On natural inflation

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We reexamine the model of natural inflation, in which the inflaton potential is flat due to shift symmetries. The original version of the model, where the inflaton is a pseudo–Nambu-Goldstone boson with potential of the form $V(\phi) = \Lambda^4[1 \pm \cos(\phi/f)]$, is studied in light of recent data. We find that the model is alive and well. Successful inflation as well as data from the Wilkinson Microwave Anisotropy Probe require $f > 0.6 m_{\rm Pl}$ (where $m_{\rm Pl} = 1.22 \times 10^{19}$ GeV) and $\Lambda \sim m_{\rm GUT}$ (where $m_{\rm GUT} \sim 10^{16}$ GeV), scales which can be accommodated in particle physics models. The detectability of tensor modes from natural inflation in upcoming microwave background experiments is discussed. We find that natural inflation predicts a tensor/scalar ratio within reach of future observations.

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I. INTRODUCTION

The inflationary Universe model was proposed [1] to solve several cosmological puzzles, the horizon, flatness, and monopole problems, via an early period of accelerated expansion. To satisfy a combination of constraints on inflationary models, in particular, sufficient inflation and microwave background anisotropy measurements [2] of density fluctuations, the potential for the inflaton field must be very flat. For a general class of inflation models involving a single slowly rolling field (including new [3], chaotic [4], and double field inflation [5]), the ratio of the height to the (width)^4 of the potential must satisfy [6]

$$\chi \equiv \Delta V/(\Delta\phi)^4 \leq \mathcal{O}(10^{-6}\text{–}10^{-8}), \tag{1}$$

where ΔV is the change in the potential $V(\phi)$, and $\Delta\phi$ is the change in the field ϕ during the slowly rolling portion of the inflationary epoch. Thus, the inflaton must be extremely weakly self-coupled, with effective quartic self-coupling constant $\lambda_\phi < \mathcal{O}(\chi)$ (in realistic models, $\lambda_\phi < 10^{-12}$). The small ratio of mass scales required by Eq. (1) quantifies how flat the inflaton potential must be and is known as the "fine-tuning" problem in inflation.

Three approaches have been taken toward this required flat potential characterized by a small ratio of mass scales. First, some simply say that there are many as yet unexplained hierarchies in physics, and inflation requires another one. The hope is that all these hierarchies will someday be explained. In these cases, the tiny coupling λ_ϕ is simply postulated *ad hoc* at tree level, and then must be fine-tuned to remain small in the presence of radiative corrections. This merely replaces a cosmological naturalness problem with unnatural particle physics. Second, models have been attempted where the smallness of λ_ϕ is protected by a symmetry, e.g., supersymmetry. In these cases (e.g., [7]), λ_ϕ may arise from a small ratio of mass scales; however, the required mass hierarchy, while stable, is itself unexplained. In addition, existing models have limitations. It would be preferable if such a hierarchy, and thus inflation itself, arose dynamically in particle physics models.

Hence, in 1990 we proposed a third approach, natural inflation [8], in which the inflaton potential is flat due to shift symmetries. Nambu-Goldstone bosons (NGB) arise whenever a global symmetry is spontaneously broken. Their potential is exactly flat due to a shift symmetry under $\phi \to \phi + \text{const}$. As long as the shift symmetry is exact, the inflaton cannot roll and drive inflation, and hence there must be additional explicit symmetry breaking. Then these particles become pseudo-Nambu-Goldstone bosons (PNGBs), with "nearly" flat potentials, exactly as required by inflation. The small ratio of mass scales required by Eq. (1) can easily be accommodated. For example, in the case of the QCD axion, this ratio is of order 10^{-64}. While inflation clearly requires different mass scales than the axion, the point is that the physics of PNGBs can easily accommodate the required small numbers.[1]

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[1]For example, in "invisible" axion models [9] with Peccei-Quinn scale $f_{PQ} \sim 10^{15}$ GeV, the axion self-coupling is $\lambda_a \sim (\Lambda_{\rm QCD}/f_{PQ})^4 \sim 10^{-64}$. [This simply reflects the hierarchy between the QCD and grand unified theory (GUT) scales, which arises from the slow logarithmic running of $\alpha_{\rm QCD}$.] Because of the nonlinearly realized global symmetry, the potential for PNGBs is exactly flat at tree level. The symmetry may be explicitly broken by loop corrections, as in schizon [10] and axion [11] models. In the case of axions, for example, the PNGB mass arises from nonperturbative gauge-field configurations (instantons) through the chiral anomaly. When the associated gauge group becomes strong at a mass scale Λ, instanton effects give rise to a periodic potential of height $\sim\Lambda^4$ for the PNGB field [12]. Since the nonlinearly realized symmetry is restored as $\Lambda \to 0$, the flatness of the PNGB potential is natural in the sense of 't Hooft [13].

We first proposed this model and performed a simple analysis in [8]. Then, in 1993, we followed with a second paper which provides a much more detailed study [14]. The results of Sec. III of the second paper, which presents a careful analysis of the dynamics of the natural inflaton, are of particular relevance here.

Many types of candidates have subsequently been explored for natural inflation. For example, [15] used shift symmetries in Kahler potentials to obtain a flat potential and drive natural chaotic inflation in supergravity. Additionally, [16] examined natural inflation in the context of extra dimensions and [17] used PNGBs from little Higgs models to drive hybrid inflation. Also, [18,19] use the natural inflation idea of PNGBs in the context of brane-world scenarios to drive inflation. Freese [20] suggested using a PNGB as the rolling field in double field inflation [5] (in which the inflaton is a tunneling field whose nucleation rate is controlled by its coupling to a rolling field). We will focus in this paper on the original version of natural inflation, in which there is a single rolling field; we will comment further on other variants of natural inflation in Sec. VI.

In the current paper, we show that the original proposal of natural inflation is live and well, contrary to recent criticisms (which we address in Sec. III). In particular, the single-field version of the model is successful for $f > 0.6 m_{\rm Pl}$ (and does not require $f \gg m_{\rm Pl}$, contrary to the claims of [16]). A second focus of the current paper is to discuss tests of natural inflation from existing and upcoming data from microwave background experiments. Recent results from the Wilkinson Microwave Anisotropy Probe (WMAP) [2] are used to constrain our model, and predictions are made for upcoming experiments such as the PLANCK satellite which will begin taking data in 2007.

We begin in Sec. II by reviewing the basic idea of natural inflation. In Sec. III we present results of the evolution of the scalar field driving inflation, including explicit numerical calculation of the evolution of the scalar field in its potential. In Sec. IV we discuss density fluctuations, and find the constraint on the potential due to comparison with WMAP data. In Sec. V we compute the tensor modes from natural inflation, and discuss their detectability in upcoming microwave background experiments. In Sec. VI, we conclude with a discussion of the pros and cons of having a model in which the width of the potential is of the order of the Planck scale.

II. INFLATION DUE TO SHIFT SYMMETRIES

Here we review the original variant of natural inflation [8], in which a single rolling field has a flat potential due to a shift symmetry and drives inflation. Whenever a global symmetry is spontaneously broken, Nambu-Goldstone bosons arise, with a potential that is exactly flat due to a remaining shift symmetry under $\phi \rightarrow \phi +$ const. If there is additional explicit symmetry breaking, these particles become pseudo-Nambu-Goldstone bosons (PNGBs), with nearly flat potentials. The resulting PNGB potential in single-field models (in four spacetime dimensions) is generally of the form

$$V(\phi) = \Lambda^4[1 \pm \cos(N\phi/f)]. \tag{2}$$

We will take the positive sign in Eq. (2) (this choice has no effect on our results) and take $N = 1$, so the potential, of height $2\Lambda^4$, has a unique minimum at $\phi = \pi f$ (we assume the periodicity of ϕ is $2\pi f$).

We show below that, for appropriately chosen values of the mass scales, namely $f \sim m_{\rm Pl}$ and $\Lambda \sim m_{\rm GUT} \sim 10^{15}$ GeV, the PNGB field ϕ can drive inflation. This choice of parameters indeed produces the small ratio of scale required by Eq. (1), with $\chi \sim (\Lambda/f)^4 \sim 10^{-13}$.

We shall assume that inflation is initiated from a state that is at least approximately thermal. In general, this is a dangerous assumption, since there is no *a priori* reason to expect homogeneity or thermal equilibrium prior to inflation. However, this assumption is in keeping with the motivation of a PNGB potential arising from a phase transition associated with spontaneous symmetry breaking. In addition, such a homogeneous, thermal initial condition could naturally arise from an earlier period of inflation associated with the breaking of the global symmetry at the scale f. For temperatures $T \leq f$, the global symmetry is spontaneously broken, and the field ϕ describes the phase degree of freedom around the bottom of a Mexican hat. Since ϕ thermally decouples at a temperature $T \sim f^2/m_{\rm Pl} \sim f$, we assume it is initially laid down at random between 0 and $2\pi f$ in different causally connected regions. Within each Hubble volume, the evolution of the field is described by

$$\ddot{\phi} + 3H\dot{\phi} + \Gamma\dot{\phi} + V'(\phi) = 0, \tag{3}$$

where Γ is the decay width of the inflaton. In the temperature range $\Lambda \leq T \leq f$, the potential $V(\phi)$ is dynamically irrelevant, because the forcing term $V'(\phi)$ is negligible compared to the Hubble-damping term. (In addition, for axion models, $\Lambda \rightarrow 0$ as $T/\Lambda \rightarrow \infty$ due to the high-temperature suppression of instantons [12].) Thus, in this temperature range, aside from the smoothing of spatial gradients in ϕ, the field does not evolve. Finally, at $T \leq \Lambda$, in regions of the Universe with ϕ initially near the top of the potential, the field starts to roll slowly down the hill toward the minimum. In those regions, the energy density of the Universe is quickly dominated by the vacuum contribution $[V(\phi) \simeq 2\Lambda^4 \geq \rho_{\rm rad} \sim T^4]$, and the Universe expands exponentially.

To successfully solve the cosmological puzzles of the standard cosmology, an inflationary model must satisfy a variety of constraints. We describe these constraints in the following sections.

III. EVOLUTION OF THE INFLATON FIELD

In this section, we present results for the evolution of the scalar field driving natural inflation. First, we review the standard slow roll (SR) analysis, and then turn to the results of an exact calculation obtained by numerically solving the equations of motion. As our result we find that sufficient inflation takes place as long as

$$f > 0.06 m_{\rm Pl}. \tag{4}$$

In Sec. IV we will derive stronger bounds on f from constraints on the spectral index of density fluctuations. Throughout, we take $m_{\rm Pl} = 1.22 \times 10^{19}$ GeV. Hereafter, we take the onset of inflation to take place at a field value $0 < \phi_1/f < \pi$, and the end of inflation to be at a field value $0 < \phi_2/f < \pi$.

A. Standard slow roll analysis

A sufficient, but not necessary, condition for inflation is that the field be slowly rolling, i.e., its motion is overdamped, $\ddot{\phi} \ll 3H\dot{\phi}$. The SR condition implies that two conditions are met:

$$|V''(\phi)| < 9H^2, \qquad \text{i.e.,} \sqrt{\frac{2|\cos(\phi/f)|}{1+\cos(\phi/f)}} < \frac{\sqrt{48\pi} f}{m_{\rm Pl}}, \tag{5}$$

and

$$\left|\frac{V'(\phi) m_{\rm Pl}}{V(\phi)}\right| < \sqrt{48\pi}, \qquad \text{i.e.,} \frac{\sin(\phi/f)}{1+\cos(\phi/f)} < \frac{\sqrt{48\pi} f}{m_{\rm Pl}}. \tag{6}$$

From Eqs. (5) and (6), the existence of a broad SR regime requires $f \geq m_{\rm Pl}/\sqrt{48\pi}$ and ends when ϕ reaches a value ϕ_2, at which one of the inequalities (5) or (6) is violated. For example, for $f = m_{\rm Pl}$, $\phi_2/f = 2.98$ (near the minimum of the potential), while for $f = m_{\rm Pl}/\sqrt{24\pi}$, $\phi_2/f = 1.9$. Clearly, as f grows, ϕ_2/f approaches π. We note that the conditions (5) and (6) are approximate relations. A more precise calculation using the slow roll parameters ϵ and η gives similar bounds. Next we present exact numerical solutions of the equations of motion to substantiate our results.

B. Numerical evolution of the scalar field

In [14], we obtained exact numerical solutions to the equations of motion for the inflaton in the natural inflation model. We briefly recapitulate results from a numerical evolution of the scalar field found in Sec. III of [14], which provides more precise results than the simple SR analysis.

We find that the exact solution roughly reproduces the results of the SR analysis presented previously. As long as $f > 0.1 m_{\rm Pl}$, the results agree to within 10%. In particular, the numerical results for the maximum field value at the start of inflation, $\phi_1^{\rm max}$, are nearly identical to the SR estimates for values of f near $m_{\rm Pl}$; they differ by ~10% for $f = m_{\rm Pl}$ and deviate significantly as f approaches $m_{\rm Pl}/\sqrt{24\pi}$ from above. Further details can be found in [14].

C. Sufficient inflation

The expansion $H = \dot{a}/a$ of the Universe is determined by the Friedmann equation,

$$H^2 = \frac{8\pi}{3m_{\rm Pl}^2}\left[V(\phi) + \frac{1}{2}\dot{\phi}^2\right]. \tag{7}$$

Inflationary expansion takes place when the potential V dominates in the energy density. To solve the flatness and horizon problems, we demand that the scale factor of the Universe inflates by at least 60 e-foldings during the SR regime,

$$\begin{aligned} N_e(\phi_1, \phi_2, f) &\equiv \ln(R_2/R_1) \\ &= \int_{t_1}^{t_2} H\,dt \\ &= \frac{-8\pi}{m_{\rm Pl}^2}\int_{\phi_1}^{\phi_2} \frac{V(\phi)}{V'(\phi)} d\phi \\ &= \frac{16\pi f^2}{m_{\rm Pl}^2} \ln\left[\frac{\sin(\phi_2/2f)}{\sin(\phi_1/2f)}\right] \geq 60. \end{aligned} \tag{8}$$

Using Eqs. (5) and (6) to determine ϕ_2 as a function of f, the constraint (8) determines the maximum value ($\phi_1^{\rm max}$) of ϕ_1 consistent with sufficient inflation. The fraction of the Universe with $\phi_1 \in [0, \phi_1^{\rm max}]$ will inflate sufficiently. The requirement that a reasonable fraction of the Universe inflate sufficiently places a bound on f.

There are two conceptually different approaches to the question of what fraction of the Universe inflates sufficiently, and hence to the bound on the scale f. The first is an "*a priori* probability." In this (more restrictive) approach, one determines the fraction of the volume of the Universe *before* inflation which will inflate sufficiently, and requires this fraction to be reasonably large. If we assume that ϕ_1 is randomly distributed between 0 and πf from one horizon volume to another, the probability of being in a region of the Universe that inflates enough is $\phi_1^{\rm max}/\pi f$. For example, for $f = 3m_{\rm Pl}$, $m_{\rm Pl}$, $m_{\rm Pl}/2$, and $m_{\rm Pl}/\sqrt{24\pi}$, the probability is 0.7, 0.2, 3×10^{-3}, and 3×10^{-41}. The fraction of the Universe that inflates sufficiently drops precipitously with decreasing f, and hence restricts f to be near $m_{\rm Pl}$. However, this approach is unnecessarily restrictive.

The second approach, namely, "*a posteriori* probability," is more sensible. Here one examines the Universe after inflation has taken place, and ascertains what fraction of the *final* volume of the Universe has inflated sufficiently to look like our own. After inflation, those initial Hubble volumes of the Universe that did inflate end up occupying a *much* larger volume than those that did not. This second approach is much less restrictive and allows a lower value of f, as shown below. We note that neither of these approaches addresses the broader (and unsolved) issue of how to rigorously define a measure on the space of initial conditions for inflation, since we are implicitly assuming homogeneity and thermal equilibrium. However, these arguments do serve to establish the plausibility and naturalness of the model.

D. *A Posteriori* Probability of Sufficient Inflation

We now calculate the *a posteriori* probability of sufficient inflation. We consider the Universe at the end of inflation, and calculate the fraction P of the volume of the Universe at that time which had inflated by at least 60 e-foldings:

$$P = 1 - \frac{\int_{\phi_1^{\max}}^{\pi f} d\phi_1 \exp[3N(\phi_1)]}{\int_{H/2\pi}^{\pi f} d\phi_1 \exp[3N(\phi_1)]}. \tag{9}$$

Here, the lower limit of integration in the denominator is the limit of validity of the semiclassical treatment of the scalar field; the initial value of ϕ must exceed its quantum fluctuations, $\phi_1 \geq \Delta\phi = H/2\pi$. This fraction P is then the *a posteriori* probability of sufficient inflation.

Our basic result [14] is that the *a posteriori* probability for inflation P is essentially unity for f larger than the critical value $f_c \simeq 0.06 m_{\rm Pl}$. As f drops below this value, the probability given by Eq. (9) rapidly approaches 0. Hence, the requirement that a significant fraction of the Universe inflate sufficiently places a lower bound on the scale

$$f > f_c \simeq 0.06 m_{\rm Pl}. \tag{10}$$

We have explicitly calculated the evolution of the scalar field in natural inflation and found that the claim of [16] that $f \gg m_{\rm Pl}$ is unnecessarily restrictive. The correct bound due to sufficient inflation is given by Eq. (10).

IV. DENSITY FLUCTUATIONS

The amplitude and spectrum of density fluctuations produced in the natural inflation model can be compared with microwave background data in order to constrain the height and width of the potential. Here we find the constraint on the potential due to comparison with WMAP data.

A. Density fluctuation amplitude

Quantum fluctuations of the inflaton field as it rolls down its potential generate adiabatic density perturbations that may lay the groundwork for large-scale structure and leave their imprint on the microwave background anisotropy [21–23]. In this context, a convenient measure of the perturbation amplitude is given by the gauge-invariant variable ζ, first studied in [24]. We follow [25] in defining the power in ζ,

$$P_\zeta^{1/2}(k) = \frac{15}{2}\left(\frac{\delta\rho}{\rho}\right)_{\rm HOR} = \frac{3}{2\pi}\frac{H^2}{\dot\phi}. \tag{11}$$

Here, $(\delta\rho/\rho)_{\rm HOR}$ denotes the perturbation amplitude (in uniform Hubble constant gauge) when a given wavelength enters the Hubble radius in the radiation- or matter-dominated era, and the last expression is to be evaluated when the same comoving wavelength crosses outside the Hubble radius during inflation. For scale-invariant perturbations, the amplitude at Hubble-radius crossing is independent of perturbation wavelength. Normalizing to the COBE [26] or WMAP [2] Cosmic Microwave Background (CMB) anisotropy measurements gives $P_\zeta^{1/2}(k) \sim 10^{-5}$. We can use this normalization to get an approximate fix on the scale Λ. Using the analytic estimates of Sec. III A, the largest amplitude perturbations on observable scales are produced 60 e-foldings before the end of inflation, where $\phi = \phi_1^{\max}$, and have amplitude

$$P_\zeta^{1/2} \simeq \frac{\Lambda^2 f}{m_{\rm Pl}^3}\frac{9}{2\pi}\left(\frac{8\pi}{3}\right)^{3/2}\frac{[1+\cos(\phi_1^{\max}/f)]^{3/2}}{\sin(\phi_1^{\max}/f)}. \tag{12}$$

We can obtain an analytic estimate of Λ as a function of f when $f \leq (3/4)m_{\rm Pl}$; in this case, it is a good approximation to take $\phi_1^{\max}/\pi f \ll 1$. As a result, in Eq. (12), we have approximately

$$P_\zeta^{1/2} \approx \frac{1.4\Lambda^2 f}{M_{\rm Pl}^3}\left(\frac{16\pi}{3}\right)^{3/2}\left(\frac{f}{\phi_1^{\max}}\right). \tag{13}$$

Now the last term in this expression is obtained by using Eq. (8) with $N(\phi_1^{\max}, \phi_2, f) = 60$:

$$\frac{\phi_1^{\max}}{f} \simeq 2\sin\left(\frac{\phi_2}{2f}\right)\exp\left[-\frac{15 m_{\rm Pl}^2}{4\pi f^2}\right]. \tag{14}$$

Applying the CMB normalization constraint to Eq. (12) gives $\Lambda \sim 10^{15}$–10^{16} GeV for $f \sim m_{\rm Pl}$. Thus, to generate the fluctuations responsible for large-scale structure, Λ should be comparable to the GUT scale, and the inflaton mass $m_\phi = \Lambda^2/f \sim (10^{11}$–$10^{13})$ GeV. We note that this is strictly only an upper bound on the scale Λ, since the perturbations responsible for large-scale structure could be formed by some other (noninflationary) mechanism.

B. Density fluctuation spectrum

Using the approximations above, we can investigate the wavelength dependence of the perturbation amplitude at Hubble-radius crossing and, in particular, study how it deviates from scale invariance (usually associated with inflation).

Let k denote the comoving wave number of a fluctuation. The comoving length scale of the fluctuation, k^{-1}, crosses outside the comoving Hubble radius $[Ha]^{-1}$ during inflation at the time when the rolling scalar field has the value ϕ_k. This occurs $N_I(k) \equiv N(\phi_k, \phi_2, f)$ e-folds before the end of inflation, where $N(\phi_k, \phi_2, f)$ is given by Eq. (8) with ϕ_1 replaced by ϕ_k. The corresponding comoving length scale (expressed in current units) is

$$k^{-1} \simeq (3000h^{-1}\ \mathrm{Mpc}) \exp[N_I(k) - 60], \qquad (15)$$

where the horizon size today is $\simeq 3000h^{-1}$ Mpc. For scales of physical interest for large-scale structure, $N_I(k) \geq 50$; for $f \leq (3/4)m_{\mathrm{Pl}}$, these scales satisfy $\phi_k/f \ll 1$. In this limit, comparing two different field values ϕ_{k_1} and ϕ_{k_2}, from Eq. (8) we have

$$\phi_{k_2} \simeq \phi_{k_1} \exp\left(-\frac{\Delta N_I m_{\mathrm{Pl}}^2}{16\pi f^2}\right), \qquad (16)$$

where $\Delta N_I = N_I(k_2) - N_I(k_1)$. Thus, using Eqs. (12) and (13), we can compare the perturbation amplitude at the two field values,

$$\frac{(P_\zeta^{1/2})_{k_1}}{(P_\zeta^{1/2})_{k_2}} \simeq \frac{\phi_{k_2}}{\phi_{k_1}} \simeq \exp\left(-\frac{\Delta N_I m_{\mathrm{Pl}}^2}{16\pi f^2}\right). \qquad (17)$$

Now, from Eq. (15), we have the relation $\Delta N_I = \ln(k_1/k_2)$ [here we have approximated $H_{k_1} \simeq H_{k_2}$; more precisely, $\Delta N_I = \ln(k_1 H_{k_2}/k_2 H_{k_1})$]. Substituting this relation into (17), we find how the perturbation amplitude at Hubble-radius crossing scales with comoving wavelength,

$$\left(\frac{\delta\rho}{\rho}\right)_{\mathrm{HOR},k} \sim (P_\zeta^{1/2})_k \sim k^{-m_{\mathrm{Pl}}^2/16\pi f^2}. \qquad (18)$$

By comparison, for a scale-invariant spectrum, the Hubble-radius amplitude would be independent of the perturbation length scale k^{-1}; the positive exponent in Eq. (18) indicates that the PNGB models with $f \leq m_{\mathrm{Pl}}$ have more relative power on large scales than do scale-invariant fluctuations.

It is useful to transcribe this result in terms of the power spectrum of the primordial perturbations at fixed time (rather than at Hubble-radius crossing). Defining the Fourier transform δ_k of the density field, from Eq. (18) the power spectrum is a power law in the wave number k, $|\delta_k|^2 \sim k^{n_s}$, where the index n_s is given by

$$n_s = 1 - \frac{m_{\mathrm{Pl}}^2}{8\pi f^2} \qquad (f \leq 3m_{\mathrm{Pl}}/4). \qquad (19)$$

For comparison, the scale-invariant Harrison-Zel'dovich-Peebles-Yu spectrum corresponds to $n_s = 1$. For values of f close to m_{Pl}, the spectrum is close to scale invariant, as expected; however, as f decreases, the spectrum deviates significantly from scale invariance—e.g., for $f = m_{\mathrm{Pl}}/\sqrt{8\pi} = 0.2m_{\mathrm{Pl}}$, the perturbations have a white noise spectrum, $n_s = 0$.

Recently, WMAP has placed bounds on the spectrum of density fluctuations. If we assume that inflationary perturbations are indeed responsible for what is being seen in the WMAP data, then these spectral bounds can be translated into bounds on the parameter f in the potential. The precise formulation of the WMAP results depends on the choice of priors. Here we take the bound on the deviation of the spectrum from scale invariant from WMAP as found by [27,28]:

$$|n_s - 1| < 0.1. \qquad (20)$$

Applying this bound to Eq. (19), we see that a strong lower bound on the scale f results:

$$f \geq 0.6m_{\mathrm{Pl}}. \qquad (21)$$

This is the strongest bound on the scale f.

V. TENSOR MODES

In addition to density fluctuation, inflation also predicts the generation of tensor (gravitational wave) fluctuations with amplitude

$$P_{\mathrm{T}}^{1/2} = \frac{H}{2\pi}. \qquad (22)$$

In this section we study these tensor modes and discuss their detectability in upcoming microwave background experiments. We also examine the possible running of the scalar index and find that it is so small as to be observationally inaccessible.

For comparison with observation, the tensor amplitude is conventionally expressed in terms of the tensor/scalar ratio r, defined as[2]

$$r \equiv \frac{P_{\mathrm{T}}^{1/2}}{P_\zeta^{1/2}} = 16\epsilon, \qquad (23)$$

where ϵ is the first slow roll parameter evaluated when the fluctuation mode crosses the horizon, $\phi = \phi_1^{\max}$:

$$\begin{aligned}\epsilon &= \frac{m_{\mathrm{Pl}}^2}{16\pi^2}\left(\frac{V'(\phi_1^{\max})}{V(\phi_1^{\max})}\right)^2\\ &= \frac{1}{16\pi^2}\left(\frac{m_{\mathrm{Pl}}}{f}\right)^2\left[\frac{\sin(\phi_1^{\max}/f)}{1+\cos(\phi_1^{\max}/f)}\right]^2\\ &\simeq \frac{1}{32\pi^2}\left(\frac{m_{\mathrm{Pl}}}{f}\right)^2\left(\frac{\phi_1^{\max}}{f}\right)^2, \qquad \phi \ll f. \end{aligned} \qquad (24)$$

In principle there are four parameters describing the scalar and tensor fluctuations: the amplitudes and spectra of both components. The amplitude of the scalar perturbations is normalized by the height of the potential (the energy density Λ^4). The tensor spectral index n_T is not an independent parameter since it is related to the tensor/scalar ratio by the inflationary consistency condition $r = -8n_{\mathrm{T}}$. The remaining free parameters are the spectral index n of the scalar density fluctuations, and the tensor amplitude (given by r).

Hence, a useful parameter space for plotting the model predictions versus observational constraints is on the

[2]Normalization of this parameter varies in the literature. We use the convention of Peiris *et al.* [29]

(r, n) plane [30,31]. Natural inflation generically predicts a tensor amplitude well below the detection sensitivity of current measurements such as WMAP. However, the situation will improve markedly in future experiments with greater sensitivity such as the Planck satellite, which will start taking data in 2007, and proposed experiments such as CMBPOL.

Figure 1 shows the predictions of natural inflation for various choices of the number of e-folds N_I and the mass scale f, together with a variety of observational constraints. Fluctuations on observable scales (up to the scale of the current horizon size) are expected to lie roughly in the range $N_I = 50$–60, depending on the reheat temperature (although the relevant range also depends on the subsequent evolution of the Universe [32]). In general, a lower value of f results in a "redder" (smaller n) spectrum and a smaller tensor fluctuation amplitude. The current observational constraint from WMAP is given by the shaded (green) region on the left-hand side of the plot: The white region is still allowed by WMAP. We have also forecast error bars for the PLANCK satellite based on a Fisher matrix analysis (see Ref. [30] for details of the calculation). Roughly, the PLANCK satellite is expected to have 1σ error bars $\sim \pm 0.05$ on the magnitude of r, and

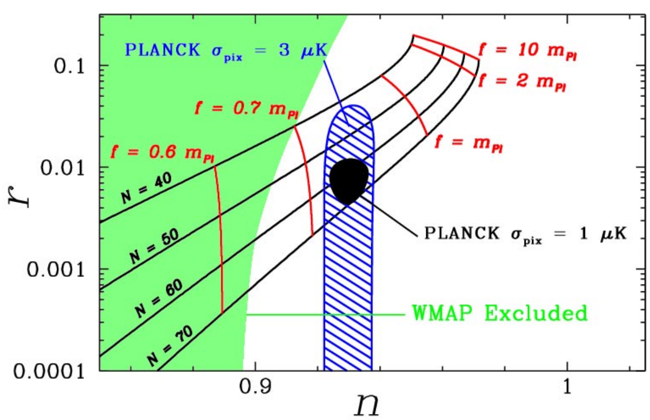

FIG. 1 (color online). The predictions of natural inflation compared with current and projected observational constraints, plotted on the (r, n_s) plane, where r is the tensor/scalar ratio and n_s is the spectral index of scalar fluctuations. The lines show the predictions of natural inflation for varying choices of the mass scale f and the number of e-folds N_I. Length scales of the order of the current horizon size correspond to $N_I \simeq 60$ for high reheat temperature. In general, a lower value of f results in a redder (smaller n_s) spectrum and a smaller tensor fluctuation amplitude. The shaded region at the left of the plot (green) is excluded to 2σ by WMAP [27]. The hatched (blue) error ellipse is the 2σ sensitivity expected for the Planck satellite. The central value is arbitrary: Only the size of the error bar is significant. The solid (black) error ellipse is the corresponding result for a hypothetical experiment with the same angular resolution as Planck but with a factor of 3 better temperature sensitivity. Such a measurement would be capable of detecting the gravitational wave fluctuations from natural inflation.

1σ errors bars $\sim \pm 0.01$ on n. The hatched (blue) region indicates the 2σ sensitivity of the PLANCK satellite if the central value is (arbitrarily) chosen to be $r \sim 0.01$. The central value is arbitrary; only the size of the error bars is significant. Similarly, we have also forecast error bars for a hypothetical experimental measurement with the same angular resolution as Planck, but with sensitivity improved by a factor of 3; the solid (black) error ellipse is the corresponding result (the 1σ errors on r here are roughly $\pm 5 \times 10^{-3}$). Hence, PLANCK should be able to detect the tensor signal from natural inflation if $f > 1.5 m_{\rm Pl}$. The next generation of experiments should be able to do so for $f > 0.7 m_{\rm Pl}$, the region allowed by WMAP data.

One property of the potential to note is that the spectral index is very weakly dependent on N_I for $f < m_{\rm Pl}$, indicating that the "running" of the spectral index $dn/d\ln k$ is negligible. A more careful calculation indicates that the running of the spectral index is less than 10^{-3} for all parameter regions considered here, and therefore for all practical purposes unobservable. This provides a powerful means of falsifying natural inflation. In particular, if indications of a strong negative running of the spectral index [33] from small-scale CMB observations such as CBI [34], ACBAR [35], and VSA [36] are borne out, this will kill the model, at least in its simplest single-field form of Eq. (2).

VI. DISCUSSION

In conclusion, natural inflation is alive and well. Recent WMAP data constrain the width of the potential to be $f > 0.6 m_{\rm Pl}$, and our predictions show that upcoming CMB observations such as the PLANCK satellite may be able to see the tensor modes.

In this section, we discuss the pros and cons of $f \sim m_{\rm Pl}$, as well as comment on some of the literature of PNGB models using shift symmetries.

Although it is *not* true that the original model of natural inflation requires $f \gg m_{\rm Pl}$ for the width of the potential, it does require f to be of order $m_{\rm Pl} \sim 10^{19}$ GeV. In fact, virtually all 4D inflationary models require $f \sim m_{\rm Pl}$ and the height of the potential $\sim m_{\rm GUT}$, in order to satisfy the simultaneous requirements of sufficient inflation and the right amplitude of density perturbations; this fact is emphasized by the conclusions of Ref. [6]. However, the height of the potential is generically of the order of the GUT scale, far enough below the Planck scale that we can safely ignore quantum gravitational effects on the background evolution. However, energy density is not the only issue.

In [37,38], it was argued that Planck-scale physics results in the violation of all global symmetries, including the Peccei-Quinn symmetry of the axion and the underlying symmetry from which we derive the PNGB inflaton. Wormholes are suggested as one mechanism for this violation, and black holes another (as a consequence of black hole no-hair theorems, the global charge of a black hole is not defined). The authors argue that, as a consequence, one is required to add all higher dimension operators (suppressed by powers of $m_{\rm Pl}$) consistent with the symmetries of the full theory, which then does not respect global symmetries. One should include terms of the form

$$\mathcal{L} = \frac{1}{2}m^2\phi^2 + \lambda\phi^4 + \sum_{n=6}^{\infty}\lambda_n\left(\frac{\phi^n}{m_{\rm Pl}^{n-4}}\right). \qquad (27)$$

Without a complete theory of quantum gravity, the validity of these arguments is not clear. If true, then the idea of using a PNGB directly as the inflaton would fail; however, the axion also could not exist and we would have no theory at all to escape the strong CP problem in QCD.

In [39], Lyth discussed the failure of effective field theory if the width of an inflationary potential approaches $m_{\rm Pl}$. Again, if inflation is to be formulated as an effective low-energy field theory, he argues that we expect additional nonrenormalizable operators in the Lagrangian to be suppressed by inverse powers of $m_{\rm Pl}$ as above. Then if observational constraints require the field to travel a distance $\Delta\phi \sim m_{\rm Pl}$, the effective field theory will begin to break down due to radiative corrections from the nonrenormalizable operators. Such a theory rapidly becomes inconsistent as $\Delta\phi \gg m_{\rm Pl}$. Motivated by the desire to evade these issues, in 1995 Kinney and Mahantappa [40] constructed natural inflation models in which symmetries suppress the mass terms and the potential is of the form $V \simeq 1 - \sin^4(\phi/f) \sim 1 - \phi^4$. Then one automatically obtains an effective width of the potential $f \ll m_{\rm Pl}$ [41]. One does so, however, at the expense of an unobservably small tensor component in the CMB.

Natural inflation has been implemented in the context of extra dimensions with varying degrees of success. Recently, Arkani-Hamed *et al.* [16] examined natural inflation in the context of extra dimensions, and also found models for which the mass terms were suppressed by a symmetry with $f \ll m_{\rm Pl}$, similar to the work of [40]. Focusing on a Wilson line in a fifth dimension, Arkani-Hamed *et al.* alternatively suggested that one might obtain large $f \gg m_{\rm Pl}$ if the inflaton is the extra component of a gauge-field propagating in the bulk.

However, Banks *et al.* examined the general question of whether it is ever possible to obtain large values of $f \gg m_{\rm Pl}$ in string theory [42]. While their study was not exhaustive, it strongly suggests that it is not possible. Generically there is a T-dual description in which small radii become large and the value of f is small ($f \ll m_{\rm Pl}$); hence the model of Arkani-Hamed *et al.* does not succeed in providing large f inflation. In addition, in a variety of regions in moduli space Banks *et al.* find that there are low action instantons which give rise to rapidly varying contributions to the potential that effectively rescale the value of f to the Planck scale. Although they do not provide an exhaustive proof, Banks *et al.* suggest the very strong statement that natural inflation cannot work in the context of string theory for $f \gg m_{\rm Pl}$. However, natural inflation with $f \sim m_{\rm Pl}$, as discussed throughout this paper, is fine.

Natural inflation has been implemented in brane-world scenarios as well. Shift symmetries have been studied in brane inflation by Firouzjahi and Tye [18] and in the work of Hsu and Kallosh [19]. The four-dimensional effective field theory description of some brane-world scenarios is likely to be described by the physics in this paper.

The shift symmetries of natural inflation have also been used in multiple field models. Freese [20] suggested using a PNGB as the rolling field in double field inflation [5] (in which the inflaton is a tunneling field whose nucleation rate is controlled by its coupling to a rolling field). Kaplan and Weiner have examined natural inflationlike models in the context of "little" fields [17].

Kawasaki *et al.* proposed a supergravity inflation model in which the inflaton potential is flat due to a shift symmetry, again utilizing the basic idea of natural inflation. Choi *et al.* have discussed thermal inflation in the context of the radiatively generated axion scale in supersymmetric axion models [43].

While arguments based on theoretical prejudice are useful guidelines for model building, the ultimate test is an observational one. It is a remarkable coincidence that the borderline between consistent and inconsistent effective field theory models for inflation is roughly the same as the borderline between whether or not tensor modes are, in a practical sense, observable [44]. In order for the tensor/scalar ratio to be large enough to be measured by foreseeable future experiments, the width of the potential must be of order $f \sim m_{\rm Pl}$ or larger. Natural inflation is still very much in the running as a candidate model for the early Universe. These models are also especially attractive from a particle physics perspective because they possess a hierarchy of scales which is stable against radiative corrections. Such a hierarchy is necessary for the suppression of density perturbations, but, in other models, is typically left to a fine-tuning of the inflaton self-coupling to order $\lambda \sim 10^{-14}$. Finally, the simplest models of natural inflation predict a large enough gravitational wave component that a detection by advanced CMB measurements will be possible.

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