Small Kerr–anti-de Sitter black holes are unstable

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(Received 4 May 2004; published 12 October 2004)

Superradiance in black hole spacetimes can trigger instabilities. Here we show that, due to superradiance, small Kerr–anti-de Sitter black holes are unstable. Our demonstration uses a matching procedure, in a long wavelength approximation.

DOI: 10.1103/PhysRevD.70.084011 PACS numbers: 04.70.–s

I. INTRODUCTION

Einstein equations describing general relativity and gravitation form a system of coupled nonlinear partial differential equations, which is extremely hard to solve, even resorting to state-of-the-art computing. Therefore exact solutions to Einstein equations, which are possible to obtain only in special instances, are of fundamental importance. They allow us to probe essential features of general relativity. For example, by having at hand the spherically symmetric Schwarzschild solution it was possible to match general relativity predictions against experimental observations. Once an exact solution is found, one must examine it in detail and investigate the physical properties of such a solution. One of the most important aspects is the stability of a given solution. In fact, if a solution is not stable, then it will most certainly not be found in nature, unless the instability time scale is much larger than the age of our universe. What does one mean by stability? In this classical context, stability means that a given initially bounded perturbation of the spacetime remains bounded for all times. For example, the Schwarzschild spacetime is stable against all kinds of perturbations, massive or massless [1]. On the other hand the Kerr spacetime, describing a rotating black hole, is stable against massless field perturbations but not against massive bosonic fields [2].

The physics behind this instability is related to a phenomenon known as superradiance, a process which has been known for several decades, and which consists basically of a scattering process which extracts energy from the scattering potential. For example, the Klein-Gordon equation for a charged scalar particle on a steplike potential already displays such a "superradiant" scattering, i.e., the energy of the reflected wave is larger than the incident one [3,4]. The first classical example of superradiant scattering, which would lead to the notion of superradiant scattering in black hole spacetimes, was given by Zel'dovich [5], by examining what happens when scalar waves impinge upon a rotating cylindrical absorbing object. Considering a wave of the form $e^{-i\omega t+im\phi}$ incident upon such a rotating object, Zel'dovich concluded that if the frequency ω of the incident wave satisfies

$$\omega < m\Omega, \tag{1}$$

where Ω is the angular velocity of the body, then the scattered wave is amplified. If the "rotating object" is a Kerr black hole, then superradiant scattering also occurs [5–7] for frequencies ω satisfying (1), but where Ω is now the angular velocity of the black hole. If one could find a way to feed the amplified scattered wave onto the black hole again, then one could in principle extract as much energy as one likes from the black hole (as long as it is less than the total rotational energy). The first proposal of this kind was in fact made by Zel'dovich [5], who suggested to surround the rotating cylinder by a reflecting mirror. In this case the wave would bounce back and forth, between the mirror and the cylinder, amplifying itself each time. A similar situation can be achieved for a Kerr black hole: surround it by a spherical mirror and excite a given multipole m wave in it. Then the total extracted energy should grow exponentially until finally the radiation pressure destroys the mirror. This is exactly the same principle behind the instability of Kerr black holes against massive bosonic perturbations, because in this case the mass of the field works as a wall near infinity [2].

The system black hole plus mirror is known as Press and Teukolsky's black hole bomb [8], which has been recently investigated in detail in [9]. It was shown in [9] that for the system to really become unstable, the mirror must have a radius larger than a certain critical value. This is because the oscillation frequencies are

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dictated by the mirror, and go like $1/r_0$, with r_0 being the mirror radius. Thus for superradiance to work, one must have by (1), $1/r_0 \lesssim m\Omega$. In principle, black holes in anti-de Sitter (AdS) space should have similar properties as those of the black hole bomb, since the boundary of anti-de Sitter spacetime behaves as a wall. In fact, a similar reasoning applied to Kerr–anti-de Sitter (Kerr-AdS) black holes leads one to verify the stability of large rotating black holes in anti-de Sitter spacetime (the stability of these large black holes was proven by Hawking and Reall [10]), and lead also to the conjecture that small Kerr-AdS black holes should be unstable [9]. The purpose of the present paper is to prove the instability of small Kerr-AdS black holes, by solving directly the wave equation for a scalar field, in the large wavelength approximation, by using matched asymptotic expansions.

II. FORMULATION OF THE PROBLEM AND BASIC EQUATIONS

We shall consider a scalar field in the vicinity of a Kerr-AdS black hole, with an exterior geometry described by the line element [11]

$$ds^2 = -\frac{\Delta_r}{\rho^2}\left(dt - \frac{a}{\Sigma}\sin^2\theta d\phi\right)^2 + \frac{\rho^2}{\Delta_r}dr^2 + \frac{\rho^2}{\Delta_\theta}d\theta^2 + \frac{\Delta_\theta}{\rho^2}\sin^2\theta\left(adt - \frac{r^2+a^2}{\Sigma}d\phi\right)^2, \tag{2}$$

with

$$\Delta_r = (r^2+a^2)\left(1+\frac{r^2}{\ell^2}\right) - 2Mr, \qquad \Sigma = 1 - \frac{a^2}{\ell^2},$$
$$\Delta_\theta = 1 - \frac{a^2}{\ell^2}\cos^2\theta, \qquad \rho^2 = r^2 + a^2\cos^2\theta, \tag{3}$$

and $\ell = \sqrt{-3/\Lambda}$ is the cosmological length associated with the cosmological constant Λ. This metric describes the gravitational field of the Kerr black hole, with mass M, angular momentum $J = Ma$, and has an event horizon at $r = r_+$ (the largest root of Δ_r). A characteristic and important parameter of a Kerr black hole is the angular velocity of its event horizon given by

$$\Omega = \frac{a}{r_+^2 + a^2}\left(1 - \frac{a^2}{\ell^2}\right). \tag{4}$$

In order to avoid singularities, the black hole rotation is constrained to be

$$a < \ell. \tag{5}$$

In the absence of sources, which we consider to be our case, the evolution of the scalar field is dictated by the Klein-Gordon equation in a Kerr-AdS spacetime, $[\nabla_\mu\nabla^\mu - \xi R - \mu^2]\Phi = 0$. Here, $R = -12/\ell^2$ is the Ricci scalar of the Kerr-AdS spacetime, ξ is a coupling constant, and μ is the mass of the scalar field. For simplicity, and without loss of generality, we choose the value of ξ and μ in order that the Klein-Gordon equation stays simply as $\nabla_\mu\nabla^\mu\Phi = 0$. To make the whole problem more tractable, it is convenient to separate the field as [12]

$$\Phi(t, r, \theta, \phi) = e^{-i\omega t + im\phi}\tilde{S}_l^m(\theta)R(r), \tag{6}$$

where $\tilde{S}_l^m(\theta)$ are the AdS spheroidal angular functions, and the azimuthal number m takes on integer (positive or negative) values. For our purposes, it is enough to consider positive ω's in (6) [6]. Inserting this in the Klein-Gordon equation, we get the following angular and radial wave equations for $\tilde{S}_l^m(\theta)$ and $R(r)$:

$$\frac{\Delta_\theta}{\sin\theta}\partial_\theta(\Delta_\theta\sin\theta\partial_\theta\tilde{S}_l^m) + \left[a^2\omega^2\cos^2\theta - \frac{m^2\Sigma^2}{\sin^2\theta} + A_{lm}\Delta_\theta\right]\tilde{S}_l^m = 0, \tag{7}$$

$$\Delta_r\partial_r(\Delta_r\partial_r R) + [\omega^2(r^2+a^2)^2 - 2Mam\omega r + a^2m^2 - \Delta_r(a^2\omega^2 + A_{lm})]R = 0, \tag{8}$$

where A_{lm} is the separation constant that allows the split of the wave equation and is found as an eigenvalue of (7). For small $a\omega$ and for small a/ℓ, the regime of interest in the next section, one has [13]

$$A_{lm} = l(l+1) + \mathcal{O}(a^2\omega^2, a^2/\ell^2). \tag{9}$$

The boundary conditions that one must impose upon the scalar field are the following. First, we require that the scalar field vanishes at $r \to \infty$ because the AdS space behaves effectively as a reflecting box, i.e., the AdS infinity works as a mirror wall (but see also [14] and references therein for another possible set of boundary conditions). Second, near the horizon $r = r_+$, the scalar field as given by (6) behaves as

$$\Phi \sim e^{-i\omega t}e^{\pm i(\omega - m\Omega)r_*}, \qquad r \to r_+, \tag{10}$$

where the tortoise r_* coordinate is defined implicitly by $dr_*/dr = (r^2+a^2)/\Delta_r$. Requiring ingoing waves at the horizon, which is the physically acceptable solution, one must impose a negative group velocity $v_{\rm gr}$ for the wave packet. Since $v_{\rm gr} = \pm 1$ we must thus use the minus sign in (10). To satisfy these two boundary conditions simultaneously, the frequencies ω must take on certain special values, which are called quasinormal frequencies (QN frequencies, $\omega_{\rm QN}$) and the associated modes are called quasinormal modes. In general, $\omega_{\rm QN}$ will be a complex quantity, signaling the decay of the field, or then its growth. Note that according to the field decomposition (6) if the imaginary part of ω is positive then the field will grow exponentially as time goes by. Thus we say that the system is unstable if the imaginary part of $\omega_{\rm QN}$ is positive.

III. ANALYTICAL CALCULATION OF THE UNSTABLE MODES

In this section, we will show that small Kerr-AdS black holes are unstable. We shall, within some approximations, compute the characteristic QN frequencies for a scalar field and show that they do have positive imaginary parts. The instability is due to the presence of an effective "reflecting mirror" at the AdS infinity, since the waves are then successively impinging on the small AdS black hole and being reflected at infinity [9]. We shall see that this interpretation agrees in all aspects with the study of the black hole bomb [9].

We assume that $1/\omega \gg M$, i.e., that the Compton wavelength of the scalar particle is much larger than the typical size of the black hole, and that the AdS black hole is small, i.e., that the size of the black hole is much smaller than the typical AdS radius, $r_+/\ell \ll 1$. We will also assume slow rotation: $a \ll M$, and $a \ll \ell$. Following a matching procedure introduced in [7,15,16], we divide the space outside the event horizon in two regions, namely, the near region, $r - r_+ \ll 1/\omega$, and the far region, $r - r_+ \gg M$. We will solve the radial equation (8) in each one of these two regions. Then, we will match the near-region and the far-region solutions in the overlapping region where $M \ll r - r_+ \ll 1/\omega$ is satisfied. When the correct boundary conditions are imposed upon the solutions, we shall get a defining equation for $\omega_{\rm QN}$, and the stability or instability of the spacetime depends basically on the sign of the imaginary component of $\omega_{\rm QN}$.

A. Near-region wave equation and solution

For small AdS black holes, $r_+/\ell \ll 1$, in the near region, $r - r_+ \ll 1/\omega$, we can neglect the effects of the cosmological constant, $\Lambda \sim 0$. Moreover, one has $r \sim r_+$, $r_+ \sim 2M$, and $\omega a^2 \sim 0$ (since $\omega \ll M^{-1}$ and $a \ll M$), and $\Delta_r \sim \Delta$ with

$$\Delta = r^2 + a^2 - 2Mr. \tag{11}$$

The near-region radial wave equation can then be written as

$$\Delta\partial_r(\Delta\partial_r R) + r_+^4(\omega - m\Omega)^2 R - l(l+1)\Delta R = 0. \tag{12}$$

To find the analytical solution of this equation, one first introduces a new radial coordinate,

$$z = \frac{r - r_+}{r - r_-}, \qquad 0 \le z \le 1, \tag{13}$$

with the event horizon being at $z = 0$. Then, one has $\Delta\partial_r = (r_+ - r_-)z\partial_z$, and the near-region radial wave equation can be written as

$$z(1-z)\partial_z^2 R + (1-z)\partial_z R + \varpi^2\frac{1-z}{z}R - \frac{l(l+1)}{1-z}R = 0, \tag{14}$$

where we have defined the superradiant factor

$$\varpi \equiv (\omega - m\Omega)\frac{r_+^2}{r_+ - r_-}. \tag{15}$$

Through the definition

$$R = z^{i\varpi}(1-z)^{l+1}F, \tag{16}$$

the near-region radial wave equation becomes

$$z(1-z)\partial_z^2 F + [(1 + i2\varpi) - (1 + 2(l+1) + i2\varpi)z]\partial_z F - [(l+1)^2 + i2\varpi(l+1)]F = 0. \tag{17}$$

This wave equation is a standard hypergeometric equation [17], $z(1-z)\partial_z^2 F + [c - (a+b+1)z]\partial_z F - abF = 0$, with

$$a = l + 1 + i2\varpi, \qquad b = l + 1, \qquad c = 1 + i2\varpi, \tag{18}$$

and its most general solution in the neighborhood of $z = 0$ is $Az^{1-c}F(a-c+1, b-c+1, 2-c, z) + BF(a, b, c, z)$. Using (16), one finds that the most general solution of the near-region equation is

$$R = Az^{-i\varpi}(1-z)^{l+1}F(a-c+1, b-c+1, 2-c, z) + Bz^{i\varpi}(1-z)^{l+1}F(a, b, c, z). \tag{19}$$

The first term represents an ingoing wave at the horizon $z = 0$, while the second term represents an outgoing wave at the horizon. We are working at the classical level, so there can be no outgoing flux across the horizon, and thus one sets $B = 0$ in (19). One is now interested in the large r, $z \to 1$, behavior of the ingoing near-region solution. To achieve this aim one uses the $z \to 1 - z$ transformation law for the hypergeometric function [17],

$$F(a-c+1, b-c+1, 2-c, z) = (1-z)^{c-a-b}\frac{\Gamma(2-c)\Gamma(a+b-c)}{\Gamma(a-c+1)\Gamma(b-c+1)}F(1-a, 1-b, c-a-b+1, 1-z) + \frac{\Gamma(2-c)\Gamma(c-a-b)}{\Gamma(1-a)\Gamma(1-b)}F(a-c+1, b-c+1, -c+a+b+1, 1-z), \tag{20}$$

and the property $F(a, b, c, 0) = 1$. Finally, noting that when $r \to \infty$ one has $1 - z = (r_+ - r_-)/r$, one obtains the large ωr behavior of the ingoing wave solution in the near region,

$$R \sim A\Gamma(1 - i2\varpi)\left[\frac{(r_+ - r_-)^{-l}\Gamma(2l+1)}{\Gamma(l+1)\Gamma(l+1-i2\varpi)} r^l + \frac{(r_+ - r_-)^{l+1}\Gamma(-2l-1)}{\Gamma(-l)\Gamma(-l-i2\varpi)} r^{-l-1}\right]. \tag{21}$$

B. Far-region wave equation and solution

In the far region, $r - r_+ \gg M$, the effects induced by the black hole can be neglected ($a \sim 0$, $M \sim 0$, $\Delta_r \sim r^2[1 + r^2/\ell^2]$) and the radial wave equation (8) reduces to the wave equation of a scalar field of frequency ω and angular momentum l in a pure AdS background,

$$(r^2 + \ell^2)\partial_r^2 R + 2\left(2r + \frac{\ell^2}{r}\right)\partial_r R + \ell^2\left[\omega^2 \frac{\ell^2}{r^2 + \ell^2} - \frac{l(l+1)}{r^2}\right]R = 0. \tag{22}$$

Notice that in the above approximation, the far-region wave equation in the Kerr-AdS black hole background is equal to the wave equation in the pure AdS background. However, one must be cautious since the boundaries of the far region in the Kerr-AdS black hole case are $r = r_+$ and $r = \infty$, while in the pure AdS case the boundaries are $r = 0$ and $r = \infty$. In what follows we will find the solution of (22), first in the pure AdS case, and then we will use this last solution to find the far-region solution of the Kerr-AdS black hole case.

The wave equation (22) can be written in a standard hypergeometric form. First we introduce a new radial coordinate,

$$x = 1 + \frac{r^2}{\ell^2}, \qquad 1 \le x \le \infty, \tag{23}$$

with the origin of the AdS space, $r = 0$, being at $x = 1$, and $r = \infty$ corresponds to $x = \infty$. Then, one has $\partial_r = 2\ell^{-1}\sqrt{x-1}\partial_x$, and the radial wave equation can be written as

$$x(1-x)\partial_x^2 R + \frac{2-5x}{2}\partial_x R - \left[\frac{\omega^2\ell^2}{4x}R + \frac{l(l+1)}{4(1-x)}\right]R = 0. \tag{24}$$

Through the definition

$$R = x^{\omega\ell/2}(1-x)^{l/2}F, \tag{25}$$

the radial wave equation becomes

$$x(1-x)\partial_x^2 F + \left[(1+\omega\ell) - \left(l + \tfrac{5}{2} + \omega\ell\right)x\right]\partial_x F - \tfrac{1}{4}(l + \omega\ell)(l + 3 + \omega\ell)F = 0. \tag{26}$$

This wave equation is a standard hypergeometric equation [17], $x(1-x)\partial_x^2 F + [\gamma - (\alpha + \beta + 1)x]\partial_x F - \alpha\beta F = 0$, with

$$\alpha = \frac{l + 3 + \omega\ell}{2}, \qquad \beta = \frac{l + \omega\ell}{2}, \qquad \gamma = 1 + \omega\ell, \tag{27}$$

and its most general solution in the neighborhood of $x = \infty$ is $Cx^{-\alpha}F(\alpha, \alpha - \gamma + 1, \alpha - \beta + 1, 1/x) + Dx^{-\beta}F(\beta, \beta - \gamma + 1, \beta - \alpha + 1, 1/x)$. Using (25), one finds that the most general solution for $R(x)$ is

$$R = Cx^{-(l+3)/2}(1-x)^{l/2}F(\alpha, \alpha - \gamma + 1, \alpha - \beta + 1, 1/x) + Dx^{-l/2}(1-x)^{l/2}F(\beta, \beta - \gamma + 1, \beta - \alpha + 1, 1/x). \tag{28}$$

Since $F(a, b, c, 0) = 1$, as $x \to \infty$ this solution behaves as $R \sim (-1)^{l/2}(Cx^{-3/2} + D)$. But the AdS infinity behaves effectively as a wall, and thus the scalar field must vanish there which implies that we must set $D = 0$ in (28). We are now interested in the small r, $x \to 1$, behavior of (28). To achieve this aim one uses the $1/x \to 1 - x$ transformation law for the hypergeometric function [17],

$$F(\alpha, \alpha - \gamma + 1, \alpha - \beta + 1, 1/x) = x^{\alpha - \gamma + 1}(x-1)^{\gamma - \alpha - \beta}\frac{\Gamma(\alpha - \beta + 1)\Gamma(\alpha + \beta - \gamma)}{\Gamma(\alpha)\Gamma(\alpha - \gamma + 1)}F(1 - \beta, 1 - \alpha, \gamma - \alpha - \beta + 1, 1 - x) + x^{\alpha}\frac{\Gamma(\alpha - \beta + 1)\Gamma(\gamma - \alpha - \beta)}{\Gamma(1 - \beta)\Gamma(\gamma - \beta)}F(\alpha, \beta, \alpha + \beta - \gamma + 1, 1 - x), \tag{29}$$

and the property $F(a, b, c, 0) = 1$. Finally, noting that when $x \to 1$ one has $x - 1 \to r^2/\ell^2$, one obtains the small ωr behavior of $R(r)$,

$$R \sim C\Gamma(5/2)\left[\frac{(-1)^{l/2}\ell^{-l}\Gamma(-l - \frac{1}{2})}{\Gamma(1 - \frac{l}{2} - \frac{\omega\ell}{2})\Gamma(1 - \frac{l}{2} + \frac{\omega\ell}{2})} r^l + \frac{(-1)^{-3l/2}\ell^{l+1}\Gamma(l + \frac{1}{2})}{\Gamma(\frac{3}{2} + \frac{l}{2} + \frac{\omega\ell}{2})\Gamma(\frac{3}{2} + \frac{l}{2} - \frac{\omega\ell}{2})} r^{-l-1}\right]. \tag{30}$$

The boundaries of the pure AdS spacetime are the origin, $r = 0$, and the effective wall at $r = \infty$. When $r \to 0$, the wave solution R diverges, since $r^{-l-1} \to \infty$ in (30). In order to have a regular solution at the origin we must then demand that $\Gamma(\frac{3}{2} + \frac{l}{2} - \frac{\omega\ell}{2}) = \infty$. This occurs when the argument of the gamma function is a nonpositive integer, $\Gamma(-n) = \infty$ with $n = 0, 1, 2, \ldots$. Therefore, the requirement of regularity of the wave solution at the origin selects the frequencies that might propagate in the AdS background. These are given by the discrete spectrum $\omega\ell = l + 3 + 2n$, which agrees with known results [18,19]. We remark that, alternatively, in order to have a regular solution at the origin we could have required $\Gamma(\frac{3}{2} + \frac{l}{2} + \frac{\omega\ell}{2}) = \infty$. This option would lead to the negative spectrum $\omega\ell = -(l + 3 + 2n)$, which of course must also be a solution. However, to simplify matters we shall deal only with positive frequencies, as was said earlier.

Now that we have found the wave solution that propagates in a pure AdS spacetime, we can discuss the far-region solution in a Kerr-AdS background. As we pointed out earlier, the main difference between the two solutions lies on the inner boundary: $r = 0$ in the pure AdS case and $r = r_+$ in the black hole case. We expect that the allowed spectrum of discrete real frequencies that can propagate in the far region of the Kerr-AdS black hole is equal to the one of the pure AdS background since, at large distances from the inner boundary, both backgrounds are similar. However, the existence of the black hole inner boundary implies that once radiation crosses this zone it will be scattered by the black hole (more precisely it will be scattered by the potential barrier outside the event horizon) and its amplitude will decrease or, eventually, since conditions for superradiance might be present, it will grow leading to an instability. Therefore, in the spirit of [2], we expect that the presence of this scattering by the black hole induces a small complex imaginary part in the allowed frequencies, $\delta = \mathrm{Im}[\omega]$, that describes the slow decay of the amplitude of the wave if $\delta < 0$, or the slowly growing instability of the mode if $\delta > 0$. Summarizing, the frequencies that can propagate in the Kerr-AdS background are given by

$$\omega_{\mathrm{QN}} = \frac{l + 3 + 2n}{\ell} + i\delta, \tag{31}$$

with n being a non-negative integer, and δ being a small quantity. The small ωr behavior of the radial wave solution in the Kerr-AdS background is described by (30), subjected to the regularity condition (31). Now, we want to extract δ from the gamma function in (30). This is done in Appendix A, yielding for small δ and for small ωr the result

$$R \sim C\Gamma(5/2)\left[\frac{(-1)^{l/2}\ell^{-l}\Gamma(-l-\frac{1}{2})}{\Gamma(-l-\frac{1}{2}-n)\Gamma(\frac{5}{2}+n)}r^{l} + i\delta\frac{\Gamma(l+1/2)}{2}\frac{(-1)^{-3l/2+n+1}\ell^{l+2}n!}{(l+2+n)!}r^{-l-1}\right]. \tag{32}$$

C. Matching conditions: properties of the unstable modes

When $M \ll r - r_+ \ll 1/\omega$, the near-region solution and the far-region solution overlap, and thus one can match the large ωr near-region solution (21) with the small ωr far-region solution (32). This matching yields

$$\delta \simeq -2i\frac{(-1)^{n+1}\ell^{-2(l+1)}}{\Gamma(l+1/2)}\frac{(r_+ - r_-)^{2l+1}}{\Gamma(n+5/2)}\frac{\Gamma(l+1-i2\varpi)}{\Gamma(-l-i2\varpi)} \times\frac{\Gamma(-2l-1)}{\Gamma(-l)}\frac{\Gamma(-l-1/2)}{\Gamma(-l-1/2-n)}\frac{\Gamma(l+1)}{\Gamma(2l+1)} \times\frac{(l+2+n)!}{n!}. \tag{33}$$

Using the property of the gamma function, $\Gamma(1 + x) = x\Gamma(x)$, we can find the values of all the gamma functions that appear in (33) yielding simply (see Appendix B)

$$\delta \simeq -\sigma\left(\frac{l+3+2n}{\ell} - m\Omega\right)\frac{r_+^2(r_+ - r_-)^{2l}}{\pi\ell^{2(l+1)}}, \tag{34}$$

with

$$\sigma \equiv \frac{(l!)^2(l+2+n)!}{(2l+1)!(2l)!n!}\frac{2^{l+4}(2l+1+n)!!}{(2l-1)!!(2l+1)!!(2n+3)!!} \times\left[\prod_{k=1}^{l}(k^2 + 4\varpi^2)\right], \tag{35}$$

and $\varpi = [(l + 3 + 2n)/\ell - m\Omega][r_+^2/(r_+ - r_-)]$. Equations (31) and (34) are the main results of this paper. We have

$$\delta \propto -(\mathrm{Re}[\omega_{\mathrm{QN}}] - m\Omega). \tag{36}$$

Thus, $\delta > 0$ for $\mathrm{Re}[\omega_{\mathrm{QN}}] < m\Omega$, and $\delta < 0$ for $\mathrm{Re}[\omega_{\mathrm{QN}}] > m\Omega$. The scalar field Φ has the time dependence $e^{-i\omega t} = e^{-i\mathrm{Re}(\omega)t}e^{\delta t}$ which implies that for $\mathrm{Re}[\omega_{\mathrm{QN}}] < m\Omega$, the amplitude of the field grows exponentially and the mode becomes unstable, with a growth time scale given by $\tau = 1/\delta$. This was the main aim of this paper, namely, to show that small, $r_+ \ll \ell$, Kerr-AdS black holes are unstable. As a check of our results we note that for $l = 0$ we have $\delta \propto r_+^2$, which is in agreement with numerical results for quasinormal modes of small Schwarzschild–anti-de Sitter black holes [18,20]. Also, it was shown numerically in [18] that for higher l poles the imaginary component decays faster with r_+, which is consistent with our result. Indeed, we see from (34) that the imaginary part should behave as r_+^{2l+2}, for nonrotating black holes.

At this point it is appropriate to discuss the domain of validity of our results. Our final result (31) says that $\mathrm{Re}[\omega_{\mathrm{QN}}] \sim 1/\ell$, and the condition for superradiance is $\mathrm{Re}[\omega_{\mathrm{QN}}] \lesssim \Omega$. Now, we have $\Omega \sim a/r_+^2$ in the slow rotation approximation. Therefore, the superradiance condition together with (31) implies that the rotation parameter must satisfy $a/\ell \gtrsim r_+^2/\ell^2$, where the small black hole condition implies $r_+/\ell \ll 1$. This sets the lower bound on a/ℓ for which instability sets in. The upper bound is fixed by the slow rotation approximation $a \ll r_+$ that we used to derive our results. Thus, within

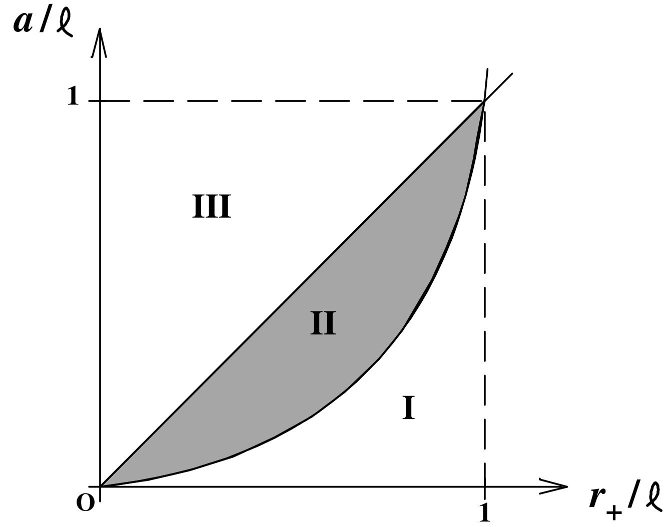

FIG. 1. Range of black hole parameters for which one has stable and unstable modes. Regularity condition implies that $a/\ell < 1$, and for small Kerr-AdS black holes we have $r_+/\ell < 1$. Region I represents a stable mode zone, while regions II and III represent black holes that can have unstable modes. To be accurate, in the approximations we used, we can only guarantee the presence of an instability in region II. There is however no reason to doubt that the instability also exists in region III. The frontier between regions I and II is the parabola $a/\ell = r_+^2/\ell^2$. To ascertain the complete instability zone, numerical work is needed.

all our approximations, we see that instability sets in for $r_+^2/\ell^2 \lesssim a/\ell \ll r_+/\ell$. There is however no reason to doubt that the instability exists all the way up to the maximal rotation case $a = \ell$. This discussion is summarized in Fig. 1.

IV. CONCLUSIONS

We have shown that small, $r_+ \ll \ell$, Kerr-AdS black holes are unstable against the scattering of a wave that satisfies the superradiant regime, $\omega < m\Omega$. This possibility was raised in [10], and heuristic arguments that favored this hypothesis were presented in [9]. We have achieved this result by analytical means in the long wavelength limit, $\omega \ll 1/r_+$, and in the slow rotation regime, $a \ll \ell$ and $a \ll r_+$. We have provided analytical estimates for growing time scales and oscillation frequencies of the corresponding unstable modes. Although we have worked only with zero spin (scalar) waves, we expect that the general features for other spins will be the same. As shown in [10], large Kerr-AdS black holes are stable.

The properties of the instabilities present in the small Kerr-AdS black hole and in the black hole-mirror system (proposed in [8] and studied in detail in [9]) are quite similar. The black hole-mirror system, also known as Press-Teukolsky's black hole bomb [8], consists of a Kerr black hole in an asymptotically flat background surrounded by a mirror placed at constant r, $r = r_0$. Superradiant scattering occurs naturally in the Kerr black hole and, in this regime, if one surrounds the black hole by a reflecting mirror, the wave will bounce back and forth between the mirror and the black hole, amplifying itself each time and leading to an instability. The analogy between this system and the Kerr-AdS black hole is clear. The AdS space behaves effectively as a box, i.e., the AdS wall with typical radius ℓ plays in this analogy the role of a mirror wall with radius $r_0 \equiv \ell$. Indeed, in the Press-Teukolsky's black hole bomb the real part of the allowed frequency is proportional to the inverse of the mirror's radius [9], $\mathrm{Re}[\omega] \propto 1/r_0$, while in the small Kerr-AdS black hole case we have found that $\mathrm{Re}[\omega] \propto 1/\ell$. Moreover, in the Press-Teukolsky's system the growth time scale of the instability satisfies [9] $\delta^{-1} = 1/\mathrm{Im}[\omega] \propto r_0^{2(l+1)}$, while in the AdS black hole we have $\delta^{-1} \propto \ell^{2(l+1)}$. Although we worked in the four dimensional case only, the general arguments that pointed to the existence of this instability allow one to predict that higher dimensional small Kerr-AdS black holes are also unstable.

APPENDIX A: THE SMALL r BEHAVIOR OF THE FAR-REGION SOLUTION

In this Appendix we present the main steps that allow us to go from (30) into (32). In order to do so, one first notes that use of (31) yields

$$\Gamma\left(\frac{3}{2} + \frac{l}{2} + \frac{\omega\ell}{2}\right)\Gamma\left(\frac{3}{2} + \frac{l}{2} - \frac{\omega\ell}{2}\right) = \Gamma(l + 3 + 2n + i\ell\delta/2) \times \Gamma(-n - i\ell\delta/2). \tag{A1}$$

Using the gamma function properties [17], $\Gamma(k+z) = (k-1+z)(k-2+z)\cdots(1+z)\Gamma(1+z)$ with $k = l+3$ and $z = n + i\ell\delta/2$, and $\Gamma(z)\Gamma(1-z) = \pi/\sin(\pi z)$ with $z = 1 + n + i\ell\delta/2$, one has (for $\delta \ll 1$) the result

$$\left[\Gamma\left(\frac{3}{2} + \frac{l}{2} + \frac{\omega\ell}{2}\right)\Gamma\left(\frac{3}{2} + \frac{l}{2} - \frac{\omega\ell}{2}\right)\right]^{-1} \simeq i(-1)^{n+1}\frac{n!}{(l+2+n)!}\frac{\ell}{2}\delta. \tag{A2}$$

Moreover, use of (31) with $\delta \sim 0$ yields

$$\Gamma\left(1 - \frac{l}{2} - \frac{\omega\ell}{2}\right)\Gamma\left(1 - \frac{l}{2} + \frac{\omega\ell}{2}\right) \simeq \Gamma\left(-l - \frac{1}{2} - n\right)\Gamma\left(\frac{5}{2} + n\right). \tag{A3}$$

Finally, inserting (A2) and (A3) into (30) yields (32).

APPENDIX B: USEFUL GAMMA FUNCTION RELATIONS

The transition from (33) into (34) is done using only the gamma function property, $\Gamma(1+x) = x\Gamma(x)$. Indeed, with it we can show that

$$\frac{\Gamma(l+1-i2\varpi)}{\Gamma(-l-i2\varpi)} = i(-1)^{l+1}2\varpi\prod_{k=1}^{l}(k^2+4\varpi^2), \qquad \frac{\Gamma(-2l-1)}{\Gamma(-l)} = (-1)^{l+1}\frac{l!}{(2l+1)!},$$
$$\frac{\Gamma(-l-1/2)}{\Gamma(-l-1/2-n)} = (-1)^n 2^{-n}\frac{(2l+1+n)!!}{(2l+1)!!}, \qquad \Gamma(l+1/2) = 2^{-l}(2l-1)!!\sqrt{\pi}, \tag{B1}$$
$$\Gamma(n+5/2) = 2^{-n-2}(2n+3)!!\sqrt{\pi}.$$

ACKNOWLEDGMENTS

The authors acknowledge José P. S. Lemos for a critical reading of the manuscript. This work was partially funded by Fundação para a Ciência e Tecnologia (FCT)—Portugal through Project No. CERN/FNU/43797/2001. The authors acknowledge financial support from FCT through Grant No. SFRH/BPD/2003.

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