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# Negative energy and stability in scalar-tensor gravity

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Linearized gravitational waves in Brans-Dicke and scalar-tensor theories carry negative energy. A gauge-invariant analysis shows that the background Minkowski space is stable at the classical level with respect to linear scalar and tensor inhomogeneous perturbations.

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## I. INTRODUCTION

It has recently been shown [1] that all the standard energy conditions can easily be violated at the classical level in the theory of a scalar field coupled nonminimally with the spacetime curvature. Although there are some ambiguities in the definition of energy density and effective pressure [2], the possibility of violating the energy conditions is undeniable. Even allowing for the possibility of exotica such as traversable wormholes made possible by the violation of the energy conditions, one would like to preserve the non-negativity of the energy density, at least at the classical level. However, even this last requirement may be violated in nonminimally coupled scalar field theory. Even worse, the problem of negative energy fluxes is not unique to nonminimally coupled theory [3]—it also occurs when linearized gravitational waves are considered in Brans-Dicke or more general scalar-tensor theories [4–7]. To summarize the issue, consider linearized gravitational waves in Brans-Dicke theory, which is described by the action

$$S^{(BD)} = \int d^4x\sqrt{-g}\left[\phi R - \frac{\omega}{\phi}g^{ab}\nabla_a\phi\nabla_b\phi - V(\phi)\right] + S^{(\mathrm{matter})}. \tag{1.1}$$

In the linearized version of the theory with $V(\phi) = 0$ the metric and scalar field are given by

$$g_{ab} = \eta_{ab} + h_{ab}, \qquad \phi = \phi_0 + \varphi, \tag{1.2}$$

where $\eta_{ab}$ is the Minkowski metric, $\phi_0$ is a constant and $\mathrm{O}(h_{ab}) = \mathrm{O}(\varphi/\phi_0) = \mathrm{O}(\epsilon)$, with $\epsilon$ a smallness parameter. The corresponding linearized field equations in a region outside sources are

$$R_{ab} = \frac{\partial_a\partial_b\varphi}{\phi_0} + \mathrm{O}(\epsilon^2), \tag{1.3}$$

$$\Box\varphi \equiv \eta^{ab}\partial_a\partial_b\varphi = 0 + \mathrm{O}(\epsilon^2). \tag{1.4}$$

It is natural to interpret the right hand side of Eq. (1.3) as an effective energy-momentum tensor $T_{ab}[\varphi]$ of the Brans-Dicke field. More generally, the interpretation of the right hand side of the vacuum Brans-Dicke field equations as an effective energy-momentum tensor is widespread in the literature and may ultimately be questionable [4,8,9]. In fact, writing the vacuum Brans-Dicke field equations as

$$G_{ab} = \frac{1}{\phi}T_{ab}[\phi] = \frac{1}{\phi}\left[\frac{\omega}{\phi}\left(\nabla_a\phi\nabla_b\phi - \frac{1}{2}g_{ab}\nabla^c\phi\nabla_c\phi\right) + \nabla_a\nabla_b\phi - g_{ab}\Box\phi - \frac{V}{2}g_{ab}\right] \tag{1.5}$$

means forcing upon them an interpretation as effective Einstein equations, which they are not—they are field equations of a theory different from general relativity. However, there is little doubt that this interpretation is appropriate in the linearized approximation. Let us consider a monochromatic component of the Fourier decomposition of $\varphi(t, \vec{x})$

$$\varphi_{\vec{k}} = \varphi_0\cos(k_c x^c). \tag{1.6}$$

The effective energy density associated with the monochromatic wave (1.6) by an observer with four-velocity $u^a$ is

$$\rho = T_{ab}[\varphi]u^a u^b = -(k_a u^a)^2\frac{\varphi_{\vec{k}}}{\phi_0}. \tag{1.7}$$

Because of the noncanonical form of $T_{ab}[\varphi]$—linear in the second derivatives instead of quadratic in the first derivatives—$\rho$ is not positive definite but oscillates with the frequency of $\varphi_{\vec{k}}$. This has the disturbing consequence that scalar-tensor waves emitted by a binary massive stellar system such as, e.g., $\mu$-Sco, carry a negative energy flux over macroscopic times (of order $3\times 10^5$ s for $\mu$-Sco). The contribution of the tensor modes $h_{ab}$ is described by the Isaacson effective tensor and is absent to order $\mathrm{O}(\epsilon)$.

The argument showing the presence of negative energy presented in the context of Brans-Dicke theory is generalized to arbitrary scalar-tensor theories of gravity with gravitational sector described by the action

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$$S^{(ST)} = \int d^4x\sqrt{-g} \times \left[\frac{f(\phi)}{2}R - \frac{\omega(\phi)}{2}g^{ab}\nabla_a\phi\nabla_b\phi - V(\phi)\right], \quad (1.8)$$

by expanding the coupling functions $f(\phi)$ and $\omega(\phi)$ around their present day values $f_0$ and $\omega_0$. From a conceptual point of view it could be objected that the consideration of Minkowski space is inconsistent with the original motivation of Brans-Dicke theory (distant matter in the universe determines the effective gravitational coupling $G_{eff} = \phi^{-1}$ here and now, according to Mach's principle). However, Minkowski space is a perfectly legitimate solution of the Brans-Dicke field equations from the mathematical point of view. Moreover, current interest in scalar-tensor gravity is not motivated by Mach's principle but rather by the fact that scalar-tensor theories mimic properties of stringy physics [4,10]. For example, the low energy limit of the bosonic string theory is Brans-Dicke gravity with parameter $\omega = -1$ [11].

The presence of negative energy fluxes is seen by certain authors as a reason to abandon the usual Jordan frame version of the theory and consider instead its Einstein frame counterpart with fixed units of time, length, and mass (see [4,12,13] for reviews). Little matters that the two conformally related formulations are equivalent if one allows the Einstein frame units of mass, time, and length to scale appropriately, as done in Dicke's original paper [14] introducing the Einstein frame version of Brans-Dicke theory. Most of the current literature considers instead a version of Brans-Dicke theory in the Einstein frame with *fixed* units of mass, length, and time. The result is a new theory physically inequivalent to the original Jordan frame; in this new theory the scalar has canonical (positive definite) kinetic energy. In this paper we do not seek escape to the Einstein frame but we work in the Jordan frame. Physicists shy away from negative energy because it is usually associated with instability and runaway solutions and intuitively this should also be the case for classical Brans-Dicke theory and its scalar-tensor generalizations. It comes therefore as a surprise that, as we show in the following, the Minkowski space taken as the background metric in Eqs. (1.2), (1.3), and (1.4) is stable against inhomogeneous perturbations (scalar and tensor) to first order and that there are no runaway solutions to this order.

## II. STABILITY OF MINKOWSKI SPACETIME

Inhomogeneous perturbations are gauge-dependent and a stability analysis using gauge-independent quantities is mandatory. These are conveniently obtained by regarding Minkowski space as a trivial case of a Friedmann-Lemaitre-Robertson-Walker (hereafter "FLRW") universe, for which there is a vast literature on gauge-independent perturbations. Recently [15] we have carried out a stability analysis of de Sitter spaces in generalized gravity theories described by the action

$$S = \int d^4x\sqrt{-g} \times \left[\frac{1}{2}\psi(\phi, R) - \frac{1}{2}\omega(\phi)g^{ab}\nabla_a\phi\nabla_b\phi - V(\phi)\right], \quad (2.1)$$

which contains the scalar-tensor action (1.8) as a special case. We employed the covariant and gauge-invariant formalism developed by Bardeen [16], Ellis, Bruni, Hwang and Vishniac [17,18] in general relativity, in a version extended to encompass generalized gravity by Hwang and Hwang and Noh [19]. The gauge-invariant variables used are the Bardeen [16] potentials $\Phi_H$ and $\Phi_A$ and the Ellis-Bruni [17] variable $\Delta\phi$ defined by

$$\Phi_H = H_L + \frac{H_T}{3} + \frac{\dot{a}}{k}\left(B - \frac{a}{k}\dot{H}_T\right), \quad (2.2)$$

$$\Phi_A = A + \frac{\dot{a}}{k}\left(B - \frac{a}{k}\dot{H}_T\right) + \frac{a}{k}\left[\dot{B} - \frac{1}{k}(a\dot{H}_T)^{\cdot}\right], \quad (2.3)$$

$$\Delta\phi = \delta\phi + \frac{a}{k}\dot{\phi}\left(B - \frac{a}{k}\dot{H}_T\right), \quad (2.4)$$

where $a$ is the scale factor of the background FLRW metric with line element

$$ds^2 = -dt^2 + a^2(t)(dx^2 + dy^2 + dz^2) \quad (2.5)$$

and $A$, $B$, $H_L$ and $H_T$ are the metric perturbations defined by

$$g_{00} = -a^2(1 + 2AY), \quad (2.6)$$

$$g_{0i} = -a^2BY_i, \quad (2.7)$$

$$g_{ij} = a^2[h_{ij}(1 + 2H_L) + 2H_TY_{ij}]. \quad (2.8)$$

Here $h_{ij}$ is the three-dimensional metric of the FLRW background and the operator $\bar{\nabla}_i$ is the covariant derivative associated with $h_{ij}$. The scalar harmonics $Y$ are the eigenfunctions of the eigenvalue problem

$$\bar{\nabla}_i\bar{\nabla}^i Y = -k^2Y, \quad (2.9)$$

while the vector and tensor harmonics $Y_i$ and $Y_{ij}$ are defined by

$$Y_i = -\frac{1}{k}\bar{\nabla}_i Y, \qquad Y_{ij} = \frac{1}{k^2}\bar{\nabla}_i\bar{\nabla}_j Y + \frac{1}{3}Y h_{ij}. \quad (2.10)$$

The general equations for inhomogeneous perturbations [19] simplify considerably in a Minkowski background, reducing to

$$\Delta\ddot{\phi} + \left[k^2 - \frac{(\psi_0'' - 2V_0'')}{2\omega_0\left(1 + \frac{3\psi_{\phi R}^2}{2\omega_0\psi_R^{(0)}}\right)}\right]\Delta\phi = 0, \quad (2.11)$$

$$\ddot{H}_T + k^2 H_T = 0, \quad (2.12)$$

$$\dot{\Phi}_H = -\frac{1}{2}\frac{\Delta\dot{\psi}_R}{\psi_R}, \quad (2.13)$$

$$\Phi_A + \Phi_H = -\frac{\Delta\psi_R}{\psi_R}, \quad (2.14)$$

where

$$\psi_0 \equiv \psi(\phi, R)\Big|_{(\phi_0,R_0)}, \qquad \psi_0' \equiv \frac{\partial\psi}{\partial\phi}\Big|_{(\phi_0,R_0)}, \qquad \psi_0'' \equiv \frac{\partial^2\psi}{\partial\phi^2}\Big|_{(\phi_0,R_0)}, \quad (2.15)$$

$$\psi_R \equiv \frac{\partial\psi}{\partial R}, \qquad \psi_R^{(0)} \equiv \frac{\partial\psi}{\partial R}\Big|_{(\phi_0,R_0)}, \qquad \psi_{\phi R} \equiv \frac{\partial^2\psi}{\partial\phi\partial R}\Big|_{(\phi_0,R_0)}, \quad (2.16)$$

and $\Delta\psi_R$ is defined analogously to Eq. (2.4). An overdot denotes differentiation with respect to the proper time $t$ of the FLRW background. In the cosmological analysis $(H_0, \phi_0)$ is the de Sitter fixed point of which one wants to study the stability and $R_0 = 12H_0^2$. Minkowski space corresponds to the trivial case $H_0 = 0$ and Eqs. (2.13) and (2.14) yield

$$\Phi_H = \Phi_A = -\frac{\Delta\psi_R}{2\psi_R^{(0)}} = -\frac{\psi_{\phi R}}{2\psi_R^{(0)}}\Delta\phi. \quad (2.17)$$

Hence, we are interested in Eqs. (2.11) and (2.12) for the scalar and tensor perturbations $\Delta\phi$ and $H_T$. It is obvious that the solutions of Eq. (2.12) are oscillating for any real value of $k$, and hence Minkowski space is always stable with respect to tensor perturbations. Let us turn now to Eq. (2.11): stability is equivalent to

$$k^2 + \frac{(2V_0'' - \psi_0'')}{2\omega_0(1 + \frac{3\psi_{\phi R}^2}{2\omega_0\psi_R^{(0)}})} \geq 0. \quad (2.18)$$

In scalar-tensor gravity $\psi(\phi, R) = f(\phi)R$ and

$$\psi_{\phi R} = \frac{df}{d\phi}\Big|_{\phi_0}, \qquad \psi_R^{(0)} = f(\phi_0), \qquad \psi_0'' = \frac{d^2 f}{d\phi^2}R\Big|_{(\phi_0,R_0)} = 0. \quad (2.19)$$

In the case of non self-interacting ($V \equiv 0$) linearized Brans-Dicke scalar $\varphi$ of Eq. (1.2), Minkowski space is always stable with respect to linear inhomogeneous perturbations. The same conclusion holds for massive scalar waves ($V_0'' = m^2 > 0$) if $\omega_0 > 0$ and $f(\phi_0) > 0$, which is the usual range of parameters in Brans-Dicke theory. Runaway potentials with $V_0'' < 0$ give unstable scalar perturbations if the wavelength is larger than a critical value—the usual phenomenon present in particle dynamics with runaway potentials.

## III. DISCUSSION AND CONCLUSIONS

Scalar-tensor theories are plagued by negative energies. Although it is not always clear how to unambigously identify energy densities [2], it is clear that negative energies are present in these theories. The situation of linearized Brans-Dicke theory considered in Sec. I is free of such ambiguities—the effective energy density of scalar waves can be clearly identified and is not positive definite. One would therefore expect the background Minkowski space to be unstable and to be destroyed by small perturbations. However this is not the case: the negative energy associated with linearized, massless, scalar-tensor gravitational waves does not cause instability or runaway solutions at the classical level—Minkowski space is stable with respect to inhomogeneous scalar and tensor perturbations at the linear order. The reason for stability can be traced to the fact that the energy density of each individual mode can be negative but is bounded from below once the wave frequency and amplitude are fixed. By contrast, one expects an instability when the energy of the system keeps decreasing and decreasing *ad infinitum.* Hence the issue is not whether the effective energy of the scalar is negative, but whether it has a lower bound or not. It is not trivial to answer this question in general because the scalar and the tensor fields are explicitly coupled in the full field equations, while they decouple to linear order. As a consequence one expects an exchange of energy and momentum between the scalar and the tensor field, and there is no fully satisfactory solution to the problem of energy localization for the gravitational field even in general relativity.

Instabilities may appear at the second or higher order or when the theory is quantized. However, it is well known that also the full equations of Brans-Dicke cosmology admit stable solutions. Their stability has been checked only with respect to homogeneous perturbations in several studies of the phase space, but the analysis goes well beyond the linear order [20].

Regarding quantization, the covariant perturbation scheme only works in the Einstein conformal frame with fixed units and is not possible in the Jordan frame ([4] and references therein). Thus it would appear that the conformal transformation to the Einstein frame is a panacea for scalar-tensor gravity. However, the new theory in the Einstein frame with fixed units of mass, length, and time is physically inequivalent to the original one in the Jordan frame [21]. Moreover, the original theory in the Jordan frame and its scalar-tensor generalizations are still accepted as viable theories and are the subject of a vast literature. Nevertheless, in order for scalar-tensor theories to be fully satisfactory at least at the classical level it would be desirable to have a better understanding of the issues of negative energies and stability beyond linear order and homogeneous and isotropic cosmology.

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