# Baryon axial charge in a finite volume

S. R. Beane[1,2] and M. J. Savage[3]

[1]Department of Physics, University of New Hampshire, Durham, New Hampshire 03824-3568, USA
[2]Jefferson Laboratory, 12000 Jefferson Avenue, Newport News, Virginia 23606, USA
[3]Department of Physics, University of Washington, Seattle, Washington 98195-1560, USA

(Received 20 May 2004; published 25 October 2004)

We compute finite-volume corrections to nucleon matrix elements of the axial-vector current. We show that knowledge of this finite-volume dependence—as well as that of the nucleon mass—obtained using lattice QCD may allow a clean determination of the chiral-limit values of the nucleon and $\Delta$-resonance axial-vector couplings.

DOI: 10.1103/PhysRevD.70.074029 PACS numbers: 12.39.Fe

## I. INTRODUCTION

The nucleon axial charge, $g_A$, is a fundamental quantity in QCD as it in some sense quantifies spontaneous chiral symmetry breaking in the low-energy hadronic theory. It is known very accurately from neutron beta decay experiments, $g_A = 1.2670 \pm 0.0030$ (in units of the vector charge $g_V$) [1] and therefore serves as an important test of nonperturbative, first-principles calculations of hadronic properties using lattice QCD. Fortunately, it is relatively straightforward to compute $g_A$ in numerical lattice QCD simulations. In spite of this, there is still no consensus in regards to $g_A$ from lattice QCD [2–9]. A trend toward underpredicting [2–4] $g_A$ led to the suggestion that there may be large finite-volume effects [10,11]. This in turn has inspired some controversy regarding the finite-volume dependence of $g_A$ near the chiral limit [12,13]. A recent quenched simulation using domain-wall fermions over several volumes finds that large finite-volume effects do seem to account for the discrepancy [9]. The somewhat tentative current state of affairs is primarily due to the fact that current computational limitations use quark masses, $m_q$, that are significantly larger than the physical values, lattice spacings, $a$, that are not significantly smaller than the physical scales of interest, and lattice sizes, $L$, that are not significantly larger than the pion Compton wavelength [14]. It is confidence in the extrapolations of these quantities that will allow a confrontation between lattice QCD predictions for $g_A$ and other hadronic observables and experiment. While the dependence of $g_A$ on the lattice parameters can be described by an effective field theory (EFT), calculability requires maintaining the hierarchy of mass scales: $|\mathbf{p}|, m_\pi \ll \Lambda_\chi \ll a^{-1}$, where $|\mathbf{p}|$ is a typical momentum in the system of interest, $m_\pi$ is the pion mass, and $\Lambda_\chi \sim 2\sqrt{2}\pi f$ is the scale of chiral symmetry breaking ($f = 132$ MeV is the pion decay constant). Lattice simulations are only now beginning to achieve the hierarchy of scales necessary to utilize a perturbative extrapolation.

Here we will be concerned primarily with the finite-volume dependence of $g_A$. In a spatial box of size $L$, momenta are quantized such that $\mathbf{p} = 2\pi\mathbf{n}/L$ with $\mathbf{n} \in \mathbf{Z}$. The EFT momentum hierarchy then requires maintenance of the additional inequality $fL \gg 1$. This bound ensures that (nonpionic) hadronic physics is completely contained inside the lattice volume. In addition, the bound $(m_\pi L)^2(fL)^2 \gg 1$ ensures that the lattice volume has no effect on spontaneous chiral symmetry breaking [15,16]. These two bounds, taken together, then imply that in order to have a perturbative EFT description $m_\pi L \gtrsim 1$. When $(m_\pi L)^2(fL)^2 \lesssim 1$, and therefore $m_\pi L \ll 1$, momentum zero modes must be treated nonperturbatively [15,16] and one is in the so-called $\epsilon$ regime.[1]

We will consider the range of pion masses,[2] $130 \lesssim m_\pi \lesssim 300$ MeV, and therefore we will take $L \gtrsim 2$ fm, keeping in mind that the EFT may be reaching the limits of its validity when this bound on $L$ is saturated, particularly when the pions are light. For the observables considered here, finite-volume effects tend to be small for $L > 4$ fm. It is therefore of interest to have control over the finite-volume dependence of hadronic observables in the range $2 < L \leq 4$ fm. Chiral perturbation theory ($\chi$PT), which provides a systematic description of low-energy QCD near the chiral limit, is the appropriate EFT to exploit the hierarchy of scales described above and to describe the dependence of hadronic observables on $L$ [15,18–20]. Recent work has investigated the finite-volume dependence in the meson [21–29] sector and in the baryon [30–35] sector.

In this paper we compute the leading finite-volume dependence of the axial-vector charge of the nucleon in heavy-baryon $\chi$PT (HB$\chi$PT), including the $\Delta$ resonance as an explicit degree of freedom. The finite-volume corrections to the axial-vector charge of the nucleon depend on the $\Delta$-nucleon mass splitting and on the chiral-limit

[1]The chiral-limit considerations of Refs. [12,13] fall in the $\epsilon$ regime. However, to our knowledge no systematic finite-volume calculation of baryon properties has been done in the $\epsilon$ regime.

[2]The current upper limit of this range has been estimated recently by one of the authors [17].

 

values of the nucleon, Δ-nucleon and Δ axial-vector charges. Traditionally, the nucleon and Δ axial couplings have been estimated using the spin-flavor SU(4) symmetry of the quark model, and in recent work [36] the authors have conjectured the chiral-limit values of these couplings. We point out that lattice QCD measurements of finite-volume effects in the axial-vector charge (and mass) of the nucleon will provide a clean determination of the nucleon and Δ-resonance axial-vector couplings.

## II. THE NUCLEON AXIAL CHARGE IN A FINITE VOLUME

At the one-loop level, the matrix elements of the axial-vector current between nucleons of flavor "$a$" and "$b$" may be written as

$$\langle N_b | j_{\mu,5} | N_a \rangle = [\Gamma_{ab} + c_{ab}] 2\overline{U}_b S_\mu U_a, \quad (1)$$

where $c_{ab}$ represents a counterterm with a single insertion of the light-quark mass matrix. The leading-order Lagrange density describing the interactions between the pions and the low-lying baryons is

$$\mathcal{L} = 2g_A \overline{N} S^\mu A_\mu N + g_{\Delta N}[\overline{T}^{abc,\nu} A^d_{a,\nu} N_b \epsilon_{cd} + \text{H.c.}] + 2g_{\Delta\Delta} \overline{T}^\nu S^\mu A_\mu T_\nu. \quad (2)$$

This Lagrange density gives rise to the diagrams in Fig. 1, which are the leading one-loop contributions to the axial-current matrix elements. In the isospin limit one finds [37]

$$\Gamma_{NN} = g_A - i\frac{4}{3f^2}\Big[4g_A^3 J_1(m_\pi, 0, \mu) + 4\Big(g_{\Delta N}^2 g_A + \frac{25}{81} g_{\Delta N}^2 g_{\Delta\Delta}\Big) J_1(m_\pi, \Delta, \mu) + \frac{3}{2} g_A R_1(m_\pi, \mu) - \frac{32}{9} g_{\Delta N}^2 g_A N_1(m_\pi, \Delta, \mu)\Big], \quad (3)$$

where $J_1(m, \Delta, \mu)$, $R_1(m, \mu)$, and $N_1(m, \Delta, \mu)$ are loop integrals defined in the Appendix and Δ is the Δ-nucleon mass splitting. All $\Gamma[\epsilon]$ poles have been subtracted. They—and their associated counterterm $c_{NN}$—need not concern us here as the finite-volume corrections do not depend on the ultraviolet behavior of the theory at leading one-loop order. All of the couplings (including $f$) in Eq. (3) take their chiral-limit values.

Using the notation $\delta_L(\vartheta) \equiv \vartheta(L) - \vartheta(\infty)$ to denote the finite-volume corrections to the quantity $\vartheta$, and the results obtained in the Appendix, the finite-volume corrections to $\Gamma_{NN}$ are

$$\delta_L(\Gamma_{NN}) \equiv \delta g_A = \frac{m_\pi^2}{3\pi^2 f^2}\Big[g_A^3 \mathbf{F_1} + \Big(g_{\Delta N}^2 g_A + \frac{25}{81} g_{\Delta N}^2 g_{\Delta\Delta}\Big)\mathbf{F_2} + g_A \mathbf{F_3} + g_{\Delta N}^2 g_A \mathbf{F_4}\Big], \quad (4)$$

where

$$\begin{aligned}
\mathbf{F_1}(m, L) &= \sum_{\mathbf{n}\neq\mathbf{0}}\Big[K_0(mL|\mathbf{n}|) - \frac{K_1(mL|\mathbf{n}|)}{mL|\mathbf{n}|}\Big];\\
\mathbf{F_2}(m, \Delta, L) &= -\sum_{\mathbf{n}\neq\mathbf{0}}\Big[\frac{K_1(mL|\mathbf{n}|)}{mL|\mathbf{n}|} + \frac{\Delta^2 - m^2}{m^2} K_0(mL|\mathbf{n}|) - \frac{\Delta}{m^2}\int_m^\infty d\beta \frac{2\beta K_0(\beta L|\mathbf{n}|) + (\Delta^2 - m^2) L|\mathbf{n}| K_1(\beta L|\mathbf{n}|)}{\sqrt{\beta^2 + \Delta^2 - m^2}}\Big];\\
\mathbf{F_3}(m, L) &= -\frac{3}{2}\sum_{\mathbf{n}\neq\mathbf{0}} \frac{K_1(mL|\mathbf{n}|)}{mL|\mathbf{n}|};\\
\mathbf{F_4}(m, \Delta, L) &= \frac{8}{9}\sum_{\mathbf{n}\neq\mathbf{0}}\Big[\frac{K_1(mL|\mathbf{n}|)}{mL|\mathbf{n}|} - \frac{\pi e^{-mL|\mathbf{n}|}}{2\Delta L|\mathbf{n}|} - \frac{\Delta^2 - m^2}{m^2\Delta}\int_m^\infty d\beta \frac{\beta K_0(\beta L|\mathbf{n}|)}{\sqrt{\beta^2 + \Delta^2 - m^2}}\Big],
\end{aligned} \quad (5)$$

and $K_\alpha(z)$ is a modified Bessel function of the second kind. The extension of this result to partially quenched QCD, including strong isospin violation, is straightforward to extract from Ref. [38] using the results of this paper. We do not give an asymptotic expression for $\delta g_A$ as we do not find it useful for $L < 10$ fm for the pion masses of interest, however, it may be found by taking the appropriate asymptotic limits of Eq. (5) using technology developed in Refs. [28,35].

## III. EXTRACTING AXIAL CHARGE FROM LATTICE QCD

### A. Model-independent considerations

The finite-volume corrections to $\Gamma_{NN}$ depend only on infrared quantities, i.e., the axial-vector charges and the pion decay constant, the meson mass, and the Δ-nucleon mass splitting. Hence, with precise determinations of $f$ (chiral-limit value), $m_\pi$, and Δ, lattice data at several

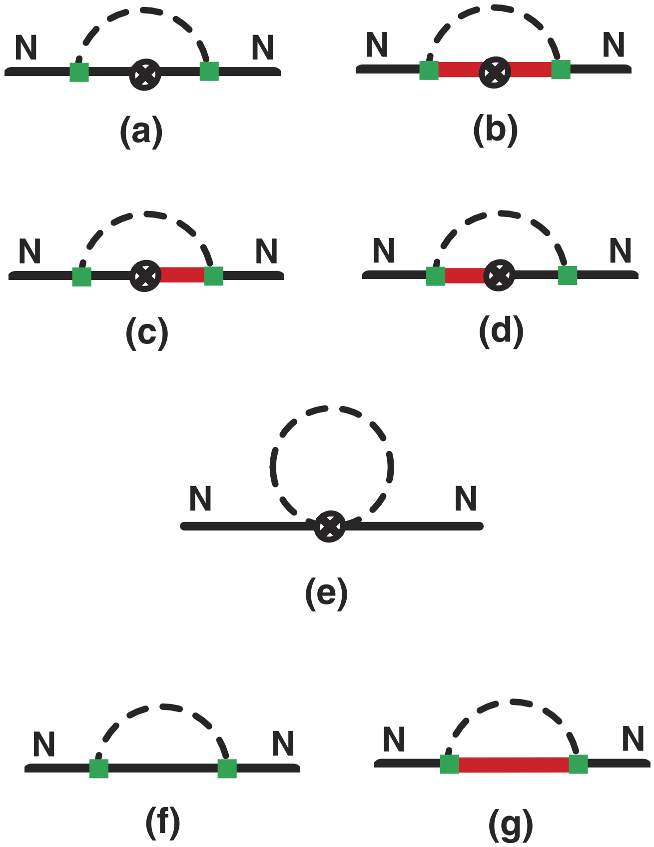


FIG. 1 (color online). One-loop graphs that contribute to the matrix elements of the axial-current in the nucleon. Solid, thick-solid, and dashed lines denote a nucleon, a $\Delta$ resonance, and a pion, respectively. The solid squares denote an axial coupling given in Eq. (2), while the crossed circle denotes an insertion of the axial-vector current operator. (a)–(e) are vertex corrections, while (f) and (g) give rise to wave function renormalization.

different values of $L$ will allow a determination of the axial-vector charges. However, in order to separate the various contributions to Eq. (4), one must ensure that the $\mathbf{F_i}$ scale differently over the relevant values of $L$. In Fig. 2 we plot $\mathbf{F_1}$ and the ratios $\mathbf{F_2}/\mathbf{F_1}$, $\mathbf{F_3}/\mathbf{F_1}$, and $\mathbf{F_4}/\mathbf{F_1}$ as functions of $L$ for various pion masses. For $\mathbf{F_2}$ and $\mathbf{F_4}$ we use $\Delta = 293$ MeV. It is clear from Fig. 2 that the ratios of the $\mathbf{F_i}$ scale differently and therefore, in principle, the coefficients of the $\mathbf{F_i}$ in Eq. (4) may be extracted from the $L$ dependence of $\delta g_A$. Of course, in practice, the exponentially suppressed finite-volume effects must compete against the typical error of a lattice QCD simulation and extraction of the axial charges will be difficult.

### B. A Conjecture and an estimate

In a recent paper by the authors [36], based on earlier work by Weinberg [39–42], it was conjectured that in the chiral limit, the helicity one-half components of the nucleon, $\Delta$ and the Roper ($N'(1440)$) fall into the reducible $(\mathbf{2}, \mathbf{3}) \oplus (\mathbf{1}, \mathbf{2})$ representation of $SU(2)_L \otimes SU(2)_R$ with maximal mixing. Denoting the mixing angle between the irreducible representations as $\psi$ (with maximal mixing corresponding to $\psi = \pi/4$), the conjecture determines the chiral-limit values $g_A = 1 + (2/3)\cos^2\psi$, $g_{\Delta N} = -2\cos\psi$, and $g_{\Delta\Delta} = -3$. Inserting these values into Eq. (4) leads to

$$\delta g_A = \frac{m_\pi^2}{3\pi^2 f^2}\Big[\mathbf{F_1} + \mathbf{F_3} + \Big(2\mathbf{F_1} + \frac{8}{27}\mathbf{F_2} + \frac{2}{3}\mathbf{F_3} + 4\mathbf{F_4}\Big)\cos^2\psi + \frac{4}{3}(\mathbf{F_1} + 2\mathbf{F_2} + 2\mathbf{F_4})\cos^4\psi + \frac{8}{27}\mathbf{F_1}\cos^6\psi\Big]. \tag{6}$$

It would be interesting to have a direct lattice determination of $\psi$ using this formula. The spin-flavor SU(4) (naive constituent quark-model) results are recovered with $\psi = 0$. However, the conjectured values (with $\psi = \pi/4$) are in much better agreement with existing experimental knowledge [36,43]. We use Eq. (6) to estimate our current knowledge of the finite-volume dependence of the nucleon axial-vector charge.[3] This expression is plotted as a function of $L$ for various pion masses in Fig. 3 for the two cases $\psi = \pi/4$ and $\psi = 0$. Variation of $\psi$ provides a measure of the experimental uncertainty associated with the chiral-limit values of the axial-vector couplings [44–46].[4] It is encouraging that the two scenarios lead to quite distinct predictions for $\delta g_A$, and therefore a precise determination of the volume dependence of $g_A$ will allow for a determination of the mixing-angle $\psi$. In both cases it is clear that for $L \gtrsim 2$ fm, finite-volume effects are at the few-percent level for all relevant pion masses.

## IV. CONCLUSIONS

It has long been known that the infinite-volume $S$ matrix can be extracted from power-law suppressed finite-volume effects that arise when a two-particle state is put in a finite volume [47–51], and very recently it has been shown that this method may be extended to include the effect of external electroweak gauge fields [52]. Therefore, a lattice calculation of the energy levels of a pion and a nucleon in a finite volume can, in principle, allow for an extraction of the axial-vector couplings. Important information may also be extracted from exponentially suppressed finite-volume effects that arise from quantum loops [20]. An important observation is that ultraviolet physics (counterterms) enters the chiral expansion for finite-volume effects beyond leading one-loop order in the expansion [32]. Finite-volume effects therefore offer a clean probe of infrared physics.

[3] We use the physical $\Delta$-nucleon mass splitting, $\Delta = 293$ MeV. In principle, this quantity should be measured by the lattice simulation.

[4] For a recent discussion of current knowledge of the chiral-limit value of $g_A$, see Ref. [17].

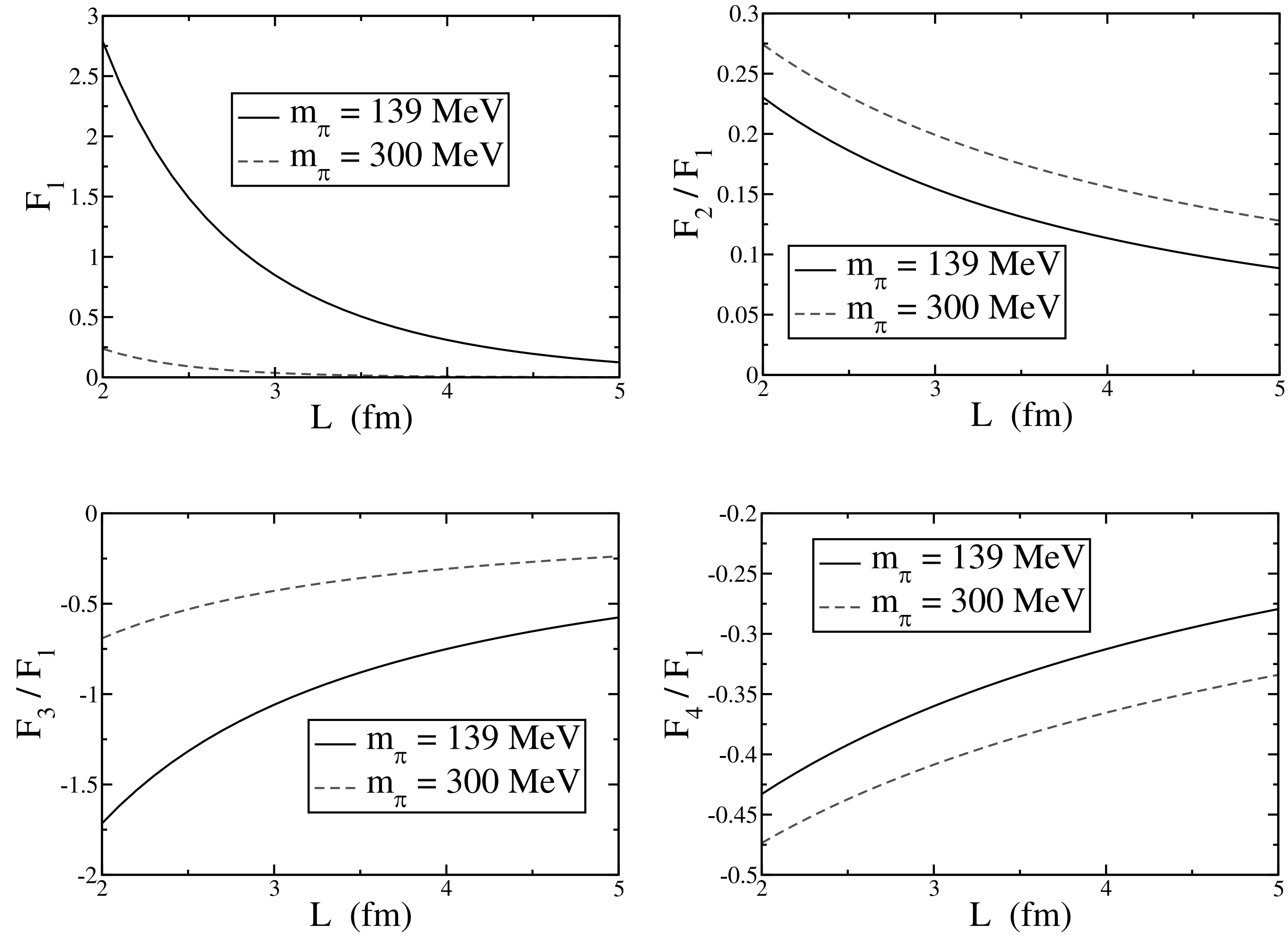


FIG. 2. Plot of $\mathbf{F_1}$ and the ratios $\mathbf{F_2}/\mathbf{F_1}$, $\mathbf{F_3}/\mathbf{F_1}$, and $\mathbf{F_4}/\mathbf{F_1}$ vs $L$. The solid and dashed lines correspond to $m_\pi = 139$ and 300 MeV, respectively, for the physical $\Delta$-nucleon mass splitting, $\Delta = 293$ MeV.

Moreover, this method is optimal for $m_\pi < \Delta$ where the $\Delta$ resonance is unstable and a direct probe of $\Delta$ properties is problematic. One should, however, keep in mind that the exponentially suppressed nature of the finite-volume corrections in the $p$ regime renders an extraction of chiral-limit axial charges a difficult task. However, we believe that it is worthy of investigation in lattice QCD simulations where one can make a definitive determination of whether our method has merit.

To conclude, we have computed the leading finite-volume corrections to nucleon matrix elements of the axial-vector current and argued that, in principle, a lattice QCD measurement of this finite-volume dependence can determine the chiral-limit values of the axial-vector charges of the nucleon and $\Delta$ resonance.

## APPENDIX: LOOP INTEGRALS

In this Appendix we review some standard one-loop integrals that arise in HB$\chi$PT [37] and give their finite-volume dependence. First we consider the generic one-loop integral

$$\begin{aligned} I_0(m, \Delta, \mu) &= \mu^\epsilon \int \frac{d^n q}{(2\pi)^n} \frac{1}{q_0 - \Delta + i\epsilon} \frac{1}{q^2 - m^2 + i\epsilon} \\ &= \frac{i}{8\pi^2}\bigg[ \Delta \log\frac{m^2}{\mu^2} - 2\Delta - \sqrt{\Delta^2 - m^2 + i\epsilon} \\ &\quad \times \log\bigg(\frac{\Delta - \sqrt{\Delta^2 - m^2 + i\epsilon}}{\Delta + \sqrt{\Delta^2 - m^2 + i\epsilon}}\bigg)\bigg] \\ &= -\frac{i}{8\pi^2} \pi \mathcal{F}(m, \Delta, \mu), \end{aligned} \tag{A1}$$

where $\pi\mathcal{F}(m, 0, \mu) = \pi m$, $\epsilon = 4 - n$, and we have subtracted the $\Delta\Gamma(\epsilon)$ divergence. Evaluating the energy integral yields

$$\begin{aligned} I_0(m, \Delta, \mu) &= \frac{i}{2}\mu^\epsilon \int_m^\infty d\beta \frac{\beta}{\sqrt{\beta^2 + \Delta^2 - m^2}} \\ &\quad \times \int \frac{d^{n-1}q}{(2\pi)^{n-1}} \frac{1}{[|\mathbf{q}|^2 + \beta^2]^{3/2}}, \end{aligned} \tag{A2}$$

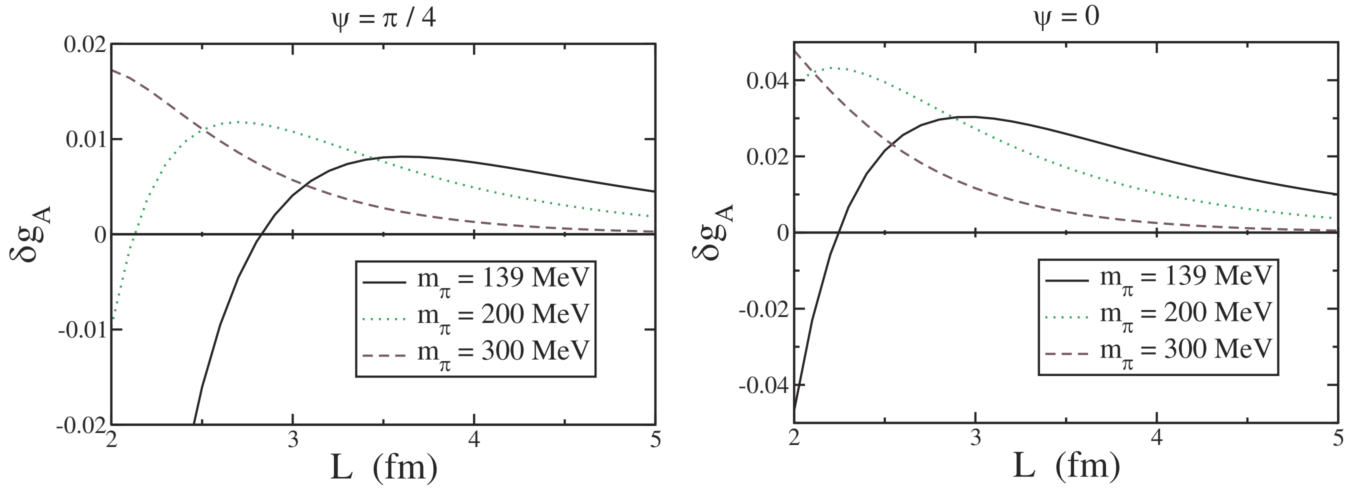


FIG. 3 (color online). The volume dependence of $g_A$ for chiral-multiplet mixing angles $\psi = \pi/4$ and $\psi = 0$. The left panel shows $\delta g_A$ vs $L$ with $\psi = \pi/4$, where the solid, dotted, and dashed lines correspond to $m_\pi = 139$, 200 , and 300 MeV, respectively. The right panel shows $\delta g_A$ vs $L$ with $\psi = 0$ [spin-flavor SU(4) values of axial-vector couplings]. The physical $\Delta$-nucleon mass splitting, $\Delta = 293$ MeV, is used for both panels.

where $\beta(\lambda)^2 = \lambda^2 + 2\lambda\Delta + m^2$ and we have performed a change of variable that is valid only for $\Delta > 0$, as the relation is noninvertible for $\Delta < 0$. Using the master relation

$$\delta_L\left(\int \frac{d^3k}{(2\pi)^3}\frac{1}{[|\mathbf{k}|^2 + \mathcal{M}^2]^\alpha}\right) = \frac{\mathcal{M}^{3-2\alpha}}{2^{1/2+\alpha}\pi^{3/2}\Gamma(\alpha)}\sum_{\mathbf{n}\neq\mathbf{0}}(\mathcal{M}L|\mathbf{n}|)^{\alpha-3/2}K_{3/2-\alpha}(\mathcal{M}L|\mathbf{n}|), \tag{A3}$$

which has been derived previously [32,35], one finds the finite-volume corrections

$$\delta_L(I_0(m,\Delta,\mu)) = \frac{i}{4\pi^2}\int_m^\infty d\beta\frac{\beta}{\sqrt{\beta^2+\Delta^2-m^2}} \times\sum_{\mathbf{n}\neq\mathbf{0}}K_0(\beta L|\mathbf{n}|). \tag{A4}$$

Notice that there is no renormalization-scale dependence. In general, the integral over $\beta$ cannot be performed analytically, however, for $\Delta = 0$

$$\delta_L(I_0(m,0,\mu)) = \frac{i}{8\pi L}\sum_{\mathbf{n}\neq\mathbf{0}}\frac{e^{-mL|\mathbf{n}|}}{|\mathbf{n}|}. \tag{A5}$$

Next we consider the integral, $I_1(m,\Delta,\mu)$, which appears in the one-loop contribution to the nucleon mass,

$$I_1(m,\Delta,\mu) = \mu^\epsilon\int\frac{d^nq}{(2\pi)^n}\frac{(S\cdot q)^2}{v\cdot q - \Delta + i\epsilon}\frac{1}{q^2 - m^2 + i\epsilon} = \frac{1}{4}[\Delta R_1(m,\mu) + (\Delta^2 - m^2)I_0(m,\Delta,\mu)], \tag{A6}$$

where

$$R_1(m,\mu) = \mu^\epsilon\int\frac{d^nq}{(2\pi)^n}\frac{1}{q^2 - m^2 + i\epsilon} = \frac{i}{16\pi^2}m^2\left[\Gamma(\epsilon) + 1 - \log\frac{m^2}{\mu^2}\right]. \tag{A7}$$

This integral contributes to the pion-tadpole diagram in Fig. 1(e). Subtracting the $m^2\Gamma(\epsilon)$ divergence, one then has

$$I_1(m,\Delta,\mu) = \frac{i}{32\pi^2}\left\{(m^2-\Delta^2)\left[\sqrt{\Delta^2-m^2+i\epsilon}\log\left(\frac{\Delta-\sqrt{\Delta^2-m^2+i\epsilon}}{\Delta+\sqrt{\Delta^2-m^2+i\epsilon}}\right) - \Delta\log\frac{m^2}{\mu^2} + 2\Delta\right] - \frac{1}{2}m^2\Delta\log\frac{m^2}{\mu^2} + \frac{1}{2}m^2\Delta\right\} = \frac{i}{32\pi^2}F(m,\Delta,\mu). \tag{A8}$$

Using the master relation, Eq. (A3), one finds

$$\delta_L[R_1(m,\mu)] = -\frac{im}{4\pi^2L}\sum_{\mathbf{n}\neq\mathbf{0}}\frac{K_1(mL|\mathbf{n}|)}{|\mathbf{n}|}. \tag{A9}$$

Finally, we find [35]

$$\delta_L(I_1(m,\Delta,\mu)) = \frac{i}{16\pi^2}\int_m^\infty d\beta\frac{\beta^3}{\sqrt{\beta^2+\Delta^2-m^2}}\sum_{\mathbf{n}\neq\mathbf{0}}\left[\frac{K_1(\beta L|\mathbf{n}|)}{\beta L|\mathbf{n}|} - K_0(\beta L|\mathbf{n}|)\right]. \tag{A10}$$

Another useful integral is $J_0(m, \Delta, \mu) = \partial I_0(m, \Delta, \mu)/\partial\Delta$,

$$J_0(m, \Delta, \mu) = \mu^\epsilon \int \frac{d^n q}{(2\pi)^n} \frac{1}{(q_0 - \Delta + i\epsilon)^2} \frac{1}{q^2 - m^2 + i\epsilon}$$

$$= \frac{i}{8\pi^2}\left[\log\frac{m^2}{\mu^2} - \frac{\Delta}{\sqrt{\Delta^2 - m^2 + i\epsilon}} \log\left(\frac{\Delta - \sqrt{\Delta^2 - m^2 + i\epsilon}}{\Delta + \sqrt{\Delta^2 - m^2 + i\epsilon}}\right)\right]. \tag{A11}$$

The finite-volume corrections are

$$\delta_L(J_0(m, \Delta, \mu)) = -\frac{iL}{4\pi^2} \int_m^\infty d\beta\left[1 - \frac{\Delta}{\sqrt{\beta^2 + \Delta^2 - m^2}}\right]\sum_{\mathbf{n}\neq\mathbf{0}} |\mathbf{n}| K_1(\beta L|\mathbf{n}|). \tag{A12}$$

The one-loop contributions to wave function renormalization, Figs. 1(f) and 1(g), and to the vertex diagrams, Figs. 1(a) and 1(b), for the axial-vector current operator depend on

$$J_1(m, \Delta, \mu) = \mu^\epsilon \int \frac{d^n q}{(2\pi)^n} \frac{(S\cdot q)^2}{(v\cdot q - \Delta + i\epsilon)^2} \frac{1}{q^2 - m^2 + i\epsilon}$$

$$= \frac{1}{4}[R_1(m, \mu) + 2\Delta I_0(m, \Delta, \mu) + (\Delta^2 - m^2) J_0(m, \Delta, \mu)]$$

$$= -\frac{3}{4}\frac{i}{16\pi^2}\left[(m^2 - 2\Delta^2)\log\frac{m^2}{\mu^2} + 2\Delta\sqrt{\Delta^2 - m^2 + i\epsilon}\log\left(\frac{\Delta - \sqrt{\Delta^2 - m^2 + i\epsilon}}{\Delta + \sqrt{\Delta^2 - m^2 + i\epsilon}}\right)\right]$$

$$= -\frac{3}{4}\frac{i}{16\pi^2} J(m, \Delta, \mu). \tag{A13}$$

The finite-volume corrections may be written as

$$\delta_L(J_1(m, \Delta, \mu)) = -\frac{i}{16\pi^2}\int_m^\infty d\beta \frac{\Delta\beta^3}{[\beta^2 + \Delta^2 - m^2]^{3/2}} \times \sum_{\mathbf{n}\neq\mathbf{0}}\left[\frac{K_1(\beta L|\mathbf{n}|)}{\beta L|\mathbf{n}|} - K_0(\beta L|\mathbf{n}|)\right]. \tag{A14}$$

Finally, the vertex diagrams, Figs. 1(c) and 1(d), for the axial-vector current operator depend on

$$N_1(m, \Delta, \mu) = \mu^\epsilon \int \frac{d^n q}{(2\pi)^n} \frac{(S\cdot q)^2}{v\cdot q - \Delta + i\epsilon} \frac{1}{v\cdot q + i\epsilon} \frac{1}{q^2 - m^2 + i\epsilon}$$

$$= \frac{1}{\Delta}[I_1(m, \Delta, \mu) - I_1(m, 0, \mu)]$$

$$= -\frac{3}{4}\frac{i}{16\pi^2}\left\{\left(m^2 - \frac{2}{3}\Delta^2\right)\log\frac{m^2}{\mu^2} + \frac{2}{3}\Delta\sqrt{\Delta^2 - m^2 + i\epsilon}\log\left(\frac{\Delta - \sqrt{\Delta^2 - m^2 + i\epsilon}}{\Delta + \sqrt{\Delta^2 - m^2 + i\epsilon}}\right)\right.$$

$$\left. + \frac{2}{3}\frac{m^2}{\Delta}\left[\pi m - \sqrt{\Delta^2 - m^2 + i\epsilon}\log\left(\frac{\Delta - \sqrt{\Delta^2 - m^2 + i\epsilon}}{\Delta + \sqrt{\Delta^2 - m^2 + i\epsilon}}\right)\right]\right\}$$

$$= -\frac{3}{4}\frac{i}{16\pi^2} K(m, \Delta, \mu). \tag{A15}$$

The finite-volume corrections are simply

$$\delta_L(N_1(m, \Delta, \mu)) = \frac{1}{\Delta}[\delta_L(I_1(m, \Delta, \mu)) - \delta_L(I_1(m, 0, \mu))], \tag{A16}$$

where one uses Eq. (A10).

## ACKNOWLEDGMENTS

The work of S. R. B. is partly supported by the DOE Contract No. DE-AC05-84ER40150, under which the Southeastern Universities Research Association (SURA) operates the Thomas Jefferson National Accelerator Facility. M. J. S. is supported in part by the U.S. Department of Energy under Grant No. DE-FG03-97ER4014.

[1] Particle Data Group Collaboration, K. Hagiwara *et al.*, Phys. Rev. D **66**, 010001 (2002).
[2] M. Fukugita *et al.*, Phys. Rev. Lett. **75**, 2092 (1995).
[3] K. F. Liu *et al.*, Phys. Rev. D **49**, 4755 (1994).
[4] M. Göckeler *et al.*, Phys. Rev. D **53**, 2317 (1996).
[5] TXL Collaboration, S. Güsken *et al.*, Phys. Rev. D **59**, 114502 (1999).
[6] LHPC Collaboration, D. Dolgov *et al.*, Phys. Rev. D **66**, 034506 (2002).
[7] S. Capitani *et al.*, Nucl. Phys. B, Proc. Suppl. **79**, 548 (1999).
[8] UKQCD Collaboration, R. Horsley *et al.*, Nucl. Phys. B, Proc. Suppl. **94**, 307 (2001).
[9] RIKEN-BNL-Columbia-KEK Collaboration, S. Sasaki, K. Orginos, S. Ohta, and T. Blum, Phys. Rev. D **68**, 054509 (2003).
[10] T. Blum and S. Sasaki, hep-lat/0002019.
[11] RBC Collaboration, S. Sasaki, T. Blum, S. Ohta, and K. Orginos, Nucl. Phys. B, Proc. Suppl. **106**, 302 (2002).
[12] R. L. Jaffe, Phys. Lett. B **529**, 105 (2002).
[13] T. D. Cohen, Phys. Lett. B **529**, 50 (2002).
[14] For a recent discussion, see K. Jansen, Nucl. Phys. B, Proc. Suppl. **129**, 3 (2004).
[15] H. Leutwyler, Phys. Lett. B **189**, 197 (1987).
[16] H. Leutwyler and A. Smilga, Phys. Rev. D **46**, 5607 (1992).
[17] S. R. Beane, Nucl. Phys. **B695**, 192 (2004).
[18] J. Gasser and H. Leutwyler, Nucl. Phys. **B307**, 763 (1988).
[19] P. Hasenfratz and H. Leutwyler, Nucl. Phys. **B343**, 241 (1990).
[20] M. Lüscher, in "Proceedings of the Cargese Summer Institute," 1983.
[21] M. F. L. Golterman and K.-C. Leung, Phys. Rev. D **56**, 2950 (1997); **58**, 097503 (1998).
[22] M. Golterman and E. Pallante, Nucl. Phys. B, Proc. Suppl. **83**, 250 (2000).
[23] C.-J. D. Lin *et al.*, Nucl. Phys. **B650**, 301 (2003).
[24] C.-J. D. Lin *et al.*, Phys. Lett. B **553**, 229 (2003).
[25] G. Colangelo, S. Dürr, and R. Sommer, Nucl. Phys. B, Proc. Suppl. **119**, 254 (2003).
[26] G. Colangelo and S. Dürr, Eur. Phys. J. C **33**, 543 (2004).
[27] D. Becirevic and G. Villadoro, Phys. Rev. D **69**, 054010 (2004).
[28] D. Arndt and C.-J. D. Lin, Phys. Rev. D **70**, 014503 (2004).
[29] G. Colangelo and C. Haefeli, Phys. Lett. B **590**, 258 (2004).
[30] QCDSF Collaboration, A. Ali Khan *et al.*, Nucl. Phys. B, Proc. Suppl. **119**, 419 (2003).
[31] QCDSF Collaboration, A. Ali Khan *et al.*, Nucl. Phys. B, Proc. Suppl. **129**, 176 (2004).
[32] QCDSF and UKQCD Collaborations, A. Ali Khan *et al.*, hep-lat/0312029.
[33] A. A. Khan *et al.*, Nucl. Phys. **B689**, 175 (2004).
[34] A. S. Kronfeld, hep-lat/0205021.
[35] S. R. Beane, Phys. Rev. D **70**, 034507 (2004).
[36] S. R. Beane and M. J. Savage, Phys. Lett. B **556**, 142 (2003).
[37] E. Jenkins and A. V. Manohar, Phys. Lett. B **259**, 353 (1991).
[38] S. R. Beane and M. J. Savage, Nucl. Phys. **A709**, 319 (2002).
[39] S. Weinberg, Phys. Rev. **177**, 2604 (1969).
[40] S. Weinberg, Phys. Rev. Lett. **19**, 1023 (1969).
[41] S. Weinberg, in *Chiral Dynamics: Theory and Experiment*, edited by A. M. Bernstein and B. R. Holstein (Springer-Verlag, Berlin, 1995).
[42] S. Weinberg, Phys. Rev. Lett. **65**, 1177 (1990).
[43] S. R. Beane and U. van Kolck, nucl-th/0212039.
[44] N. Fettes, V. Bernard, and U. G. Meissner, Nucl. Phys. **A669**, 269 (2000).
[45] N. Fettes, Ph.D. thesis, Bonn U. (Report No. JUL-3814, 2000).
[46] T. R. Hemmert, M. Procura, and W. Weise, Phys. Rev. D **68**, 075009 (2003).
[47] K. Huang and C. N. Yang, Phys. Rev. **105**, 767 (1957).
[48] M. Lüscher, Commun. Math. Phys. **105**, 153 (1986).
[49] M. Lüscher, Nucl. Phys. **B354**, 531 (1991).
[50] S. R. Beane, P. F. Bedaque, A. Parreño, and M. J. Savage, Phys. Lett. B **585**, 106 (2004).
[51] P. F. Bedaque, nucl-th/0402051.
[52] W. Detmold and M. J. Savage, hep-lat/0403005.