacetime. The vector fields $\{\bar{n}, \bar{l}\}$

$$\bar{n} = \left(\frac{\partial}{\partial \bar{u}} - \bar{A}^a_+ \frac{\partial}{\partial y^a}\right) - \bar{h}\left(\frac{\partial}{\partial \bar{v}} - \bar{A}^a_- \frac{\partial}{\partial y^a}\right), \tag{5.6}$$

$$\bar{l} = \left(\frac{\partial}{\partial \bar{v}} - \bar{A}^a_- \frac{\partial}{\partial y^a}\right) \tag{5.7}$$

are null with respect to the background metric, and satisfy the same normalization conditions as before,

$$<\bar{n}, \bar{n}>_{\rm ref} = 0, <\bar{l}, \bar{l}>_{\rm ref} = 0, <\bar{n}, \bar{l}>_{\rm ref} = -1. \tag{5.8}$$

If $\bar{U}$ is chosen as

$$\bar{U} := -\frac{1}{8\pi} \mathcal{L}_{\bar{n}} \mathcal{A}, \tag{5.9}$$

such that it satisfies the u-independent condition

$$\frac{\partial \bar{U}}{\partial u} = 0, \tag{5.10}$$

then (5.4) becomes

$$U(u, v) = \frac{1}{8\pi} \mathcal{L}_{(n-\bar{n})} \mathcal{A}. \tag{5.11}$$

This expression is entirely geometrical, stating that $U(u, v)$ is determined by the rate of change of the area $\mathcal{A}$ of N_2 along the difference $n - \bar{n}$ of the in-going null geodesics, and becomes zero when

$$n = \bar{n}. \tag{5.12}$$

B. Quasilocal Linear Momentum

One can also express $P(u, v)$ geometrically. Let us notice that

$$\frac{1}{16\pi} \oint d^2y(\pi_h) = -\frac{1}{8\pi} \oint d^2y e^{\sigma} D_- \sigma = -\frac{1}{8\pi} \oint d^2y e^{\sigma} \mathcal{L}_l \sigma = -\frac{1}{8\pi} \mathcal{L}_l \mathcal{A}. \tag{5.13}$$

Therefore, if we choose the subtraction term $\bar{P}$ as

$$\bar{P} := \frac{1}{8\pi} \mathcal{L}_{\bar{l}} \mathcal{A} \tag{5.14}$$

such that it satisfies the u-independent condition

$$\frac{\partial}{\partial u} \bar{P} = 0, \tag{5.15}$$

then (3.28) becomes

$$P(u, v) = -\frac{1}{8\pi} \mathcal{L}_{(l-\bar{l})} \mathcal{A}. \tag{5.16}$$

Thus, $P(u, v)$ is given by the minus of the rate of change of the area $\mathcal{A}$ of N_2 along the difference $l - \bar{l}$ of the out-going null geodesics, and becomes zero when

$$l = \bar{l}. \tag{5.17}$$

C. Quasilocal Angular Momentum

Let us now find the geometrical expression of $L(u, v; \xi)$. If we notice that the Lie bracket of n and l is given by

$$[n, l]_{\rm L} = -F^a_{+-} \partial_a + (D_- h) l, \tag{5.18}$$

then we have

$$\begin{aligned} \frac{1}{16\pi} \oint d^2y(\xi^a \pi_a) &= \frac{1}{16\pi} \oint d^2y e^{\sigma} \xi_a F^a_{+-} \\ &= -\frac{1}{16\pi} \oint d^2y e^{\sigma} \xi_a [n, l]^a_{\rm L} \\ &= -\frac{1}{16\pi} \oint d^2y e^{\sigma} < \xi, [n, l]_{\rm L} >, \end{aligned} \tag{5.19}$$

where we used

$$\xi_a = e^{\sigma} \rho_{ab} \xi^b, \tag{5.20}$$

and the fact that l is orthogonal to ξ,

$$l^a \xi_a = 0. \tag{5.21}$$

If we choose the subtraction term $\bar{L}$ as

$$\bar{L} := \frac{1}{16\pi} \oint d^2y e^{\sigma} < \xi, [\bar{n}, \bar{l}]_{\rm L} >_{\rm ref}, \tag{5.22}$$

and require that the condition

$$\frac{\partial}{\partial u} \bar{L} = 0 \tag{5.23}$$

hold, then (3.29) becomes

$$L(u, v; \xi) = -\frac{1}{16\pi} \oint d^2y e^{\sigma} (<\xi, [n, l]_{\rm L}> - < \xi, [\bar{n}, \bar{l}]_{\rm L} >_{\rm ref}). \tag{5.24}$$

Thus, $L(u, v; \xi)$ is given by the integral over N_2 of the projection of the twist $[n, l]_{\rm L}$ onto $\xi = \xi^a \partial_a$ modulo the subtraction term $\bar{L}$. Notice that ξ^a is an arbitrary function of $\{v, y^b\}$, so that the vector field ξ need not satisfy any Killing's equations. Thus, ξ is an arbitrary vector field tangent to N_2, defining the direction of rotation at each point of that surface.

If $L(u, v; \xi)$ is to be regarded as an acceptable candidate of quasilocal angular momentum, its value must be zero for any two-surface embedded in the flat Minkowski spacetime. Our expression clearly satisfies this criterion, as can be seen by the following observation[42]. Let ϕ be a diffeomorphism from a given spacetime M_{3+1} to itself, and ϕ_* be the push-forward associated with ϕ. For any vector fields X, Y defined on M_{3+1}, the inner product and the Lie bracket are preserved by the mapping ϕ_*[42],

$$<\phi_* X, \phi_* Y> = \phi_* < X, Y>, \tag{5.25}$$

$$[\phi_* X, \phi_* Y]_{\rm L} = \phi_* [X, Y]_{\rm L}. \tag{5.26}$$

Suppose that M_{3+1} is the flat Minkowski spacetime, and let n and l be the null vector fields at each point of N_2 of the flat Minkowski spacetime. Then these null vector fields remain null and the normalization condition is preserved under the mapping ϕ_*, since we have

$$<\phi_* n, \phi_* n> = \phi_* < n, n> = 0, \tag{5.28}$$

$$<\phi_* l, \phi_* l> = \phi_* < l, l> = 0, \tag{5.29}$$

$$<\phi_* n, \phi_* l> = \phi_* < n, l> = -1.$$

The following identity

$$[\phi_* n, \phi_* l]_{\rm L} = \phi_* [n, l]_{\rm L} = 0 \tag{5.30}$$

is also true, since two null vector fields at any points in the flat Minkowski spacetime must commute. Therefore, $[n, l]_{\rm L} = 0$ holds on any two-surface (and its deformations) in the flat Minkowski spacetime. Thus, $L(u, v; \xi)$ is zero on any two-surface N_2 in the flat Minkowski spacetime, modulo the subtraction term that can be trivially put to zero.

Notice that $L(\xi)$ is linear in ξ. That is, for any ξ given by

$$\xi = a\xi_1 + b\xi_2, \tag{5.31}$$

where a, b are constants, we have

$$L(\xi) = aL(\xi_1) + bL(\xi_2), \tag{5.32}$$

which shows that the quasilocal angular momentum is additive. Asymptotic properties of quasilocal angular momentum and its flux will be studied in Sec. VI.

D. Quasilocal Rotational Energy

Like quasilocal angular momentum, the value of any reasonable candidate of quasilocal rotational energy must be zero for any two-surface in the flat Minkowski spacetime. The quasilocal rotational energy W_R defined in (3.38) trivially satisfies this criterion. Since W_R can be written as

$$W_R = -\frac{1}{16\pi}\int_0^u du \oint d^2y A_+^a \partial_+ (e^{2\sigma}\rho_{ab}[n, l]_{\rm L}^b), \tag{5.33}$$

it is zero when

$$[n, l]_{\rm L}^a = 0, \tag{5.34}$$

which is true for any n and l of the flat Minkowski spacetime.

VI. ASYMPTOTICALLY FLAT LIMITS

In this section I will show that the limits of quasilocal balance equations in the asymptotically flat zones are the Bondi energy-loss, linear momentum-loss, and angular momentum-loss equations[13,14]. Moreover, the integral W_R turns out to be proportional to the total angular momentum squared in this limit, which is a strong indication that it is to be regarded as a quasilocal generalization of the Carter's constant[25–27] for a generic gravitational field.

The general form of asymptotically flat metrics[13,19,31–33] is given by

$$\begin{aligned} ds^2 \longrightarrow & -2dudv - \Big(1 - \frac{2m}{v} + \cdots\Big)du^2 + \Big(\frac{4ma\sin^2\vartheta}{v} \\ & - \frac{4ma^3\sin^2\vartheta\cos^2\vartheta}{v^3} + \cdots\Big)dud\varphi \\ & + v^2\Big(1 + \frac{a^2\cos^2\vartheta}{v^2} + \cdots\Big)d\vartheta^2 + v^2\sin^2\vartheta\Big(1 + \frac{a^2}{v^2} \\ & + \cdots\Big)d\varphi^2 + \sin^2\vartheta\Big(\frac{4ma^3}{v^3} + \frac{8m^2a^3}{v^4} + \cdots\Big)dvd\varphi \\ & - \Big(\frac{a^2\sin^2\vartheta}{v^2} + \cdots\Big)dv^2, \end{aligned} \tag{6.1}$$

as $v \longrightarrow \infty$. From this expansion, asymptotic fall-off rates of the metric coefficients are found to be

$$e^{\sigma} = v^2\sin\vartheta\Big\{1 + O\Big(\frac{1}{v^2}\Big)\Big\}, \tag{6.2}$$

$$\rho_{\vartheta\vartheta} = (\sin\vartheta)^{-1}\Big\{1 + \frac{C(u, \vartheta, \varphi)}{v} + O\Big(\frac{1}{v^2}\Big)\Big\}, \tag{6.3}$$

$$\rho_{\varphi\varphi} = \sin\vartheta\Big\{1 - \frac{C(u, \vartheta, \varphi)}{v} + O\Big(\frac{1}{v^2}\Big)\Big\}, \tag{6.4}$$

$$\rho_{\vartheta\varphi} = O\Big(\frac{1}{v^2}\Big), \tag{6.5}$$

$$2h = 1 - \frac{2m}{v} + O\Big(\frac{1}{v^2}\Big), \tag{6.6}$$

$$A_+^{\varphi} = \frac{2ma}{v^3} + O\Big(\frac{1}{v^4}\Big), \tag{6.7}$$

$$A_-^{\varphi} = \frac{2ma^3}{v^5} + O\Big(\frac{1}{v^6}\Big), \tag{6.8}$$

$$A_{\pm}^{\vartheta} = O\Big(\frac{1}{v^6}\Big), \tag{6.9}$$

and their derivatives are given by

$$\partial_+\sigma = O\Big(\frac{1}{v^2}\Big), \tag{6.10}$$

$$\partial_{-}\sigma = \frac{2}{v} + O\left(\frac{1}{v^2}\right), \tag{6.11}$$

$$\partial_{+}\rho_{ab} = O\left(\frac{1}{v}\right), \tag{6.12}$$

$$\partial_{-}\rho_{ab} = O\left(\frac{1}{v^2}\right), \tag{6.13}$$

$$\mathcal{L}_{\xi}\rho_{ab} = O\left(\frac{1}{v}\right), \tag{6.14}$$

$$\pi^{ab} = -\frac{1}{2}e^{\sigma}\rho^{ac}\rho^{bd}(\partial_{+}\rho_{cd}) + O(1), \tag{6.15}$$

$$\pi_h = -4v\sin\vartheta + O(1), \tag{6.16}$$

$$\pi_\sigma = -2v\sin\vartheta + O(1), \tag{6.17}$$

$$\pi_\varphi = 6ma\sin^3\vartheta + O\left(\frac{1}{v}\right), \tag{6.18}$$

$$\pi_\vartheta = O\left(\frac{1}{v^2}\right). \tag{6.19}$$

Therefore, n and l become, asymptotically,

$$n \longrightarrow \frac{\partial}{\partial u} - \left(\frac{1}{2} - \frac{m}{v}\right)\frac{\partial}{\partial v}, \tag{6.20}$$

$$l \longrightarrow \frac{\partial}{\partial v}. \tag{6.21}$$

The natural reference spacetime at the asymptotic infinity is the flat Minkowski spacetime,

$$d\bar{s}^2 = -2d\bar{u}d\bar{v} - d\bar{u}^2 + r^2(d\vartheta^2 + \sin^2\vartheta d\varphi^2), \tag{6.22}$$

where $N_2 = S_2$. Thus the embedding degrees of freedom of S_2 into the flat Minkowski spacetime are given by

$$\bar{A}_{\pm}^{\,a} = 0, \qquad 2\bar{h} = 1, \tag{6.23}$$

so that $\bar{n}$ and $\bar{l}$ are given by

$$\bar{n} = \frac{\partial}{\partial \bar{u}} - \frac{1}{2}\frac{\partial}{\partial \bar{v}}, \tag{6.24}$$

$$\bar{l} = \frac{\partial}{\partial \bar{v}}. \tag{6.25}$$

A. The Bondi Energy-Loss Relation

In the asymptotic region $v \to \infty$ where $v =$ constant hypersurface is timelike, the r.h.s. of (3.34) represents the canonical energy-flux carried by gravitational radiation crossing N_2. Then the l.h.s. should be identified as the instantaneous rate of change in the gravitational energy of the region enclosed by N_2. Energy-flux in general does not have a definite sign, since it includes the energy-flux carried by the in-coming as well as the out-going radiation across N_2. In the asymptotically flat region, however, energy-flux turns out to be negative-definite, representing the physical situation that there is no in-coming flux coming from the infinity[43].

Let us show that the balance equation (3.34) indeed reduces to the Bondi energy-loss formula at the null infinity. Let $\bar{U}$ be given by (5.9), and use $\bar{n}$ in (6.24). Then we find that

$$\bar{U} = \frac{\bar{v}}{2}. \tag{6.26}$$

Let us suppose that the coordinates $\{u, v, y^a\}$ approach the coordinates of the Minkowski spacetime $\{\bar{u}, \bar{v}, \vartheta, \varphi\}$ as $v \to \infty$,

$$u \longrightarrow \bar{u},\ v \longrightarrow \bar{v},\ y^a \longrightarrow \{\vartheta, \varphi\}. \tag{6.27}$$

Then $\bar{U}$ trivially satisfies the condition (5.10), since

$$\frac{\partial}{\partial u}\bar{U} \longrightarrow \frac{\partial}{\partial \bar{u}}\bar{U} = 0. \tag{6.28}$$

If we use the asymptotic formula

$$n - \bar{n} \longrightarrow \frac{m}{v}\frac{\partial}{\partial v}, \tag{6.29}$$

then the total energy is given by

$$U_{\rm B}(u) := \lim_{v\to\infty} U(u, v) = \lim_{v\to\infty}\frac{1}{8\pi}\mathcal{L}_{(n-\bar{n})}\mathcal{A} = m(u), \tag{6.30}$$

which is just the Bondi mass at the null infinity.

Asymptotic limit of the balance equation (3.34) is found to be

$$\begin{aligned}\frac{d}{du}U_{\rm B}(u) &= -\lim_{v\to\infty}\frac{1}{32\pi}\oint_{S_2} d\Omega v^2\rho^{ac}\rho^{bd}(\partial_{+}\rho_{bc})(\partial_{+}\rho_{ad}) \\ &= -\lim_{v\to\infty}\frac{1}{32\pi}\oint_{S_2} d\Omega v^2(j^a_{+b}j^b_{+a}) \leq 0,\end{aligned} \tag{6.31}$$

where j^a_{+b} is the *shear current* defined as

$$j^a_{+b} := \rho^{ac}\partial_{+}\rho_{bc} \quad (j^a_{+a} = 0), \tag{6.32}$$

which represents traceless shear degrees of freedom of gravitational radiation. The relation (6.31) is just the Bondi energy-loss formula with the correct normalization coefficient[43]. Notice that the energy-flux is bilinear in j^a_{+b}. Equivalently, it can be written as

$$\frac{d}{du}U_{\rm B}(u) = -\frac{1}{16\pi}\oint_{S_2} d\Omega(\partial_{+}C)^2 \leq 0, \tag{6.33}$$

if one uses the expansion of ρ_{ab} given by (6.3), (6.4), and (6.5).

B. The Bondi Linear Momentum and Linear Momentum-Flux

The total linear momentum $P_{\mathrm{B}}(u)$ and its flux are trivially zero,

$$P_{\mathrm{B}}(u) := \lim_{v\to\infty} P(u, v) = -\lim_{v\to\infty} \frac{1}{8\pi} \mathcal{L}_{(l-\bar{l})}\mathcal{A} = 0, \quad (6.34)$$

$$\frac{d}{du} P_{\mathrm{B}}(u) = 0, \quad (6.35)$$

since we have

$$l - \bar{l} \longrightarrow 0. \quad (6.36)$$

That the net-flux of the total linear momentum is zero can be also seen by evaluating each term of H in (3.14) in the asymptotic limit. Let us notice that although the fourth, sixth, and seventh term in (3.14) are not zero individually, they add up to zero asymptotically,

$$\begin{aligned} &\frac{1}{2h} e^{-\sigma} \rho_{ab}\rho_{cd}\pi^{ac}\pi^{bd} + \frac{1}{2h}\pi^{ab}(D_+\rho_{ab}) \\ &\quad + \frac{1}{8h} e^{\sigma}\rho^{ab}\rho^{cd}(D_+\rho_{ac})(D_+\rho_{bd}) \\ &\quad = \frac{h}{2} e^{\sigma}\rho^{ab}\rho^{cd}(D_-\rho_{ac})(D_-\rho_{bd}) = O\left(\frac{1}{v^2}\right), \end{aligned} \quad (6.37)$$

where we used the definition of π^{ab} in (3.5). The third and fifth terms become zero, respectively,

$$-\frac{1}{2} e^{-2\sigma}\rho^{ab}\pi_a\pi_b = O\left(\frac{1}{v^4}\right), \quad (6.38)$$

$$\frac{1}{2}\pi_h(D_+\sigma) = O\left(\frac{1}{v}\right). \quad (6.39)$$

The remaining nonvanishing terms are given by

$$\lim_{v\to\infty} -\frac{1}{16\pi}\oint_{S_2} d^2y\left(\frac{1}{2} e^{-\sigma}\pi_h\pi_\sigma\right) = -1, \quad (6.40)$$

$$\lim_{v\to\infty} \frac{1}{16\pi}\oint_{S_2} d^2y\left(\frac{1}{4} h e^{-\sigma}\pi_h^2\right) = \frac{1}{2}, \quad (6.41)$$

$$\lim_{v\to\infty} \frac{1}{16\pi}\oint_{S_2} d^2y e^{\sigma} R_2 = \frac{1}{4}\chi, \quad (6.42)$$

where $\chi = 2$ for S_2. Since these terms add up to zero, it follows that the net momentum-flux H is zero at the null infinity.

C. The Bondi Angular Momentum and Angular Momentum-Flux

Likewise, the total angular momentum $L_{\mathrm{B}}(u; \xi)$ is defined as the asymptotic limit of quasilocal angular momentum $L(u, v; \xi)$,

$$\begin{aligned} L_{\mathrm{B}}(u;\xi) &:= \lim_{v\to\infty} L(u, v; \xi) \\ &= -\lim_{v\to\infty}\frac{1}{16\pi}\oint d^2y e^{\sigma}(<\xi, [n, l]_{\mathrm{L}}> - \\ &\qquad < \xi, [\bar{n}, \bar{l}]_{\mathrm{L}} >_{\mathrm{ref}}). \end{aligned} \quad (6.43)$$

Let ξ be asymptotic to the azimuthal Killing vector field

$$\xi = \xi^a\partial_a \longrightarrow \frac{\partial}{\partial\varphi}. \quad (6.44)$$

From the expansions (6.2), $\cdots$, (6.9), we find that

$$e^{\sigma} \longrightarrow v^2\sin\vartheta, \quad (6.45)$$

$$\xi_\varphi \longrightarrow v^2\sin^2\vartheta, \quad (6.46)$$

$$\xi_\vartheta \longrightarrow 0, \quad (6.47)$$

where we used the definition of ξ_a in (5.20). The Lie bracket $[n, l]_{\mathrm{L}}$ is found to be

$$[n, l]_{\mathrm{L}}^{\varphi} \longrightarrow -\frac{6ma}{v^4} + O\left(\frac{1}{v^5}\right), \quad (6.48)$$

$$[n, l]_{\mathrm{L}}^{\vartheta} \to O\left(\frac{1}{v^6}\right). \quad (6.49)$$

Since $[\bar{n}, \bar{l}]_{\mathrm{L}} = 0$, we find that

$$L_{\mathrm{B}}(u;\xi) = \frac{1}{16\pi}\int_0^{2\pi} d\varphi\int_0^{\pi} d\vartheta(6ma)\sin^3\vartheta = ma, \quad (6.50)$$

which is just the total angular momentum of the Kerr spacetime.

The Bondi angular momentum-loss relation will be obtained by taking the limit of the quasilocal balance equation (3.24),

$$\begin{aligned} \frac{dL_{\mathrm{B}}}{du} &= \lim_{v\to\infty}\frac{1}{16\pi}\oint_{S_2} d^2y(\pi^{ab}\mathcal{L}_\xi\rho_{ab} + \pi_\sigma\mathcal{L}_\xi\sigma \\ &\qquad - h\mathcal{L}_\xi\pi_h - A_+^a\mathcal{L}_\xi\pi_a). \end{aligned} \quad (6.51)$$

Let us evaluate each term of this equation. The first term has a finite limit, which is

$$\begin{aligned} \oint_{S_2} d^2y\pi^{ab}\mathcal{L}_\xi\rho_{ab} &\longrightarrow -\frac{1}{2}\oint_{S_2} d\Omega v^2\rho^{ac}\rho^{bd}(\partial_+\rho_{cd}) \\ &\qquad \times(\mathcal{L}_\varphi\rho_{ab}), \end{aligned} \quad (6.52)$$

where we used the notation

$$\mathcal{L}_\varphi := \mathcal{L}_{\partial/\partial\varphi}. \quad (6.53)$$

Notice that

$$\pi_\sigma\mathcal{L}_\xi\sigma \longrightarrow \{-2v\sin\vartheta + O(1)\}\mathcal{L}_\varphi\sigma. \quad (6.54)$$

But σ becomes, asymptotically,

$$\sigma \longrightarrow 2\ln v + \ln|\sin\vartheta| + \ln\left\{1 + O\left(\frac{1}{v^2}\right)\right\}, \qquad (6.55)$$

so that we have

$$\mathcal{L}_\varphi \sigma = O\left(\frac{1}{v^2}\right). \qquad (6.56)$$

Thus the second term becomes zero,

$$\oint_{S_2} d^2y \pi_\sigma \mathcal{L}_\xi \sigma = O\left(\frac{1}{v}\right) \longrightarrow 0. \qquad (6.57)$$

Notice also that

$$\begin{aligned} h\mathcal{L}_\xi \pi_h &= \mathcal{L}_\xi(h\pi_h) - \pi_h \mathcal{L}_\xi h \\ &\longrightarrow \mathcal{L}_\xi(h\pi_h) - 4\sin\vartheta \mathcal{L}_\xi m + O\left(\frac{1}{v}\right) \\ &= \mathcal{L}_\xi(h\pi_h - 4m\sin\vartheta) + O\left(\frac{1}{v}\right). \end{aligned} \qquad (6.58)$$

Thus the third term also becomes zero,

$$\oint_{S_2} d^2y h \mathcal{L}_\xi \pi_h = O\left(\frac{1}{v}\right) \longrightarrow 0. \qquad (6.59)$$

The fourth term dies off much faster, since

$$\oint_{S_2} d^2y A_+^a \mathcal{L}_\xi \pi_a = O\left(\frac{1}{v^3}\right) \longrightarrow 0. \qquad (6.60)$$

From (6.52), (6.57), (6.59), and (6.60), we find that (6.51) becomes

$$\frac{dL_{\rm B}}{du} = -\lim_{v\to\infty} \frac{1}{32\pi} \oint_{S_2} d\Omega v^2 \rho^{ac}\rho^{bd}(\partial_+\rho_{cd})(\mathcal{L}_\varphi \rho_{ab}), \qquad (6.61)$$

which is just the Bondi angular momentum-loss relation with the correct normalization coefficient. It is worth noting that (6.61) is the coordinate-dependent expression of the angular momentum-flux discussed in [22–24]. If we use the asymptotic expansion of ρ_{ab} given by (6.3), (6.4), and (6.5), then this relation can be expressed as

$$\frac{dL_{\rm B}}{du} = -\frac{1}{16\pi} \oint_{S_2} d\Omega (\partial_+ C)(\mathcal{L}_\varphi C). \qquad (6.62)$$

D. Gravitational Carter's Constant

Let us find what W_R becomes in this limit. The Eq. (3.39) becomes

$$\begin{aligned} \frac{d}{du} W_R &\longrightarrow \frac{3}{4\pi v^3} \int_0^{2\pi} d\varphi \int_0^\pi d\vartheta \sin^3\vartheta (ma) \frac{d(ma)}{du} \\ &= \frac{1}{v^3} \frac{d}{du} (ma)^2. \end{aligned} \qquad (6.63)$$

If we choose the constant of u-integration as zero, then we have

$$\lim_{v\to\infty} v^3 W_R = (ma)^2, \qquad (6.64)$$

which is just the total angular momentum squared for the Kerr spacetime. Thus, W_R may be regarded as a quasilocal generalization of the Carter's constant[25–28], and physically, could be interpreted as gravitational contribution to the quasilocal rotational energy.

VII. RELATION BETWEEN ENERGY-LOSS AND ANGULAR MOMENTUM-LOSS

In general, if a given system undergoes an energy-losing process, then it always accompanies angular momentum-loss, unless the system remains spherically symmetric throughout the whole process. Since we already found general expressions of quasilocal energy-flux and angular momentum-flux, it is natural to ask what relation exists between them, if there is any. The relation is that the angular momentum-flux (3.24) and energy-flux (3.34) transform into each other

$$\frac{\partial}{\partial u} L(u, v; \xi) \longleftrightarrow \frac{\partial}{\partial u} U(u, v), \qquad (7.1)$$

under the exchange of the derivatives

$$\mathcal{L}_\xi \longleftrightarrow \frac{\partial}{\partial u} \qquad (7.2)$$

in the flux integrals.

Let us examine the implication of this symmetry for spacetimes *close* to a background spacetime that possesses two commuting Killing vector fields. Let us choose the coordinates of the background spacetime as $\{\bar{u}, \bar{v}, \vartheta, \varphi\}$, and let $\{\partial/\partial\bar{u}, \partial/\partial\varphi\}$ be two Killing vector fields of the background spacetime. Let us also suppose that the coordinates $\{u, v, y^a\}$ approach the coordinates of the background spacetime $\{\bar{u}, \bar{v}, \vartheta, \varphi\}$ as $v \to \infty$,

$$u \longrightarrow \bar{u}, \quad v \longrightarrow \bar{v}, \quad y^a \longrightarrow \{\vartheta, \varphi\}. \qquad (7.3)$$

If we perturb this background spacetime by adding a small amount of gravitational waves, then we may regard these waves as propagating in the background spacetime, carrying a small amount of energy and angular momentum. Let us write $q^I = \{h, \sigma, A_+^a, \rho_{ab}\}$ and $\pi_I = \{\pi_h, \pi_\sigma, \pi_a, \pi^{ab}\}$ about an exact solution $\{\bar{q}^I, \bar{\pi}_I\}$ of the Einstein's equations,

$$q^I = \bar{q}^I(\bar{v}, \vartheta) + \delta q^I, \qquad (7.4)$$

$$\pi_I = \bar{\pi}_I(\bar{v}, \vartheta) + \delta\pi_I, \qquad (7.5)$$

where $\{\delta q^I, \delta\pi_I\}$ represents gravitational waves propagating on the background spacetime. The dependence of a given mode on $\bar{u}$ and φ is given by

$$\delta q^I = Q^I(\bar{v}, \vartheta) e^{i\omega\bar{u} + im_z\varphi} + \text{c.c.}, \qquad (7.6)$$

$$\delta\pi_I = \Pi_I(\bar{v}, \vartheta) e^{i\omega\bar{u} + im_z\varphi} + \text{c.c.}, \qquad (7.7)$$

where $\{Q^I, \Pi_I\}$ are the amplitudes of the wave that has the frequency ω and the azimuthal quantum number m_z ($m_z = 0, \pm 1, \cdots$). Now, if we use (7.3) and the fact that $\{\partial/\partial \bar{u}, \partial/\partial \varphi\}$ are the timelike and azimuthal Killing vector fields of the background spacetime, respectively, then from the symmetry (7.1) and (7.2) we obtain the following relation[44]

$$\frac{dU}{d\bar{u}} = \frac{\omega}{m_z}\frac{dL_z}{d\bar{u}}, \tag{7.8}$$

which is a well-known relation between energy-loss and angular momentum-loss for perturbations around the stationary and axi-symmetric spacetime. Thus, the symmetry (7.1) and (7.2) is the sought-for relation between the energy-loss and angular momentum-loss for a generic gravitational radiation that has no isometries.

VIII. DISCUSSIONS

The key ingredient of the (2,2) fibre bundle formalism discussed so far is the observation that the out-going null vector field defines a natural time flow. With the affine parameter of the out-going null geodesic as the time function of the theory, the canonical variables were introduced and the nonvanishing gravitational Hamiltonian was spelled out. Then I obtained the Hamilton's equations of motion from the Hamiltonian by the variational principle, which are the evolution equations along the out-going null geodesics with respect to the affine parameter. Thus, in this paper, I have shown that the Einstein's equations split into 12 first-order Hamilton's equations of motion and the four quasilocal balance equations or constraint equations that implement the Hamilton's evolution equations.

I also found coordinate-independent and geometric expressions of quasilocal gravitational energy, linear momentum, and angular momentum for any two-surface. The corresponding fluxes of gravitational field were found to assume the canonical form of energy-momentum-flux,

$$T_{0\alpha}\eta^{\alpha} \sim \sum_I \pi_I \mathcal{L}_\eta q^I, \tag{8.1}$$

just as in standard field theories. I have shown that the quasilocal balance equations correctly reproduce the well-known Bondi relations at the null infinity of asymptotically flat spacetimes. However, because of the breakdown of the coordinate system due to potential occurrence of caustics after a finite propagation along the out-going null geodesics, there could be difficulties in extending the Hamilton's evolution equations beyond the caustics. In principle, however, it is still possible to approach the null infinity by using a new coordinate system after the breakdown of the old one, and perhaps one could use the quasilocal balance equations across the caustics and search for "weak" solutions[45]. However, it must be mentioned that, the farther out one goes, the less likely is the chance for caustics to occur due to the weakness of gravity near the infinity. If one is interested in the strong gravity region near black holes, or black hole dynamics itself, then the caustics might cause serious problems since they are much more likely to occur as we approach strong gravity region along the in-going null geodesics.

Quasilocal angular momentum was defined in this paper for spacetimes that have no isometries, and was found to be zero for any two-surface in the flat Minkowski spacetime. It was found to have the additive property, being a linear functional of a vector field ξ that defines the rotation at each point of the two-surface. One might be interested in studying symmetry properties of this quasilocal angular momentum, and look for some generalization of the BMS symmetries at a finite region[19].

In addition, I obtained a quasilocal generalization of the Carter's constant of gravitational field, and interpreted it as gravitational contribution to the quasilocal rotational energy. The Carter's constant is known to exist when the system under study has two commuting Killing vector fields, such as the Einstein-Maxwell system and the Einstein's equations coupled to a scalar field. For a generic gravitational field that has no isometries, no analog of the Carter's constant is known. In this paper, I presented a candidate for the generalized Carter's constant which becomes zero for any two-surface in the flat Minkowski spacetime, and reduces to the total angular momentum squared in the asymptotic region of the Kerr spacetime. It is interesting to see how this quasilocal Carter's constant generalizes when the electromagnetic and scalar fields are present.

There could be a number of applications of the quasilocal balance equations in astrophysics. The most important and challenging problem seems to be the calculation of back-reaction on the geometry of black holes as a consequence of the emission or absorption of gravitational radiation. One could also use the quasilocal balance equations in searching for consistent boundary conditions at a finite boundary in numerical relativity, since the boundary data at a finite boundary must satisfy the quasilocal balance equations. These problems are left for future works.

Another issue in this (2,2) formalism is the well-posedness of the initial value problem. When the initial three dimensional hypersurface is chosen spacelike, there is no problem in the well-posedness of the initial value problem since the null direction can be viewed as a limit of timelike direction. But there are several other choices of initial surfaces, such as the double null initial surfaces and the initial/boundary value problems where the boundary can be either timelike or null. One of the difficulties associated with the characteristic or initial/boundary value problem is that one has to know the "flows of information" across the characteristic or the timelike boundary that belongs to the future. In these hybrid formulations of the Einstein's equations, not so many articles that aim to study the well-posedness of the field equations appeared. However, a series of the recent papers by Frittelli[46] shows that, for a certain choice of first-order variables for the characteristic problem of the *linearized* Einstein's equations, the system can be cast into manifestly well-posed form. For the nonlinear characteristic problems, the notion of well-posedness is still not available. It is interesting to examine whether the first-order variables in this paper might have any relevance in establishing the well-posedness of the nonlinear characteristic initial value problem.

Finally, there are problems related to the gauge invariance of this (2,2) fibre bundle formalism. It is obvious that this formalism is tied to a particular gauge, and the nonvanishing Hamiltonian is obtained as a consequence of selecting a particular time function, namely, choosing the affine parameter along the null direction as the time function. But one should notice that, in the standard ADM formalism, it is also possible to obtain another nonvanishing Hamiltonian if one chooses a time function such as the Gauss normal time coordinate[29]. Moreover, if one quantizes the theory in a particular gauge, the resulting quantum theory will depend on that gauge, losing the spacetime diffeomorphism invariance that one wishes to carry over to the quantum regime. In view of the present situation that there is not any single complete version of sensible quantum theory of gravity, however, this gauge problem does not seem to be an urgent problem. Clearly, quantizing the full Einstein's gravity is beyond the scope of the present paper.

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APPENDIX: HAMILTON'S EQUATIONS OF MOTION

In this Appendix, I will show that, the Einstein's equations, (2.15), (2.16), (2.18), and (2.19), which are second-order in D_- derivatives, are the Hamilton's equations of motion,

$$D_- q^I = \frac{\delta K}{\delta \pi_I}, \tag{A1}$$

$$D_- \pi_I = -\frac{\delta K}{\delta q^I}, \tag{A2}$$

if the boundary conditions

$$\delta\sigma = \delta\rho_{ab} = 0 \tag{A3}$$

are assumed at the endpoints of u-integration in K, where the Hamiltonian K is given by

$$K = \int du \oint d^2y \{H + \lambda(\det\rho_{ab} - 1)\}, \tag{A4}$$

and the Hamiltonian density H is

$$\begin{aligned} H = &-\frac{1}{2}\mathrm{e}^{-\sigma}\pi_\sigma\pi_h + \frac{1}{4}h\mathrm{e}^{-\sigma}\pi_h^2 - \frac{1}{2}\mathrm{e}^{-2\sigma}\rho^{ab}\pi_a\pi_b \\ &+ \frac{1}{2h}\mathrm{e}^{-\sigma}\rho_{ac}\rho_{bd}\pi^{ab}\pi^{cd} + \frac{1}{2}\pi_h(D_+\sigma) \\ &+ \frac{1}{2h}\pi^{ab}(D_+\rho_{ab}) + \frac{1}{8h}\mathrm{e}^{\sigma}\rho^{ab}\rho^{cd}(D_+\rho_{ac})(D_+\rho_{bd}) \\ &+ \mathrm{e}^{\sigma}R_2. \end{aligned} \tag{A5}$$

1. Variations with Respect to π_h and h

It is trivial to see that the equation

$$D_- h = \frac{\delta K}{\delta \pi_h} \tag{A6}$$

is identical to the Eq. (3.6), and the equation

$$D_- \pi_h = -\frac{\delta K}{\delta h} \tag{A7}$$

can be written as

$$\begin{aligned} D_-\pi_h = &-\frac{1}{4}\mathrm{e}^{-\sigma}\pi_h^2 + \frac{1}{2h^2}\mathrm{e}^{-\sigma}\rho_{ab}\rho_{cd}\pi^{ac}\pi^{bd} \\ &+ \frac{1}{2h^2}\pi^{ab}D_+\rho_{ab} + \frac{1}{8h^2}\mathrm{e}^{\sigma}\rho^{ab}\rho^{cd}(D_+\rho_{ac}) \\ &\times (D_+\rho_{bd}). \end{aligned} \tag{A8}$$

Let us show that the Eq. (A8) is just the Eq. (2.15), using the Eq. (A6). Notice that each term in (2.15) becomes

$$\text{(i)} \quad 2\mathrm{e}^{\sigma}D_-^2\sigma = -\frac{1}{2}\mathrm{e}^{-\sigma}\pi_h^2 - D_-\pi_h, \tag{A9}$$

$$\text{(ii)} \quad \mathrm{e}^{\sigma}(D_-\sigma)^2 = \frac{1}{4}\mathrm{e}^{-\sigma}\pi_h^2, \tag{A10}$$

(iii)

$$\begin{aligned} &\frac{1}{2}\mathrm{e}^{\sigma}\rho^{ab}\rho^{cd}(D_-\rho_{ac})(D_-\rho_{bd}) \\ &= \frac{1}{2h^2}\mathrm{e}^{-\sigma}\rho_{ab}\rho_{cd}\pi^{ac}\pi^{bd} + \frac{1}{2h^2}\pi^{ab}D_+\rho_{ab} \\ &\quad + \frac{1}{8h^2}\mathrm{e}^{\sigma}\rho^{ab}\rho^{cd}(D_+\rho_{ac})(D_+\rho_{bd}). \end{aligned} \tag{A11}$$

From (A9), (A10), and (A11), it follows that the Eq. (2.15) is identical to the Eq. (A8).

2. Variations with Respect to π_σ and σ

It is trivial to show that the equation

$$D_-\sigma = \frac{\delta K}{\delta \pi_\sigma} \tag{A12}$$

is identical to the Eq. (3.7). In the variation

$$D_-\pi_\sigma = -\frac{\delta K}{\delta \sigma}, \tag{A13}$$

the less trivial part is the following one,

$$\begin{aligned} \delta \int du \oint d^2y(\pi_h D_+\sigma) &= \int du \oint d^2y \pi_h D_+ \delta\sigma \\ &= -\int du \oint d^2y(D_+\pi_h)\delta\sigma + \int du \oint d^2y D_+(\pi_h \delta\sigma) \\ &= -\int du \oint d^2y(D_+\pi_h)\delta\sigma + \int du \frac{d}{du}\bigg\{\oint d^2y \pi_h \delta\sigma\bigg\}. \end{aligned} \tag{A14}$$

Therefore, if we assume the boundary condition

$$\delta\sigma = 0 \tag{A15}$$

at the endpoints of u-integration, then we have

$$\frac{1}{2}\frac{\delta}{\delta\sigma}\bigg\{\int du \oint d^2y \pi_h D_+\sigma\bigg\} = -\frac{1}{2}D_+\pi_h. \tag{A16}$$

The remaining variations are straightforward, so that (A13) becomes

$$\begin{aligned} D_-\pi_\sigma = &-\frac{1}{2}\mathrm{e}^{-\sigma}\pi_\sigma\pi_h + \frac{1}{4}h\mathrm{e}^{-\sigma}\pi_h^2 - \mathrm{e}^{-2\sigma}\rho^{ab}\pi_a\pi_b \\ &+ \frac{1}{2h}\mathrm{e}^{-\sigma}\rho_{ab}\rho_{cd}\pi^{ac}\pi^{bd} + \frac{1}{2}D_+\pi_h \\ &- \frac{1}{8h}\mathrm{e}^{\sigma}\rho^{ab}\rho^{cd}(D_+\rho_{ac})(D_+\rho_{bd}). \end{aligned} \tag{A17}$$

In order to show that the Eq. (A17) is the same as the Eq. (2.18), we need to express the derivatives of D_- and D_-^2 in (2.18), using the conjugate momenta. Notice that the first term in (2.18) becomes

$$\begin{aligned} 2\mathrm{e}^{\sigma}D_-^2 h &= \mathrm{e}^{\sigma}D_-\{-\mathrm{e}^{-\sigma}\pi_\sigma + D_+\sigma + h\mathrm{e}^{-\sigma}\pi_h\} \\ &= -\mathrm{e}^{-\sigma}\pi_h\pi_\sigma - D_-\pi_\sigma + \mathrm{e}^{\sigma}D_-D_+\sigma \\ &\quad + \frac{1}{2}\pi_h D_+\sigma + h\mathrm{e}^{-\sigma}\pi_h^2 + hD_-\pi_h. \end{aligned} \tag{A18}$$

Since the third term in the r.h.s. of (A18) can be written as

$$\begin{aligned} \mathrm{e}^{\sigma}D_-D_+\sigma &= \mathrm{e}^{\sigma}D_+D_-\sigma + \partial_a(\mathrm{e}^{\sigma}F^a_{+-}) \\ &= -\frac{1}{2}D_+\pi_h + \frac{1}{2}\pi_h D_+\sigma + \partial_a(\mathrm{e}^{-\sigma}\rho^{ab}\pi_b), \end{aligned} \tag{A19}$$

(A18) becomes

$$\begin{aligned} \text{(i)}\quad 2\mathrm{e}^{\sigma}D_-^2 h = &-D_-\pi_\sigma - \mathrm{e}^{-\sigma}\pi_h\pi_\sigma - \frac{1}{2}D_+\pi_h \\ &+ \pi_h D_+\sigma + \frac{3}{4}h\mathrm{e}^{-\sigma}\pi_h^2 \\ &+ \frac{1}{2h}\mathrm{e}^{-\sigma}\rho_{ab}\rho_{cd}\pi^{ac}\pi^{bd} + \frac{1}{2h}\pi^{ab}D_+\rho_{ab} \\ &+ \frac{1}{8h}\mathrm{e}^{\sigma}\rho^{ab}\rho^{cd}(D_+\rho_{ac})(D_+\rho_{bd}) \\ &+ \partial_a(\mathrm{e}^{-\sigma}\rho^{ab}\pi_b), \end{aligned} \tag{A20}$$

where we used the equation of motion of π_h given by (A8). It is straightforward to express the remaining terms in (2.18) using the canonical variables. They are given by

(ii)

$$2\mathrm{e}^{\sigma}(D_-h)(D_-\sigma) = \frac{1}{2}\mathrm{e}^{-\sigma}\pi_h\pi_\sigma - \frac{1}{2}\pi_h D_+\sigma - \frac{1}{2}h\mathrm{e}^{-\sigma}\pi_h^2, \tag{A21}$$

$$\text{(iii)}\quad \mathrm{e}^{\sigma}D_+D_-\sigma = -\frac{1}{2}D_+\pi_h + \frac{1}{2}\pi_h D_+\sigma, \tag{A22}$$

(iv)

$$\mathrm{e}^{\sigma}D_-D_+\sigma = -\frac{1}{2}D_+\pi_h + \frac{1}{2}\pi_h D_+\sigma + \partial_a(\mathrm{e}^{-\sigma}\rho^{ab}\pi_b), \tag{A23}$$

$$\text{(v)}\quad \mathrm{e}^{\sigma}(D_+\sigma)(D_-\sigma) = -\frac{1}{2}\pi_h D_+\sigma, \tag{A24}$$

(vi)

$$\begin{aligned} &\frac{1}{2}\mathrm{e}^{\sigma}\rho^{ab}\rho^{cd}(D_+\rho_{ac})(D_-\rho_{bd}) \\ &\quad = \frac{1}{2h}\pi^{ab}D_+\rho_{ab} + \frac{1}{4h}\mathrm{e}^{\sigma}\rho^{ab}\rho^{cd}(D_+\rho_{ac})(D_+\rho_{bd}), \end{aligned} \tag{A25}$$

$$\text{(vii)}\quad \mathrm{e}^{2\sigma}\rho_{ab}F^a_{+-}F^b_{+-} = \mathrm{e}^{-2\sigma}\rho^{ab}\pi_a\pi_b. \tag{A26}$$

If we plug (A20), $\cdots$, (A26) into (2.18), then the Eq. (2.18) becomes

$$\begin{aligned} D_-\pi_\sigma &+ \frac{1}{2}\mathrm{e}^{-\sigma}\pi_\sigma\pi_h - \frac{1}{4}h\mathrm{e}^{-\sigma}\pi_h^2 + \mathrm{e}^{-2\sigma}\rho^{ab}\pi_a\pi_b \\ &- \frac{1}{2h}\mathrm{e}^{-\sigma}\rho_{ab}\rho_{cd}\pi^{ac}\pi^{bd} - \frac{1}{2}D_+\pi_h \\ &+ \frac{1}{8h}\mathrm{e}^{\sigma}\rho^{ab}\rho^{cd}(D_+\rho_{ac})(D_+\rho_{bd}) \\ &- 2h\mathrm{e}^{\sigma}\bigg\{(D_-^2\sigma) + \frac{1}{2}(D_-\sigma)^2 \\ &+ \frac{1}{4}\rho^{ab}\rho^{cd}(D_-\rho_{ac})(D_-\rho_{bd})\bigg\} = 0. \end{aligned} \tag{A27}$$

But the term in the bracket {} in (A27) is zero if we use

(2.15), and this shows that the Eqs. (2.18) and (A17) are identical.

3. Variations with Respect to π_a and A^a_+

The equation

$$D_- A^a_+ = \frac{\delta K}{\delta \pi_a} \tag{A28}$$

is

$$D_- A^a_+ = -e^{-2\sigma}\rho^{ab}\pi_b, \tag{A29}$$

which is the defining equation (3.8) of π_a, since the covariant derivative of A^a_+ is given by

$$D_- A^a_+ := F^a_{-+}. \tag{A30}$$

In order to write down the equation

$$D_- \pi_a = -\frac{\delta K}{\delta A^a_+}, \tag{A31}$$

one needs to do the following variations. Notice that

$$\begin{aligned}\delta \int du \oint d^2y(\pi_h D_+\sigma) &= \int du \oint d^2y\{\pi_h \delta(\partial_+\sigma - A^a_+\partial_a\sigma - \partial_a A^a_+)\} \\ &= \int du \oint d^2y\{(-\pi_h\partial_a\sigma + \partial_a\pi_h)\delta A^a_+ - \partial_a(\pi_h \delta A^a_+)\}.\end{aligned} \tag{A32}$$

Thus we have

$$\frac{\delta}{\delta A^a_+}\int du \oint d^2y \frac{1}{2}(\pi_h D_+\sigma) = -\frac{1}{2}\pi_h\partial_a\sigma + \frac{1}{2}\partial_a\pi_h. \tag{A33}$$

Let us also notice that, for an arbitrary field ζ^{ab}, the following is true,

$$\begin{aligned}\int du \oint d^2y \zeta^{ab}\delta(D_+\rho_{ab}) &= \int du \oint d^2y \zeta^{ab}\{-(\delta A^c_+)\partial_c\rho_{ab} - (\partial_a \delta A^c_+)\rho_{cb} - (\partial_b\delta A^c_+)\rho_{ac} + (\partial_c\delta A^c_+)\rho_{ab}\} \\ &= \int du \oint d^2y[\{-\zeta^{ab}\partial_c\rho_{ab} + \partial_a(\zeta^{ab}\rho_{cb}) + \partial_b(\zeta^{ab}\rho_{ac}) - \partial_c(\zeta^{ab}\rho_{ab})\}\delta A^c_+ \\ &\quad - \partial_a(\zeta^{ab}\rho_{cb}\delta A^c_+) - \partial_b(\zeta^{ab}\rho_{ac}\delta A^c_+) + \partial_c(\zeta^{ab}\rho_{ab}\delta A^c_+)].\end{aligned} \tag{A34}$$

Let us consider the following two cases, (a) and (b). (a) If we choose

$$\zeta^{ab} := \frac{1}{2h}\pi^{ab}, \tag{A35}$$

then from (A34) we obtain

$$\begin{aligned}&\frac{\delta}{\delta A^a_+}\int du \oint d^2y \frac{1}{2h}\pi^{bc}(D_+\rho_{bc}) \\ &\quad = -\frac{1}{2h}\pi^{bc}\partial_a\rho_{bc} + \partial_b\left(\frac{1}{h}\pi^{bc}\rho_{ca}\right),\end{aligned} \tag{A36}$$

where we used the identity,

$$\rho_{ab}\pi^{ab} = 0. \tag{A37}$$

(b) If we choose

$$\zeta^{ab} := \frac{1}{4h}e^{\sigma}\rho^{ae}\rho^{bf}(D_+\rho_{ef}), \tag{A38}$$

which now depends on A^a_+, then (A34) becomes

$$\begin{aligned}&\frac{\delta}{\delta A^a_+}\int du \oint d^2y \frac{1}{8h}e^{\sigma}\rho^{bd}\rho^{ce}(D_+\rho_{bc})(D_+\rho_{de}) \\ &\quad = -\frac{1}{4h}e^{\sigma}\rho^{bd}\rho^{ce}(D_+\rho_{de})(\partial_a\rho_{bc}) \\ &\qquad + \partial_b\left(\frac{1}{2h}e^{\sigma}\rho^{bc}D_+\rho_{ca}\right),\end{aligned} \tag{A39}$$

where we used the identity

$$\rho^{ab}D_+\rho_{ab} = 0. \tag{A40}$$

From the Eqs. (A33), (A36), and (A39), we find that the Eq. (A31) becomes

$$\begin{aligned}D_-\pi_a =& \frac{1}{2}\pi_h\partial_a\sigma - \frac{1}{2}\partial_a\pi_h \\ &+ \frac{1}{2h}\left\{\pi^{bc} + \frac{1}{2}e^{\sigma}\rho^{bd}\rho^{ce}(D_+\rho_{de})\right\}(\partial_a\rho_{bc}) \\ &- \partial_b\left\{\frac{1}{h}\pi^{bc}\rho_{ca} + \frac{1}{2h}e^{\sigma}\rho^{bc}(D_+\rho_{ca})\right\}.\end{aligned} \tag{A41}$$

Using the definitions of the momenta (3.6), $\cdots$, (3.9), one can easily show that the Eq. (2.16) is identical to (A41).

4. Variations with Respect to π^{ab} and ρ_{ab}

It is trivial to see that the equation

$$D_-\rho_{ab} = \frac{\delta K}{\delta \pi^{ab}} \tag{A42}$$

is just the Eq. (3.9). Let us show that the equation

$$D_-\pi^{ab} = -\frac{\delta K}{\delta\rho_{ab}} \tag{A43}$$

is identical to the Eq. (2.19). If we vary terms in K which do not contain $D_+\rho_{ab}$, then we have

$$\frac{\delta}{\delta\rho_{ab}}\int du\oint d^2y\Big(-\frac{1}{2}e^{-2\sigma}\rho^{cd}\pi_c\pi_d + \frac{1}{2h}e^{-\sigma}\rho_{ce}\rho_{df}\pi^{ce}\pi^{df}\Big) = \frac{1}{2}e^{-2\sigma}\rho^{ac}\rho^{bd}\pi_c\pi_d + \frac{1}{h}e^{-\sigma}\rho_{cd}\pi^{ac}\pi^{bd}. \tag{A44}$$

Varying terms linear in $D_+\rho_{ab}$, we find that

$$\delta\int du\oint d^2y\Big(\frac{1}{2h}\pi^{ab}D_+\rho_{ab}\Big) = -\int du\oint d^2y D_+\Big(\frac{1}{2h}\pi^{ab}\Big)\delta\rho_{ab} + \int du\frac{d}{du}\times\Big\{\oint d^2y\Big(\frac{1}{2h}\pi^{ab}\delta\rho_{ab}\Big)\Big\}. \tag{A45}$$

If we assume the boundary condition

$$\delta\rho_{ab} = 0, \tag{A46}$$

then we have

$$\frac{\delta}{\delta\rho_{ab}}\int du\oint d^2y\Big(\frac{1}{2h}\pi^{cd}D_+\rho_{cd}\Big) = -D_+\Big(\frac{1}{2h}\pi^{ab}\Big). \tag{A47}$$

Now let us define

$$S^a_b := \rho^{ac}D_+\rho_{cb}. \tag{A48}$$

Then we have

$$\frac{1}{8h}e^{\sigma}\rho^{ab}\rho^{cd}(D_+\rho_{ac})(D_+\rho_{bd}) = \frac{1}{8h}e^{\sigma}S^b_aS^a_b, \tag{A49}$$

and the variation of S^a_b is given by

$$\delta S^a_b = -\rho^{ac}S^d_b\delta\rho_{cd} + \rho^{ac}D_+\delta\rho_{cb}. \tag{A50}$$

Therefore, we have

$$\begin{aligned}\delta\int du\oint d^2y\Big\{\frac{1}{8h}e^{\sigma}\rho^{ab}\rho^{cd}(D_+\rho_{ac})(D_+\rho_{bd})\Big\} &= \int du\oint d^2y\Big(\frac{1}{4h}e^{\sigma}S^b_a\delta S^a_b\Big)\\ &= \int du\oint d^2y\Big\{-\frac{1}{4h}e^{\sigma}\rho^{ac}S^d_cS^b_d\delta\rho_{ab} + \frac{1}{4h}e^{\sigma}\rho^{ac}S^b_cD_+\delta\rho_{ab}\Big\}\\ &= \int du\oint d^2y\Big\{-\frac{1}{4h}e^{\sigma}\rho^{ac}S^d_cS^b_d - D_+\Big(\frac{1}{4h}e^{\sigma}\rho^{ac}S^b_c\Big)\Big\}\delta\rho_{ab}.\end{aligned} \tag{A51}$$

Therefore, we find that

$$\frac{\delta}{\delta\rho_{ab}}\int du\oint d^2y\Big\{\frac{1}{8h}e^{\sigma}\rho^{ce}\rho^{df}(D_+\rho_{cd})(D_+\rho_{ef})\Big\} = -\frac{1}{4h}e^{\sigma}\rho^{ac}\rho^{bd}\rho^{ef}(D_+\rho_{ce})(D_+\rho_{df}) - D_+\Big(\frac{1}{4h}e^{\sigma}\rho^{ac}\rho^{bd}D_+\rho_{cd}\Big). \tag{A52}$$

Finally, we have to vary the Lagrange multiplier term in (4.1). It is given by

$$\frac{\delta}{\delta\rho_{ab}}\int du\oint d^2y\lambda(\det\rho_{cd} - 1) = \lambda\rho^{ab}. \tag{A53}$$

The scalar curvature term $e^{\sigma}R_2$ is a topological density that does not contribute to the metric variation. From (A44), (A47), (A52), and (A53), we have

$$D_-\pi^{ab} + \frac{1}{2}e^{-2\sigma}\rho^{ac}\rho^{bd}\pi_c\pi_d + \frac{1}{h}e^{-\sigma}\rho_{cd}\pi^{ac}\pi^{bd} - D_+\Big\{\frac{1}{2h}\pi^{ab} + \frac{1}{4h}e^{\sigma}\rho^{ac}\rho^{bd}(D_+\rho_{cd})\Big\} - \frac{1}{4h}e^{\sigma}\rho^{ac}\rho^{bd}\rho^{ef}(D_+\rho_{ce})(D_+\rho_{df}) + \lambda\rho^{ab} = 0. \tag{A54}$$

The Lagrange multiplier λ is determined by taking the trace of (A54). Notice that for any traceless field χ^{ab} such that

$$\rho_{ab}\chi^{ab} = 0, \tag{A55}$$

we have

$$\rho_{ab}D_\pm\chi^{ab} = -\chi^{ab}D_\pm\rho_{ab}. \tag{A56}$$

Thus, for χ^{ab} defined as

$$\chi^{ab} := \frac{1}{2h}\pi^{ab} + \frac{1}{4h}e^{\sigma}\rho^{ac}\rho^{bd}D_+\rho_{cd}, \tag{A57}$$

one has

$$-\rho_{ab}D_+\Big(\frac{1}{2h}\pi^{ab} + \frac{1}{4h}e^{\sigma}\rho^{ac}\rho^{bd}D_+\rho_{cd}\Big) = \frac{1}{2h}\pi^{ab}D_+\rho_{ab} + \frac{1}{4h}e^{\sigma}\rho^{ac}\rho^{bd}(D_+\rho_{ab})(D_+\rho_{cd}). \tag{A58}$$

Therefore, the trace of the Eq. (A54) becomes

$$\begin{aligned}0 &= 2\lambda - \pi^{ab}D_-\rho_{ab} + \frac{1}{2}e^{-2\sigma}\rho^{ab}\pi_a\pi_b + \frac{1}{h}e^{-\sigma}\rho_{ab}\rho_{cd}\pi^{ac}\pi^{bd} + \frac{1}{2h}\pi^{ab}D_+\rho_{ab}\\ &= 2\lambda + \frac{1}{2}e^{-2\sigma}\rho^{ab}\pi_a\pi_b + \pi^{ab}\Big(-D_-\rho_{ab} + \frac{1}{h}e^{-\sigma}\rho_{ac}\rho_{bd}\pi^{cd} + \frac{1}{2h}D_+\rho_{ab}\Big)\\ &= 2\lambda + \frac{1}{2}e^{-2\sigma}\rho^{ab}\pi_a\pi_b,\end{aligned} \tag{A59}$$

where we used the Eq. (A42). Thus λ is given by

$$\lambda = -\frac{1}{4}\mathrm{e}^{-2\sigma}\rho^{cd}\pi_c\pi_d. \tag{A60}$$

Thus we find that the Eq. (A54) finally becomes

$$D_-\pi^{ab} = -\frac{1}{2}\mathrm{e}^{-2\sigma}\rho^{ac}\rho^{bd}\pi_c\pi_d + \frac{1}{4}\mathrm{e}^{-2\sigma}\rho^{ab}\rho^{cd}\pi_c\pi_d - \frac{1}{h}\mathrm{e}^{-\sigma}\rho_{cd}\pi^{ac}\pi^{bd} + D_+\left\{\frac{1}{2h}\pi^{ab} + \frac{1}{4h}\mathrm{e}^{\sigma}\rho^{ac}\rho^{bd}(D_+\rho_{cd})\right\} + \frac{1}{4h}\mathrm{e}^{\sigma}\rho^{ac}\rho^{bd}\rho^{ef}(D_+\rho_{ce})(D_+\rho_{df}). \tag{A61}$$

One can show that this equation is the same as the Eq. (2.19). To show this, let us multiply the Eq. (A61) by $\rho_{am}\rho_{bn}$, and use the definitions of the conjugate momenta (3.3), $\cdots$, (3.5). Then each term in (A61) becomes as follows,

(i)
$$\begin{aligned}\rho_{am}\rho_{bn}(D_-\pi^{ab}) &= \rho_{am}\rho_{bn}D_-\left\{-\frac{1}{2}\mathrm{e}^{\sigma}\rho^{ac}\rho^{bd}(D_+\rho_{cd}) + h\mathrm{e}^{\sigma}\rho^{ac}\rho^{bd}(D_-\rho_{cd})\right\} = -\frac{1}{2}\mathrm{e}^{\sigma}(D_-\sigma)(D_+\rho_{mn}) \\ &\quad + \frac{1}{2}\mathrm{e}^{\sigma}\rho^{cd}(D_-\rho_{mc})(D_+\rho_{nd}) + \frac{1}{2}\mathrm{e}^{\sigma}\rho^{cd}(D_-\rho_{nc})(D_+\rho_{md}) - \frac{1}{2}\mathrm{e}^{\sigma}(D_-D_+\rho_{mn}) + \mathrm{e}^{\sigma}(D_-h)(D_-\rho_{mn}) \\ &\quad + h\mathrm{e}^{\sigma}(D_-\sigma)(D_-\rho_{mn}) - 2h\mathrm{e}^{\sigma}\rho^{cd}(D_-\rho_{mc})(D_-\rho_{nd}) + h\mathrm{e}^{\sigma}(D_-^2\rho_{mn}),\end{aligned} \tag{A62}$$

(ii)
$$\frac{1}{2}\mathrm{e}^{-2\sigma}\pi_m\pi_n - \frac{1}{4}\mathrm{e}^{-2\sigma}\rho_{mn}\rho^{cd}\pi_c\pi_d = \frac{1}{2}\mathrm{e}^{2\sigma}\rho_{mc}\rho_{nd}F^c_{+-}F^d_{+-} - \frac{1}{4}\mathrm{e}^{2\sigma}\rho_{mn}\rho_{cd}F^c_{+-}F^d_{+-}, \tag{A63}$$

(iii)
$$\begin{aligned}\frac{1}{h}\mathrm{e}^{-\sigma}\rho_{cd}\pi^{ac}\pi^{bd}\rho_{am}\rho_{bn} &= \frac{1}{4h}\mathrm{e}^{\sigma}\rho^{cd}(D_+\rho_{mc})(D_+\rho_{nd}) - \frac{1}{2}\mathrm{e}^{\sigma}\rho^{cd}(D_+\rho_{mc})(D_-\rho_{nd}) \\ &\quad - \frac{1}{2}\mathrm{e}^{\sigma}\rho^{cd}(D_+\rho_{nc})(D_-\rho_{md}) + h\mathrm{e}^{\sigma}\rho^{cd}(D_-\rho_{mc})(D_-\rho_{nd}),\end{aligned} \tag{A64}$$

(iv)
$$\begin{aligned}-\rho_{am}\rho_{bn}D_+\left\{\frac{1}{2h}\pi^{ab} + \frac{1}{4h}\mathrm{e}^{\sigma}\rho^{ac}\rho^{bd}(D_+\rho_{cd})\right\} &= -\frac{1}{2}\rho_{am}\rho_{bn}D_+\{\mathrm{e}^{\sigma}\rho^{ac}\rho^{bd}(D_-\rho_{cd})\} \\ &= -\frac{1}{2}\mathrm{e}^{\sigma}(D_+\sigma)(D_-\rho_{mn}) + \frac{1}{2}\mathrm{e}^{\sigma}\rho^{cd}(D_+\rho_{mc})(D_-\rho_{nd}) \\ &\quad + \frac{1}{2}\mathrm{e}^{\sigma}\rho^{cd}(D_+\rho_{nc})(D_-\rho_{md}) - \frac{1}{2}\mathrm{e}^{\sigma}(D_+D_-\rho_{mn}),\end{aligned} \tag{A65}$$

(v)
$$-\frac{1}{4h}\mathrm{e}^{\sigma}\rho^{ac}\rho^{bd}\rho^{ef}(D_+\rho_{ce})(D_+\rho_{df})\rho_{am}\rho_{bn} = -\frac{1}{4h}\mathrm{e}^{\sigma}\rho^{cd}(D_+\rho_{mc})(D_+\rho_{nd}). \tag{A66}$$

After a little algebra, we find that the Eq. (A61) is identical to the Eq. (2.19). Thus, assuming the boundary conditions (A3), I have shown that the 12 Hamilton's equations of motion, (A1) and (A2), are just the first-order form of the six Einstein's equations (2.15), (2.16), (2.18), and (2.19). Therefore, the Hamilton's equations of motion, (A1) and (A2), together with the four divergence-type equations (3.15), (3.16), and (3.17), are completely equivalent to the full Einstein's equations (2.13), $\cdots$, (2.19).

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