Relativistic gravitation theory for the modified Newtonian dynamics paradigm

Jacob D. Bekenstein*,†
Racah Institute of Physics, Hebrew University of Jerusalem, Givat Ram, Jerusalem 91904 Israel
(Received 29 March 2004; published 13 October 2004)

The modified Newtonian dynamics (MOND) paradigm of Milgrom can boast of a number of successful predictions regarding galactic dynamics; these are made without the assumption that dark matter plays a significant role. MOND requires gravitation to depart from Newtonian theory in the extragalactic regime where dynamical accelerations are small. So far relativistic gravitation theories proposed to underpin MOND have either clashed with the post-Newtonian tests of general relativity, or failed to provide significant gravitational lensing, or violated hallowed principles by exhibiting superluminal scalar waves or an *a priori* vector field. We develop a relativistic MOND inspired theory which resolves these problems. In it gravitation is mediated by metric, a scalar, and a 4-vector field, all three dynamical. For a simple choice of its free function, the theory has a Newtonian limit for nonrelativistic dynamics with significant acceleration, but a MOND limit when accelerations are small. We calculate the β and γ parameterized post-Newtonian coefficients showing them to agree with solar system measurements. The gravitational light deflection by nonrelativistic systems is governed by the same potential responsible for dynamics of particles. To the extent that MOND successfully describes dynamics of a system, the new theory's predictions for lensing by that system's visible matter will agree as well with observations as general relativity's predictions made with a dynamically successful dark halo model. Cosmological models based on the theory are quite similar to those based on general relativity; they predict slow evolution of the scalar field. For a range of initial conditions, this last result makes it easy to rule out superluminal propagation of metric, scalar, and vector waves.

DOI: 10.1103/PhysRevD.70.083509 PACS numbers: 95.35.+d, 04.80.Cc, 95.30.Sf, 98.62.Sb

I. INTRODUCTION

In the extragalactic regime, where Newtonian gravitational theory would have been expected to be an excellent description, accelerations of stars and gas, as estimated from Doppler velocities and geometric considerations, are as a rule much larger than those due to the Newtonian field generated by the visible matter in the system [1,2]. This is the "missing mass" problem [3] or "acceleration discrepancy" [4]. It is fashionable to infer from it the existence of much dark matter in systems ranging from dwarf spheroidal galaxies with masses $\sim 10^6\ M_\odot$ to great clusters of galaxies in the $10^{13}\ M_\odot$ regime [3,5]. And again, galaxies and clusters of galaxies are found to gravitationally lense background sources. When interpreted within general relativity (GR), this lensing is anomalously large unless one assumes the presence of dark matter in quantities and with distribution similar to those required to explain the accelerations of stars and gas. Thus extragalactic lensing has naturally been regarded as confirming the presence of the dark matter suggested by the dynamics.

But the putative dark matter has never been identified despite much experimental and observational effort [6]. This raises the possibility that the acceleration discrepancy as well as the gravitational lensing anomaly may reflect departures from Newtonian gravity and GR on galactic and larger scales. Now alternatives to GR are traditionally required to possess a Newtonian limit for small velocities and potentials; thus the acceleration discrepancy also raises the possibility that the correct relativistic gravitational theory may be of a kind not generally considered hitherto.

In the last two decades, Milgrom's modified Newtonian dynamics (MOND) paradigm [7–9] has gained recognition as a successful scheme for unifying much of extragalactic dynamics phenomenology without invoking "dark matter." In contrast with earlier suggested modifications of Newton's law of universal gravitation [10–13], MOND is characterized by an acceleration scale $\mathfrak{a}_0$, not a distance scale, and its departure from Newtonian predictions is acceleration dependent:

$$\tilde{\mu}(|\mathbf{a}|/\mathfrak{a}_0)\mathbf{a} = -\nabla\Phi_N. \tag{1}$$

Here Φ_N is the usual Newtonian potential of the visible matter, while $\tilde{\mu}(x) \approx x$ for $x \ll 1$ and $\tilde{\mu}(x) \to 1$ for $x \gg 1$. Milgrom estimated $\mathfrak{a}_0 \approx 1 \times 10^{-8}\ \mathrm{cm\,s^{-2}}$ from the empirical data. In the laboratory and the solar system where accelerations are strong compared to $\mathfrak{a}_0$, formula (1) goes over to the Newtonian law $\mathbf{a} = -\nabla\Phi_N$.

Milgrom constructed formula (1) to agree with the fact that rotation curves of disk galaxies become flat outside

*Electronic address: bekenste@vms.huji.ac.il
†URL: http://www.phys.huji.ac.il/~bekenste/

their central parts. That far out a galaxy of mass M exhibits an approximately spherical Newtonian potential. The scales are such that $|\nabla\Phi_N| \approx GMr^{-2} \ll \mathfrak{a}_0$ in this region, and so Eq. (1) with $\tilde{\mu}(x) \approx x$ gives $|\mathbf{a}| \approx (GM\mathfrak{a}_0)^{1/2}r^{-1}$ which has the r dependence appropriate for the centripetal acceleration v_c^2/r of a radius independent rotational velocity v_c—an asymptotically flat rotation curve. In fact one obtains the relation $M = (G\mathfrak{a}_0)^{-1}v_c^4$ which leads to the *prediction* that for any class of galaxies with a constant mass to luminosity ratio Υ in a specified spectral band, the luminosity in that band should scale as v_c^4. And indeed, there exists an empirical law of just this form: the Tully-Fisher law [14] (TFL) relating near infrared (H-band) luminosity L_H of a spiral disk galaxy to its rotation velocity $L_H \propto v_c^4$ with the proportionality factor being constant within each galactic morphology class.

This version of the TFL was established only after MOND was enunciated [15]. It is in harmony with the MOND prediction in two ways. First, the infrared light of a galaxy comes mostly from cool dwarf stars which make up most of its mass (hence giving a tight correlation between M of the predicted relation and L_H of the empirical law). Second, the proportionality coefficient varies from class to class as would be expected from the observed correlation between Υ of a galaxy and its morphology.

In the alternative dark matter paradigm (which casts no doubt on standard gravity theory), flat rotation curves are explained by assuming that every disk galaxy is nested inside a roundish spherical halo of dark matter [16] whose mass density drops approximately like r^{-2}. The halo is supposed to dominate the gravitational field in the outer parts of the galaxy. This makes the Newtonian potential approximately logarithmic with radius in those regions, thereby leading to an asymptotically flat rotation curve. In practice the dark halo resolution works only after fine tuning. It is an observational fact that for bright spiral galaxies the rotation curve in the optically bright region is well explained in Newtonian gravity by the observed matter [17]. But, as mentioned, in the outer regions the visible matter's contribution must be dwarfed by the halo's. So fine tuning is needed between the dark halo parameters (velocity dispersion and core radius) and the visible disk ones [18,19].

This fine tuning problem is exacerbated by the TFL $L_H \propto v_c^4$. Because the infrared luminosity comes from the visible matter in the galaxy, but the rotation velocity is mostly set by the halo, the TFL also requires fine tuning between halo and disk parameters. The standard dark matter explanation of the r^{-2} profile of a halo is that it arises naturally from primordial cosmological perturbations [20]. The visible galaxy is regarded as forming by dissipational collapse of gas into the potential well of the halo. The fine tuning mentioned is then viewed as resulting from the adjustment of the halo to the gravitation of the incipient disk [19,21]. But the TFL is observationally a very sharp correlation; in fact, it is the basis for one of the most reliable methods for gauging distances to spiral galaxies. Such sharpness is hardly to be expected from statistical processes of the kind envisaged in galaxy formation, a point emphasized by Sanders [22]. So in the dark matter picture the TFL is something of a mystery.

There are other MOND successes. Milgrom predicted early that in galaxies with surface mass density well below $\mathfrak{a}_0G^{-1}$, the acceleration discrepancy should be especially large [8]. In dwarf spirals this property was established empirically years later [23], and it is now known to be exhibited by a large number of low surface brightness galaxies [24]. Another example: MOND successfully predicts the detailed shape of a rotation curve from the observed matter (stars and gas) distribution on the basis of a single free parameter Υ down to correlating features in the velocity field with those seen in the light distribution [25–28]. This is especially true in the case of low surface mass density disk galaxies for which MOND's predictions are independent of the specific choice of $\tilde{\mu}(x)$ [29], and these MOND theoretical rotation curves fit the observed curves of a number of low surface brightness dwarf galaxies [27,30,31] very well. By contrast, the dark halo paradigm requires one or two free parameters apart from Υ to approximate the success of the MOND predictions [32]. In fact, even when the empirical data is analyzed within the dark halo paradigm, it displays the preferred acceleration scale $\mathfrak{a}_0$ of MOND [33].

Occasionally doubt has been cast on MOND's ability to describe clusters of galaxies properly [34]. Many of these exhibit accelerations not small on scale $\mathfrak{a}_0$, yet conventional analysis suggests they contain much dark matter in opposition to what MOND would suggest. Sanders has recently reanalyzed the problem [35] with the conclusion that these clusters may contain much as yet undiscovered baryonic matter in the core which should be classed as "visible" in connection with MOND. Other MOND successes, outside the province of disk galaxies, have been reviewed elsewhere [22,32,36].

So the simple MOND formula (1) is very successful. But it is not a theory. Literally taken, the MOND recipe for acceleration violates the conservation of momentum (and of energy and of angular momentum) [7]. And MOND entails a paradox: why does the center of mass of a star orbit in its galaxy with anomalously large acceleration given by Eq. (1) with $\tilde{\mu} \ll 1$, while each parcel of gas composing it is subject to such high acceleration that it should, by the same formula, be accelerated Newtonially [7]? In short, the MOND formula is not a consistent theoretical scheme. Neither is MOND, as initially stated, complete. For example, it does not specify how to calculate gravitational lensing by galaxies and clusters of galaxies. As is well known, in standard gravity theory light deflection is well described only by relativistic theory (GR). And whereas Newtonian cosmological models work well for part of the cosmological evolution, MOND cosmological models built in analogy with their Newtonian counterparts, though sometimes agreeing with phenomenology [34], can yield peculiar predictions [37] (but see Ref. [38]). In short, a complete, consistent theoretical underpinning of the MOND paradigm which accords with observed facts, and is also relativistic, has been lacking.

This lack is being resolved in measured steps. A first step was the Lagrangian reformulation of MOND [39] called AQUAL (aquadratic Lagrangian theory) (see Sec. II A). AQUAL cures the nonconservation problems and resolves the paradox of the galactic motion of an object whose parts accelerate strongly relative to one another; it does so in accordance with a conjecture of Milgrom [7]. And for systems with high symmetry, AQUAL reduces exactly to the MOND formula (1).

A relativistic generalization of AQUAL is easy to construct with help of a scalar field which together with the metric describes gravity [39] (see Sec. II C 1 below). It reduces to MOND approximately in the weak acceleration regime, to Newtonian gravity for strong accelerations, and can be made consistent with the post-Newtonian solar system tests for GR. But relativistic AQUAL is acausal: waves of the scalar field can propagate superluminally in the MOND regime (see the appendix of Ref. [39] or Appendix A here). The problem can be traced to the aquadratic kinetic part of the Lagrangian of the theory which mimics that in the original AQUAL. A theory involving a second scalar field, phase coupled gravity theory (PCG), was thus developed to bypass the problem [4,40,41] (see Sec. II C 2 below). PCG may be better behaved causally than relativistic AQUAL [42], but it brings woes of its own. It is marginally in conflict with the observed perihelion precession of Mercury [4], and in common with relativistic AQUAL, PCG predicts extragalactic gravitational lensing which is too weak if there is indeed no dark matter. This last problem is traceable to a feature common to PCG and relativistic AQUAL: the physical metric is conformal to the metric appearing in the Einstein-Hilbert action [43].

One way to sidestep this problem without discarding the MOND features is to exploit the direction defined by the gradient of the first scalar field to relate the physical metric to the Einstein metric by a disformal transformation (see Ref. [43] or Sec II C 3 below). But it turns out that with this relation the requirement of causal propagation acts to depress gravitational lensing [44], rather than enhancing it as is observationally required. The persistence of the lensing problem in modified gravitational theories has engendered a folk theorem to the effect that it is impossible for a relativistic theory to simultaneously incorporate the MOND dynamics, observed gravitational lensing and correct post-Newtonian behavior without calling on dark matter [45–48].

Needless to say, this theorem cannot be proved [49]. Indeed, by the simple device of relating the physical and Einstein metrics via a disformal transformation based on a *constant* time directed 4-vector, Sanders [50] has constructed an AQUAL-like "stratified" relativistic theory which gives the correct lensing while ostensibly retaining the MOND phenomenology and consistency with the post-Newtonian tests. Admittedly Sanders's stratified theory is a preferred frame theory and as such is outside the traditional framework for gravitational theories. But it does point out a trail to further progress.

The present paper introduces T*e*V*e*S, a new relativistic gravitational theory devoid of *a priori* fields, whose nonrelativistic weak acceleration limit accords with MOND while its nonrelativistic strong acceleration regime is Newtonian. T*e*V*e*S is based on a metric and dynamic scalar and 4-vector fields (one each); it naturally involves one free function, a length scale, and two positive dimensionless parameters, k and K. T*e*V*e*S passes the usual solar system tests of GR, predicts gravitational lensing in agreement with the observations (without requiring dark matter), does not exhibit superluminal propagation, and provides a specific formalism for constructing cosmological models.

In Sec. II we summarize the foundations on which a workable relativistic formulation of MOND must stand. We follow this with a brief critical review of relativistic AQUAL, PCG and disformal metric theories, some of whose elements we borrow. Sec. III A builds the action for T*e*V*e*S while Sec. III B derives the equations for the metric, scalar, and vector fields. In Sec. III C we demonstrate that T*e*V*e*S has a GR limit for a range of small k and K. This is shown explicitly for cosmology (Sec. III C 1) and for quasistatic situations like galaxies (Sec. III C 2). All the above applies for any choice of the free function; in Sec. III E we make a simple choice for it which facilitates further elaboration. For spherically symmetric systems the nonrelativistic MOND limit is derived in Sec. IV B, while the Newtonian limit is recovered for modestly small k in Sec. IV C. The above conclusions are extended to nonspherical systems in Sec. IV D. Sec. V shows that the theory passes the usual post-Newtonian solar system tests if the K parameter is chosen small. Sec. VI demonstrates that for given dynamics, T*e*V*e*S gives the same gravitational lensing as does a dynamically successful dark halo model within GR. In Sec. VII we discuss T*e*V*e*S cosmological models with flat spaces showing that they are very similar to the corresponding GR models (apart from the question of cosmological dark matter which is left open), and demonstrating that the scalar field evolves little and so can be taken to be small and positive. As discussed next in Sec. VIII, this last conclusion serves to rule out superluminal propagation in T*e*V*e*S.

II. THEORETICAL FOUNDATIONS FOR THE MOND PARADIGM

A. AQUAL: Nonrelativistic field reformulation of MOND

However successful empirically when describing motions of test particles, e.g., stars in the collective field of a galaxy, formula (1) is not fully correct. It is easily checked that a pair of particles accelerating one in the field of the other according to (1) does *not* conserve momentum. Thus the MOND formula by itself is not a theory. It is, however, a simple matter to construct a fully satisfactory *nonrelativistic* theory for MOND ([39]). Suppose we retain the Galilean and rotational invariance of the Lagrangian density which gives Poisson's equation, but drop the requirement of linearity of the equation. Then we come up with

$$\mathcal{L} = -\frac{\mathfrak{a}_0^2}{8\pi G} f\left(\frac{|\nabla\Phi|^2}{\mathfrak{a}_0^2}\right) - \rho\Phi. \tag{2}$$

Here ρ is the mass density, $\mathfrak{a}_0$ is a scale of acceleration introduced for dimensional consistency, and f is some function. Newtonian theory (Poisson's equation) corresponds to the choice $f(y) = y$. From Eq. (2) follows the gravitational field equation

$$\nabla \cdot [\tilde{\mu}(|\nabla\Phi|/\mathfrak{a}_0)\nabla\Phi] = 4\pi G\rho, \tag{3}$$

where $\tilde{\mu}(\sqrt{y}) \equiv df(y)/dy$. Because of its AQUAdratic Lagrangian, the theory has been called AQUAL [4]. The form of f and the value of $\mathfrak{a}_0$ must be supplied by phenomenology. We assume

$$f(y) \longrightarrow \begin{cases} y & y \gg 1; \\ \frac{2}{3}y^{3/2} & y \ll 1. \end{cases} \tag{4}$$

For systems with spherical, cylindrical or planar geometry, Eq. (3) can be integrated once immediately. With the usual prescription for the acceleration,

$$\mathbf{a} = -\nabla\Phi, \tag{5}$$

the solution corresponds precisely to the MOND formula (1). This is no longer true for lower symmetry. However, numerical integration reveals that Eq. (1) is approximately true, in most cases to respectable accuracy [51].

The mentioned inexactness of Eq. (1) for systems such as a discrete collection of particles is at the root of the mentioned violation of the conservation laws. Because AQUAL starts from a Lagrangian, it respects all the usual conservation laws (energy, momentum and angular momentum), as can be checked directly [39]. This supplies the appropriate perspective for the mentioned failings of MOND. AQUAL also supplies the tools for showing that Newtonian behavior of the constituents of a large body, e.g., a star, is consistent with non-Newtonian dynamics of the latter's center of mass in the weak collective field of a larger system, e.g., a galaxy.

To summarize, whenever parts of a system devoid of high symmetry move with accelerations weak on scale $\mathfrak{a}_0$, the field $\nabla\Phi$ which defines their accelerations is to be calculated by solving the AQUAL Eq. (3). AQUAL then becomes the nonrelativistic field theory on which to model the relativistic formulation of the MOND paradigm.

B. Principles for relativistic MOND

A relativistic MOND theory seems essential if gravitational lensing by extragalactic systems and cosmology are to be understood without reliance on dark matter. What principles should the relativistic embodiment of the MOND paradigm adhere to? The following list is culled from those suggested by Bekenstein [4,43], Sanders [52], and Romatka [53].

1. Principles

Action principle.— The theory must be derivable from an action principle. This is the only way known to guarantee that the necessary conservation laws of energy, linear, and angular momentum are incorporated automatically. It is simplest to take the action as an integral over a local Lagrangian density. A nonlocal action has been tried [47], but the resulting theory fails on account of gravitational lensing.

Relativistic invariance.—Innumerable elementary particle experiments provide direct evidence for the universal validity of special relativity. The action should thus be a relativistic scalar so that all equations of the theory are relativistically invariant. Implied in this is the correspondence of the theory with special relativity when gravitation is negligible. This proviso rules out preferred frame theories.

Equivalence principle.—As demonstrated with great accuracy (one part in 10^{12}) by the Eötvös-Dicke experiments [54], free particles with negligible self-gravity fall in a gravitational field along universal trajectories (weak equivalence principle). For slow motion (the case tested by the experiments), the equation $\mathbf{a} = -\nabla\Phi$, which encapsulates the universality, is equivalent to the geodesic equation in a (curved) metric $\tilde{g}_{\alpha\beta}$ with $\tilde{g}_{tt} \approx -1 - 2\Phi$. For light propagating in a static gravitational field, such a metric would predict that all frequencies as measured with respect to (w.r.t.) observers at rest in the field undergo a redshift measured by Φ. This is experimentally verified [55] to one part in 10^4. It thus appears that a curved metric $\tilde{g}_{\alpha\beta}$ describes those properties of spacetime in the presence of gravitation that are sensed by material objects. According to Schiff's conjecture [54,56], this implies that the theory must be a metric theory, i.e., that in order to account for the effects of gravitation, all *nongravitational* laws of physics, e.g., electromagnetism, weak interactions, etc., must be expressed in their usual laboratory forms but with the metric $\tilde{g}_{\alpha\beta}$ replacing the Lorentz metric. This is the Einstein equivalence principle [54].

Causality.—So as not to violate causality and thereby compromise the logical consistency of the theory, the equations deriving from the action should not permit superluminal propagation of any measurable field or of energy and linear and angular momenta. Superluminal here means exceeding the speed which is invariant under the Lorentz transformations. By Lorentz invariance of Maxwell's equations this is also the speed of light. In curved space, where curvature can cause waves to develop tails, the maximal speed is that of wave fronts, typically that of the high frequency components.

Positivity of energy.—Fields in the theory should never carry negative energy. From the quantum point of view this is a precaution against instability of the vacuum. This principle is usually taken to mean that the energy density of each field should be nonnegative at each event (local positivity). The fact that the gravitational field itself cannot be generically assigned an energy density shows that this popular conception is overly stringent. A more useful statement of positivity of energy is that any bounded system must have positive energy (global positivity instead of the stronger local positivity). For example, the gravitational field can carry negative energy density locally (at least in the Newtonian conception), yet for pure gravity and in some cases in the presence of matter, a complete gravitating system is subject to the positive energy theorems [57]. Also, there are examples of scalar fields whose local energy density is of indefinite sign, yet a complete stationary system of such fields with sources has positive mass [58]. Of course, local positivity implies global positivity.

Departures from Newtonian gravity.—The theory should exhibit a preferred scale of *acceleration* below which departures from Newtonian gravity should set in, even at low velocities.

2. Requirements

The relativistic embodiment of MOND should predict a number of well-established phenomena. For example, we expect the following:

Agreement with the extragalactic phenomenology.—The nonrelativistic limit of the theory should make predictions in agreement with those of the AQUAL equation, which is known to subsume much extragalactic phenomenology. This is checked for T*e*V*e*S in Sec. IV B.

Agreement with phenomenology of gravitational lenses.—The theory should predict correctly the lensing of electromagnetic radiation by extragalactic structures which is responsible for gravitational lenses and arcs. In particular, it should give predictions similar to those of GR within the dark matter paradigm. This point is established for T*e*V*e*S in Sec. VI.

Concordance with the solar system.—The theory should make predictions in agreement with the various solar system tests of relativity [54]: deflection of light rays, time delay of radar signals, precessions of the perihelia of the inner planets, the absence of the Nordtvedt effect in the lunar orbit, the nullness of aether drift, etc. T*e*V*e*S is confronted with the first three tests in Sec. V.

Concordance with binary pulsar tests.—The theory should make predictions in harmony with the observed pulse times of arrival from the various binary pulsars. These contain information about relativistic time delay, periastron precession, and the orbit's decay due to gravitational radiation. They thus constitute a test of the strong *potential* limit of the theory.

Harmony with cosmological facts.—The theory should give a picture of cosmology in harmony with basic empirical facts such as the Hubble expansion, its time scales for various eras, existence of the microwave background, light element abundances from primordial nucleosynthesis, etc., The similarity of cosmological evolution in GR and in T*e*V*e*S is established in Sec. VII, though the problem of how to eliminate cosmological dark matter with T*e*V*e*S is left open.

C. Some antecedent relativistic theories

It is now in order to briefly review *some* of the previous attempts to give a relativistic theory of MOND. This will introduce the concepts to be borrowed by T*e*V*e*S and help to establish the notation and conventions that we shall follow. A metric signature $+2$ and units with $c = 1$ are used throughout this paper. Greek indices run over four coordinates while Latin ones run over the spatial coordinates alone.

1. Relativistic AQUAL

It is well known that theories constructed, for example, by using a local function of the scalar curvature as Lagrangian density, have a purely Newtonian limit for weak potentials. So if we steer away from nonlocal actions, then AQUAL behavior cannot arise from merely modifying the gravitational action. The theory one seeks has to involve degrees of freedom other than the metric.

In the first relativistic theory with MOND aspirations, relativistic AQUAL [39], the physical metric $\tilde{g}_{\alpha\beta}$ was taken as conformal to a primitive (Einstein) metric $g_{\alpha\beta}$, i.e., $\tilde{g}_{\alpha\beta} = e^{2\psi} g_{\alpha\beta}$ with ψ a real scalar field. In order not to break violently with GR, which is well tested in the solar system (and to some extent in cosmology), the gravitational action was taken as the Einstein-Hilbert's one built out of $g_{\alpha\beta}$. The MOND phenomenology was implanted by taking for the Lagrangian density for ψ

$$\mathcal{L}_\psi = -\frac{1}{8\pi G L^2}\tilde{f}(L^2 g^{\alpha\beta}\psi_{,\alpha}\psi_{,\beta}), \qquad (6)$$

where $\tilde{f}$ is some function (not known *a priori*), and L is a constant with dimensions of length introduced for dimensional consistency. Note that when $\tilde{f}(y) = y$, $\mathcal{L}_\psi$ is just the Lagrangian density for a linear scalar field, but in general $\mathcal{L}_\psi$ is aquadratic.

To implement the universality of free fall, one must write all Lagrangians of matter fields using a single metric, which is taken as $\tilde{g}_{\alpha\beta}$ (not $g_{\alpha\beta}$ which choice would make the theory GR). Thus, for example, the action for a particle of mass m is taken as

$$S_m = -m\int e^{\psi}(-g_{\alpha\beta}dx^\alpha dx^\beta)^{1/2}. \qquad (7)$$

Hence test particle motion is nongeodesic w.r.t. $g_{\alpha\beta}$ but, of course, geodesic w.r.t. $\tilde{g}_{\alpha\beta}$. Evidently this last is the metric measured by clocks and rods, hence the physical metric. Addition of a constant to ψ merely multiplies all masses by a constant (irrelevant global redefinition of units), so that the theory is insensitive to the choice of zero of ψ.

For slow motion in a quasistatic situation with nearly flat metric $g_{\alpha\beta}$, and in a weak field ψ, $e^{\psi}(-g_{\alpha\beta}dx^\alpha dx^\beta)^{1/2} \approx (1 + \Phi_N + \psi - \mathbf{v}^2/2)dt$, here $\Phi_N = -(g_{tt} + 1)/2$ is the Newtonian potential determined by the mass density ρ through the linearized Einstein equations for $g_{\alpha\beta}$, and $\mathbf{v}$ is the velocity defined w.r.t. the Minkowski metric which is close to $g_{\alpha\beta}$. Thus the particle's Lagrangian is $m(\mathbf{v}^2/2 - \Phi_N - \psi)$; this leads to the equation of motion

$$\mathbf{a} \approx -\nabla(\Phi_N + \psi). \qquad (8)$$

How is ψ determined? For stationary weak fields the Lagrangian *density* for ψ, including a point source of physical mass M at $\mathbf{r} = 0$, is from the above discussion and Eqs. (6) and (7),

$$\mathcal{L}_\psi = -\frac{1}{8\pi G L^2}\tilde{f}[L^2(\nabla\psi)^2] - \psi M\delta(\mathbf{r}). \qquad (9)$$

Comparing Eqs. (9) and (2) we conclude that ψ here corresponds to Φ of mass M as computed from AQUAL's Eq. (3), provided we take $\tilde{f} = f$ and $L = 1/\mathfrak{a}_0$. Whenever $|\nabla\psi| \gg |\nabla\Phi_N$ (Φ_N is the Newtonian potential of the same mass distribution), the equation of motion (8) reduces to (5), and we obtain MOND-like dynamics. For the choice of MOND function (4) the said strong inequality is automatic in the deep MOND regime, $|\nabla\psi| \ll \mathfrak{a}_0$, because $\tilde{\mu} \ll 1$ there.

In the regime $|\nabla\psi| \gg \mathfrak{a}_0$, $\tilde{\mu} \approx 1$ and $f(y) \approx y$ so that ψ reduces to Φ_N. It would seem from Eq. (8) that a particle's acceleration is then twice the correct Newtonian value. However, this just means that the measurable Newton's constant G_N is twice the bare G appearing in $\mathcal{L}_\psi$ or in Einstein's equations. It is thus clear, regarding dynamics, that the relativistic AQUAL theory has the appropriate MOND and Newtonian limits depending on the strength of $\nabla\psi$.

But relativistic AQUAL has problems. Early on [4,39,42] it was realized that ψ waves can propagate faster than light. This acausal behavior can be traced to the aquadratic form of the Lagrangian, as explained in Appendix A. A second problem [43,53] issues from the conformal relation $\tilde{g}_{\alpha\beta} = e^{2\psi}g_{\alpha\beta}$. Light propagates on the null cones of the physical metric; by the conformal relation these coincide with the light cones of the Einstein metric. This last is calculated from Einstein's equations with the visible matter and ψ field as sources. Thus so long as the ψ field contributes comparatively little to the energy-momentum tensor, it cannot affect light deflection, which will thus be due to the visible matter alone. But in reality, galaxies and clusters of galaxies are observed to deflect light stronger than the visible mass in them would suggest. Thus relativistic AQUAL fails to accurately describe light deflection in situations in which GR requires dark matter. It is thus empirically falsified.

Relativistic AQUAL bequeaths to T*e*V*e*S the use of a scalar field to connect Einstein and physical metrics, a field which satisfies an equation reminiscent of the nonrelativistic AQUAL Eq. (3).

2. Phase coupling gravitation

The Phase Coupled Gravity (PCG) theory was proposed [4,40,42] in order to resolve relativistic AQUAL's acausality problem. It retains the two metrics related by $\tilde{g}_{\alpha\beta} = e^{2\psi}g_{\alpha\beta}$, but envisages ψ as one of a pair of mutually coupled real scalar fields with the Lagrangian density (our definitions here differ slightly from those in Ref. [4])

$$\mathcal{L}_{\psi,A} = -\frac{1}{2}[g^{\alpha\beta}(A_{,\alpha}A_{,\beta} + \eta^{-2}A^2\psi_{,\alpha}\psi_{,\beta}) + \mathcal{V}(A^2)] \qquad (10)$$

Here η is a real parameter and $\mathcal{V}$ a real valued function. The coupling between A and ψ is designed to bring about AQUAL-like features for small $|\eta|$. The theory receives its name because matter is coupled to ψ, which is proportional to the phase of the self-interacting complex field $\chi = Ae^{\imath\psi/\eta}$.

Variation of $\mathcal{L}_{\psi,A}$ w.r.t. A leads to (all covariant derivatives and index raising w.r.t. $g_{\alpha\beta}$)

$$A^{\cdot\alpha}{}_{;\alpha} - \eta^{-2}A\psi_{,\alpha}\psi^{,\alpha} - A\mathcal{V}'(A^2) = 0. \qquad (11)$$

In the variation w.r.t. ψ we must include the Lagrangian density of a source, say a point mass M at rest at $\mathbf{r} = 0$ [c.f. S_m in Eq. (7)]:

$$(A^2 g^{\alpha\beta}\psi_{,\beta})_{;\alpha} = \eta^2 e^{\psi}M\delta(\mathbf{r}). \qquad (12)$$

The connection with AQUAL is now clear. For sufficiently small $|\eta|$ the $A^{\cdot\alpha}{}_{;\alpha}$ term in Eq. (11) becomes negligible, and the other two establish an algebraic relation between $\psi_{,\alpha}\psi^{,\alpha}$ and A^2. Substituting this in Eq. (12) gives the AQUAL type of equation for ψ that would derive from $\mathcal{L}_\psi$ in Eq. (6).

The PCG Lagrangian's advantage over that of the relativistic AQUAL's is precisely that it involves first derivatives only in quadratic form. This would seem to rule out the superluminality generating X^α dependent terms discussed in Appendix A. In practice things are more complicated. A detailed local analysis employing the eikonal approximation [42] shows that there are superluminal ψ perturbations, for example, when $\mathcal{V}'' < 0$. However, the same analysis shows that such superluminality occurs only when the background solution is itself locally unstable. This makes the said causality violation moot.

One way to obtain MOND phenomenology from PCG is to choose $\mathcal{V}(A^2) = -\frac{1}{3}\varepsilon^{-2}A^6$ with ε a constant with dimension of energy. Although with this choice $\mathcal{V}'' < 0$ which makes for unstable backgrounds, we only need this form for small A; $\mathcal{V}$ can take different form for large argument. Then in a static situation with nearly flat $g_{\alpha\beta}$ and weak ψ, Eqs. (11) and (12) reduce to

$$\nabla^2 A - \eta^{-2}A(\nabla\psi)^2 + \varepsilon^{-2}A^5 = 0, \tag{13}$$

$$\nabla \cdot (A^2\nabla\psi) = \eta^2 M\delta(r). \tag{14}$$

The spherically symmetric solution of Eqs. (13) and (14) is

$$A = (\kappa\varepsilon/r)^{1/2}; \quad d\psi/dr = (\eta\varpi/4\kappa r), \tag{15}$$

$$\varpi \equiv (\eta M/\pi\varepsilon); \quad \kappa \equiv 2^{-3/2}\left(1 + \sqrt{1 + 4\varpi^2}\right)^{1/2}. \tag{16}$$

One may evidently still use Eq. (8):

$$a_r = -GM/r^2 - (\eta^2 M/4\pi\varepsilon\kappa r). \tag{17}$$

Thus a $1/r$ force competes with the Newtonian one. For small M it starts to dominate at a fixed radius scale r_c, just as in Tohline's [59] and Kuhn-Kruglyak's [60] non-Newtonian gravity theories. Here $r_c = 2\pi G\varepsilon/\eta^2$. By contrast for $M \gg M_c \equiv \frac{1}{2}\pi\varepsilon/\eta$, $\kappa \approx \frac{1}{2}\sqrt{\varpi}$ and the $1/r$ force scales as $M^{1/2}$ and begins to dominate when the Newtonian acceleration drops below the fixed acceleration scale

$$\mathfrak{a}_0 \equiv \eta^3/(4\pi G\varepsilon). \tag{18}$$

For $a_r \ll \mathfrak{a}_0$ the circular velocity whose centripetal acceleration balances the $1/r$ force is $v_c = (G\mathfrak{a}_0 M)^{1/4}$, precisely as in MOND. Thus $\mathfrak{a}_0$ here is to be identified with Milgrom's constant $\mathfrak{a}_0$. We conclude that, with a suitable choice of parameters, PCG with a sextic potential recovers the main features of MOND: asymptotically flat rotation curves and the TFL for disk galaxies. Specifically, the choice $\eta = 10^{-8}$ and $\varepsilon = 10^{53}$ erg gives $\mathfrak{a}_0 = 8.7 \times 10^{-9}$ cm s^{-2}, $M_c = 8.7 \times 10^6 M_\odot$ and $r_c = 5.2 \times 10^{19}$ cm. Now since r_c is larger than the Hubble scale, the Tohline-Kuhn-Kruglyak $1/r$ force is comparatively unimportant. Hence for $M \gg 10^7 M_\odot$ we should have MOND, and for low masses almost Newtonian behavior. This is about right: globular star clusters at $10^4 - 10^5 M_\odot$ show no missing mass problem.

However, the above parameters are bad from the point of view of the solar system tests of gravity, as summarized in Appendix B. But the gravest problem with PCG is that it, just as AQUAL, provides insufficient light deflection [43]. Here again, the conformal relation between Einstein and physical metric is to blame. T*e*V*e*S incorporates PCG's Lagrangian density (10) in the limit of small η in which A becomes nondynamical.

3. Theories with disformally related metrics

The light deflection problem can be solved only by giving up the relation $\tilde{g}_{\alpha\beta} = e^{-2\psi}g_{\alpha\beta}$. It was thus suggested [43] to replace this conformal relation by a disformal one, namely

$$\tilde{g}_{\alpha\beta} = e^{-2\psi}(\mathcal{A}g_{\alpha\beta} + \mathcal{B}L^2\psi_{,\alpha}\psi_{,\beta}), \tag{19}$$

with $\mathcal{A}$ and $\mathcal{B}$ functions of the invariant $g^{\mu\nu}\psi_{,\mu}\psi_{,\nu}$ and L a constant length unrelated, of course, to that in Eq. (6). This relation already allows ψ to deflect light via the $\psi_{,\alpha}\psi_{,\beta}$ term in the physical metric. However, it was found [44] that if one insists on causal propagation of both light and gravitational waves w.r.t. the light cones of the physical metric, then the sign required of $\mathcal{B}$ is opposite that required to enhance the light deflection coming from the metric $g_{\alpha\beta}$ alone. Thus the cited disformal relation between metrics, if respecting causality, will give weaker light deflection than would $g_{\alpha\beta}$ were it the physical metric.

This last observation of Ref. [44] has given rise to a folk belief that relativistic gravity theories which attempt to supplant dark matter's dynamical effects necessarily reduce light deflection rather than enhancing it [34,46–48]. However, as remarked by Sanders, the mentioned problem disappears if the term $\psi_{,\alpha}\psi_{,\beta}$ is replaced by $\mathfrak{U}_\alpha\mathfrak{U}_\beta$, where $\mathfrak{U}_\alpha$ is a constant 4-vector which, at least in the solar system and within galaxies, points in the time direction [50]. Specifically Sanders takes $\tilde{g}_{\alpha\beta} = e^{-2\psi}g_{\alpha\beta} - 2\mathfrak{U}_\alpha\mathfrak{U}_\beta\sinh(2\psi)$.

This stratified gravitation theory is reported to do well in the confrontation with the solar system tests and to possess the right properties to explain the coincidence between $\mathfrak{a}_0$ of MOND and the Hubble scale [7]. But its vector $\mathfrak{U}_\alpha$ is an *a priori* nondynamical element whose direction is selected in an unspecified way by the cosmological background. This means the theory is a preferred frame theory (although it is reported to be protected on this account against falsification in the solar system and other strong acceleration systems by its AQUAL behavior [50]). This is obviously a conceptual shortcoming which T*e*V*e*S removes, but the latter's debt to the stratified theory should be underlined.

III. FUNDAMENTALS OF T*e*V*e*S

A. Fields and actions

T*e*V*e*S is based on three dynamical gravitational fields: an Einstein metric $g_{\mu\nu}$ with a well defined inverse $g^{\mu\nu}$, a timelike 4-vector field $\mathfrak{U}_\mu$ such that

$$g^{\alpha\beta}\mathfrak{U}_\alpha\mathfrak{U}_\beta = -1, \tag{20}$$

and a scalar field ϕ; there is also a nondynamical scalar field σ (the acronym T*e*V*e*S recalls the theory's Tensor-Vector-Scalar content). The physical metric in T*e*V*e*S, just as in Sanders's stratified theory, is obtained by stretching the Einstein metric in the spacetime directions orthogonal to $\mathfrak{U}^\alpha \equiv g^{\alpha\beta}\mathfrak{U}_\beta$ by a factor $e^{-2\phi}$ while shrinking it by the same factor in the direction parallel to $\mathfrak{U}^\alpha$:

$$\tilde{g}_{\alpha\beta} = e^{-2\phi}(g_{\alpha\beta} + \mathfrak{U}_\alpha\mathfrak{U}_\beta) - e^{2\phi}\mathfrak{U}_\alpha\mathfrak{U}_\beta, \tag{21}$$

$$= e^{-2\phi}g_{\alpha\beta} - 2\mathfrak{U}_\alpha\mathfrak{U}_\beta \sinh(2\phi). \tag{22}$$

It is easy to verify that the inverse physical metric is

$$\tilde{g}^{\alpha\beta} = e^{2\phi}g^{\alpha\beta} + 2\mathfrak{U}^\alpha\mathfrak{U}^\beta \sinh(2\phi), \tag{23}$$

where $\mathfrak{U}^\alpha$ will *always* mean $g^{\alpha\beta}\mathfrak{U}_\beta$.

The geometric part of the action S_g is formed from the Ricci tensor $R_{\alpha\beta}$ of $g_{\mu\nu}$ just as in GR:

$$S_g = (16\pi G)^{-1}\int g^{\alpha\beta}R_{\alpha\beta}(-g)^{1/2}d^4x. \tag{24}$$

Here g means the determinant of metric $g_{\alpha\beta}$. This choice is made in order to keep T*e*V*e*S close to GR in some sense to be clarified below.

In terms of two constant positive parameters k and ℓ the action for the pair of scalar fields is taken to be of roughly PCG form,

$$S_s = -\frac{1}{2}\int\Big[\sigma^2 h^{\alpha\beta}\phi_{,\alpha}\phi_{,\beta} + \frac{1}{2}G\ell^{-2}\sigma^4 F(kG\sigma^2)\Big] \times (-g)^{1/2}d^4x, \tag{25}$$

where $h^{\alpha\beta} \equiv g^{\alpha\beta} - \mathfrak{U}^\alpha\mathfrak{U}^\beta$ and F is a free dimensionless function (it is related to PCG's potential $\mathcal{V}$). No overall coefficient is required for the kinetic term; were it included, it could be absorbed into a redefinition of σ and thereby in k and ℓ. Because ϕ is obviously dimensionless, the dimensions of σ^2 are those of G^{-1}. Thus k is a dimensionless constant (it could be absorbed into the definition of F, but we choose to exhibit it) while ℓ is a constant length.

Because no kinetic σ terms appear, the "equation of motion" of σ takes the form of an algebraic relation between it and the invariant $h^{\alpha\beta}\phi_{,\alpha}\phi_{,\beta}$, and when this is substituted for σ in S_s, the phenomenologically successful AQUAL type action for ϕ appears. We could, of course, have written this last action directly. The present route is more suggestive of the possible origin of the action; for example, S_s resembles the action for a complex self-interacting scalar field $\eta\sigma\exp(\imath\phi/\eta)$ in the limit of small η. The term $-\sigma^2\mathfrak{U}^\alpha\mathfrak{U}^\beta\phi_{,\alpha}\phi_{,\beta}$ here included in the scalar's action is new; its role is to eliminate superluminal propagation of the ϕ field, a recalcitrant problem in AQUAL-type theories.

The action of the vector $\mathfrak{U}^\alpha$ is taken to have the form

$$S_v = -\frac{K}{32\pi G}\int[g^{\alpha\beta}g^{\mu\nu}\mathfrak{U}_{[\alpha,\mu]}\mathfrak{U}_{[\beta,\nu]} - 2(\lambda/K) \times (g^{\mu\nu}\mathfrak{U}_\mu\mathfrak{U}_\nu + 1)](-g)^{1/2}d^4x, \tag{26}$$

where antisymmetrization in a pair of indices is indicated by surrounding them by square brackets, e.g. $A_{[\mu}B_{\nu]} = A_\mu B_\nu - A_\nu B_\mu$. In Eq. (26) λ is a spacetime dependent Lagrange multiplier enforcing the normalization Eq. (20) (we shall calculate λ later), while K is a dimensionless constant since $\mathfrak{U}^\alpha$ is dimensionless. Thus T*e*V*e*S has two dimensionless parameters k and K in addition to the dimensional constants G and ℓ. The kinetic terms in Eq. (26) are chosen antisymmetric not because of any desire for gauge symmetry, which is broken by the form of the physical metric anyway, but because this choice precludes appearance of second derivatives of $\mathfrak{U}_\alpha$ in the energy-momentum tensor of T*e*V*e*S (see next subsection). The action S_v is a special case of that in Jacobson and Mattingly's generalization of GR with a preferred frame [61].

In accordance with the equivalence principle, the matter action in T*e*V*e*S is obtained by transcribing the flat spacetime Lagrangian $\mathcal{L}(\eta_{\mu\nu}, f^\alpha, \partial_\mu f^\alpha, \cdots)$ for fields written schematically f^α as

$$S_m = \int \mathcal{L}(\tilde{g}_{\mu\nu}, f^\alpha, f^\alpha_{|\mu}, \cdots)(-\tilde{g})^{1/2}d^4x, \tag{27}$$

where the covariant derivatives denoted by $|$ are taken w.r.t. $\tilde{g}_{\mu\nu}$. This has the effect that the spacetime delineated by matter dynamics has the metric $\tilde{g}_{\mu\nu}$. The appearance of $(-\tilde{g})^{1/2}$ here requires us to specify its relation to $(-g)^{1/2}$. In Appendix C we show that

$$(-\tilde{g})^{1/2} = e^{-2\phi}(-g)^{1/2}. \tag{28}$$

By coupling to matter only through $\tilde{g}_{\alpha\beta}$, the field $\mathfrak{U}_\alpha$ is totally different from the Lee-Yang 4-vector field with gravitation strength interaction [62], whose existence is ruled out by the equivalence principle tests as well as by cosmological symmetry arguments [62,63].

B. Basic equations

We shall obtain the basic equations by varying the total action $S = S_g + S_s + S_v + S_m$ with respect to the basic fields $g^{\alpha\beta}$, ϕ, σ and $\mathfrak{U}_\alpha$. To this end we must be explicit about how $\tilde{g}_{\alpha\beta}$, which enters into S_m, varies with the basic fields. Taking increments of Eq. (23) we get

$$\begin{aligned}\delta\tilde{g}^{\alpha\beta} &= e^{2\phi}\delta g^{\alpha\beta} + 2\sinh(2\phi)\mathfrak{U}_\mu\delta g^{\mu(\alpha}\mathfrak{U}^{\beta)} \\ &+2[e^{2\phi}g^{\alpha\beta} + 2\mathfrak{U}^\alpha\mathfrak{U}^\beta\cosh(2\phi)]\delta\phi \\ &+2\sinh(2\phi)\mathfrak{U}^{(\alpha}g^{\beta)\mu}\delta\mathfrak{U}_\mu, \end{aligned} \tag{29}$$

where symmetrization in a pair of indices is indicated by surrounding them by round brackets, e.g. $A^{(\mu}B^{\nu)} = A^\mu B^\nu + A^\nu B^\mu$.

1. Equations for the metric

When varying S w.r.t. $g^{\alpha\beta}$ we recall that $\delta S_g = (16\pi G)^{-1}G_{\alpha\beta}(-g)^{1/2}\delta g^{\alpha\beta}$ ($G_{\alpha\beta}$ denotes the Einstein tensor of $g_{\alpha\beta}$) while

$$\delta S_m = -\frac{1}{2}\tilde{T}_{\alpha\beta}(-\tilde{g})^{1/2}\delta\tilde{g}^{\alpha\beta} + \ldots, \tag{30}$$

where the ellipsis denotes variations of the f^α fields and $\tilde{T}_{\alpha\beta}$ stands for the physical energy-momentum tensor defined with the metric $\tilde{g}_{\alpha\beta}$. We get

$$G_{\alpha\beta} = 8\pi G[\tilde{T}_{\alpha\beta} + (1 - e^{-4\phi})\mathfrak{U}^\mu\tilde{T}_{\mu(\alpha}\mathfrak{U}_{\beta)} + \tau_{\alpha\beta}] + \Theta_{\alpha\beta}, \tag{31}$$

where

$$\begin{aligned}\tau_{\alpha\beta} \equiv \sigma^2\Big[&\phi_{,\alpha}\phi_{,\beta} - \frac{1}{2}g^{\mu\nu}\phi_{,\mu}\phi_{,\nu}g_{\alpha\beta} \\ &- \mathfrak{U}^\mu\phi_{,\mu}\Big(\mathfrak{U}_{(\alpha}\phi_{,\beta)} - \frac{1}{2}\mathfrak{U}^\nu\phi_{,\nu}\,g_{\alpha\beta}\Big)\Big] \\ &- \frac{1}{4}G\ell^{-2}\sigma^4F(kG\sigma^2)g_{\alpha\beta}, \end{aligned} \tag{32}$$

$$\begin{aligned}\Theta_{\alpha\beta} \equiv K\Big(&g^{\mu\nu}\mathfrak{U}_{[\mu,\alpha]}\mathfrak{U}_{[\nu,\beta]} - \frac{1}{4}g^{\sigma\tau}g^{\mu\nu}\mathfrak{U}_{[\sigma,\mu]}\mathfrak{U}_{[\tau,\nu]}g_{\alpha\beta}\Big) \\ &-\lambda\mathfrak{U}_\alpha\mathfrak{U}_\beta, \end{aligned} \tag{33}$$

When varying $g^{\alpha\beta}$ in S_v we have used Eq. (20) to drop a term proportional to $g_{\alpha\beta}$.

2. Scalar equation

Variation of σ in S_s gives the relation between σ and $\phi_{,\alpha}$ [$F' \equiv dF(\mu)/d\mu$],

$$-kG\sigma^2F - 1/2(kG\sigma^2)^2F' = k\ell^2h^{\alpha\beta}\phi_{,\alpha}\phi_{,\beta}. \tag{34}$$

In carrying out the variation w.r.t. ϕ it must be remembered that this quantity enters in S_m exclusively through $\tilde{g}^{\alpha\beta}$, so that use must be made of Eqs. (29) and (30):

$$[\sigma^2h^{\alpha\beta}\phi_{,\alpha}]_{;\beta} = [g^{\alpha\beta} + (1 + e^{-4\phi})\mathfrak{U}^\alpha\mathfrak{U}^\beta]\tilde{T}_{\alpha\beta}, \tag{35}$$

In view of Eq. (34) this is an equation for ϕ only, with $\tilde{T}_{\alpha\beta}$ as source.

Suppose we define a function $\mu(y)$ by

$$-\mu F(\mu) - \frac{1}{2}\mu^2F'(\mu) = y, \tag{36}$$

so that $kG\sigma^2 = \mu(k\ell^2h^{\alpha\beta}\phi_{,\alpha}\phi_{,\beta})$. We may now recast Eq. (35) as

$$\begin{aligned}&[\mu(k\ell^2h^{\mu\nu}\phi_{,\mu}\phi_{,\nu})h^{\alpha\beta}\phi_{,\alpha}]_{;\beta} \\ &\quad = kG[g^{\alpha\beta} + (1 + e^{-4\phi})\mathfrak{U}^\alpha\mathfrak{U}^\beta]\tilde{T}_{\alpha\beta}. \end{aligned} \tag{37}$$

This equation is reminiscent of the relativistic AQUAL scalar equation [see Appendix A, Eq. (A1)], albeit with the replacement $g^{\alpha\beta} \mapsto h^{\alpha\beta}$ in the left-hand side (l.h.s.) In quasistatic situations we may replace $h^{\alpha\beta}$ by $g^{\alpha\beta}$ so that Eq. (37) has the same structure as the AQUAL equation.

3. Vector equation

Variation of S w.r.t. $\mathfrak{U}_\alpha$ and use of Eq. (29) gives the vector equation

$$\begin{aligned}&K\mathfrak{U}^{[\alpha;\beta]}{}_{;\beta} + \lambda\mathfrak{U}^\alpha + 8\pi G\sigma^2\mathfrak{U}^\beta\phi_{,\beta}g^{\alpha\gamma}\phi_{,\gamma} \\ &\quad = 8\pi G(1 - e^{-4\phi})g^{\alpha\mu}\mathfrak{U}^\beta\tilde{T}_{\mu\beta}. \end{aligned} \tag{38}$$

As mentioned, λ here is a Lagrange multiplier. It can be solved for by contracting the previous equation with $\mathfrak{U}_\alpha$. Substituting it back gives

$$\begin{aligned}&K(\mathfrak{U}^{[\alpha;\beta]}{}_{;\beta} + \mathfrak{U}^\alpha\mathfrak{U}_\gamma\mathfrak{U}^{[\gamma;\beta]}{}_{;\beta}) + 8\pi G\sigma^2[\mathfrak{U}^\beta\phi_{,\beta}g^{\alpha\gamma}\phi_{,\gamma} \\ &\quad + \mathfrak{U}^\alpha(\mathfrak{U}^\beta\phi_{,\beta})^2] \\ &\quad = 8\pi G(1 - e^{-4\phi})[g^{\alpha\mu}\mathfrak{U}^\beta\tilde{T}_{\mu\beta} + \mathfrak{U}^\alpha\mathfrak{U}^\beta\mathfrak{U}^\gamma\tilde{T}_{\gamma\beta}]. \end{aligned} \tag{39}$$

This equation has only three independent components since both sides of it are orthogonal to $\mathfrak{U}_\alpha$. It thus determines three components of $\mathfrak{U}^\alpha$ with the fourth being determined by the normalization (20). Like any other partial differential equation, the vector equation does not by itself determine $\mathfrak{U}_\alpha$ uniquely.

C. General relativity limit

T*e*V*e*S has three parameters: k, ℓ and K. Here we show first that in several familiar contexts the limit $k \to 0$, $\ell \propto k^{-3/2}$, $K \propto k$ of it corresponds to standard GR for any form of the function F. Many of the intermediate results will be useful in Sec. V and VII. We then expand on a remark by Milgrom that the GR limit actually follows under more general circumstances: $K \to 0$ and $\ell \to \infty$.

Whenever a specific matter content is needed, we shall assume the matter to be an ideal fluid. Its energy-momentum tensor has the familiar form

$$\tilde{T}_{\alpha\beta} = \tilde{\rho}\tilde{u}_\alpha\tilde{u}_\beta + \tilde{p}(\tilde{g}_{\alpha\beta} + \tilde{u}_\alpha\tilde{u}_\beta), \tag{40}$$

where $\tilde{\rho}$ is the proper energy density, $\tilde{p}$ the pressure and $\tilde{u}_\alpha$ the 4-velocity, all three expressed in the physical metric. We may profitably simplify Eq. (37) in any case when for symmetry reasons $\tilde{u}_\alpha$ is collinear with $\mathfrak{U}_\alpha$. In order that the velocity be normalized w.r.t. $\tilde{g}_{\alpha\beta}$, we must take in that case $\tilde{u}_\alpha = e^{\phi}\mathfrak{U}_\alpha$ from which follows

$$\tilde{g}_{\alpha\beta} + \tilde{u}_\alpha\tilde{u}_\beta = e^{-2\phi}(g_{\alpha\beta} + \mathfrak{U}_\alpha\mathfrak{U}_\beta). \tag{41}$$

Substituting this in $\tilde{T}_{\alpha\beta}$ allows us to rewrite Eq. (37) as

$$[\mu(k\ell^2 h^{\mu\nu}\phi_{,\mu}\phi_{,\nu})h^{\alpha\beta}\phi_{,\alpha}]_{;\beta} = kG(\tilde{\rho} + 3\tilde{p})e^{-2\phi}. \tag{42}$$

This form is suitable for the analysis of cosmology as well as static systems.

1. Cosmology

Not only important in itself, cosmology is relevant for setting boundary conditions in the study of T*e*V*e*S in the solar system and other localized weak gravity situations. We shall confine our remarks to Friedmann-Robertson–Walker (FRW) cosmologies, for which the metric can be given the form

$$g_{\alpha\beta}dx^\alpha dx^\beta = -dt^2 + a(t)^2[d\chi^2 + f(\chi)^2(d\theta^2 + \sin^2\theta d\varphi^2)]. \tag{43}$$

Here $f(\chi) \equiv \sin\chi, \chi, \sinh\chi$ for closed, flat, and open spaces, respectively.

In applying Eq. (37) we shall assume that the fields ϕ, σ and $\mathfrak{U}^\alpha$ partake of the symmetries of the FRW spacetime. Thus we take these fields to depend solely on t. Also since there are no preferred spatial directions, $\mathfrak{U}^\alpha$ must point in the cosmological time direction: $\mathfrak{U}^\alpha = \delta_t{}^\alpha$ (that this is possible distinguishes $\mathfrak{U}^\alpha$ from the Lee-Yang field which is ruled out in FRW cosmology [63]). Obviously this is a case where $\tilde{u}_\alpha = e^{\phi}\mathfrak{U}_\alpha$; the scalar equation then takes the form

$$a^{-3}\partial_t[a^3\mu(-2k\ell^2\dot{\phi}^2)\dot{\phi}] = -\frac{1}{2}kG(\tilde{\rho} + 3\tilde{p})e^{-2\phi}, \tag{44}$$

where an overdot signifies $\partial/\partial t$. The first integral is

$$\mu(-2k\ell^2\dot{\phi}^2)\dot{\phi} = \frac{-k}{2a^3}\int_0^t G(\tilde{\rho} + 3\tilde{p})e^{-2\phi}a^3 dt. \tag{45}$$

As is customary in scalar-tensor theories, we have dropped an additive integration constant; this has the effect of ameliorating any divergence of $\dot{\phi}$ as $a \to 0$. In fact we can see that the right-hand side (r.h.s.) of the equation behaves there as $k(\tilde{\rho} + 3\tilde{p})e^{-2\phi}t$. We observe that as $k \to 0$ with $\ell \propto k^{-3/2}$, $\dot{\phi}$ will behave as k with the argument of μ staying constant. Thus regardless of the form of μ, we have $\dot{\phi} \sim k$. It is thus consistent to assume that ϕ itself is of $\mathcal{O}(k)$ throughout cosmological history. This despite the possible divergence of $\dot{\phi}$ at the cosmological singularity, since the rate of that divergence is also proportional to k, as we have just seen. Recalling that $kG\sigma^2 = \mu$, we conclude that σ^2 is of $\mathcal{O}(k^{-1})$ in the cosmological solutions (otherwise μ would vary with k whereas its argument stayed constant).

Let us check whether our assumption that $\mathfrak{U}^\alpha = \delta_t{}^\alpha$ is consistent with the vector Eq. (38). The choice $\mathfrak{U}^\alpha = \delta_t{}^\alpha$ makes $\mathfrak{U}^{[\alpha;\beta]} = 0$. For a comoving ideal fluid $\mathfrak{U}^\beta\tilde{T}_{\alpha\beta} = -e^{2\phi}\tilde{\rho}\mathfrak{U}_\alpha$. Thus the spatial components of the vector Eq. (38) vanish identically, while the temporal one informs us that

$$\lambda = 8\pi G[\sigma^2\dot{\phi}^2 - 2\tilde{\rho}\sin(2\phi)]. \tag{46}$$

Our previous comments make it clear that λ is of $\mathcal{O}(k)$.

Turning to the gravitational Eqs. (31)–(33) we first note that in the limit $\{k \to 0,\ \ell \propto k^{-3/2},\ K \propto k\}$, $\tau_{\alpha\beta}$ and $\Theta_{\alpha\beta}$ are both $\mathcal{O}(k)$. It follows that $G_{\alpha\beta} = 8\pi G\tilde{T}_{\alpha\beta} + \mathcal{O}(k)$. Since the difference between $\tilde{g}_{\alpha\beta}$ and $g_{\alpha\beta}$ is also of $\mathcal{O}(k)$, it is obvious that $\tilde{G}_{\alpha\beta} = 8\pi G\tilde{T}_{\alpha\beta} + \mathcal{O}(k)$ so that any cosmological model based on T*e*V*e*S differs from the corresponding one in GR only by terms of $\mathcal{O}(k)$. In FRW cosmology T*e*V*e*S has GR as its limit when $k \to 0$ with $\ell \propto k^{-3/2}$ and $K \propto k$.

2. Quasistatic localized system

We now turn to systems such as the solar system, or a neutron star, which may be thought of as quasistatic situations in asymptotically flat spacetime (at least up to subcosmological distances). We shall idealize them as truly static systems with time-independent metrics of the form

$$g_{\alpha\beta}dx^\alpha dx^\beta = g_{tt}(x^k)dt^2 + g_{ij}(x^k)dx^i dx^j, \tag{47}$$

and no energy flow. The scalar and vector equations have a variety of joint solutions. We shall single out the physical one by requiring the boundary condition that $\phi \to$ const. at spatial infinity, the constant being just the value of ϕ from the cosmological model in which our localized system is embedded. Likewise, we shall require that $\mathfrak{U}^\alpha \to \delta_t{}^\alpha$ so that the vector field matches the cosmological field at "spatial infinity".

We first show that $\mathfrak{U}^\alpha = N\xi^\alpha$, with $\xi^\alpha = \delta_t{}^\alpha$ the Killing vector associated with the static character of the spacetime, is an acceptable solution (with $N \equiv (-g_{\alpha\beta}\xi^\alpha\xi^\beta)^{-1/2}$, $\mathfrak{U}^\alpha$ is properly normalized). Let us consider the expression $g^{\alpha\mu}\mathfrak{U}^\beta\tilde{T}_{\mu\beta} + \mathfrak{U}^\alpha\mathfrak{U}^\beta\mathfrak{U}^\gamma\tilde{T}_{\gamma\beta}$ appearing in the source of the vector Eq. (39) for this choice of $\mathfrak{U}^\alpha$. Its $\alpha = t$ component is $N(\tilde{T}^t{}_t + \mathfrak{U}_t\mathfrak{U}^t\tilde{T}^t{}_t) = 0$, while the $\alpha = i$ component is $N[g^{ij}\tilde{T}_{jt} + \mathfrak{U}^i(\mathfrak{U}^t)^2\tilde{T}_{tt}]$

which also vanishes because $\tilde{T}_{jt} = 0$ (no energy flow). Turn now to the l.h.s. of Eq. (39). Because $\mathfrak{U}^\alpha$ has only a (time-independent) temporal component, $\mathfrak{U}^\alpha \phi_{,\alpha} = 0$, and the only nonvanishing components of $\mathfrak{U}^{[\alpha,\beta]}$ are the jt ones, and they depend only on the x^j. Hence $\mathfrak{U}^{[i,\beta]}{}_{;\beta} = 0$ so that the $\alpha = i$ components of the l.h.s. of the equation vanish. What is left of the $\alpha = t$ component is $K(\mathfrak{U}^{[t,\beta]}{}_{;\beta} + \mathfrak{U}^t \mathfrak{U}_t \mathfrak{U}^{[t,\beta]}{}_{;\beta})$ which vanishes by the normalization of $\mathfrak{U}^\alpha$. Hence $\mathfrak{U}^\alpha = N\xi^\alpha$ satisfies the vector equation for any k and K. We have not succeeded in proving that this is the unique solution, but this seems to be a reasonable supposition.

Now, as $k \to 0$, the scalar Eq. (37) reduces to $(\mu h^{\alpha\beta}\phi_{,\alpha})_{;\beta} = 0$. Multiplying this by $\phi(-g)^{1/2}$, discarding all time derivatives, and integrating over space gives, after an integration by parts and application of the boundary condition at infinity, that $\int \mu g^{\alpha\beta}\phi_{,\alpha}\phi_{,\beta}(-g)^{1/2}d^3x = 0$. Because for any static metric, g^{ij} is positive definite and, when defined, $\mu > 0$, this equation is satisfied only by $\phi = \text{const.}$ throughout. But for $k \to 0$, the cosmological model has $\phi \to 0$. Hence as $k \to 0$, $\phi \to 0$ in all the space.

Returning to the full scalar Eq. (37) and recalling that $\ell \propto k^{-3/2}$, it is easy to see that for small but finite k the *gradient* of ϕ scales as k. From the last paragraph it then follows that $\phi = \mathcal{O}(k)$. These last conclusions are actually independent of the form of μ because its argument goes to a nonzero constant in the limit $k \to 0$. We recall [see Eq. (34)] that as $k \to 0$, $\sigma^2 \propto k^{-1}$. Thus the scalars' energy-momentum tensor $\tau_{\alpha\beta}$ is of $\mathcal{O}(k)$ (recall $\ell \propto k^{-3/2}$). From the $\alpha = t$ component of Eq. (38) we see that $\lambda = \mathcal{O}(k) + \mathcal{O}(K)$. Hence $\Theta_{\alpha\beta} = \mathcal{O}(k) + \mathcal{O}(K)$. In addition, the term in the gravitational Eqs. (31) proportional to $1 - e^{-4\phi}$ is itself of $\mathcal{O}(k)$; hence we have $G_{\alpha\beta} = 8\pi G\tilde{T}_{\alpha\beta} + \mathcal{O}(k) + \mathcal{O}(K)$. Since the difference between $\tilde{g}_{\alpha\beta}$ and $g_{\alpha\beta}$ is of $\mathcal{O}(\phi)$, namely $\mathcal{O}(k)$, it is obvious that $\tilde{G}_{\alpha\beta} = 8\pi G\tilde{T}_{\alpha\beta} + \mathcal{O}(k) + \mathcal{O}(K)$. Thus for quasistatic situations also, T*e*V*e*S has GR as its limit when $k \to 0$ with $\ell \propto k^{-3/2}$ and $K \propto k$.

In conclusion, the limit $\{k \to 0,\ \ell \propto k^{-3/2},\ K \propto k\}$ of T*e*V*e*S is GR, both in cosmology and in quasistatic localized systems.

D. Generic general relativity limit

Milgrom [64] has remarked that GR actually follows from T*e*V*e*S in the more general limit $K \to 0$ and $\ell \to \infty$ with k arbitrary. This is easily seen after the change of variables $\phi \mapsto \phi_* \equiv \ell\phi$, $\sigma \mapsto \sigma_* \equiv \sqrt{k}\sigma$, whereby only $\tilde{g}_{\alpha\beta}$ and S_s are changed:

$$\tilde{g}_{\alpha\beta} = e^{-2\phi_*/\ell} g_{\alpha\beta} - 2\mathfrak{U}_\alpha \mathfrak{U}_\beta \sinh(2\phi_*/\ell), \tag{48}$$

$$S_s = -\frac{1}{2k^2\ell^2}\int\left[k\sigma_*{}^2 h^{\alpha\beta}\phi_{*,\alpha}\phi_{*,\beta} + \frac{1}{2}G\sigma_*{}^4 F(G\sigma_*{}^2)\right] \times (-g)^{1/2}d^4x, \tag{49}$$

Thus as $\ell \to \infty$ the scalar action disappears and ϕ_* decouples from the theory. In addition, with $K \to 0$, the vector's action S_v disappears apart from the term with λ. All this means that the r.h.s. of the Einstein Eqs. (31) retains only the $\tilde{T}_{\alpha\beta}$ and $\lambda\mathfrak{U}_\alpha\mathfrak{U}_\beta$ terms. But according to the vector Eq. (38), from which the terms with differentiated ϕ_* and $\mathfrak{U}_\alpha$ have dropped out, $\lambda \to 0$ because $(1 - e^{-4\phi_*/\ell}) \to 0$. Accordingly, we get the usual Einstein equations. Since $g_{\alpha\beta}$ and $\tilde{g}_{\alpha\beta}$ coincide as $\ell \to \infty$, we get exact GR.

In this paper we shall assume that $k \ll 1$ and $K \ll 1$ without restricting ℓ. Empirical bounds on k and K are discussed in Secs. IV C and V.

E. The choice of F

Because we have no theory for the functions $F(\mu)$ or $y(\mu)$, there is great freedom in choosing them. In this paper we shall adopt, as an example, the form

$$y = \frac{3}{4}\frac{\mu^2(\mu - 2)^2}{1 - \mu}, \tag{50}$$

plotted in Fig. 1. As y ranges from 0 to ∞, $\mu(y)$ increases monotonically from 0 to unity; for small y, $\mu(y) \approx \sqrt{y/3}$. For negative y the function $\mu(y)$ is double-valued. As y decreases from 0, one branch decreases monotonically from $\mu = 2$ and tends to unity as $y \to -\infty$, while the second increases monotonically from $\mu = 2$ and diverges as $y \to -\infty$. We adopt the second (far right) branch as the physical one.

What features of the above $y(\mu)$ are essential for the following sections? The denominator in Eq. (50) is included so that μ shall asymptote to unity for $y \to \infty$ (the Newtonian limit, c.f. Sec. IV C). The factor μ^2 ensures

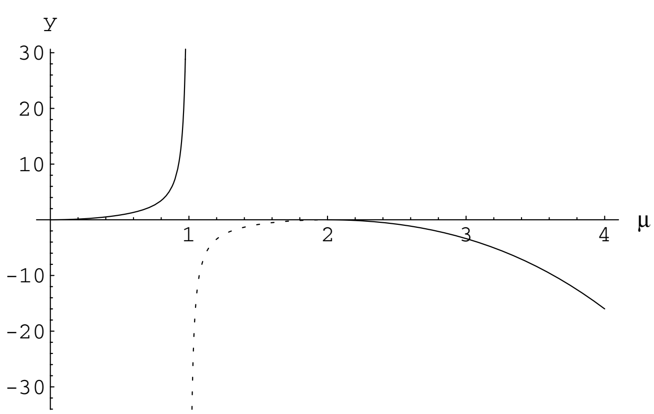

FIG. 1. The function $y(\mu)$ as relevant for quasistationary systems $0 < \mu < 1$ and for cosmology $2 < \mu < \infty$.

that the MOND limit is contained in the theory (see Sec. IV B), while the factor $(\mu - 2)^2$ ensures there exists a monotonically decreasing branch of $\mu(y)$ which covers the whole of the range $y \in [0, -\infty)$ (relevant to cosmology, c.f. Sec. VII) and only it.

Integrating Eq. (36) with $y(\mu)$ we obtain (see Fig. 2)

$$F = \frac{3}{8}\frac{\mu(4 + 2\mu - 4\mu^2 + \mu^3) + 2\ln[(1-\mu)^2]}{\mu^2}, \quad (51)$$

where we ignore a possible integration constant (which will, however, be useful in Sec. VII F below). Obviously $F < 0$ in the range $\mu \in (0, 1)$ (relevant for quasistationary systems) but $F > 0$ for $\mu > 2$ (the cosmological range). Where negative, F contributes negative energy density in the energy-momentum tensor (32). Despite this there seems to be no collision with the requirement of positive overall energy density (see Secs. V and VII A).

IV. NONRELATIVISTIC LIMIT OF T*e*V*e*S

Sec. III C 2 shows that in quasistatic systems T*e*V*e*S approaches GR in the limit $\{k \to 0, \ell \sim k^{-3/2}, K \sim k\}$. But in what limit do we recover standard Newtonian gravity? And where is MOND, which is antagonistic to Newtonian gravity, in all this? This section shows that with our choice of F, both Newtonian and MOND limits emerge from T*e*V*e*S for small gravitational potentials, but that MOND requires in addition small gravitational *fields*, just as expected from Milgrom's original scheme.

A. Quasistatic systems

We are here concerned with a quasistatic, *weak* potential and *slow* motion situation, such as a galaxy or the solar system. As in Sec. III C 2, quasistatic means we can neglect time derivatives in comparison with spatial ones. Let us assume that the metric $g_{\alpha\beta}$ is nearly flat and that $|\phi| \ll 1$. Then linearization of Eq. (31) in terms of the Newtonian potential V generated by the energy content

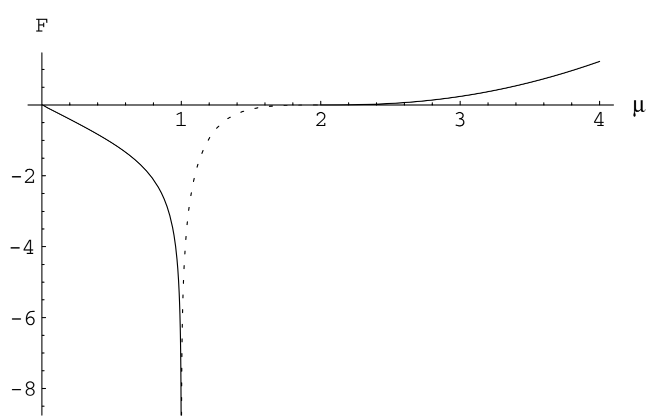

FIG. 2. The function $F(\mu)$ as relevant for quasistationary systems, $0 < \mu < 1$, and for cosmology, $2 < \mu < \infty$.

on its r.h.s. gives $g_{tt} = -(1 + 2V) + \mathcal{O}(V^2)$. From the prescription given in Sec. III C 2, $\mathfrak{U}_\alpha = -[1 + V + \mathcal{O}(V^2)]\delta_{t\alpha}$. It follows from Eq. (22) that to $\mathcal{O}(\phi)$ and $\mathcal{O}(V)$, $\tilde{g}_{tt} = -(1 + 2V + 2\phi)$. Thus in T*e*V*e*S the total potential governing all nonrelativistic motion is $\Phi = V + \phi$. We should remark that if asymptotically $\phi \to \phi_c \neq 0$, the $\tilde{g}_{tt}$ does not there correspond to a Minkowski metric. This is remedied by rescaling the time (or spatial) coordinates by factors e^{ϕ_c} (or $e^{-\phi_c}$). With respect to the new coordinates the metric is then asymptotically Minkowskian. In this paper we assume throughout that $|\phi_c| \ll 1$; Sec. VII shows this is consistent with cosmological evolution of ϕ.

How is Φ related to Φ_N, the Newtonian gravitational potential generated by the mass density $\tilde{\rho}$ according to Poisson's equation with gravitational constant G? To relate ϕ to Φ_N we first set temporal derivatives in Eq. (42) to zero which means replacing $h^{\alpha\beta}\phi_{,\alpha} \to g^{\alpha\beta}\phi_{,\alpha}$:

$$[\mu(k\ell^2 g^{\mu\nu}\phi_{,\mu}\phi_{,\nu})g^{\alpha\beta}\phi_{,\alpha}]_{;\beta} = kG(\tilde{\rho} + 3\tilde{p})e^{-2\phi}. \quad (52)$$

This equation is still exact. Next we replace $g^{\alpha\beta} \to \eta^{\alpha\beta}$ as well as $e^{-2\phi} \to 1$. This is the nonrelativistic approximation. Further, to be consistent we must neglect $\tilde{p}$ compared to $\tilde{\rho}$; keeping the former would be tantamount to accepting that V is not small. Thus

$$\nabla \cdot \left[\mu\left(k\ell^2(\nabla\phi)^2\right)\nabla\phi\right] = kG\tilde{\rho}. \quad (53)$$

This is just the AQUAL Eq. (3) with a suitable reinterpretation of the function μ. Now comparing Eq. (53) with Poisson's equation we see that

$$k^{-1}\mu|\nabla\phi| = \mathcal{O}(|\nabla\Phi_N|), \quad (54)$$

This will be made more precise below in situations with symmetry.

We now show that it is consistent to take $V = C\Phi_N$, with C a constant close to unity (to be determined). The starting point are the modified Einstein Eqs. (31). With F as in (51), $F < 0$ simultaneously with $F' < 0$ for $0 < \mu < 1$; it follows from Eq. (36) that $\mu|F| < y$. Now the F term on the r.h.s. of Eq. (31) is $-2\pi G^2\ell^{-2}\sigma^4 F(kG\sigma^2)g_{\alpha\beta} = -2\pi k^{-2}\ell^{-2}\mu^2 F(\mu)g_{\alpha\beta}$. Similarly, since $\phi_{,t} = 0$ here, the terms on the r.h.s. involving $\phi_{,\alpha}$ are of order $8\pi G\sigma^2 h^{\gamma\delta}\phi_{,\gamma}\phi_{,\delta}g_{\alpha\beta} = 8\pi k^{-2}\ell^{-2}\mu y(\mu)g_{\alpha\beta}$. Thus by our earlier remark the ϕ derivative terms in $\tau_{\alpha\beta}$ dominate the F term, and by Eq. (54) they are of order $8\pi k\mu^{-1}(\nabla\Phi_N)^2$. But $(\nabla\Phi_N)^2$ is precisely the type of source (Newtonian gravitational energy or stress density) needed to compute the first nonlinear or $\mathcal{O}(\Phi_N^2)$ contributions to the metric. As we shall see in Sec. VII, we need $k \sim 10^{-2}$, so that if all we desire is to compute the metric to $\mathcal{O}(\Phi_N)$, and μ is not very small, then all of $\tau_{\alpha\beta}$ may be neglected.

Further, since $\mathfrak{U}_\alpha = -[1 + V + \mathcal{O}(V^2)]\delta_{t\alpha}$, the $\mathfrak{U}_{[\alpha,\beta]}{}^2$ terms in $\Theta_{\alpha\beta}$ have the form $(C\nabla\Phi_N)^2$; we drop them for the same reason that we dropped the $\mathcal{O}(\Phi_N^2)$ term in $\tau_{\alpha\beta}$. It follows that in the weak potential approximation the spatio-temporal and spatial-spatial components of Einstein's equations are exactly the same as in GR because the term proportional to $1 - e^{-4\phi}$ can be dropped by virtue of the slow motion condition which suppresses the spatio-temporal components of $T_{\alpha\beta}$. The temporal-temporal component of Einstein's equations depends on λ, and is thus another story. From Eqs. (38) and (40) and the observation that $\mathfrak{U}^\alpha\phi_{,\alpha} = 0$,

$$\lambda = K\mathfrak{U}_\alpha\mathfrak{U}^{[\alpha;\beta]}{}_{;\beta} - 16\pi G\tilde{\rho}\sin(2\phi). \tag{55}$$

With our $\mathfrak{U}_\alpha$ the first term is $K\mathfrak{U}_t\mathfrak{U}^{[t;\beta]}{}_{;\beta} = KC\nabla^2\Phi_N + KC^2\mathcal{O}(\nabla\Phi_N{}^2)$, where by Poisson's equation $\nabla^2\Phi_N = 4\pi G\tilde{\rho}$. Further, as we shall see in Sec. V, ϕ is always very close to its aforementioned asymptotic value ϕ_c (which is just ϕ's very slowly varying cosmological value). Dropping the $C^2\mathcal{O}(\nabla\Phi_N{}^2)$ contribution for the same reason as above gives

$$\lambda \approx 8\pi G[KC/2 - 2\sin(2\phi_c)]\tilde{\rho}. \tag{56}$$

Substituting this in Eq. (33) and combining the result with the $(1 - e^{-4\phi_c})$ term in the G_{tt} equation Eq. (31), we see that $(e^{-2\phi_c} - KC/2)\tilde{\rho}$ replaces the source $\tilde{\rho}$ appropriate in the weak potential approximation to GR. By linearizing the G_{tt} equation as done in GR, we conclude that

$$V = (e^{-2\phi_c} - KC/2)\Phi_N, \tag{57}$$

which verifies the claim that V is proportional to Φ_N. Indeed, since the proportionality constant here must be identical with C, we have $C = (1 + K/2)^{-1}e^{-2\phi_c}$. Since we shall show in Sec. V that $K \ll 1$ and in Sec. VII that it is consistent to assume $|\phi_c| \ll 1$, we shall replace C everywhere by $\Xi \equiv 1 - K/2 - 2\phi_c$. In particular

$$\Phi = \Xi\Phi_N + \phi. \tag{58}$$

In summary, Eq. (58), which is subject to corrections of $\mathcal{O}(\Phi_N{}^2)$, quantifies the difference at the nonrelativistic level between T*e*V*e*S and GR, a difference which is in harmony with our conclusion in Sec. III C 2. We shall use it until we turn to post-Newtonian corrections. The condition "μ is not very small" which we imposed above to be able to neglect the $\tau_{\alpha\beta}$ contribution to the gravitational equations is not restrictive. For the Newtonian limit we shall see that $\mu \approx 1$. And when $\mu \ll 1$ (extreme MOND limit relevant for extragalactic phenomena), the consequent corrections of $\mathcal{O}(\Phi_N{}^2)$ (with large coefficient) to V are entirely ignorable because this potential is then dominated by ϕ in the expression for Φ, c.f. Eq. (59).

B. The MOND limit: spherical symmetry

First for orientation we assume a spherically symmetric situation. Then from Eq. (53) together with Gauss's theorem we infer that

$$\nabla\phi = (k/4\pi\mu)\nabla\Phi_N. \tag{59}$$

In view of Eq. (58) we have

$$\tilde{\mu}\nabla\Phi = \nabla\Phi_N, \tag{60}$$

with

$$\tilde{\mu} \equiv (\Xi + k/4\pi\mu)^{-1}. \tag{61}$$

Consider the case $\mu \ll 1$ for which $\mu[k\ell^2(|\nabla\phi|)^2] \approx (k/3)^{1/2}\ell|\nabla\phi|$ (see Sec. III E). Eliminating $\nabla\Phi_N$ between Eqs. (59) and (60) and defining

$$\mathfrak{a}_0 \equiv \frac{(3k)^{1/2}}{4\pi\Xi\ell}, \tag{62}$$

we obtain a quadratic equation for μ with positive root

$$\mu = (k/8\pi\Xi)(-1 + \sqrt{1 + 4|\nabla\Phi|/\mathfrak{a}_0}). \tag{63}$$

This is obviously valid only when $|\nabla\Phi| \ll (4\pi/k)^2\mathfrak{a}_0$ since otherwise μ is not small. From Eq. (61) we now deduce the MOND function

$$\tilde{\mu} = \frac{1}{\Xi}\frac{-1 + \sqrt{1 + 4|\nabla\Phi|/\mathfrak{a}_0}}{1 + \sqrt{1 + 4|\nabla\Phi|/\mathfrak{a}_0}}. \tag{64}$$

For $|\nabla\Phi| \ll \mathfrak{a}_0$ (which is consistent with the above restriction since $k \ll 1$) this equation gives to lowest order in K and ϕ_c

$$\tilde{\mu} \approx |\nabla\Phi|/\mathfrak{a}_0. \tag{65}$$

Thus if we identify our $\mathfrak{a}_0$ with Milgrom's constant, Eq. (60) with this $\tilde{\mu}$ coincides with the MOND formula (1) in the extreme low acceleration regime. Therefore, T*e*V*e*S recovers MOND's successes in regard to low surface brightness disk galaxies, dwarf spheroidal galaxies, and the outer regions of spiral galaxies. For all these the low acceleration limit of Eq. (1) is known to summarize the phenomenology correctly.

Now suppose $|\nabla\Phi|$ varies from an order below $\mathfrak{a}_0$ up to a couple of orders above it. This respects the condition $|\nabla\Phi| \ll (4\pi/k)^2\mathfrak{a}_0$. Then Eq. (64) shows $\tilde{\mu}$ to grow monotonically from about 0.1 to 0.9. Then Eq. (60) is essentially formula (1) in the intermediate MOND regime. This regime is relevant for the disks of massive spiral galaxies well outside the central bulges but not quite in their outer reaches. It is known that the precise form of $\tilde{\mu}$ makes little difference for the task of predicting detailed rotation curves from surface photometry.

We see that T*e*V*e*S reproduces the MOND paradigm encapsulated in Eq. (1) for not too large values of $|\nabla\Phi|/\mathfrak{a}_0$. What happens for very large $|\nabla\Phi|/\mathfrak{a}_0$?

C. The Newtonian limit: spherical symmetry

According to our choice of $y(\mu)$, Eq. (50), the limit $\mu \to 1$ corresponds to $y \to \infty$, that is to say $|\nabla\phi| \to \infty$. By Eqs. (59)–(61) we simultaneously have $|\nabla\Phi| \to \infty$ and $\tilde{\mu} \to (\Xi + k/4\pi)^{-1}$. Defining the Newtonian gravitational constant by

$$G_N = (\Xi + k/4\pi)G, \tag{66}$$

we see from Eq. (60) that $\nabla\Phi$ is obtained from $\nabla\Phi_N$ by just replacing $G \to G_N$ in it. In other words, in the nonrelativistic and arbitrarily large $|\nabla\Phi|$ regime, T*e*V*e*S is equivalent to Newtonian gravity but with a "renormalized" value of the gravitational constant. Now Ξ is really a surrogate of $C = (1 + K/2)^{-1}e^{-2\phi_c}$, a positive quantity. Hence G_N is always positive.

But how close are dynamics to Newtonian for *large but finite* $|\nabla\Phi|/\mathfrak{a}_0$? Expanding the r.h.s. of Eq. (50) in the neighborhood of $\mu = 1$ gives

$$y = \frac{3/4}{1-\mu} + \mathcal{O}(1-\mu). \tag{67}$$

We also have by Eqs. (59) and (60) that $y \equiv k\ell^2|\nabla\phi|^2 \approx (k^3\ell^2/16\pi^2)|\nabla\Phi|^2$ where we have dropped corrections of higher order in $(k/4\pi)$. Dropping the $\mathcal{O}(1-\mu)$ term in $y(\mu)$ and eliminating ℓ in favor of $\mathfrak{a}_0$ (with $\Xi = 1$) we get

$$\mu \approx 1 - \frac{64\pi^4}{k^4}\frac{\mathfrak{a}_0{}^2}{|\nabla\Phi|^2}. \tag{68}$$

Thus to trust the approximation $\mu \approx 1$ we must have $|\nabla\Phi|/\mathfrak{a}_0 \gg 8\pi^2 k^{-2}$. Using Eqs. (68) and (61) we obtain, again after dropping higher order terms in k, that

$$\tilde{\mu} \approx \frac{G}{G_N}\left(1 - \frac{16\pi^3}{k^3}\frac{\mathfrak{a}_0^2}{|\nabla\Phi|^2}\right). \tag{69}$$

Here the factor (G/G_N) just reflects the mentioned "renormalization" of the gravitational constant; it is the next factor which interests us as a measure of departures from strict Newtonian behavior. For example, if $k = 0.03$ there is a 5.3×10^{-9} fractional enhancement of the Sun's Newtonian field at Earth's orbit where $|\nabla\Phi| = 0.59\ \mathrm{cm\,s^{-2}}$. This is probably unobservable today. At Saturn's orbit where $|\nabla\Phi| = 0.0065\ \mathrm{cm\,s^{-2}}$ the fractional correction is 4.3×10^{-5}, corresponding to an excess acceleration $2.8 \times 10^{-7}\ \mathrm{cm\,s^{-2}}$ (at this point μ departs from unity by only 0.018 so that Eq. (68) is still reliable). Although this departure from Newtonian predictions seems serious, it should be remembered that navigational data from the Pioneer 10 and 11 spacecrafts seem to disclose a constant acceleration in excess of Newtonian of about $8 \times 10^{-8}\ \mathrm{cm\,s^{-2}}$ between Uranus's orbit and the trans-Plutonian region [65]. It is, however, unclear whether the correction in Eq. (69), sensitive as it is to the choice of F, has anything to do with the "Pioneer anomaly."

D. Nonspherical systems

We now consider *generically* asymmetric systems. Since any system has a region where μ differs from unity and is variable, Eq. (59) is not the general solution of Eq. (53) and must be replaced by

$$\nabla\phi = (k/4\pi\mu)(\nabla\Phi_N + \nabla \times \boldsymbol{h}), \tag{70}$$

where $\boldsymbol{h}$ is some regular vector field which is determined up to a gradient by the condition that the curl of the r.h.s. of Eq. (70) vanish.

The freedom inherent in $\boldsymbol{h}$ allows it to be made divergenceless. Then by Gauss's theorem $\boldsymbol{h}$ must fall off faster than $1/r^2$ and $\nabla \times \boldsymbol{h}$ faster than $1/r^3$ at large distances. On physical grounds $|\nabla \times \boldsymbol{h}|$ is expected to be of the same order as $|\nabla\Phi_N|$ well inside the matter. But since the latter quantity falls off as $1/r^2$ well outside the matter, the curl term in Eq. (70) must rapidly become negligible well outside the system. We thus expect the discussion in Sec. IV B to apply well outside any nonspherical galaxy just as it applies anywhere inside a spherical one. The interior and near exterior of such a galaxy, where $\nabla \times \boldsymbol{h}$ is still important, must be treated by numerical methods which would be no different than those developed by Milgrom within the old AQUAL theory [51].

Needless to say, an asymmetric system so dense that the Newtonian regime (μ approximately constant) obtains in its interior, e.g., an oblate globular cluster like ω Centauri, can be described everywhere without an $\boldsymbol{h}$. For in the interior $\boldsymbol{h}$ is not needed since even in its absence the curl of the r.h.s. of Eq. (70) vanishes (approximately). And μ begins to differ substantially from unity only well outside the system where we know from our previous argument that any $\boldsymbol{h}$ is becoming negligible. Hence both Newtonian and MOND regimes of the system may be described as in Secs. IV B and IV C.

In summary, we see that the extragalactic predictions of the MOND Eq. (1) are recovered from T*e*V*e*S; at the same time T*e*V*e*S hints at non-Newtonian behavior in the reaches of the solar system, though the effect is sensitive to the choice of F in the theory.

V. THE POST-NEWTONIAN CORRECTIONS

The upshot of the discussion at the end of Sec. III C 2 is that in the solar system (regarded as a static system—with rotation neglected—embedded in a FRW cosmological background), $\tilde{G}_{\alpha\beta} = 8\pi G\tilde{T}_{\alpha\beta} + \mathcal{O}(k) + \mathcal{O}(K)$. Here we compute the consequent $\mathcal{O}(k) + \mathcal{O}(K)$ corrections to the Schwarzschild metric

$$g_{\alpha\beta}dx^\alpha dx^\beta = -\frac{(1 - Gm/2\varrho)^2}{(1 + Gm/2\varrho)^2}dt^2 + (1 + Gm/2\varrho)^4 \times[d\varrho^2 + \varrho^2(d\theta^2 + \sin^2\theta d\varphi^2)], \qquad (71)$$

that describes the exterior of a spherically mass m, and determine the post-Newtonian parameters of T*e*V*e*S which we compare with those of GR.

Rather than just extending the Newtonian limit calculation of Sec. IV C, we start from scratch. First we write the spherically symmetric and static metric of the Sun (inside and outside it) as

$$g_{\alpha\beta}dx^\alpha dx^\beta = -e^\nu dt^2 + e^\varsigma[d\varrho^2 + \varrho^2(d\theta^2 + \sin^2\theta d\varphi^2)], \qquad (72)$$

with $\nu = \nu(\varrho)$ and $\varsigma = \varsigma(\varrho)$. Just as for metric (71), outside the Sun these functions should admit the expansions (α_i and β_i are dimensionless constants)

$$e^\nu = 1 - r_g/\varrho + \alpha_2(r_g/\varrho)^2 + \cdots, \qquad (73)$$

$$e^\varsigma = 1 + \beta_1 r_g/\varrho + \beta_2(r_g/\varrho)^2 + \cdots, \qquad (74)$$

where r_g is a length scale to be determined (see Appendix D). The magnitude of the coefficient of the r_g/ϱ term in Eq. (73) has been absorbed into r_g; its sign must be negative, as shown, because gravity is attractive. From the fact that T*e*V*e*S approaches GR for small k and K, we may infer that r_g is close to $2G$ times the system's Newtonian mass. This is made precise below.

Taking $\phi = \phi(\varrho)$ and $\tilde{T}_{\alpha\beta}$ from Eq. (40), we may write the scalar Eq. (42) as

$$\varrho^{-2}e^{-(\nu+3\varsigma)/2}[\mu e^{(\nu+\varsigma)/2}\varrho^2\phi']' = kGe^{-2\phi}(\tilde{\rho} + 3\tilde{p}). \qquad (75)$$

Here $'$ stands for $d/d\varrho$. The first integral of Eq. (75) is

$$\phi' = \frac{kGe^{-(\nu+\varsigma)/2}}{\mu\varrho^2}\int_0^\varrho(\tilde{\rho} + 3\tilde{p})e^{\nu/2+3\varsigma/2-2\phi}\varrho^2 d\varrho, \qquad (76)$$

where the integration constant has been chosen so that ϕ is regular at $\varrho = 0$.

Supposing the matter's boundary is at $\varrho = R$, we define the (positive) "scalar mass"

$$m_s \equiv 4\pi\int_0^R(\tilde{\rho} + 3\tilde{p})e^{\nu/2+3\varsigma/2-2\phi}\varrho^2 d\varrho. \qquad (77)$$

Because for a nonrelativistic fluid $\tilde{p} \ll \tilde{\rho}$, m_s must be close to the Newtonian mass. In fact, as shown in Appendix D, m_s and an appropriately defined gravitational mass m_g differ only by a fraction of $\mathcal{O}(Gm_g/R)$ which amounts to 10^{-5} for the inner solar system. For $\varrho > R$ we may expand ϕ' as

$$\phi' = \frac{kGm_s}{4\pi\mu}\left[\frac{1}{\rho^2} + \frac{(1 - \beta_1)r_g}{2\varrho^3} + \mathcal{O}(\varrho^{-4})\right]. \qquad (78)$$

It is obvious from this that ϕ *decreases inward.* Its asymptotic value, as will be explained in Sec. VII, is positive and of $\mathcal{O}(k)$. The decrement in ϕ down to "radius" ϱ is, according to Eq. (76), or its integral Eq. (92) below, of $\mathcal{O}(kGm_s/4\pi\varrho)$. In any weakly gravitating system, $Gm_s/\varrho \ll 1$ and for strongly gravitating systems like a neutron star, Gm_s/ϱ is still well below unity (black holes require a special discussion which we defer to another occasion). Thus ϕ remains positive and small throughout space for all systems, and for the solar system, in particular. This will have repercussions for the causality question examined in Sec. VIII.

Since we are not here interested in purely MOND corrections, we shall take $\mu = 1$ in Eq. (78) as well as in the terms in $\tau_{\alpha\beta}$, Eq. (32), which explicitly involve ϕ derivatives. The μ in the F term of $\tau_{\alpha\beta}$ is not so easily disposed of because with our choice of F, and indeed with any viable one, F must be singular at $\mu = 1$. If neglecting the F term in $8\pi G\tau_{\alpha\beta}$ can be justified, then using Eq. (78) we may compute from Eq. (32) that for $\varrho > R$

$$8\pi G\tau_{tt} = 8\pi G\tau_{\varrho\varrho} = \frac{kG^2m_s^2}{4\pi\varrho^4} + \mathcal{O}(\varrho^{-5}). \qquad (79)$$

Now by the approximation (68) the ratio of the F term in $8\pi G\tau_{\alpha\beta}$ to these last terms is

$$\frac{8\pi^2\mu^2|F(\mu)|\varrho^4}{k^3\ell^2G^2{m_s}^2} = \frac{128\pi^4\mathfrak{a}_0^2\mu^2|F(\mu)|}{3k^4|\nabla\Phi_N|^2} \approx \frac{2}{3}(1 - \mu)|F(\mu)|, \qquad (80)$$

which numerically does not exceed 0.04 for $\mu > 0.99$. This justifies Eq. (79) in any region where MOND effects are totally negligible. However, as pointed out in Sec. IV C, at Saturn's orbit μ already departs from unity by 2%. In such cases the contribution of the F term to $\tau_{\alpha\beta}$ must be taken into account, and its post-Newtonian effects compared with the MOND departure from strict Newtonian behavior calculated in Sec. IV C. Here we shall only be concerned with inner solar system dynamics where μ is very close to unity. Because τ_{tt} is dominated by the derivative terms, the energy density contributed by the scalar fields is evidently positive.

Clearly in our situation (see Sec. III C 2)

$$\mathfrak{U}^\alpha = \{e^{-\nu/2}, 0, 0, 0\}. \qquad (81)$$

Using this in Eqs. (33) and (38) we find for $\varrho > R$ that

$$\lambda = \frac{K(2 - \beta_1 - 4\alpha_2){r_g}^2}{4\varrho^4} + \mathcal{O}(\varrho^{-5}), \qquad (82)$$

$$\Theta_{tt} = \frac{K(2\beta_1 - 3 + 8\alpha_2)r_g{}^2}{8\varrho^4} + \mathcal{O}(\varrho^{-5}), \tag{83}$$

$$\Theta_{\varrho\varrho} = -\frac{Kr_g{}^2}{8\varrho^4} + \mathcal{O}(\varrho^{-5}). \tag{84}$$

With this we now turn to Einstein's Eqs. (31) for all ϱ. By virtue of $\mathfrak{U}^\alpha$'s form here, the tt and $\varrho\varrho$ components simplify to

$$-e^{\nu-\varsigma}\left(\varsigma'' + \frac{1}{4}\varsigma'^2 + 2\varsigma'/\varrho\right) = 8\pi G[(2e^{-4\phi} - 1)\tilde{T}_{tt} + \tau_{tt}] + \Theta_{tt}, \tag{85}$$

$$\frac{1}{4}\varsigma'^2 + \frac{1}{2}\varsigma'\nu' + (\varsigma' + \nu')/\varrho = 8\pi G[\tilde{T}_{\varrho\varrho} + \tau_{\varrho\varrho}] + \Theta_{\varrho\varrho}. \tag{86}$$

First we solve these for $\varrho > R$ where $\tilde{T}_{\alpha\beta} = 0$. From Eqs. (73) and (74) it follows that

$$\nu' = r_g/\varrho^2 + (1 - 2\alpha_2)r_g{}^2/\varrho^3 + \cdots, \tag{87}$$

$$\varsigma' = -\beta_1 r_g/\varrho^2 + (\beta_1{}^2 - 2\beta_2)r_g{}^2/\varrho^3 + \cdots. \tag{88}$$

Substituting these together with Eqs. (73), (74), (79) and (83) in Eqs. (84)–(86), matching coefficients of like powers of $1/\varrho$, and solving the three resulting algebraic conditions gives to lowest order in k and K

$$\beta_1 = 1, \tag{89}$$

$$\alpha_2 = \frac{1}{2} + \frac{1}{4}K, \tag{90}$$

$$\beta_2 = \frac{3}{8} - \frac{3}{16}K - \frac{kG^2m_s{}^2}{8\pi r_g{}^2}. \tag{91}$$

Using these results we show in Appendix D that $r_g = 2Gm_g[1 + \mathcal{O}(kGm_g/R) + \mathcal{O}(KGm_g/R)]$ with m_g, the gravitational mass, defined by Eq. (D4). The relative correction here is $\ll 10^{-5}$ for the inner solar system. We also remark that with the values (89)–(91) the energy density contributed by Θ_{tt} is positive [see Eq. (83)].

For solar system tests of T*e*V*e*S we must know the physical metric $\tilde{g}_{\mu\nu}$. According to Eqs. (22) and (81), $\tilde{g}_{tt} = -e^{2\phi+\nu}$, $\tilde{g}_{\varrho\varrho} = \tilde{g}_{\theta\theta}/\varrho^2 = g_{\varphi\varphi}/\varrho^2\sin^2\theta = e^{-2\phi+\varsigma}$, so we need ϕ. Integration of Eq. (78) in light of Eq. (89) gives

$$\phi(\varrho) = \phi_c - \frac{kGm_s}{4\pi\varrho} + \mathcal{O}(\varrho^{-3}), \tag{92}$$

whereupon

$$e^{\pm 2\phi} = e^{\pm 2\phi_c}\left[1 \mp \frac{kGm_s}{2\pi\varrho} + \frac{k^2G^2m_s^2}{8\pi^2\varrho^2} + \mathcal{O}(\varrho^{-3})\right]. \tag{93}$$

The integration constant ϕ_c is evidently the cosmological value of ϕ at the epoch in question. This value changes slowly over solar system time scales, so we can ignore its drift for most purposes. Thus by taking the advantage of the isotropic form of the metric (72), and rescaling the t and ϱ coordinates appropriately, we absorb the factors $e^{2\phi_c}$ and $e^{-2\phi_c}$ that would otherwise appear in $\tilde{g}_{\mu\nu}$ so that it can asymptote to Minkowskian form as expected. With this precaution one can calculate as if ϕ_c vanished. It must be stressed that this strategy works at a particular cosmological era.

Accordingly,

$$\tilde{g}_{tt} = -1 + 2G_N m\varrho^{-1} - 2\beta G_N{}^2 m^2\varrho^{-2} + \mathcal{O}(\varrho^{-3}), \tag{94}$$

$$\tilde{g}_{\varrho\varrho} = 1 + 2\gamma G_N m\varrho^{-1} + \mathcal{O}(\varrho^{-2}), \tag{95}$$

$$G_N m \equiv r_g/2 + (kGm_s/4\pi), \tag{96}$$

$$\beta \equiv 1 + \frac{Kr_g{}^2}{8G_N{}^2m^2} \approx 1 + K/2, \tag{97}$$

$$\gamma \equiv 1. \tag{98}$$

As previously, G_N is defined by Eq. (66). Recalling the relations between r_g, m_g and m_s (Appendix D), we find that $m = m_g[1 + \mathcal{O}(kGm_g/R) + \mathcal{O}(KGm_g/R)]$, i.e., in the inner solar system m and m_g differ fractionally by $\ll 10^{-5}$. Setting $r_g = 2Gm_g = 2Gm$ gives the second form of β. Our results for β and γ are consistent with those obtained by Eling and Jacobson [66] for the relevant case of the Jacobson-Mattingly theory.

The β and γ are the standard post-Newtonian coefficients measurable by the classical tests of gravity [54]. They are both unity in GR. The light deflection and time delay tests are sensitive only to γ which is also unity in T*e*V*e*S, so that these effects cannot distinguish between T*e*V*e*S and GR. The perihelion precession of the planets is proportional to $\frac{1}{3}(2 + 2\gamma - \beta)$ which is unity in GR but $1 - K/6$ in T*e*V*e*S. The measured precession is known to agree with GR's prediction to about three parts in 10^3. Thus K must be smaller than 10^{-2}. There seems to be no observation that prevents us from taking K much smaller, but it must be nonzero so that the $\mathfrak{U}^\alpha$ dynamics can act to align this vector with the time direction.

The β and γ are *not* the only parameterized post-Newtonian coefficients. Future work should look at those coefficients having to do with preferred frame effects, as well as at the Nortvedt effect, which should not be null in T*e*V*e*S.

VI. GRAVITATIONAL LENSING IN T*e*V*e*S

In Sec. V we touched upon gravitational lensing in the Newtonian regime. Here we show that in the low acceleration regime, T*e*V*e*S predicts gravitational lensing of the correct magnitude to explain the observations of intergalactic lensing without any dark matter. First by following the essentially exact method of Ref. [44], we show this for a spherically symmetric structure; in nature many elliptical galaxies and galaxy clusters are well modeled as spherically symmetric. We then use linearized theory to give a short proof of the same result for asymmetric systems. Our discussion refers to lensing of both rays that pass through the system and those that skirt it, and is thus a generalization of the implicit result about light deflection in Sec. V in more than one way.

A. Spherically symmetric systems

We adopt the Einstein metric (72); the physical metric is obtained by replacing $e^\nu \to e^{\nu+2\phi}$ and $e^\varsigma \to e^{\varsigma-2\phi}$ in it. Consider a light ray which propagates in the equatorial plane of the metric (which may, of course, be chosen to suit any light ray). The 4-velocity $\dot{x}^\alpha$ of the ray (derivative taken with respect to some suitable parameter) must satisfy

$$-e^{\nu+2\phi}\dot{t}^2 + e^{\varsigma-2\phi}(\dot{\varrho}^2 + \varrho^2\dot{\varphi}^2) = 0. \tag{99}$$

From the metric's stationarity follows the conservation law $e^{\nu+2\phi}\dot{t} = E$ where E is a constant characteristic of the ray. From spherical symmetry it follows that $e^{\varsigma-2\phi}\varrho^2\dot{\varphi} = L$ where L is another constant property of the ray. Let us write $\dot{\varrho} = (d\varrho/d\varphi)\dot{\varphi}$. Now eliminating $\dot{t}$ and $\dot{\varphi}$ from Eq. (99) in favor of E and L and dividing by E^2 yields

$$-e^{-\nu-2\phi} + (b/\varrho)^2 e^{-\varsigma+2\phi}[\varrho^{-2}(d\varrho/d\varphi)^2 + 1] = 0, \tag{100}$$

where $b \equiv L/E$. By going to infinity where the metric factors approach unity one sees that b is just the ray's impact parameter with respect to the matter distribution's center at $\varrho = 0$. This last equation has the quadrature

$$\varphi = \int^{\varrho}\left[e^{\varsigma-\nu-4\varphi}\left(\frac{\varrho}{b}\right)^2 - 1\right]^{-1/2}\frac{d\varrho}{\varrho}. \tag{101}$$

Were the physical metric exactly flat, this relation would describe a line with φ varying from 0 to π as ϱ decreased from infinity to its value $\varrho_{\rm turn}$ at the turning point and then returned to infinity. Hence the deflection of the ray due to gravity is

$$\Delta\varphi = 2\int_{\varrho_{turn}}^{\infty}\left[e^{\varsigma-\nu-4\varphi}\left(\frac{\varrho}{b}\right)^2 - 1\right]^{-1/2}\frac{d\varrho}{\varrho} - \pi. \tag{102}$$

This last integral is difficult. So let us take advantage of the weakness of extragalactic fields which allow us to assume that ν, ς and ϕ are all small compared to unity. Then the above result is closely approximated by

$$\Delta\varphi = -4\frac{\partial}{\partial\alpha}\int_{\varrho_{turn}}^{\infty}\left[(1+\varsigma-\nu-4\varphi)\left(\frac{\varrho}{b}\right)^2 - \alpha\right]^{1/2} \times \frac{d\varrho}{\varrho}\bigg|_{\alpha=1} - \pi. \tag{103}$$

The rewriting in terms of an α derivative allows us to Taylor expand the radical in the small quantity $\varsigma - \nu - 4\varphi$ without incurring a divergence of the integral at its lower limit. The zeroth order of the expansion yields a well-known integral which cancels the π. Thus, to first order in small quantities

$$\Delta\varphi = -\frac{2}{b}\frac{\partial}{\partial\alpha}\int_{b\sqrt{\alpha}}^{\infty}\frac{(\varsigma-\nu-4\phi)\varrho d\varrho}{(\varrho^2-\alpha b^2)^{1/2}}\bigg|_{\alpha=1}. \tag{104}$$

At this point it pays to integrate by parts:

$$\Delta\varphi = -\frac{2}{b}\frac{\partial}{\partial\alpha}\bigg[\lim_{\varrho\to\infty}(\varsigma-\nu-4\phi)(\varrho^2-\alpha b^2)^{1/2} - \int_{b\sqrt{\alpha}}^{\infty}(\varsigma'-\nu'-4\phi')(\varrho^2-\alpha b^2)^{1/2}d\varrho\bigg]\bigg|_{\alpha=1}. \tag{105}$$

Since ν, ς and ϕ all decrease asymptotically as ϱ^{-1}, the integrated term, being α independent, contributes nothing. Carrying out the α derivative, and introducing the usual Cartesian x coordinate along the initial ray by $x \equiv \pm(\varrho^2 - b^2)^{1/2}$, we have

$$\Delta\varphi = \frac{b}{2}\int_{-\infty}^{\infty}\frac{\nu'-\varsigma'+4\phi'}{\varrho}dx. \tag{106}$$

A factor 1/2 appears because we have included the integral in Eq. (105) twice, once with ϱ decreasing to, and once with ϱ increasing from b. The integral is now performed over an infinite straight line following the original ray.

The difference between GR with dark matter and T*e*V*e*S in this respect is that with dark matter one would have $\phi = 0$ and would compute ν and ς from Einstein's equations including dark matter as source, whereas in T*e*V*e*S one has a nontrivial ϕ and computes ν and ς on the basis of the visible matter alone.

We may simplify the above result by means of Einstein's Eq. (86). We shall neglect the ς'^2 and $\varsigma'\nu'$ terms because they are of second order, and thus smaller than ν'/ϱ by a factor $G\cdot$mass$/\varrho$ which amounts to v^2, with v the typical orbital velocity in the system. Using the residual terms we eliminate ς' from Eq. (106):

$$\Delta\varphi = b\int_{-\infty}^{\infty}\frac{\nu'+2\phi'}{\varrho}dx - 4\pi Gb\int_{-\infty}^{\infty}(\tilde{T}_{\varrho\varrho} + \tau_{\varrho\varrho} + \Theta_{\varrho\varrho}/8\pi G)dx. \tag{107}$$

Now by Sec. IVA, $\nu = 2V + \mathcal{O}(V^2)$ and $\Phi = V + \phi$. Hence with fractional corrections of $\mathcal{O}(V^2)$,

$$\Delta\varphi = 2b\int_{-\infty}^{\infty}\frac{\Phi'}{\varrho}dx - 4\pi Gb\int_{-\infty}^{\infty}(\tilde{T}_{\varrho\varrho} + \tau_{\varrho\varrho} + \Theta_{\varrho\varrho}/8\pi G)dx. \tag{108}$$

The first integral here depends exclusively on the potential Φ which determines nonrelativistic motion. That is, the observed stellar or galactic dynamics will uniquely fix this part of $\Delta\varphi$. For this reason the first term makes the same predictions for lensing by nonrelativistic systems in TeVeS as in GR (where $\Phi = \Phi_N$, the last calculated assuming dark matter). We next show that for nonrelativistic systems the second integral is negligible.

In astrophysical matter the radial pressure $\tilde{T}_{\varrho\varrho}$ is of order $\tilde{\rho}$ times the local squared random velocity of the matter particles (stars, gas clouds, galaxies). Thus $\int \tilde{T}_{\varrho\varrho} dx = \langle v^2\rangle \int \tilde{\rho} dx$ with $\langle v^2\rangle$ a suitably averaged v^2. But by Poisson's equation $4\pi G\tilde{\rho} = \nabla\cdot\nabla\Phi_N \sim {\Phi_N}'/\varrho = \tilde{\mu}\Phi'/\varrho$ where we have also used Eq. (60). Thus the term with the integral over $\tilde{T}_{\varrho\varrho}$ is smaller than the first term in Eq. (108) by a factor of $\mathcal{O}(\tilde{\mu}\langle v^2\rangle)$. In GR (for which effectively $\tilde{\mu} = 1$) this factor is no larger than 10^{-5} for all extragalactic systems which have a missing mass problem; inTeVeS it is even smaller because typically $\tilde{\mu} \ll 1$ for such systems.

Turning now to $\tau_{\varrho\varrho}$ we recall from Sec. IVA that in the quasistatic situation in question, the F part is dominated by the term quadratic in ϕ derivatives. Using Eqs. (59) and (60) we work out that $4\pi G\tau_{\varrho\varrho} \approx (k\tilde{\mu}/8\pi\mu)\Phi'{\Phi_N}'$. Evidently $\Phi' \sim \Phi/\varrho$, and since $\Phi = \mathcal{O}(v^2)$ and $(k\tilde{\mu}/8\pi\mu) < \frac{1}{2}$, the contribution of $\tau_{\varrho\varrho}$ to the second term of Eq. (108) is no larger than that coming from $\tilde{T}_{\varrho\varrho}$.

Finally we note that the λ term in $\Theta_{\varrho\varrho}$ vanishes in a quasistatic situation because then $\mathfrak{U}_\alpha \approx -(1+\Phi_N)\delta_{\alpha t}$. And from this last formula we estimate $|\Theta_{\varrho\varrho}| \approx \frac{1}{2}K({\Phi_N}')^2 \sim K\tilde{\mu}^2|\Phi\Phi'|/\varrho$. Since $\tilde{\mu} < 1$ and by Sec. V we must take $K < 10^{-2}$, it is clear that the contribution of $\Theta_{\varrho\varrho}$ is much smaller than that coming from $\tilde{T}_{\varrho\varrho}$. From all the above the light ray deflection in TeVeS is

$$\Delta\varphi = 2b[1 + \mathcal{O}(\tilde{\mu}v^2)]\int_{-\infty}^{\infty}\frac{\Phi'}{\varrho}dx. \tag{109}$$

In GR with dark matter the same formula is valid with $\mathcal{O}(\tilde{\mu}v^2)$ replaced by $\mathcal{O}(v^2)$. Since these corrections are beyond foreseeable accuracy of extragalactic astronomy, it is clear that for given dynamics (given Φ), both theories predict identical lensing. We shall elaborate on this statement shortly.

B. Asymmetric systems

We now turn to systems with no particular spatial symmetry. The weakness of the gravitational potentials typical of nonrelativistic systems entitles us to use linearized theory [67] in which the metric is viewed as a perturbed Lorentz metric:

$$g_{\alpha\beta} = \eta_{\alpha\beta} + \bar{h}_{\alpha\beta} - \frac{1}{2}\eta_{\alpha\beta}\eta^{\gamma\delta}\bar{h}_{\gamma\delta}, \tag{110}$$

with $|\bar{h}_{\alpha\beta}| \ll 1$. By small coordinate transformations one enforces the gauge conditions $\eta^{\beta\delta}\bar{h}_{\gamma\delta,\beta} = 0$; as a consequence to first order in the $\bar{h}$ fields

$$G_{\alpha\beta} = -\frac{1}{2}\eta^{\gamma\delta}\partial_\gamma\partial_\delta\bar{h}_{\alpha\beta}, \tag{111}$$

so that Einstein's equations take the form of wave equations in flat spacetime with the r.h.s. of Eq. (31) as sources. Of course there are motions and changes in galaxies and clusters of galaxies, but the associated changes in the metric are mostly very slow. Thus we confine ourselves to quasistationary situations where we can drop time derivatives (but not yet the g_{ti} components since galaxies do rotate). This tells us that

$$G_{tt} = -\frac{1}{2}\nabla^2\bar{h}_{tt} = 8\pi G[\tilde{T}_{tt} + 2(1-e^{-4\phi})\mathfrak{U}^\mu\tilde{T}_{\mu t}\mathfrak{U}_t + \tau_{tt}] + \Theta_{tt}. \tag{112}$$

The various parts of the source here were explored in Sec. IVA; from that discussion it follows that

$$\bar{h}_{tt} = -4V = -4\Xi\Phi_N. \tag{113}$$

In regard to the spatio-temporal source components of Eq. (31), we observe that the $\tilde{T}_{it}$ is an $\mathcal{O}(v)$ below $\tilde{T}_{tt}$ (momentum density is velocity times mass density). Further, the dominant contributions to τ_{ti} are $\bar{h}_{ti}$ multiplied by $\sigma^2\eta^{jk}\phi_{,j}\phi_{,k}$ and by $(G/\ell^2)\sigma^4F$. Of these the first dominates (see Sec. IVA), and it is small on the scale of $\tilde{\rho}$ both because it is of second order (c.f. Sec. V), and because $|\bar{h}_{ti}| \ll 1$. We can guess that $\mathfrak{U}_i$ is at most of order $\bar{h}_{ti}$ (it would vanish in a truly static situation), and since by Eq. (56) λ is below $8\pi G\tilde{\rho}$ by factors of $\mathcal{O}(K)$ and $\mathcal{O}(\phi_c)$, we see that the $\lambda\mathfrak{U}_t\mathfrak{U}_i$ term contribution to Θ_{ti} is small compared to $8\pi G\tilde{\rho}$. Similarly, the $Kg^{\mu\nu}\mathfrak{U}_{[\mu,t]}\mathfrak{U}_{[\nu,i]}$ contribution to Θ_{ti}, being of second order in $V_{,i}$ and first order in $\bar{h}_{ti}$, or first order in $V_{,i}$ and first order in $\bar{h}_{ti,j}$ (aside of carrying the small coefficient K), must be very small. We conclude that the source of the spatio-temporal Einstein equation can be neglected, so that to the accuracy of Eq. (113), $\bar{h}_{ti} \approx 0$.

Things are similar for the spatial-spatial components. We have already remarked that $\tilde{T}_{ij}$ is an $\mathcal{O}(v^2)$ below $\tilde{T}_{tt}$. The τ_{ij} consists of a term quadratic in $\phi_{,i}$ and one with a F factor which has been argued to be smaller. Hence τ_{ij} is small. Again the $Kg^{\mu\nu}\mathfrak{U}_{[\mu,i]}\mathfrak{U}_{[\nu,j]}$ contributions to Θ_{ij} are quadratic in $V_{,i}$ and suppressed by the K coefficient, so they are also small. And the λ, which we remarked above to be small, is multiplied by two factors $\bar{h}_{ti}$, and so is also small. So by the same logic as above we neglect the sources of the spatial-spatial components $\bar{h}_{ij}$ and conclude that $\bar{h}_{ij} \approx 0$.

Substituting all these results in Eq. (110) we obtain

$$g_{\alpha\beta} = (1-2V)\eta_{\alpha\beta} - 4V\delta_{\alpha t}\delta_{\beta t}. \tag{114}$$

The absence of g_{ti} in this approximation makes the situation truly static (rather than just stationary); hence $\mathfrak{U}^\alpha = \delta^\alpha_t$. Calculating the physical metric from Eq. (22) with

$e^{\pm 2\phi} \approx 1 \pm 2\phi$ we have

$$\tilde{g}_{\alpha\beta} = (1 - 2V - 2\phi)\eta_{\alpha\beta} - 4(V + \phi)\delta_{\alpha t}\delta_{\beta t}, \quad (115)$$

which is equivalent to

$$\tilde{g}_{\alpha\beta}dx^\alpha dx^\beta = -(1 + 2\Phi)dt^2 + (1 - 2\Phi)\delta_{ij}dx^i dx^j, \quad (116)$$

with $\Phi = V + \phi$ as in Sec. IVA.

Metric (116) has the same form as the GR metric for weak gravity [67]. Thus in T*e*V*e*S just as in GR the same potential governs dynamics and gravitational lensing. This accords with the conclusion of Sec. VI A for the spherically symmetry case. What does this mean in practice? In GR Φ's role is played by the Newtonian potential due to the visible matter together with the putative dark matter; in T*e*V*e*S Φ is the sum of the scalar field and the renormalized Newtonian potential generated by the visible matter alone. These two prescriptions for Φ need not agree *a priori*, but as we argued in Sec. IV B, nonrelativistic dynamics in T*e*V*e*S are approximately of MOND form, and MOND's predictions have been found to agree with much of galaxy dynamics phenomenology. We thus expect T*e*V*e*S's predictions for gravitational lensing by galaxies and some clusters of galaxies to be as good as those of dark halo models within GR. But, of course, the early MOND formula (1), and T*e*V*e*S with our choice (51) for $F(\mu)$ both claim that asymptotically the potential Φ of an isolated galaxy grows logarithmically with distance indefinitely. Dark halo models do not. So T*e*V*e*S for a specific choice of F is in principle falsifiable. Dark matter is less falsifiable because of the essentially unlimited choice of halo models and choices of their free parameters. One should also remember that gravitational lensing affords the opportunity to map the Φ to greater distances than can dynamics; for unlike the latter, lensing can be measured outside the gas or galaxy distribution. Using this Φ both GR and T*e*V*e*S would predict the same dynamics for stars or galaxies, while disagreeing on the implied distribution of mass.

VII. COSMOLOGICAL EVOLUTION OF ϕ

A. Persistence of cosmological expansion

This section (where we write ϕ rather than ϕ_c) shows that for a range of initial conditions, FRW cosmological models with flat spaces in T*e*V*e*S expand forever, have $0 \leq \phi \ll 1$ throughout, and their law of expansion is very similar to that in GR. The second point is crucial for our discussion of causality in Sec. VIII.

First using Eq. (22) we transform metric (43) to the physical metric

$$\tilde{g}_{\alpha\beta}dx^\alpha dx^\beta = -d\tilde{t}^2 + \tilde{a}(\tilde{t})^2[d\chi^2 + f(\chi)^2(d\theta^2 + \sin^2\theta d\varphi^2)], \quad (117)$$

$$d\tilde{t} = e^\phi dt; \qquad \tilde{a} = e^{-\phi}a. \quad (118)$$

In what follows we take the initial moment, conventionally written as $\tilde{t} = 0$, at the end of the quantum era with $\tilde{a}(0)$ a very small scale; furthermore we take the zero of t to coincide with $\tilde{t} = 0$. For illustration we assume the initial conditions $\dot{\phi}(0) = 0$ (an overdot always denotes $\partial/\partial t$) and $0 < \phi_0 \equiv \phi(0) \ll 1$. Hence a also starts off from a very small scale, a_0, and can only increase initially.

We now show that the spatially flat ($f(\chi) \equiv \chi$) FRW models in T*e*V*e*S persist and cannot recollapse, i.e., $\tilde{a}$ has no finite maximum. As in Sec. III C 1 we have $\mathfrak{U}^\alpha = \delta_t{}^\alpha$ which causes $\mathfrak{U}^{[\alpha;\beta]}$ to vanish. As a consequence $\Theta_{\alpha\beta} = -\lambda\delta^t_\alpha\delta^t_\beta$ with λ given by Eq. (46). Since $\phi = \phi(t)$, Eq. (32) gives $\tau_{tt} = 2\sigma^2\dot{\phi}^2 + G(4\ell^2)^{-1}\sigma^4F(\mu)$. As mentioned in Sec. III C 1, $\mathfrak{U}^\beta\tilde{T}_{\alpha\beta} = -\tilde{\rho}e^{2\phi}\mathfrak{U}_\alpha$. Using $g^{\alpha\beta}\mathfrak{U}_\alpha\mathfrak{U}_\beta = -1$ gives us $\tilde{T}_{tt} + (1 - e^{-4\phi})\mathfrak{U}^\alpha\tilde{T}_{\alpha(t}\mathfrak{U}_{t)} = (2e^{-4\phi} - 1)\tilde{\rho}e^{2\phi}$. Substituting all the above in the tt component of Eq. (31), we get the following analog of Friedmann's equation:

$$\frac{\dot{a}^2}{a^2} = \frac{8\pi G}{3}\tilde{\rho}e^{-2\phi} + \frac{8\pi G\sigma^2\dot{\phi}^2}{3} + \frac{2\pi}{3k^2\ell^2}\mu^2F(\mu) = \frac{8\pi G}{3}\tilde{\rho}e^{-2\phi} + \frac{4\pi}{3k^2\ell^2}\left[-\mu y(\mu) + \frac{1}{2}\mu^2F(\mu)\right]. \quad (119)$$

With the choice (50) for $y(\mu)$ we have $\mu > 0$, $y(\mu) < 0$ and $F > 0$ in the cosmological domain. Thus the scalar fields contribute positive energy density and the r.h.s. of Eq. (119) is positive definite ($\tilde{\rho} < 0$ is physically unacceptable). It follows that $\dot{a}$ cannot vanish for any t, so that by our earlier remark it must always be positive. Now the relations (118) imply that

$$d\tilde{a}/d\tilde{t} = e^{-2\phi}(\dot{a} - a\dot{\phi}). \quad (120)$$

We shall show in the sequel that although $\dot{\phi}$ can be positive, it is always the case that $|\dot{\phi}| \ll \dot{a}/a$. As a consequence $d\tilde{a}/d\tilde{t}$ is always strictly positive: in T*e*V*e*S a FRW model with flat spaces cannot recollapse.

The fact that $\dot{\phi}$ is given by an integral over time [see Eq. (45)] means that in a cosmological phase transition, where $\tilde{\rho}$ may change suddenly, $\dot{\phi}$ (and of course ϕ) will nevertheless evolve continuously in time. It follows that F will also evolve continuously in time [see Eq. (36)]. A consequence of Eq. (119) is that any jump in $\tilde{\rho}$ will be reflected in a similar jump in $(\dot{a}/a)^2$ or in the square of the Hubble function $\tilde{H} \equiv \tilde{a}^{-1}d\tilde{a}/d\tilde{t}$.

B. The proto-radiation era

Contemporary cosmology regards the inflationary era as preceded by a brief radiation dominated era, the proto-radiation era, in which the physical scale factor $\tilde{a}$ expands by just a few orders following the quantum gravity regime. As in any radiation dominated regime, here the equation of state is $\tilde{\rho} = 3\tilde{p}$ with both $\tilde{p}$ and $\tilde{\rho}$ varying as $\tilde{a}^{-4}$. It follows from Eq. (45) that throughout the era

$$\mu\dot{\phi} = -\frac{k}{a^3}\int_0^t G\tilde{\rho}e^{-2\phi}a^3 dt. \tag{121}$$

Because in the cosmological regime $\mu > 2$, we have $\dot{\phi} < 0$ throughout this era. Thus as promised $d\tilde{a}/d\tilde{t}$ in Eq. (120) is positive. Using the constancy of $(G\tilde{\rho})^{1/2}a^2e^{-2\phi}$ we can now write

$$\mu\dot{\phi} = -\frac{k(G\tilde{\rho})^{1/2}e^{-2\phi}}{a}\int_0^t (G\tilde{\rho})^{1/2}a dt. \tag{122}$$

Tentatively assuming that $|\phi| \ll 1$ throughout the era we may, according to Eq. (119), bound both instances of $(G\tilde{\rho})^{1/2}$ from above by $(3/8\pi)^{1/2}\dot{a}/a$. The consequent integral is then trivial, and since a_0 is essentially zero we get

$$\mu|\dot{\phi}| < (3k/8\pi)(\dot{a}/a). \tag{123}$$

Thus $|\dot{\phi}| < (3k/16\pi)(\dot{a}/a)$; since $k \ll 1$, we have by Eq. (120) that $d\tilde{a}/d\tilde{t} \approx \dot{a}$.

We can now show that the cosmological evolution during the proto-radiation era is very similar to that within GR. For the choice (51) both F and F' are positive in the cosmological domain (see Fig. 2). It follows from Eq. (36) that $\mu^2F < -\mu y$ (recall that $y < 0$), so the last term on the r.h.s. of the Friedmann equation is less than half the second. Next we use $y = -2k\ell^2\dot{\phi}^2$ to infer from Eq. (123) that

$$\frac{4\pi}{3k^2\ell^2}\mu|y| < \frac{3k}{8\pi\mu}\left(\frac{\dot{a}}{a}\right)^2. \tag{124}$$

But this means that the scalar field contributions to the Friedmann equation are small compared to its l.h.s. Specifically, to within a fractional correction of $\mathcal{O}(k/16)$ (actually smaller than this because μ will turn out to be large), the relation between $\tilde{H}$ and $\tilde{\rho}$ is the same as in GR.

The fact that the scalar field contributions to the Friedmann equation are small compared to its l.h.s. also means that inequality (123) is nearly saturated, as must be its kin (124). Then

$$\mu^2|y(\mu)| \approx \frac{1}{6}(3k/4\pi)^4(\dot{a}/a)^2\mathfrak{a}_0{}^{-2}. \tag{125}$$

But $a/\dot{a}$ is a very short scale (in standard cosmological models $\tilde{H}^{-1} \sim 10^{-35}$ s in the proto-radiation era) while $\mathfrak{a}_0^{-1} \sim 3\times 10^{18}$ s. Thus $\mu^2 y(\mu) \gg 1$. Since by Eq. (50) this is possible only for $\mu \gg 1$, we can sharpen our earlier conclusion from Eq. (123): $|\dot{\phi}| \ll (3k/8\pi)\dot{a}/a$. Now it is even clearer that a and $\tilde{a}$ (as well as t and $\tilde{t}$) are essentially equal, so that the expansion in this era proceeds just as in GR. Further, integrating this last inequality gives

$$|\phi_{pr} - \phi_0| \ll (3k/8\pi)\ln(a_{pr}/a_0), \tag{126}$$

where the subscript "pr" stands for the end of the proto-radiation era. Since this era spans just a few e-foldings of the scale a, the logarithm here is of order unity. Hence ϕ is almost frozen at its initial value ϕ_0, provided this last is not extremely small. By choosing as initial condition $0 < \phi_0 \ll 1$, as we proposed, but avoiding extremely small ϕ_0, we get $0 < \phi \ll 1$ throughout the proto-radiation era, as assumed earlier. Thus our assumption was consistent.

C. The inflationary era

The equation of state during inflation is $\tilde{p} = -\tilde{\rho} = \text{const}$. Then (45) tells us that

$$\mu\dot{\phi} = \frac{k}{a^3}\int_{t_{pr}}^t G\tilde{\rho}e^{-2\phi}a^3 dt + \mu_{pr}\dot{\phi}_{pr}\left(\frac{a_{pr}}{a}\right)^3. \tag{127}$$

The integration constant prefacing the last term is fixed by the condition that μ and $\dot{\phi}$ be continuous through the proto-radiation inflation divide. It is clear that after rapid expansion has suppressed the last (negative) term here, $\dot{\phi}$ becomes positive. Because $\tilde{\rho}$ is constant, we may pull a factor $(G\tilde{\rho})^{1/2}$ out of the integral. Then by Eq. (119) and assuming everywhere that $e^{-\phi} \approx 1$ (which we verify below), we have $(G\tilde{\rho})^{1/2}e^{-2\phi} < (3/8\pi)^{1/2}\dot{a}/a$ both in and outside the integral. Thus

$$\mu\dot{\phi} < \frac{3k\dot{a}}{8\pi a^4}\int_{t_{pr}}^t a^2\dot{a}dt + \mu_{pr}\dot{\phi}_{pr}\left(\frac{a_{pr}}{a}\right)^3 \tag{128}$$

$$= \frac{k\dot{a}}{8\pi a}\left(1 - \frac{a_{pr}^3}{a^3}\right) - \frac{3k}{8\pi}\left(\frac{\dot{a}}{a}\right)_{pr}\left(\frac{a_{pr}}{a}\right)^3. \tag{129}$$

where we have used Eq. (123) as an equality as the end of the proto-radiation era. Thus during inflation

$$-(3k/8\pi)(\dot{a}/a)_{pr} < \mu\dot{\phi} < (k/8\pi)(\dot{a}/a). \tag{130}$$

The l.h.s. here comes from the last term in Eq. (127) in light of inequality (123). In the passage from the proto-radiation era, which involves a phase transition, $\tilde{\rho}$ can change by a factor of order unity, but then settles down to a constant. Thus by Eq. (119) $\dot{a}/a$ remains *at least* of the same order of magnitude as $(\dot{a}/a)_{pr}$. Hence inequality (130) translates into one of the same form as (123) but valid during inflation. As in Sec. VII B, this tells us that $d\tilde{a}/d\tilde{t} \approx \dot{a}$ also during inflation. And the argument following inequality (123) can now be repeated to show that the $-\mu y$ and μ^2F terms in Friedmann's equation amount to relative corrections of $\mathcal{O}(k/16)$ (actually smaller), so that inflation in T*e*V*e*S proceeds very much like in GR.

Repeating the argument leading to Eq. (129) in light of this last conclusion and the added realization that the a^{-3} terms disappear very rapidly, we conclude that during the

$\dot{\phi} > 0$ part of inflation, inequality (123) is very nearly saturated. One can then rederive Eq. (125) as in Sec. VII B. Because the inflation timescale is again very short compared to $\mathfrak{a}_0^{-1}$, the argument yielding Eq. (126) can be repeated with slight modifications to show that during inflation $\mu \gg 1$, and consequently

$$\phi_i - \phi_{pr} \ll (3k/8\pi)\ln(a_i/a_{pr}), \tag{131}$$

where a subscript "i" stands for the end of inflation. Thus, although in standard models inflation can span up to 70 e-foldings of a, the r.h.s. of this inequality is very small compared to unity. We conclude that inflation manages to raise ϕ above its value at the end of the proto-radiation era by a very small fraction of unity. This justifies our replacement of $e^{-\phi}$ by unity in deriving Eq. (129).

In what follows we shall denote by $\tilde{H}_i$, μ_i, ϕ_i and $\dot{\phi}_i$ the values of the Hubble parameter, $\mu(-2k\ell^2\dot{\phi}^2)$, ϕ and $\dot{\phi}$, respectively, at the end of inflation, $t = t_i$, where $a = a_i$.

D. The radiation era

In the ensuing radiation era the equation of state switches back to $3\tilde{p} = \tilde{\rho}$ with both $\tilde{p}$ and $\tilde{\rho}$ varying as $\tilde{a}^{-4}$. Thus the integral in Eq. (45) is

$$\mu\dot{\phi} = -\frac{k}{a^3}\int_{t_i}^{t} G\tilde{\rho}e^{-2\phi}a^3 dt + \mu_i\dot{\phi}_i\left(\frac{a_i}{a}\right)^3, \tag{132}$$

with the integration constant $\mu_i\dot{\phi}_i$ set so $\mu\dot{\phi}$ at the radiation's era outset equals that at inflation's end. Although initially $\dot{\phi} > 0$, clearly the integral will eventually dominate the last term making $\dot{\phi}$ negative thereafter.

Now according to Eq. (129), $\mu_i\dot{\phi}_i < (k/8\pi)(\dot{a}/a)_i$. Because of the approximate continuity of $(\dot{a}/a)$ across the inflation-radiation eras divide [which itself follows from the approximate continuity of $\tilde{\rho}$ and Eq. (119)], and from the fact that $(\dot{a}/a)$ falls off no faster than $(a_i/a)^2$ in the radiation era, Eq. (132) gives

$$\mu\dot{\phi} < (k/8\pi)(\dot{a}/a)_i(a_i/a)^3 < (k/8\pi)(\dot{a}/a). \tag{133}$$

On the other hand, from $\tilde{\rho}\tilde{a}^4 = \text{const.}$ we can move a factor $(G\tilde{\rho})^{1/2}a^2e^{-2\phi}$ out of the integral in Eq. (132). Using again $(G\tilde{\rho})^{1/2} < (3/8\pi)^{1/2}\dot{a}/a$ from Eq. (119) (if we assume provisionally that $e^{-\phi} \approx 1$) both in and outside the integral, we have

$$\mu\dot{\phi} > -\left(\frac{3k\dot{a}}{8\pi a^2}\right)\int_{t_i}^{t}(\dot{a}/a)a dt + \mu_i\dot{\phi}_i\left(\frac{a_i}{a}\right)^3. \tag{134}$$

The integral is $a(t) - a_i$. Hence

$$\begin{aligned}\mu\dot{\phi} &> (-3k/8\pi)(1 - a_i/a)(\dot{a}/a) + \mu_i\dot{\phi}_i(a_i/a)^3\\ &> -(3k/8\pi)(\dot{a}/a).\end{aligned} \tag{135}$$

In view of Eqs. (133) and (135), inequality (123) is again valid here. Because $\mu > 2$ we get again from Eq. (120) that $d\tilde{a}/d\tilde{a} \approx \dot{a}/a$. We may now reproduce inequality (124) and show as in Sec. VII B that to within a fractional correction of $\mathcal{O}(k/16)$, the relation between $\tilde{H}$ and $\tilde{\rho}$ is the same as in GR.

Because of this last result, Eq. (133) and the rapid decay of a_i/a in Eq. (135), we may conclude that when $\dot{\phi} < 0$, inequality (123) is nearly saturated. We may then rederive Eq. (125) as before. Now in conventional cosmology at redshift z during the radiation era $\tilde{H} \sim 3 \times 10^{-20}(1+z)^2\text{s}^{-1}$, which by previous inference closely approximates $\dot{a}/a$ in our model. We thus obtain $\mu^2|y(\mu)| \approx 5 \times 10^{-6}k^4(1+z)^4$. Taking $k \sim 0.03$ on the basis of Sec. IV C we see that at the end of the radiation era ($z \approx 10^4$), $\mu^2|y(\mu)| \approx 4 \times 10^4$ which corresponds to $\mu \approx 10$. For earlier times $\mu \propto (1+z)^{4/5}$ so that it rises to 10^{19} at the beginning of the era at $z \approx 10^{27}$. Going back to inequality (123) we see that in the last three e-foldings of the era $\phi(t) - \phi_i > -8 \times 10^{-4}$ with the previous 50 e-foldings contributing an even smaller decrease. Our assumption that $e^{-\phi} \approx 1$ was evidently justified if ϕ_0 is taken small compared to unity, yet sufficiently positive to keep $\phi(t)$ positive throughout the era.

We shall denote by μ_r, ϕ_r and $\dot{\phi}_r$ the values of $\mu(-2k\ell^2\dot{\phi}^2)$, ϕ and $\dot{\phi}$, respectively, at the end of the radiation era, $t = t_r$ where $a = a_r$.

E. The matter era

In the matter era $\tilde{p} \approx 0$ and $\tilde{\rho}$ varies as $\tilde{a}^{-3}$. Integrating Eq. (45) gives, c.f. Eq. (132)

$$\mu\dot{\phi} = \frac{-k}{2a^3}\int_{t_r}^{t} G\tilde{\rho}e^{-2\phi}a^3 dt + \mu_r\dot{\phi}_r\left(\frac{a_r}{a}\right)^3. \tag{136}$$

It is clear that $\dot{\phi}$ continues to be negative throughout the matter era. Using $\tilde{\rho}a^3e^{-3\phi} = \text{const.}$ and setting henceforth $e^{\phi} = 1$ (whose consistency will be checked below), we explicitly perform the integral in Eq. (136) from t_r to t:

$$\mu\dot{\phi} = -\frac{1}{2}kG\tilde{\rho}(t - t_r) + \mu_r\dot{\phi}_r(a_r/a)^3. \tag{137}$$

Integrating the inequality $(G\tilde{\rho}a^3)^{1/2} < (3/8\pi)^{1/2}a^{1/2}\dot{a}$ coming from Eq. (119) we get $(G\tilde{\rho})^{1/2}(t - t_r) < (2/3) \times (3/8\pi)^{1/2}$. Both together give $G\tilde{\rho}(t - t_r) < (\dot{a}/4\pi a)$, which when substituted in Eq. (137) finally gives

$$\mu\dot{\phi} > (-k/8\pi)(\dot{a}/a) + \mu_r\dot{\phi}_r(a_r/a)^3. \tag{138}$$

Now according to Eq. (135) $\mu_r\dot{\phi}_r > (-3k/8\pi)(\dot{a}/a)_r$. Thus at the beginning of the matter era, where $a = a_r$, the lower bound on the second term on the r.h.s. of inequality (138) maybe as much as 3 times larger in magnitude than the first term, yet it decays as a^{-3} while the first term cannot do so faster than $a^{-3/2}$ [see Friedmann's

Eq. (119)]. Hence within about one e-folding of a, the first term comes to dominate the r.h.s., and over most of the matter era

$$\mu|\dot{\phi}| < (k/8\pi)(\dot{a}/a). \quad (139)$$

From this follows a tighter version of bound (124) which again demonstrates that the scalar field terms in Einstein's equations are rather negligible. The fact that (139) may be exceeded by a factor of a few early in the matter era is no reason to exclude that epoch from the just mentioned conclusion: the rather large μ at the end of the radiation era ($\mu \sim 10$)—and a bit beyond—acts to suppress that factor. Using by now well worn logic we conclude that in the matter era as well, the relation between $\tilde{H}$ and $\tilde{\rho}$ is almost the same as in GR.

Integrating inequality (139) with the use of $\mu > 2$ (the first e-folding's relatively larger contribution is suppressed by the larger μ which holds sway then), we get

$$\phi(t) - \phi_r > -(k/16\pi)\ln(a/a_r). \quad (140)$$

Because the matter era thus far has spanned nine e-foldings, ϕ has decreased by less than 0.0054 during this era.

Note that we have not addressed the cosmological matter problem. In TeVeS the expansion is driven by just $\tilde{\rho}$, the visible matter's density, whereas the observations require that the source of Friedmann's equation which falls off like $\tilde{a}^{-3}$ should be larger by a factor of perhaps 6. There are at least two possible avenues for dealing with this embarrassment. First, we have stuck to a particular $F(\mu)$; possibly a more realistic $F(\mu)$ would change late cosmological evolution enough to resolve the problem. Second, we have insisted on ϕ being small. This is a consistent solution as we have shown, but it is perhaps not the unique solution. Plainly non-negligible values of ϕ can affect the Friedmann equation significantly.

F. The accelerating expansion

Lately data from distant supernovae indicate that in recent times ($z < 0.5$) the cosmological expansion has began to accelerate, namely, that $d\tilde{H}/d\tilde{t} > -\tilde{H}^2$. The data are best interpreted in GR by accepting the existence a positive cosmological constant $\Lambda \approx 2\tilde{H}^2_{\rm today}$ [68]. One can incorporate such accelerating epoch in the TeVeS Einstein Eqs. (31) by adding to $\mu^2 F(\mu)$—purely phenomenologically—a constant (μ-independent) term of magnitude $\approx \Lambda k^2\ell^2/2\pi$. Such constant part, which corresponds to the integration constant involved in solving Eq. (36), leaves $y(\mu)$ unchanged, merely shifting the curve for $F(\mu)$ in Fig. 2 up. Furthermore, according to Eq. (62) and the empirical connection $\mathfrak{a}_0 \sim H_0$ [7], the added constant is $\sim 3k^3/16\pi^2$, that is very small. It cannot thus affect the discussion in earlier sections, and in particular F continues to make a positive contribution to the energy both in static systems and in cosmology.

The appearance of the cosmological constant in F has almost no effect on the value of ϕ. To see why note that Λ does not directly affect the scalar Eq. (42), but only the Friedmann Eq. (119). Hence Eq. (137) is still valid. As the expansion accelerates, a begins to grow exponentially with t. Both terms on the r.h.s. of Eq. (137) thus fall off drastically, and ϕ becomes "stuck" at the value it had soon after the onset of acceleration. Consolidating the results of Secs. VII B, VII C, VII D, and VII E with our conclusion we see that the range of initial conditions $0.007 < \phi_0 \ll 1$ insures that $\phi > 0$ and $e^\phi \approx 1$ throughout cosmological evolution.

VIII. CAUSALITY IN TeVeS

TeVeS's predecessors, AQUAL and PCG, permitted superluminal propagation of scalar waves on a static background. In the case of PCG with a convex potential this occurs hand in hand with an instability of the background, so it is unclear if true causality violation occurs. How does TeVeS handle causality issues?

The question is complicated here by the existence of two metrics, $g_{\alpha\beta}$ and $\tilde{g}_{\alpha\beta}$, whose null cones do not coincide (except where $\phi = 0$). Which of the two cones is the relevant one for causal considerations? We shall take the view that since common rods and clocks are material systems with negligible self-gravity, the coordinates to which the Lorentz transformations of special relativity refer are those of local orthonormal frames of the physical metric $\tilde{g}_{\alpha\beta}$ and not of $g_{\alpha\beta}$. It is by ascertaining that in no such physical Lorentz frame can physical signals travel back in time that we shall certify the causal behavior of TeVeS. Now Lorentz transformations involve a parameter, the critical speed "c". We shall identify this with the speed of electromagnetic disturbances so that, as customary, the speed of light is the same in all Lorentz frames. Since we have built special relativity into TeVeS by insisting that all nongravitational physics equations (including Maxwell's equation) take their standard form when written with $\tilde{g}_{\alpha\beta}$, this procedure is consistent. In fact, all signals associated with particles of all sorts are subluminal or travel at light's speed with respect to $\tilde{g}_{\alpha\beta}$.

There remains the question of whether gravitational perturbations (tensor, vector or scalar) can ever exit $\tilde{g}_{\alpha\beta}$'s null cone. The analysis given below is quite different for tensor and vector perturbations on the one hand, and scalar perturbations on the other. One point in common, however, is that causality is guaranteed only in spacetime regions for which $\phi > 0$. As shown in Sec. VII, there is gamut of reasonable cosmological models for which ϕ is indeed positive throughout the expansion.

A. Propagation of tensor and vector disturbances is causal

The characteristics of both Einstein's Eqs. (31) and the vector Eq. (38) lie on the null cone of $g_{\alpha\beta}$ because all terms in them with two derivatives are the usual ones in Einstein's and gauge field's equations. Accordingly, we do not expect metric and vector perturbations to travel outside the null cone of the Einstein metric $g_{\alpha\beta}$. However, the interesting question is rather what is the speed of a wave of this class in terms of the physical metric $\tilde{g}_{\alpha\beta}$?

In the eikonal approximation the wavevector κ_α of metric perturbations, that is the four-gradient of the characteristic function, will satisfy $g^{\alpha\beta}\kappa_\alpha\kappa_\beta = 0$. Hence Eq. (23) gives

$$\tilde{g}^{\alpha\beta}\kappa_\alpha\kappa_\beta - 2(\mathfrak{U}^\alpha\kappa_\alpha)^2\sinh(2\phi) = 0. \tag{141}$$

We consider a generic situation where $\mathfrak{U}^\alpha$ may have both temporal and spatial components. The normalization (20) implies by Eq. (22) that $\tilde{g}_{\alpha\beta}\mathfrak{U}^\alpha\mathfrak{U}^\beta = -e^{2\phi}$. Thus in an appropriately oriented local Lorentz frame $\mathfrak{L}$ of the metric $\tilde{g}_{\alpha\beta}$ we may parametrize $\mathfrak{U}^\alpha$ by

$$\mathfrak{U}^\alpha = e^\phi(1-V^2)^{-1/2}\{1, -V, 0, 0\}, \tag{142}$$

with $-1 < V < 1$. This V is actually the ordinary velocity (measured by the physical metric) of $\mathfrak{L}$ w.r.t. the privileged frame in which the matter is at rest (whether in cosmology or in a local static configuration), namely, that in which $\mathfrak{U}^\alpha = \{e^\phi, 0, 0, 0\}$. This is evident by considering a Lorentz transformation from the matter rest frame to the coordinates appropriate to frame $\mathfrak{L}$.

In view of the above, Eq. (141) reduces to

$$0 = A\omega^2 + 2B\boldsymbol{\kappa}_\parallel\omega + D{\boldsymbol{\kappa}_\parallel}^2 - (1-V^2){\boldsymbol{\kappa}_\perp}^2, \tag{143}$$

$$A \equiv e^{4\phi} - V^2, \tag{144}$$

$$B \equiv V(e^{4\phi} - 1), \tag{145}$$

$$D \equiv -1 + V^2 e^{4\phi}, \tag{146}$$

with $\omega = -\kappa_t$ and $\boldsymbol{\kappa}_\parallel$ and $\boldsymbol{\kappa}_\perp$ the spatial components of κ_α collinear and normal to $\mathfrak{U}_i$ (the space part of $\mathfrak{U}_\alpha$), respectively. For arbitrary V (143) is an anisotropic inhomogeneous dispersion relation (ω depends on position through ϕ as well as on direction of the wave vector). However, in the rest frame of the matter ($V = 0$), it is isotropic (though still position dependent through ϕ) with group (or phase) speed equal to

$$v_0 = e^{-2\phi}. \tag{147}$$

The condition for tensor and vector perturbations not to propagate superluminally ($v_0 \leq 1$ as judged in the physical metric) is thus that $\phi > 0$, which as we saw, is satisfied in a wide range of cosmological models (see Sec. VII) as well as quasistatic systems embedded in them (Sec. V). Normally this conclusion could be carried over to all Lorentz frames without further calculations. But because T*e*V*e*S admits a locally privileged frame, that in which $\mathfrak{U}^\alpha = e^\phi\{1, 0, 0, 0\}$, we investigate this conclusion in more detail for any $V^2 < 1$.

Solving Eq. (143) for ω gives

$$\omega = (-B\boldsymbol{\kappa}_\parallel \pm S)A^{-1}, \tag{148}$$

$$S \equiv [C\boldsymbol{\kappa}_\parallel^2 + A(1-V^2)\boldsymbol{\kappa}_\perp^2]^{1/2}, \tag{149}$$

$$C \equiv B^2 - AD = (1-V^2)^2 e^{4\phi}. \tag{150}$$

The condition $\phi > 0$ makes A here strictly positive. It is possible for the above expression for ω to change sign, so for given $\boldsymbol{\kappa}$ we must agree to always choose the branch of the square root that makes ω positive (negative ω with opposite sign $\boldsymbol{\kappa}$ is the same mode, of course). In what follows we call the modes with upper (lower) signs of the square root $+$ ($-$) modes. For the components of *group* velocity collinear and orthogonal to $\mathfrak{U}_i$, respectively, we derive

$$\mathbf{v}_\parallel = \partial\omega/\partial\boldsymbol{\kappa}_\parallel = (-B \pm CS^{-1}\boldsymbol{\kappa}_\parallel)A^{-1}, \tag{151}$$

$$\mathbf{v}_\perp = \partial\omega/\partial\boldsymbol{\kappa}_\perp = \pm(1-\mathrm{V}^2)\mathrm{S}^{-1}\boldsymbol{\kappa}_\perp. \tag{152}$$

Since these expressions are homogeneous of degree zero in $\boldsymbol{\kappa}$, there is no dispersion, but for $V \neq 0$ the propagation is anisotropic. For small ϕ one has analytically

$$v = 1 - 2(1 \pm V\cos\vartheta)^2(1-V^2)^{-1}\phi + \mathcal{O}(\phi^2), \tag{153}$$

where $v \equiv (\mathbf{v}_\parallel^2 + |\mathbf{v}_\perp|^2)^{1/2}$ and ϑ is the angle between $\boldsymbol{\kappa}$ and $\mathfrak{U}_i$. Thus for moderate V the group speed v is subluminal, but obviously formula (153) becomes unreliable for V close to unity.

For arbitrary V it is profitable, as remarked by Milgrom, to write v in terms of ω. In fact a straightforward calculation gives

$$1 - v^2 = S^{-2}C(\boldsymbol{\kappa}_\parallel^2 + \boldsymbol{\kappa}_\perp^2 - \omega^2), \tag{154}$$

from which it is clear that v can become superluminal only if the (isotropic) phase speed $\omega(\boldsymbol{\kappa}_\parallel^2 + \boldsymbol{\kappa}_\perp^2)^{-1/2}$ does the same simultaneously. Since the latter was found subluminal at $V = 0$, we have only to ask if there is some $V < 1$ for which $\omega = (\boldsymbol{\kappa}_\parallel^2 + \boldsymbol{\kappa}_\perp^2)^{1/2}$; we might then suspect there is superluminal propagation for larger V. Suppose we substitute this last value of ω in Eq. (143) together with those of A, B, and D. Collecting terms one can put the condition for the transition to superluminality in the form

$$(e^{4\phi} - 1)\left(V\boldsymbol{\kappa}_\parallel + \sqrt{\boldsymbol{\kappa}_\parallel^2 + \boldsymbol{\kappa}_\perp^2}\right)^2 = 0. \tag{155}$$

As we saw in Sec. VII, for a broad class of cosmological models $\phi > 0$ throughout the expansion, and as Sec. V testifies, variation of ϕ in the vicinity of localized masses embedded in such a cosmology is far short of what is required to turn the sign of ϕ. It is thus clear that even in the case $\boldsymbol{\kappa}_\| < 0$, condition (155) cannot be satisfied for $V < 1$. Hence superluminal propagation of vector and tensor perturbations is forbidden.

How does v vary with V? When $\boldsymbol{\kappa}_\perp \neq 0$, we find numerically the following behavior. For the + mode with $\boldsymbol{\kappa}_\| \leq 0$, $\mathbf{v}_\| < 0$ for all V, and after experiencing a shallow maximum at modest V, v reaches a minimum at V very near unity, which is the deeper and farther from $V = 1$ the larger $|\boldsymbol{\kappa}_\perp/\boldsymbol{\kappa}_\||$. As V grows further, v rises and approaches unity for $V \to 1$. If $\boldsymbol{\kappa}_\| > 0$, $\mathbf{v}_\|$ starts positive for small V but eventually turns negative at a rather large V which grows with $|\boldsymbol{\kappa}_\perp/\boldsymbol{\kappa}_\||$. As V grows further, v reaches a minimum, which gets shallower with growing $|\boldsymbol{\kappa}_\perp/\boldsymbol{\kappa}_\||$, and then begins to rise. At a critical V the positive $\boldsymbol{\kappa}_\|$ + mode terminates. However, the − mode with *negative* $\boldsymbol{\kappa}_\|$ takes over onward from the critical V; it features $\mathbf{v}_\| < 0$, and for it v rises with V and approaches unity as $V \to 1$. The − mode with $\boldsymbol{\kappa}_\| > 0$ is always unphysical.

For $\boldsymbol{\kappa}_\perp = 0$ and $\boldsymbol{\kappa}_\| < 0$ the + mode has $\mathbf{v}_\| < 0$ throughout, and v rises monotonically with V approaching unity as $V \to 1$. For $\boldsymbol{\kappa}_\| > 0$ that same mode has $\mathbf{v}_\| > 0$ and v decreasing with increasing V up to a $V = V_c \approx e^{-2\phi}$ at which point both $\mathbf{v}_\|$ and v vanish. The terminated sequence is continued by the − mode with $\boldsymbol{\kappa}_\| < 0$ for which $\mathbf{v}_\| < 0$ and v rises monotonically with V from zero at $V = V_c$ and approaches unity as $V \to 1$.

B. Propagation of scalar perturbations is also causal

The terms with second derivatives in the scalar Eq. (37) have a nonstandard form reminiscent of those in relativistic AQUAL (see Appendix A). Do scalar perturbations propagate across $\tilde{g}_{\alpha\beta}$'s null cone, that is do they travel faster than electromagnetic waves? We now show that the answer is negative. In the scalar Eq. (37) in free space we break ϕ into background and perturbation $\phi = \phi_B + \delta\phi$, but ignore perturbations of $g_{\alpha\beta}$ and $\mathfrak{U}_\alpha$. For convenience we shall call ϕ_B simply ϕ. To first order in $\delta\phi$ we get [c.f. Eqs. (A2)–(A4)]

$$0 = (h^{\alpha\beta} + 2\xi H^\alpha H^\beta)\delta\phi_{;\alpha\beta} + \cdots, \qquad (156)$$

$$H^\alpha \equiv (h^{\mu\nu}\phi_{,\mu}\phi_{,\nu})^{-1/2}h^{\alpha\beta}\phi_{,\beta}, \qquad (157)$$

$$\xi \equiv d\ln\mu(y)/d\ln y, \qquad (158)$$

where the ellipsis denotes terms with $\delta\phi$ differentiated only once. We have *temporarily* assumed that H^α is spacelike. Using Eq. (23) we reexpress (156) in terms of the physical metric:

$$[e^{-2\phi}\tilde{g}^{\alpha\beta} - (2 - e^{-4\phi})\mathfrak{U}^\alpha\mathfrak{U}^\beta + 2\xi H^\alpha H^\beta]\delta\phi_{;\alpha\beta} + \cdots = 0. \qquad (159)$$

1. Quasistatic background

For a quasistatic background, e.g., a quiescent galaxy, H^α is indeed a purely space vector in coordinates that reflect the time symmetry. By (157) H^α is normalized to unity w.r.t. metric $g_{\alpha\beta}$ and to $e^{-2\phi}$ w.r.t. $\tilde{g}_{\alpha\beta}$. In a local Lorentz frame of $\tilde{g}_{\alpha\beta}$ at rest w.r.t. to those coordinates and appropriately oriented, a generic H^α will have the form $e^{-\phi}\{0, s, 0, \sqrt{1-s^2}\}$, with s the cosine of the angle between H_i and the positive x axis. Then in a Lorentz frame moving w.r.t. the first one at velocity V in the positive x direction

$$H^\alpha = e^{-\phi}(1 - V^2)^{-1/2}\{-Vs, s, 0, \sqrt{(1-s^2)(1-V^2)}\}. \qquad (160)$$

In this same frame $\mathfrak{U}^\alpha$ is given by Eq. (142).

In the eikonal approximation (c.f. Appendix A) one replaces in a Lorentz frame $\delta\phi_{;\alpha\beta} \mapsto -\mathrm{K}_\alpha \mathrm{K}_\beta \delta\phi$ and drops first derivatives. Again interpreting $-\kappa_t$ as ω this gives a generalization of (143), namely

$$0 = \hat{A}\omega^2 + 2(B_\|\boldsymbol{\kappa}_\| + B_\perp\boldsymbol{\kappa}_\perp)\omega + \hat{D}\boldsymbol{\kappa}_\|{}^2 - (1 - V^2)(\boldsymbol{\kappa}_|^2 + E\boldsymbol{\kappa}_\perp^2) + 2B_\perp V^{-1}\boldsymbol{\kappa}_\|\boldsymbol{\kappa}_\perp, \qquad (161)$$

$$\hat{A} \equiv 2e^{4\phi} - (1 + 2\xi s^2)V^2, \qquad (162)$$

$$B_\| \equiv V(2e^{4\phi} - 1 - 2\xi s^2), \qquad (163)$$

$$B_\perp \equiv -2V\xi s\sqrt{(1-s^2)(1-V^2)}, \qquad (164)$$

$$\hat{D} \equiv 2V^2 e^{4\phi} - (1 + 2\xi s^2), \qquad (165)$$

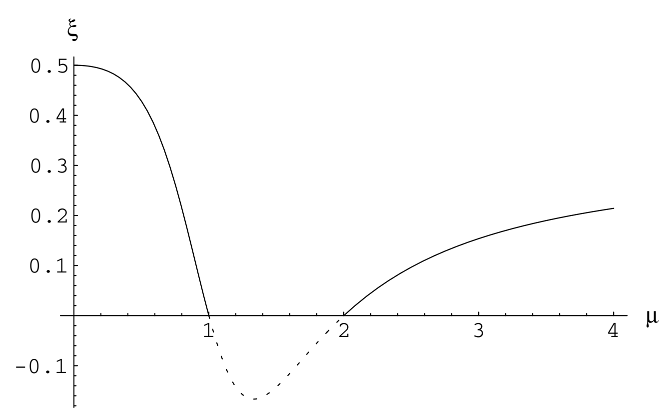

FIG. 3. The logarithmic slope $\xi(\mu)$ as relevant for quasistationary systems, $0 < \mu < 1$, and for cosmology, $2 < \mu < \infty$.

$$E \equiv 1 + 2\xi(1 - s^2), \tag{166}$$

where $\boldsymbol{\kappa}_\parallel$ is the component in the x direction, $\boldsymbol{\kappa}_\perp$ is that in a direction orthogonal to x in the plane spanning the x axis and H_i, and $\boldsymbol{\kappa}_\lfloor$ is the component orthogonal to that plane (we use vector symbols for components to keep with previous notation).

For $V = 0$ (rest frame of matter) there is nothing to distinguish the x axis from H_i's direction, so without restricting generality we may set $s = 1$ and speak jointly of $\boldsymbol{\kappa}_\perp$ and $\boldsymbol{\kappa}_\lfloor$ as a *vector* $\boldsymbol{\kappa}_\perp$. Then the group speed $v = |\partial\omega/\partial\boldsymbol{\kappa}|^{1/2}$ turns out to be

$$v_0 = \frac{e^{-2\phi}}{\sqrt{2}}\left[\frac{(1+2\xi)^2\boldsymbol{\kappa}_\parallel^2 + \boldsymbol{\kappa}_\perp^2}{(1+2\xi)\boldsymbol{\kappa}_\parallel^2 + \boldsymbol{\kappa}_\perp^2}\right]^{1/2}. \tag{167}$$

From Sec. III E we compute the logarithmic slope

$$\xi(\mu) = (\mu - 1)(\mu - 2)/(3\mu^2 - 6\mu + 4), \tag{168}$$

whose graph is shown in Fig. 3.

In particular, $\xi \leq \frac{1}{2}$ in a quasistatic region. In the deep MOND regime $\mu(y) \approx \sqrt{y/3}$ so $\xi \approx \frac{1}{2}$, while in the high acceleration limit $\mu(y) \approx 1$ so $\xi \approx 0$. Consequently, in the deep MOND regime, $v_0 \leq e^{-2\phi}$ with equality for $\boldsymbol{\kappa}_\perp = 0$. In the Newtonian regime $v_0 = 2^{-1/2}e^{-2\phi}$ for all $\boldsymbol{\kappa}$. Finally, in the intermediate regime $2^{-1/2}e^{-2\phi} \leq v_0 \leq (1+2\xi)^{1/2}2^{-1/2}e^{-2\phi}$, with lower and upper equality for $\boldsymbol{\kappa}_\parallel = 0$ and $\boldsymbol{\kappa}_\perp = 0$, respectively. Summarizing, scalar waves propagate subluminally in the frame in which the matter is at rest, provided, of course, $\phi > 0$.

Since the vector $\mathfrak{U}^\alpha$ defines a privileged Lorentz frame, the form of the wave Eq. (159) is different in different frames. Thus we must check explicitly that the subluminal propagation of scalar waves remains valid in all Lorentz frames. Since the analytic expressions for general $\boldsymbol{\kappa}$ are cumbersome, we have done so numerically for small positive ϕ. For small V the group speed starts at the value (167). If $\boldsymbol{\kappa}_\parallel < 0$, v for the + mode rises with increasing V approaching unity as $V \to 1$. By contrast, if $\boldsymbol{\kappa}_\parallel > 0$, v at first decreases with increasing V only to reach a minimum which can be quite narrow and deep for $\boldsymbol{\kappa}_\parallel/|\boldsymbol{\kappa}|$ near unity. Beyond the minimum is a critical V past which the + mode with positive $\boldsymbol{\kappa}_\parallel$ is no longer possible. It is replaced by the − mode with opposite sign of $\boldsymbol{\kappa}_\parallel$, whose v rises as V rises beyond the critical V, approaching unity for $V \to 1$.

In summary, provided $\phi > 0$ as guaranteed (see Sec. V) for the vicinity of masses embedded in the cosmologies studied in Sec. VII, no case of superluminal propagation is observed for scalar perturbations on a quasistatic background.

2. Cosmological background

Consider now propagation of scalar perturbations in FRW cosmology. Here $\mathfrak{U}^\alpha$ remains pointed in the time direction, and takes the form (142) in a local Lorentz frame of the physical metric which moves w.r.t. the matter at velocity V in the x direction. Since H^α is now timelike, one must change the sign of the argument of the square root in definition (157). Definition (158) then requires a switch in sign of the ξ term in Eq. (156). We may evidently write $\phi_{,\alpha} = \zeta\mathfrak{U}_\alpha$ (with ζ spacetime dependent). It follows from definition (142) that $H^\alpha = \sqrt{2}\mathfrak{U}^\alpha$ independent of ζ. Using all this in the modified wave Eq. (159), we obtain in the said Lorentz frame, after an eikonal approximation, a dispersion relation of the form (143) with the coefficients A, B, and C modified according to the rule $e^{4\phi} \to (2 + 4\xi)e^{4\phi}$. Thus in the frame $\mathcal{L}$ where the matter is at rest ($V = 0$) we now find the isotropic group speed, c.f. Eq. (147),

$$v_0 = (2 + 4\xi)^{-1/2}e^{-2\phi}, \tag{169}$$

so that according to Fig. 3, for $\phi > 0$, v_0 never exceeds $1/\sqrt{2}$.

For $V > 0$ we use the analysis leading to Eqs. (154) and (155) with the substitution $e^{4\phi} \to (2 + 4\xi)e^{4\phi}$ to conclude that the passage to superluminality is forbidden. Numerical plots disclose a behavior of $v(V)$ very similar to the one for tensor waves. For + type modes with $\boldsymbol{\kappa}_\parallel < 0$, v grows monotonically approaching unity for $V \to 1$. For $\boldsymbol{\kappa}_\parallel > 0$ modes there is a minimum of v at some high V, the narrower and deeper the larger $\boldsymbol{\kappa}_\parallel/|\boldsymbol{\kappa}|$. A mode of this type exists only up to a critical V beyond the minimum, and is thereafter taken over by the − type mode whose $\boldsymbol{\kappa}_\parallel$ is of opposite sign, and for which v approaches unity as $V \to 1$.

C. Caveats

Summing up, propagation of weak perturbations of the tensor, vector or scalar gravitational fields of T*e*V*e*S is always subluminal with respect to the physical metric. We have checked this in detail only for waves propagating on pure cosmological backgrounds or on quasistatic backgrounds. Furthermore, the analysis looked at perturbations of one field while keeping the others "frozen" at their background values. A more advanced analysis would have examined propagation of joint tensor-vector-scalar modes. This said, no mechanism is evident for the formation of causal loops. This under the condition $\phi > 0$ which, as we have seen, is widely obeyed in flat-space cosmological models and quasistatic systems embedded therein.

ACKNOWLEDGMENTS

I thank Mordehai Milgrom, Robert Sanders, and Arthur Kosowsky for a number of critical comments and for pointing out algebraic errors as well as easier ways to do things.

APPENDIX A: ACAUSALITY IN RELATIVISTIC AQUAL

This comes about because the wave equation for free propagation of ψ deriving from the $\mathcal{L}_\psi$ in Eq. (6) (covariant derivatives are w.r.t. $g_{\alpha\beta}$),

$$[\tilde{f}'(L^2 g^{\mu\nu}\psi_{,\mu}\psi_{,\nu})g^{\alpha\beta}\psi_{,\beta}]_{;\alpha} = 0, \tag{A1}$$

leads to the following linear equation for propagation of small perturbations $\delta\psi$ on the background $\{g^{\alpha\beta}, \psi_{\rm B}\}$:

$$0 = (g^{\alpha\beta} + 2\xi X^\alpha X^\beta)\delta\psi_{;\alpha\beta} + \cdots, \tag{A2}$$

$$X^\alpha \equiv (g^{\mu\nu}\psi_{{\rm B},\mu}\psi_{{\rm B},\nu})^{-1/2} g^{\alpha\beta}\psi_{{\rm B},\beta}, \tag{A3}$$

$$\xi \equiv d\ln\tilde{f}'(y)/d\ln y. \tag{A4}$$

In Eq. (A2) the ellipsis stands for terms where $\delta\psi$ is differentiated only once.

For a static background X^α is a unit purely space vector $\mathbf{X}$. In an appropriately oriented Cartesian coordinate system in a local Lorentz frame, it will point in the x direction. In such frame Eq. (A2) takes the form

$$0 = -\delta\psi_{,tt} + (1+2\xi)\delta\psi_{,xx} + \delta\psi_{,yy} + \delta\psi_{,zz} + \cdots. \tag{A5}$$

In the eikonal approximation appropriate for short wavelengths, one sets $\psi = Ae^{\imath\varphi}$ and neglects terms with derivatives of A or of $k_\alpha \equiv \varphi_{,\alpha}$. Then Eq. (A5) gives

$$\omega = -k_t = [(1+2\xi){k_x}^2 + {k_y}^2 + {k_z}^2]^{1/2}. \tag{A6}$$

The group speed $v_{\rm g} = |\partial\omega/\partial\mathbf{k}|^{1/2}$ turns out to be

$$v_{\rm g} = \left[\frac{(1+2\xi)^2{k_x}^2 + {k_y}^2 + {k_z}^2}{(1+2\xi){k_x}^2 + {k_y}^2 + {k_z}^2}\right]^{1/2}. \tag{A7}$$

In the deep MOND regime $[\tilde{f}(y) = \frac{2}{3}y^{3/2}]$, $2\xi = 1$ while in the high acceleration limit $[\tilde{f}(y) \approx y]$, $\xi \approx 0$. Thus whatever the choice of $\tilde{f}$, $0 < \xi < 1$ over some range of y (acceleration). There $v_{\rm g} > 1$ if $\mathbf{k}$ is not exactly orthogonal to $\mathbf{X}$ (distances and times measured w.r.t. metric $g_{\alpha\beta}$). On the other hand, light waves travel on light cones of $\tilde{g}_{\alpha\beta}$ while metric waves do so on null cones of $g_{\alpha\beta}$. The two metrics are conformally related so their null cones coincide: light and metric waves travel with unit speed. Thus most ψ waves are superluminal, in violation of the causality principle [see Sec. II B].

APPENDIX B: PROBLEMS FOR PCG IN SOLAR SYSTEM

The permissible ranges of η and ε are strongly constrained by the solar system. It can be shown [4] that the $1/r$ force in Eq. (17) causes the Kepler "constant" of planetary orbits with periods P and semimajor axes $\tilde{a}$ to vary slightly with $\tilde{a}$:

$$4\pi^2\tilde{a}^3/P^2 = GM_\odot(1 + \mathfrak{a}_0\tilde{a}/\kappa\eta). \tag{B1}$$

Assuming $M_\odot \ll M_c$, we get $\kappa = \frac{1}{2}$ so that as we pass from planet to planet, the constant varies by a fraction $\sim 2\times 10^{-15}/\eta$. The inner planet periods P are known to better than one part in 10^8. Thus $\eta > 2\times 10^{-7}$.

A stronger constraint comes the perihelia precessions of the planets. The anomalous force in Eq. (17) generates an extra precession [4] which in Mercury's case (eccentricity 0.206 and $\tilde{a} = 6\times 10^{12}$ cm) amounts to $3\times 10^{-8}\eta^{-1}$ arcsec/century. With $\eta = 2\times 10^{-7}$ this already amounts to 0.35% of the Einstein precession, which is measured to about that accuracy. Trying to assume $M_\odot > M_c$ just aggravates the problem. And we are not at liberty to raise η further because for fixed $\mathfrak{a}_0$, M_c scales as η^2. Thus, for example, with $\eta = 3\times 10^{-7}$, the MOND limit of PCG would not apply to galaxies with $M < 8\times 10^9$, a range including many dwarf spirals with missing mass problems! Hence the perihelion precession marginally rules out PCG with a sextic potential.

APPENDIX C: RELATION BETWEEN DETERMINANTS g AND $\tilde{g}$

From Eqs. (22) and (23) it follows that

$$\tilde{g}^{\mu\nu}g_{\nu\alpha} = e^{2\phi}[\delta^\mu{}_\alpha + (1 - e^{-4\phi})\mathfrak{U}^\mu\mathfrak{U}_\alpha], \tag{C1}$$

Viewing this as multiplication of two matrices, we take the determinant:

$$\begin{aligned}\tilde{g}^{-1}g &= e^{8\phi}{\rm Det}\mathcal{K}(\phi, \mathcal{U});\\ \mathcal{K}(\phi, \mathcal{U}) &\equiv \mathcal{I} + (1 - e^{-4\phi})\mathcal{U},\end{aligned} \tag{C2}$$

where $\mathcal{I}$ is the unit matrix whose components are $\delta^\mu{}_\alpha$ while $\mathcal{U}$ is a matrix with components $\mathfrak{U}^\mu\mathfrak{U}_\alpha$. Now both $\tilde{g}$ and g are scalar densities, so that their ratio must be a true scalar. Hence ${\rm Det}\mathcal{K}(\phi, \mathcal{U})$ is a scalar.

In a local Lorentz frame in which the unit timelike vector $\mathfrak{U}^\alpha$ has components $\{1, 0, 0, 0\}$, $\mathcal{U}$'s only nonvanishing component is $\mathcal{U}^0{}_0 = -1$. Therefore, ${\rm Det}\mathcal{K} = [1 - (1 - e^{-4\phi})]\times 1\times 1\times 1 = e^{-4\phi}$. Substituting this in Eq. (C2) we recover Eq. (28).

APPENDIX D: RELATIONS BETWEEN m_s, m_g, AND r_g

To determine r_g one must delve into the region $\varrho < R$. Assuming that the ideal fluid modeling the matter is at rest in the global coordinates, we may write its 4-velocity as $\tilde{u}_\alpha = e^\phi\mathfrak{U}_\alpha = -e^{\phi+\nu/2}\delta_\alpha{}^t$ (see Sec. III C). Let us return to Eq. (85), substitute $\tilde{T}_{tt}$ from Eq. (40) and reorganize the left hand side to obtain

$$\varrho^{-2}e^{\nu-5\mathfrak{s}/4}(\varrho^2\mathfrak{s}'e^{\mathfrak{s}/4})' = -8\pi G\mathfrak{P}, \tag{D1}$$

$$\mathfrak{P} \equiv \tilde{\rho}e^\nu(2e^{-2\phi} - e^{2\phi}) + \tau_{tt} + \Theta_{tt}/8\pi G, \tag{D2}$$

Integration gives for $\varrho > R$

$$\varsigma' e^{\varsigma/4} = -\frac{2Gm_g}{\varrho^2} - \frac{1}{\varrho^2}\int_R^{\varrho}(8\pi G\tau_{tt} + \Theta_{tt})e^{5\varsigma/4-\nu}\varrho^2 d\varrho, \quad \text{(D3)}$$

$$m_g \equiv 4\pi \int_0^R \mathfrak{P} e^{5\varsigma/4-\nu}\varrho^2 d\varrho, \quad \text{(D4)}$$

where the integral in Eq. (D3) does not contain $\tilde{p}$ since it extends only outside the fluid.

How much does the "gravitational mass" m_g differ from the scalar mass m_s? For a star the volume integral of $\tilde{p}$ is of order the random kinetic energy, which by the Newtonian virial theorem is of order of the gravitational energy $\sim Gm_g/R$. According to Eqs. (73) and (74), and (92) this is also the order of the fractional correction to m_s or to m_g coming from the metric factors and e^{ϕ}. We have not worked out τ_{tt} or Θ_{tt} in the interior, but from Eqs. (79) and (83) we may estimate that the τ_{tt} and $\Theta_{tt}/8\pi G$ terms contribute to m_g terms of $\mathcal{O}(kGm_s{}^2/R)$ and $\mathcal{O}(Kr_g{}^2/GR)$, respectively. Because we assume small k and K, these last two terms are obviously subdominant contributions. We may conclude that m_g and m_s differ by a fraction of order Gm_g/R which is 10^{-5} for the solar system.

Let us now calculate $\varsigma' e^{\varsigma/4}$ at $\varrho = R$ using Eq. (74) and (89), and (91) and equate the result to $-2Gm_g/R^2$ as stipulated by Eq. (D3):

$$r_g - \frac{3Kr_g{}^2}{8R} - \frac{kG^2m_s{}^2}{4\pi R} + \mathcal{O}(r_g{}^3/R^2) = 2Gm_g. \quad \text{(D5)}$$

For the Sun $r_g/R \sim Gm_s/R \sim 10^{-5}$; we see that $r_g \approx 2Gm_g$ with fractional accuracy much better than 10^{-5}.

[1] J. Oort, Bull. Astron. Inst. Neth. **6**, 249 (1932); **15**, 45 (1960).

[2] F. Zwicky, Helv. Phys. Acta **6**, 110 (1933); see also S. Smith, Astrophys. J. **83**, 23 (1936).

[3] *Dark Matter in the Universe*, edited by G. R. Knapp and J. Kormendy (Reidel, Dordrecht, 1987).

[4] J. D. Bekenstein, in *Second Canadian Conference on General Relativity and Relativistic Astrophysics*, edited by A. Coley, C. Dyer and T. Tupper (World Scientific, Singapore, 1988), p. 68.

[5] *The Birth and Early Evolution of our Universe: Proceedings of Nobel Symposium #79, Gräftåvallen, Östersund, Sweden*, edited by J. Nilsson, B. Gustafsson, and B.-S. Skagerstam [Phys. Scr. T36, (1991)].

[6] J. Ellis, in Ref. [5], p. 142; M. Turner, in Ref. [5], p. 167; C. Munoz, hep-ph/0309346.

[7] M. Milgrom, Astrophys. J. **270**, 365 (1983).

[8] M. Milgrom, Astrophys. J. **270**, 371 (1983).

[9] M. Milgrom, Astrophys. J. **270**, 384 (1983).

[10] R. Berendzen, R. Hart, and D. Seeley, *Man Discovers the Galaxies* (Columbia University, New York, 1987).

[11] J. Jeans, Mon. Not. R. Astron. Soc. **84**, 60 (1923).

[12] F. Zwicky, *Morphological Astronomy* (Springer, Berlin, 1957).

[13] A. Finzi, Mon. Not. R. Astron. Soc. **127**, 21 (1963).

[14] R. B. Tully and J. R. Fisher, Astron. Astrophys. **54**, 661 (1977).

[15] M. Aaronson and J. Mould, Astrophys. J. **265**, 1 (1983); M. Aaronson, G. Bothun, J. Mould, J. Huchra, R. A. Schommer, and N. E. Cornell, Astrophys. J. **302**, 536 (1986); W. L. Freedman, Astrophys. J. **355**, L35 (1990); R. H. Sanders and M. A.W. Verheijen, Astrophys. J. **503**, 97 (1998).

[16] J. P. Ostriker, P. J. E. Peebles, and A. Yahil, Astrophys. J. **193**, L1 (1974).

[17] A. Kalnajs, in *The Internal Kinematics and Dynamics of Galaxies*, edited by E. Athanassoula (Reidel, Dordrecht, 1983), p. 87.

[18] J. N. Bahcall, and S. Cassertano, Astrophys. J. **293**, L7 (1985).

[19] T. S. van Albada and R. Sancisi, Philos. Trans. R. Soc. London A **320**, 447 (1986).

[20] P. J. Quinn, J. K. Salmon, and W. H. Zurek, Nature (London) **322**, 329 (1986); J. Silk in *A Unified View of the Micro- and Macro-Cosmos*, edited by A. de Rujula, D.V. Nanopoulos, and P. A. Shaver (World Scientific, Singapore, 1987).

[21] G. Blumenthal, S. M. Faber, R. A. Flores, and J. R. Primack, Astrophys. J. **301**, 27 (1986); J. Barnes, in *Nearly Normal Galaxies*, edited by S. M. Faber (Springer, New York, 1987), p. 154; B. S. Ryden and J. E. Gunn, Astrophys. J. **318**, 15 (1987).

[22] R. H. Sanders, Astron. Astrophys. Rev. **2**, 1 (1990).

[23] M. Jobin and C. Carignan, Astron. J. **100**, 648 (1990); C. Carignan, R. Sancisi, and T. S. van Albada, Astron. J. **95**, 37 (1988); C. Carignan and S. Beaulieu, Astrophys. J. **347**, 192 (1989).

[24] S. S. McGaugh and W. J. G. de Blok, Astrophys. J. **508**, 132 (1998).

[25] S. M. Kent, Astron. J. **93**, 816 (1987).

[26] M. Milgrom, Astrophys. J. **333**, 689 (1988).

[27] K. G. Begeman, A. H. Broeils, and R. H. Sanders, Mon. Not. R. Astron. Soc. **249**, 523 (1991).

[28] G. Gentile, P. Salucci, U. Klein, D. Vergani, and P. Kalberla, astro-ph/0403154.

[29] M. Milgrom and E. Braun, Astrophys. J. **334**, 130 (1988).

[30] S. S. McGaugh and E. de Blok, Astrophys. J. **499**, 66 (1998).

[31] A. Begum and J. N. Chengalur, Astron. Astrophys. **413**, 525 (2004).

[32] R. H. Sanders and S. S. McGaugh, Annu. Rev. Astron. Astrophys. **40**, 263 (2002).
[33] S. S. McGaugh, astro-ph/0403610.
[34] A. Aguirre, in *Proceedings of the IAU Symposium 220 "Dark matter in galaxies"*, edited by S. Ryder, D. J. Pisano, M. Walker, and K. Freeman (Astronomical Society of the Pacific, San Francisco, 2004), 17.
[35] R. H. Sanders, Mon. Not. R. Astron. Soc. **342**, 901 (2003).
[36] M. Milgrom, Astrophys. J. **287**, 571, (1984); R. H. Sanders, Astron. Astrophys. **284**, L31 (1994); M. Milgrom and R. H. Sanders, Astrophys. J. **599**, L25 (2003); R. Scarpa, G. Marconi, and R. Gilmozzi, Astron. Astrophys. **405**, L15 (2003).
[37] J. E. Felten, Astrophys. J. **286**, 3 (1984).
[38] R. H. Sanders, Mon. Not. R. Astron. Soc. **296**, 1009 (1998).
[39] J. D. Bekenstein and M. Milgrom, Astrophys. J. **286**, 7 (1984).
[40] J. D. Bekenstein, Phys. Lett. B **202**, 497 (1988).
[41] R. H. Sanders, Mon. Not. R. Astron. Soc. **235**, 105 (1988).
[42] J. D. Bekenstein, in *Developments in General Relativity, Astrophysics and Quantum Theory*, edited by F. I. Cooperstock, L. P. Horwitz, and J. Rosen (IOP Publishing, Bristol, 1990), p. 156.
[43] J. D. Bekenstein, in *Proceedings of the Sixth Marcel Grossman Meeting on General Relativity*, edited by H. Sato and T. Nakamura (World Scientific, Singapore, 1992), p. 905.
[44] J. D. Bekenstein and R. H. Sanders, Astrophys. J. **429**, 480 (1994).
[45] V. V. Zhytnikov and J. M. Nester, Phys. Rev. Lett. **73**, 2950 (1994).
[46] A. Edery, Phys. Rev. Lett. **83**, 3990 (1999), and rebuttal J. D. Bekenstein, M. Milgrom and R. H. Sanders, Phys. Rev. Lett. **85**, 1346 (2000).
[47] M. E. Soussa and R. P. Woodard, Classical Quantum Gravity **20**, 2737 (2003).
[48] M. E. Soussa and R. P. Woodard, Phys. Lett. B **578**, 253 (2004).
[49] M. E. Soussa, astro-ph/0310531.
[50] R. H. Sanders, Astrophys. J. **480**, 492 (1997).
[51] M. Milgrom, Astrophys. J. **302**, 617 (1986).
[52] R. H. Sanders, Mon. Not. R. Astron. Soc. **223**, 559 (1986).
[53] R. Romatka, PhD dissertation (University of Munich, 1992).
[54] C. Will, *Theory and Experiment in Gravitational Physics* (Cambridge University, Cambridge, England, 1986).
[55] R. F. C. Vessot and M. A. Levine, Phys. Rev. Lett. **45**, 2081 (1980).
[56] L. Schiff, Am. J. Phys. **28**, 340 (1960).
[57] E. Witten, Commun. Math. Phys. **80**, 381 (1981).
[58] J. D. Bekenstein, Int. J. Theor. Phys. **13**, 317 (1975).
[59] J. E. Tohline, in *Internal Kinematics and Dynamics of Galaxies*, edited by A. Athanassoula (Reidel, Dordrecht, 1982), p. 205.
[60] J. R. Kuhn and L. Kruglyak, Astrophys. J. **313**, 1 (1987).
[61] T. Jacobson and D. Mattingly, Phys. Rev. D **64**, 024028 (2001).
[62] T. D. Lee and C. N. Yang, Phys. Rev. **98**, 1501 (1955).
[63] R. H. Dicke, *The Theoretical Significance of Experimental Relativity* (Gordon and Breach, New York, 1964).
[64] M. Milgrom (private communication).
[65] M. M. Nieto and S. G. Turyshev, gr-qc/0308017.
[66] C. Eling and T. Jacobson, Phys. Rev. D **69**, 064005 (2004).
[67] C. Misner, K. S. Thorne and J. A. Wheeler, *Gravitation* (Freeman, San Francisco, 1973).
[68] A. G. Riess, *et al.*, astro-ph/0402512.