Anisotropic Goldstone bosons of strong-coupling lattice QCD at high density

Barak Bringoltz and Benjamin Svetitsky
School of Physics and Astronomy, Raymond and Beverly Sackler Faculty of Exact Sciences, Tel Aviv University, 69978 Tel Aviv, Israel
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We calculate the spectrum of excitations in strong-coupling lattice QCD in a background of fixed baryon density, at a substantial fraction of the saturation density. We employ a next-nearest-neighbor fermion formulation that possesses the $SU(N_f) \times SU(N_f)$ chiral symmetry of the continuum theory. We find two types of massless excitations: type I Goldstone bosons with linear dispersion relations and type II Goldstone bosons with quadratic dispersion relations. Some of the type I bosons originate as type II bosons of the nearest-neighbor theory. Bosons of either type can develop anisotropic dispersion relations, depending on the value of N_f and the baryon density.

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I. INTRODUCTION

In a previous paper [1] we constructed a framework for calculating the effects of a background baryon density in Hamiltonian lattice QCD at strong coupling. We used strong-coupling perturbation theory to write an effective Hamiltonian for color singlet objects [2,3]. At lowest order we obtained an antiferromagnetic Hamiltonian that describes meson physics with a fixed baryon background distribution. (Baryons move only at higher order.) The Hamiltonian was then transformed to the path integral of a nonlinear σ model. The latter is most easily studied at large N_c.

The global symmetry group of the σ model depends on the fermion kernel of the lattice QCD Hamiltonian. For N_f flavors of naive fermions we get an interaction between nearest-neighbor (NN) sites that is invariant under $U(N)$ with

$$N = 4N_f. \tag{1.1}$$

This symmetry is too large and is indicative of species doubling. We add next-nearest-neighbor (NNN) interactions to the kernel and reduce the symmetry to

$$U(N_f)_L \times U(N_f)_R, \tag{1.2}$$

which is almost the symmetry of the continuum theory. The unwanted $U(1)_A$ is inevitable if one starts with a local, chirally symmetric theory [4]. It can easily be broken by hand in the σ model and we ignore it.

In [1] we studied the NN theory and found that the ground state breaks $U(N)$ spontaneously. The breakdown pattern depends on the baryon density. In [5] it was shown that the excitations of the NN theory divide into two types: type I Goldstone bosons with linear dispersion relations and type II Goldstone bosons with quadratic dispersion relations. These excitations fit the pattern described by Nielsen and Chadha [6] and studied by Leutwyler [7]. Type II Goldstone bosons, typical of ferromagnets, are prominent in work on effective field theories for dense QCD [8,9]. Their existence is made possible by the nonzero chemical potential (or baryon density), which breaks Lorentz invariance by selecting a preferred frame. It is not the lattice in itself that creates them: At zero baryon density the NN theory possesses only Type I bosons [3,10], like any (unfrustrated) antiferromagnet.

The NNN interactions may be treated as a perturbation that removes some of the global degeneracy of the NN vacuum. In Ref. [11] we found that for all baryon densities studied, the ground state breaks the NNN theory's axial symmetries. In all cases with nonzero baryon density, the discrete rotational symmetry is broken as well. In this paper we investigate how the NNN interactions affect the Goldstone boson spectrum, completing our picture of the lattice theory that has the symmetry of continuum QCD.

II. NON-LINEAR σ MODEL

We give here a brief description of the elements comprising the σ model. More details may be found in [1].

The σ field at site $\boldsymbol{n}$ is an $N \times N$ Hermitian, unitary matrix given by a $U(N)$ rotation of the reference matrix Λ,

$$\sigma_{\boldsymbol{n}} = U_{\boldsymbol{n}} \Lambda U_{\boldsymbol{n}}^{\dagger}, \tag{2.1}$$

with

$$\Lambda = \begin{pmatrix} \mathbf{1}_m & 0 \\ 0 & -\mathbf{1}_{N-m} \end{pmatrix}. \tag{2.2}$$

The σ field thus represents an element of the coset space $U(N)/[U(m) \times U(N-m)]$. The number m can vary from site to site and is determined by the local baryon number $B_{\boldsymbol{n}}$ according to

$$m = B_{\boldsymbol{n}} + N/2. \tag{2.3}$$

The Euclidean action is

$$S = \frac{N_c}{2} \int d\tau \Big[-\sum_{\boldsymbol{n}} \mathrm{Tr} \Lambda U_{\boldsymbol{n}}^{\dagger} \partial_\tau U_{\boldsymbol{n}} + \frac{J_1}{2} \sum_{\boldsymbol{n}i} \mathrm{Tr}(\sigma_{\boldsymbol{n}} \sigma_{\boldsymbol{n}+\hat{\imath}}) + \frac{J_2}{2} \sum_{\boldsymbol{n}i} \mathrm{Tr}(\sigma_{\boldsymbol{n}} \alpha_i \sigma_{\boldsymbol{n}+2\hat{\imath}} \alpha_i) \Big]. \tag{2.4}$$

Here α_i is the 4×4 Dirac matrix, times the unit matrix in flavor space. The NN term is invariant under the global $U(N)$ transformation $U_{\mathbf{n}} \to VU_{\mathbf{n}}$ (or $\sigma_{\mathbf{n}} \to V\sigma_{\mathbf{n}}V^\dagger$) while the NNN term is only invariant if $V^\dagger \alpha_i V = \alpha_i$ for all i. This restricts V to the form

$$V = \exp[i(\theta_V^a + \gamma_5 \theta_A^a)\lambda^a], \tag{2.5}$$

where λ^a are flavor generators. This is a chiral transformation in $U(N_f) \times U(N_f)$. [The $U(1)$ corresponding to baryon number is realized trivially on $\sigma_{\mathbf{n}}$.]

Apart from its $U(N)$ internal symmetry, the NN theory is symmetric under the octahedral group of discrete spatial rotations. The NNN term couples spatial rotations to $U(N)$ transformations, leaving the theory invariant under the combined transformation

$$\sigma_{\mathbf{n}} \to R^\dagger \sigma_{\mathbf{n}'} R, \qquad \mathbf{n}' = \mathcal{R}\mathbf{n}. \tag{2.6}$$

Here $\mathcal{R}$ is a 90° lattice rotation and R represents it according to

$$R = \exp\left[i\frac{\pi}{4}\begin{pmatrix} \sigma_j & 0 \\ 0 & \sigma_j \end{pmatrix}\right] \otimes \mathbf{1}_{N_f}. \tag{2.7}$$

This is just the transformation law of Dirac fermions under these discrete rotations.

If the NNN fermion kernel is taken to be a truncated SLAC derivative [12], then both couplings J_1 and J_2 are positive, and $J_2 = J_1/8$. If we argue, however, that the strong-coupling Hamiltonian is derived by block-spin transformations applied to a short-distance Hamiltonian, then we cannot say much about the couplings that appear in it. We will assume that couplings in the effective Hamiltonian fall off rapidly with distance; indeed we will assume that

$$0 < J_2 \ll J_1/N_c. \tag{2.8}$$

This means that we take as our starting point the (globally degenerate) vacuum determined in [1] for the NN theory with $O(1/N_c)$ corrections. The NNN interaction is a perturbation on this vacuum and its excitations.

III. THE GROUND STATE

Here we give a short reprise of the results of [1,11] for the ground states of the NN and NNN σ models with a uniform baryon density $B_{\mathbf{n}} = B > 0$ (i.e., a uniform value of $m > N/2$).

The overall factor of N_c in Eq. (2.4) allows a systematic treatment in orders of $1/N_c$. In leading order, the ground state is found by minimizing the action, which gives field configurations that are time-independent and that minimize the interaction. Minimizing the NN interactions results in a locally degenerate ground state: We assign $\sigma = \Lambda$ on the even sites and let the σ field on each of the odd sites wander freely in $U(m)/[U(2m-N) \times U(N-m)]$, a submanifold of $U(N)/[U(m) \times U(N-m)]$. Since the odd sites are independent, the degeneracy is exponential in the volume.[1] In Ref. [1] we showed that $O(J_1/N_c)$ fluctuations generate a ferromagnetic interaction among the odd sites, causing them to align to a common value (“order from disorder” [5]).

The resulting ground state has a Néel structure. The even sites break $U(N)$ to $U(m) \times U(N-m)$ and then the odd sites break the symmetry further to $U(2m-N) \times U(N-m) \times U(N-m)$. We can write explicitly

$$\sigma_{\text{even}} = U\Lambda_{\text{even}}U^\dagger = U\begin{pmatrix} \mathbf{1}_m & 0 \\ 0 & -\mathbf{1}_{N-m} \end{pmatrix}U^\dagger,$$
$$\sigma_{\text{odd}} = U\Lambda_{\text{odd}}U^\dagger = U\begin{pmatrix} \mathbf{1}_{2m-N} & 0 & 0 \\ 0 & -\mathbf{1}_{N-m} & 0 \\ 0 & 0 & \mathbf{1}_{N-m} \end{pmatrix}U^\dagger. \tag{3.1}$$

The matrix $U \in U(N)$ represents the global degeneracy due to spontaneous symmetry breaking.

We showed in Ref. [11] that the NNN interactions partially remove this global degeneracy. We made the ansatz

$$U = \frac{1}{\sqrt{2}}\begin{pmatrix} u & u \\ -u & u \end{pmatrix}, \tag{3.2}$$

and showed that it yields a ground state. When $m \geq 3N/4$, the matrix u is free to take any value within $U(N/2)$, but a $U(2m - 3N/2) \times U(N-m) \times U(N-m)$ subgroup of $U(N/2)$ acts trivially in Eq. (3.1) (i.e., matrices in this subgroup give the same field configuration as choosing $u = \mathbf{1}$). Vacua that are associated with different *nontrivial* choices of u are in general inequivalent, and give different realizations of the $U(N_f) \times U(N_f)$ symmetry of the theory. Since these vacua are not related by symmetry transformations, there is nothing to prevent lifting of the degeneracy in higher orders in $1/N_c$. In the sequel, we set $u = \mathbf{1}_{N/2}$. This gives the vacuum with the largest symmetry accessible via the ansatz (3.2).

For $m < 3N/4$, u was found numerically by minimizing the NNN energy (2.4). In view of what happens for $m \geq 3N/4$ this may be only one point in a degenerate manifold of ground states.[2] We emphasize that the degeneracy of these vacua is not related to the global $U(N_f) \times U(N_f)$ chiral symmetry. It is an accidental global degeneracy of the ground state.

The symmetries of these ground states are summarized in Table I, reproduced from [11]. In general, both chiral symmetry and discrete lattice rotations are broken; in some cases a symmetry under rotations around the z axis survives. The spontaneous breaking of rotational symmetry is inevitable in the Néel ground state selected

[1]This degeneracy is not removed by the NNN interactions. That is why we consider the $O(1/N_c)$ corrections first.

[2]We have found, in fact, one case where a *different* ansatz gives a more symmetric ground state than Eq. (3.2). This is the case $(N = 12, m = 8)$, i.e., $(N_f = 3, B = 2)$.

TABLE I. Breaking of $SU(N_f)_L \times SU(N_f)_R \times U(1)_A$ for all baryon densities (per site) accessible for $N_f \leq 3$.

N_f	$\|B\|$	Unbroken symmetry	Broken charges
	0	$\cdots$	1
1	1	$\cdots$	1
	2	$U(1)_A$	0
	0	$SU(2)_V$	4
	1	$U(1)_{I_3}$	6
2	2	$SU(2)_V$	4
	3	$U(1)_{I_3}$	6
	4	$SU(2)_L \times SU(2)_R \times U(1)_A$	0
	0	$SU(3)_V$	9
	1	$U(1)_Y \times SU(2)_V$	13
	2	$U(1)_Y$	16
3	3	$SU(3)_V$	9
	4	$U(1)_{I_3} \times U(1)_Y$	15
	5	$U(1)_{I_3} \times U(1)_Y \times U(1)_{A'}$	14
	6	$SU(3)_L \times SU(3)_R \times U(1)_A$	0

by the NN term, since it cannot be symmetric under the discrete rotation (2.6).[3] The Néel state is in turn forced upon us by the assumption (2.8); if the NNN coupling is sufficiently strong then it might send the theory into a ground state with more complex structure in a $2 \times 2 \times 2$ unit cell, symmetric under Eq. (2.6).

Note that if we remove the unphysical axial $U(1)$ symmetry from the σ model, all its realizations will also drop from Table I, namely, there will be no unbroken axial $U(1)$ symmetries (third column) and no Goldstone bosons corresponding to a broken axial $U(1)$ (fourth column).

IV. SPECTRUM OF EXCITATIONS

The Goldstone bosons of the NN theory were discussed in [5]. As mentioned, they divide into two types. There are $2(N-m)^2$ bosons of type I with $\omega \sim J_1|\boldsymbol{k}|$ at low momenta; these are generalized antiferromagnetic spin waves (and are the only excitations at zero density). There are also $2(2m-N)(N-m)$ bosons of type II, that derive their energy from quantum fluctuations in $O(1/N_c)$. These are generalized *ferromagnetic* magnons with $\omega \sim (J_1/N_c)|\boldsymbol{k}|^2$.

The two types of Goldstone bosons belong to different representations of the unbroken subgroup $U(2m-N) \times U(N-m) \times U(N-m)$. This means that they cannot mix to any order in $1/N_c$. The type I–type II classification is robust in the NN theory.

Now we calculate the effects of the NNN interactions on the spectrum. In view of Eq. (2.8), the NN contributions to the propagators, found in [5], remain unchanged. In particular we can take over the self-consistent determination of the self-energy of the type II bosons. We need consider the NNN contributions to the propagators in tree level only. We proceed to calculate these for $m \geq 3N/4$. In these cases, the calculations simplify (much as in [11]) and we perform them analytically.[4] We believe that the spectra of the other cases have similar features.

In the NN theory the σ fields represent fluctuations around the vacuum (3.1) with $U = 1$. We parametrize them [5] as

$$\sigma_{\text{even}} = \begin{pmatrix} 1-2\chi\chi^\dagger & -2\chi\pi^\dagger & -2\chi S \\ -2\pi\chi^\dagger & 1-2\pi\pi^\dagger & -2\pi S \\ -2S\chi^\dagger & -2S\pi^\dagger & -1+2\phi^\dagger\phi \end{pmatrix}, \tag{4.1}$$

and

$$\sigma_{\text{odd}} = \begin{pmatrix} 1-2\chi\chi^\dagger & -2\chi S & 2\chi\pi^\dagger \\ -2S\chi^\dagger & -1+2\phi^\dagger\phi & 2S\pi^\dagger \\ 2\pi\chi^\dagger & 2\pi S & 1-2\pi\pi^\dagger \end{pmatrix}. \tag{4.2}$$

Here ϕ is an $m \times (N-m)$ complex matrix field written as

$$\phi = \begin{pmatrix} \chi \\ \pi \end{pmatrix}, \tag{4.3}$$

and $S \equiv \sqrt{1-\phi^\dagger\phi}$. The field π is an $(N-m) \times (N-m)$ complex matrix, representing the type I Goldstone bosons. χ is a $(2m-N) \times (N-m)$ complex matrix that represents the type II bosons. If $\phi = 0$, we have $\sigma_{\text{even,odd}} = \Lambda_{\text{even,odd}}$, which is the ground state of the NN theory. We adapt Eqs. (4.1) and (4.2) to the NNN theory by rotating them,

$$\sigma \to U\sigma U^\dagger, \tag{4.4}$$

with U as given in Eq. (3.2). Now $\phi = 0$ corresponds to the ground state of the NNN theory.

We substitute Eqs. (4.1) and (4.2) into the action (2.4). The rotation U disappears from the kinetic term and from the NN interaction—they are both $U(N)$ invariant. This means that the bare spectra found in [5], when U was absent, remain intact.

We write the NNN energy as

$$E_{\text{nnn}} = \frac{N_c J_2}{4} \sum_{aNi} \text{Tr}\,\alpha_i \sigma_{a,\boldsymbol{N}} \alpha_i \sigma_{a,\boldsymbol{N}+2\hat{\imath}}, \tag{4.5}$$

where $a = $ (even, odd) and $\boldsymbol{N}$ denotes a site on the corresponding fcc sublattice. We rescale $\phi \to \phi/\sqrt{N_c}$ and expand Eq. (4.5) to second order,

[3] In fact the Néel state is already symmetric under $\boldsymbol{n} \to \mathcal{R}\boldsymbol{n}$. The particular alignment chosen by the NNN term is not invariant under the internal rotation $\sigma_{\boldsymbol{n}} \to R^\dagger \sigma_{\boldsymbol{n}'} R$.

[4] An exception is $(N = 12, m = 10)$, where we have no analytic solution. See below.

$$E_{\rm nnn}=N_cE_0+\frac{J_2}{4}\sqrt{N_c}\sum_{aNi}{\rm Tr}\bar{\alpha}_i\Lambda_a\bar{\alpha}_i(\Delta^{(1)}_{aN}+\Delta^{(1)}_{a,N+2\hat{\imath}})+\frac{J_2}{4}\sum_{aNi}{\rm Tr}\bar{\alpha}_i\Delta^{(1)}_{aN}\bar{\alpha}_i\Delta^{(1)}_{a,N+2\hat{\imath}}+\frac{J_2}{4}\sum_{aNi}{\rm Tr}\bar{\alpha}_i\Lambda_a\bar{\alpha}_i(\Delta^{(2)}_{aN}+\Delta^{(2)}_{a,N+2\hat{\imath}})+O\left(\frac{1}{\sqrt{N_c}}\right). \tag{4.6}$$

We have defined $\bar{\alpha}_i=U^\dagger\alpha_iU$, and $\Delta^{(1,2)}$ correspond to the linear and quadratic deviations of the σ fields from their ground state values. The latter are given by

$$\Delta^{(1)}_e=\begin{pmatrix}0&0&-2\chi\\0&0&-2\pi\\-2\chi^\dagger&-2\pi^\dagger&0\end{pmatrix},\quad \Delta^{(2)}_e=\begin{pmatrix}-2\chi\chi^\dagger&-2\chi\pi^\dagger&0\\-2\pi\chi^\dagger&-2\pi\pi^\dagger&0\\0&0&+2\phi^\dagger\phi\end{pmatrix} \tag{4.7}$$

on the even sites, and

$$\Delta^{(1)}_o=V\begin{pmatrix}0&0&-2\chi\\0&0&-2\pi\\-2\chi^\dagger&-2\pi^\dagger&0\end{pmatrix}V^\dagger,\quad \Delta^{(2)}_o=V\begin{pmatrix}-2\chi\chi^\dagger&-2\chi\pi^\dagger&0\\-2\pi\chi^\dagger&-2\pi\pi^\dagger&0\\0&0&+2\phi^\dagger\phi\end{pmatrix}V^\dagger \tag{4.8}$$

on the odd sites. Here

$$V=\begin{pmatrix}1&0&0\\0&0&1\\0&-1&0\end{pmatrix} \tag{4.9}$$

is the matrix that rotates $\Lambda_{\rm even}$ to $\Lambda_{\rm odd}$. It is easy to show that the terms linear in $\Delta^{(1)}$ vanish.

In view of the block structure of $\Lambda_{\rm even,odd}$ and of U, as given in Eqs. (3.1) and (3.2), and of α_i, it is convenient to decompose χ for $m\geq 3N/4$ as

$$\chi=\begin{pmatrix}\chi_1\\\chi_2\end{pmatrix} \tag{4.10}$$

Here χ_1 has $N/2$ rows, and χ_2 has $2m-3N/2$ rows. Both have $N-m$ columns. Substituting into Eq. (4.6) and omitting the ground state energy we find that the $O(1)$ contribution of the NNN energy depends only on χ_1. The π and χ_2 fields do not enter the NNN energy at this order and remain of type I and of type II, respectively.

We now define a new $N\times N$ matrix $\hat{\chi}$ that contains only χ_1,

$$\hat{\chi}=\begin{pmatrix}0&0&\chi_1\\0&0&0\\\chi_1^\dagger&0&0\end{pmatrix}, \tag{4.11}$$

and use it to write

$$E_{\rm nnn}=\frac{J_2}{4}\sum_{\substack{N\in{\rm fcc}\\ i}}\{4{\rm Tr}[\bar{\alpha}_i\hat{\chi}^e_N\bar{\alpha}_i\hat{\chi}^e_{N+2\hat{\imath}}+\bar{\alpha}_i(V\hat{\chi}^o_NV^\dagger)\bar{\alpha}_i(V\hat{\chi}^o_{N+2\hat{\imath}}V^\dagger)]-4{\rm Tr}[\bar{\alpha}_i\Lambda_e\bar{\alpha}_i\Lambda_e(\hat{\chi}^e_N)^2+\bar{\alpha}_i\Lambda_o\bar{\alpha}_i\Lambda_o(V\hat{\chi}^o_NV^\dagger)^2]\}. \tag{4.12}$$

Next we expand $\hat{\chi}=\chi^\eta\Gamma^\eta$, where χ^η are real and Γ^η are the Hermitian generators of $U(N)$, normalized to

$$ {\rm tr}[\Gamma^\eta\Gamma^{\eta'}]=\delta^{\eta\eta'}. \tag{4.13}$$

The form (4.11) of $\hat{\chi}$ implies that $\chi^\eta\neq 0$ for those generators whose elements $(\Gamma^\eta)_{\alpha\beta}$ are nonzero for $\alpha\in[1,N/2]$ and $\beta\in[m+1,N]$ or vice versa. Thus

$$E_{\rm nnn}=\sum_{aN}\sum_{\eta,\eta'}\left[\chi^{a\eta}_N(N_a)^{\eta\eta'}\chi^{a\eta'}_N+\sum_i\chi^{a\eta}_N(M_{ai})^{\eta\eta'}\chi^{a\eta'}_{N+2\hat{\imath}}\right], \tag{4.14}$$

where

$$\begin{aligned}(N_e)^{\eta\eta'}&=-J_2\sum_i{\rm Tr}[\Gamma^\eta\Gamma^{\eta'}\bar{\alpha}_i\Lambda_a\bar{\alpha}_i\Lambda_a],\\(M_{ei})^{\eta\eta'}&=J_2{\rm Tr}[\Gamma^\eta\bar{\alpha}_i\Gamma^{\eta'}\bar{\alpha}_i],\\(N_o)^{\eta\eta'}&=-J_2\sum_i{\rm Tr}[V\Gamma^\eta\Gamma^{\eta'}V^\dagger\bar{\alpha}_i\Lambda_a\bar{\alpha}_i\Lambda_a],\\(M_{oi})^{\eta\eta'}&=J_2{\rm Tr}[V\Gamma^\eta V^\dagger\bar{\alpha}_iV\Gamma^{\eta'}V^\dagger\bar{\alpha}_i].\end{aligned} \tag{4.15}$$

With a Fourier transform,

$$\chi^{a\eta}_N=\sqrt{\frac{2}{N_s\beta}}\sum_{\substack{{\bf k}\in BZ\\ \omega}}\chi^{a\eta}_ke^{i{\bf k}\cdot{\bf N}+i\omega\tau}, \tag{4.16}$$

we write the energy in momentum space as

$$E_{\rm nnn}=\sum_{\substack{{\bf k}\in BZ\\ \omega>0}}\chi^{e\dagger}_k[N_e+N_e^T+M_e(\boldsymbol{k})+M_e^\dagger(\boldsymbol{k})]\chi^e_k+\chi^{o\dagger}_{-k}[N_o+N_o^T+M_o(-\boldsymbol{k})+M_o^\dagger(-\boldsymbol{k})]\chi^o_{-k}. \tag{4.17}$$

Here $M_a(\boldsymbol{k})=\sum_iM_{ai}e^{ik_i}$.

The NN action, including the time derivative and $O(1/N_c)$ self-energy, was written down in [5] in terms of the Fourier transform $\tilde{\chi}$ of the $(2m-N)\times(N-m)$ matrix field χ [Eq. (4.10)]:

$$S_{\rm nn}=\sum_{\omega,\boldsymbol{k}}{\rm Tr}[(i\omega-\Sigma_{1,\boldsymbol{k}})\tilde{\chi}^{e\dagger}_k\tilde{\chi}^e_k+(-i\omega-\Sigma_{1,\boldsymbol{k}})\tilde{\chi}^{o\dagger}_{-k}\tilde{\chi}^o_{-k}-\Sigma_{2,\boldsymbol{k}}(\tilde{\chi}^e_k\tilde{\chi}^{oT}_{-k}+c.c.)]. \tag{4.18}$$

The self-energies Σ_a are of order J_1/N_c and depend on N and m. We set $\chi_2=0$ and repeat the steps leading to Eq. (4.17) to write $S_{\rm nn}$ in terms of χ^η_k,

$$S_{\rm nn}=\sum_k\chi^{e\dagger}_kK_{ee}\chi^e_k+\chi^{o\dagger}_{-k}K_{oo}\chi^o_{-k}+\chi^{eT}_kK_{eo}\chi^o_{-k}, \tag{4.19}$$

with the matrices K_{ee}, K_{oo}, and K_{eo} given by

$$(K_{ee})^{\eta\eta'}=\frac{i\omega}{2}{\rm Tr}[\Lambda_e\Gamma^\eta\Gamma^{\eta'}]-\frac{1}{2}\Sigma_{1,\boldsymbol{k}}\delta^{\eta\eta'}, \tag{4.20}$$

$$(K_{oo})^{\eta\eta'} = -\frac{i\omega}{2}\mathrm{Tr}[\Lambda_e\Gamma^\eta\Gamma^{\eta'}] - \frac{1}{2}\Sigma_{1,\boldsymbol{k}}\delta^{\eta\eta'}, \tag{4.21}$$

$$(K_{eo})^{\eta\eta'} = -\Sigma_{2,\boldsymbol{k}}\mathrm{Tr}[\Gamma^\eta\Gamma^{\eta' T}]. \tag{4.22}$$

Equation (4.19) is to be added to Eq. (4.17) to give the quadratic action of the type II Goldstone bosons χ_1.

Diagonalizing the quadratic form is straightforward but tedious. As noted above, only a subset of the generators Γ^η of $U(N)$ appear in the expansion of Eq. (4.11). Since χ_1 has dimensions $N/2 \times (N-m)$, there are $N(N-m)/2$ pairs of generators in the sum, which we write (similar to the Pauli matrices σ_x, σ_y) as $\tilde{\Gamma}^\eta_x, \tilde{\Gamma}^\eta_y$, with $\eta = 1, \ldots, N(N-m)/2$. Their coefficients are similarly written as χ^η_x, χ^η_y. Thus for each k, η we have

$$\chi^\eta_k = \begin{pmatrix} \chi^{e\eta}_{xk} \\ \chi^{e\eta}_{yk} \\ \chi^{o\eta *}_{x,-k} \\ \chi^{o\eta *}_{y,-k} \end{pmatrix}, \tag{4.23}$$

and the action is

$$S = \sum_{\omega>0}\sum_{\boldsymbol{k}} \chi^\dagger_k G^{-1}_k \chi_k. \tag{4.24}$$

The inverse propagator is

$$G^{-1}_k = \begin{pmatrix} -\Sigma_{1,\boldsymbol{k}} + n_e + m_e & -\omega & -\Sigma_{2,\boldsymbol{k}} & 0 \\ \omega & -\Sigma_{1,\boldsymbol{k}} + n_e - m_e & 0 & \Sigma_{2,\boldsymbol{k}} \\ -\Sigma_{2,\boldsymbol{k}} & 0 & -\Sigma_{1,\boldsymbol{k}} + n_o + m_o & -\omega \\ 0 & \Sigma_{2,\boldsymbol{k}} & \omega & -\Sigma_{1,\boldsymbol{k}} + n_o - m_o \end{pmatrix}, \tag{4.25}$$

where

$$(n_a)^{\eta\eta'} = -2J_2\sum_i \mathrm{tr}[(\Gamma^\eta_x)^2\bar{\alpha}_i\Lambda_a\bar{\alpha}_i\Lambda_a]\delta^{\eta\eta'}, \tag{4.26}$$

$$(m_a)^{\eta\eta'} = 2J_2\sum_i \mathrm{tr}[\Gamma^\eta_x\bar{\alpha}_i\Gamma^{\eta'}_x\bar{\alpha}_i]\cos k_i. \tag{4.27}$$

$\Sigma_{1,2}$ and ω contain a factor of $\delta_{\eta,\eta'}$. If we write the 4×4 matrix G^{-1}_k in terms of 2×2 blocks,

$$G^{-1}_k = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \tag{4.28}$$

then its determinant is easily calculated via

$$|G^{-1}_k| = |C||B - AC^{-1}D| = \begin{vmatrix} \omega^2 + (n_e + m_e - \Sigma_1)(n_o + m_o - \Sigma_1) - \Sigma_2^2 & \omega(n_e + m_e - n_o + m_o) \\ \omega(n_o + m_o - n_e + m_e) & \omega^2 + (n_e - m_e - \Sigma_1)(n_o - m_o - \Sigma_1) - \Sigma_2^2 \end{vmatrix}. \tag{4.29}$$

For $N = 4N_f \leq 12$ (and $m \geq 3N/4$), the matrices n_e, n_o, m_e, m_o all commute, except for the case ($N = 12$, $m = 10$). Dropping this last from consideration, we are left with the values of (N_f, B) listed in Table II. For each case, the simultaneous diagonalization of $n_{e,o}$ and $m_{e,o}$ gives the eigenvalues shown. The zeros of the determinant (4.29) determine the spectrum $\omega(\boldsymbol{k})$, giving (after $\omega \to i\omega$)

$$\omega^2 = \Sigma_1^2 - \Sigma_2^2 + \frac{1}{2}(n_e^2 - m_e^2 + n_o^2 - m_o^2) - \Sigma_1(n_e + n_o) \pm\left\{\left[\Sigma_1(n_o - n_e) + \frac{1}{2}(n_e^2 - m_e^2 - n_o^2 + m_o^2)\right]^2 + \Sigma_2^2[(m_e + m_o)^2 - (n_e - n_o)^2]\right\}^{1/2}. \tag{4.30}$$

Because of the symmetries of Eq. (4.30), the spectra of the various cases shown in Table II fall into four classes. We examine each class in turn.

Class 1: Here there is no contribution at all from the NNN interaction. As shown in [5], $\Sigma_{1,2}$ are proportional to J_1/N_c and for small momenta they are quadratic in $|\boldsymbol{k}|$; the same holds for the NN energy $\sqrt{\Sigma_1^2 - \Sigma_2^2}$. These fields remain isotropic type II Goldstone bosons as in the NN theory.

TABLE II. Simultaneous eigenvalues of $n_{e,o}$ and $m_{e,o}$ (in units of $2J_2$) for all values of N_f and B considered. The resulting spectra fall into four classes. Here $v_z = \cos k_z$ and $v_\perp = 2\sin\left(\frac{k_x+k_y}{2}\right)\sin\left(\frac{k_x-k_y}{2}\right)$.

N_f	B	n_e	n_o	m_e	m_o	multiplicity	class
1	1	1	2	v_z	$v_\perp$		4
		2	1	$v_\perp$	v_z		4
2	2	1	2	v_z	$v_\perp$	×3	4
		2	1	$v_\perp$	v_z	×3	4
		1	2	$-v_z$	$-v_\perp$		4
		2	1	$-v_\perp$	$-v_z$		4
2	3	0	1	0	v_z		2
		1	0	v_z	0		2
		0	2	0	$v_\perp$		3
		2	0	$v_\perp$	0		3
3	3	1	2	v_z	$v_\perp$	×6	4
		2	1	$v_\perp$	v_z	×6	4
		1	2	$-v_z$	$-v_\perp$	×3	4
		2	1	$-v_\perp$	$-v_z$	×3	4
3	5	0	0	0	0	×2	1
		0	1	0	v_z		2
		1	0	v_z	0		2
		0	2	0	$v_\perp$		3
		2	0	$v_\perp$	0		3

Class 2: Fields that correspond to the minus sign in Eq. (4.30) remain type II, but with anisotropic dispersion laws of the form

$$\omega^2 = c^2 \boldsymbol{k}^4 \left(1 + a\delta \frac{k_z^2}{k^2}\right). \tag{4.31}$$

The plus sign in Eq. (4.30) gives a linear dispersion law, again anisotropic,

$$\omega^2 = 4c_1 J_2 [k_x^2 + k_y^2 + (1 + b\delta) k_z^2] \tag{4.32}$$

The anisotropy in both cases is proportional to the ratio $\delta \equiv J_2/(12J_1/N_c)$ of NNN to NN couplings. The coefficients c and c_1 are defined as

$$c = \left(\frac{d}{d\boldsymbol{k}^2}\sqrt{\Sigma_1^2 - \Sigma_2^2}\right)_{\boldsymbol{k}=0}, \tag{4.33}$$

$$c_1 = -\left(\frac{d\Sigma_1}{d\boldsymbol{k}^2}\right)_{\boldsymbol{k}=0} > 0. \tag{4.34}$$

They are of order J_1/N_c. The coefficients

$$a = \frac{N_c}{6J_1}\frac{c_1^2 - c^2}{2c_1c^2} \quad \text{and} \quad b = \frac{N_c}{6J_1}\frac{2}{c_1} \tag{4.35}$$

are of order 10^2 for the cases at hand.

Classes 3 and 4: Taking $\boldsymbol{k} = 0$ in Eq. (4.30) we find that the fields that correspond to the plus sign get a mass equal to $2J_2$. This is a result of the explicit breaking of the $U(4N_f)$ symmetry by the NNN interaction terms; these particles are no longer Goldstone bosons. The massless bosons in Class 3, corresponding to the minus sign, are type II bosons described by

$$\omega^2 = c_1^2 \boldsymbol{k}^4. \tag{4.36}$$

(This is different from Class 1 where $\omega^2 = c^2\boldsymbol{k}^4$.) The massless bosons in Class 4 are again anisotropic, obeying Eq. (4.32), and are of type I.

These dispersion relations are correct to $O(\delta)$ for momenta of $O(\delta)$ or smaller. In all cases the dispersion relation to $O(1)$ for $\boldsymbol{k}^2 \gg \delta$ is quadratic and isotropic, unchanged from the NN result presented in [5]. Since δ is a small parameter, this means that in most of the Brillouin zone the propagator maintains its NN form. This is the reason why the self-consistent calculations in [5] that yield $\Sigma_{1,2}$ do not change when we add the NNN interactions.

V. SUMMARY AND DISCUSSION

In this work we have studied the nonlinear sigma model derived in [1] for the description of lattice QCD with a large density of baryons. The model has NN and NNN interactions. Building on the results given in [1,5] for the NN theory, and on the study of the NNN ground state presented in [11], we have determined the dispersions relations for the Goldstone bosons in the NNN theory.

We find that the physics of the NN theory is mostly undisturbed by the NNN interaction. At leading order, the properties of the type I bosons (π) and of some of the type II bosons (χ_2) do not change. The type II bosons grouped in the χ_1 field suffer a variety of fates, falling into four classes that appear for different values of N_f and m. Class 1 bosons are unaffected by the NNN perturbation. Class 2 bosons split into type I and type II, and all gain anisotropic contributions of $O(\delta)$ to their energies. Some of the Class 3 bosons become massive while others remain unaffected. In Class 4, some become massive while the others become anisotropic type I bosons.

The symmetry of the theory, in all cases, is severely broken by the NNN terms—from $SU(4N_f)$ to $SU(N_f) \times SU(N_f) \times U(1)_A$. Not surprisingly, a simple count shows that the total number of massless real fields is far greater than the number of spontaneously broken generators of $SU(N_f) \times SU(N_f) \times U(1)_A$, as shown in Table I. The particular NNN interaction we use is simply unable to generate masses *in lowest order* for many of the particles unprotected by Goldstone's theorem. This is partly reflected in the accidental degeneracy of the ground state, which we mentioned below Eq. (3.2). Just as this degeneracy should be lifted in higher orders in $1/N_c$ [beginning with $O(J_2/N_c)$], the corresponding massless excitations should develop masses. The only particles protected from mass generation are the minimal number needed to satisfy Goldstone's theorem (or the Nielsen-Chadha variant).

Another effect that is missing is the mixing of type I and type II Goldstone bosons, which is certainly permitted when the NNN interaction is turned on. In [5] we proved that such a mixing is forbidden in the NN theory, since the two types of boson belong to different representations of the unbroken subgroup. The classification in the NNN theory is less restrictive, and permits mixing of the bosons. Whether mixing occurs is a dynamical issue that can be settled only by calculating to higher order in $1/N_c$.

To conclude we note that other recent work [8,9] on the high density regime of QCD—in the continuum—also predicts type II Goldstone bosons and anisotropic dispersion. There, the starting point is an effective field theory that describes the low energy dynamics of QCD with nonzero chemical potential μ. For $\mu \neq 0$, Lorentz invariance is broken, and the field equations become nonrelativistic. This leads to the emergence of type II Goldstone bosons. In addition, the ground state in [9] can support a nonzero expectation value of vector fields. This breaks rotational symmetry and makes some of the dispersion relations anisotropic.

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