Quark-mass dependence of T_c in QCD: Working up from $m = 0$ or down from $m = \infty$?

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We analyze the dependence of the QCD transition temperature on the quark (or pion) mass. We find that a linear sigma model, which links the transition to chiral symmetry restoration, predicts a much stronger dependence of T_c on m_π than seen in present lattice data for $m_\pi \gtrsim 0.4$ GeV. On the other hand, working down from $m_\pi = \infty$, an effective Lagrangian for the Polyakov loop requires only small explicit symmetry breaking, $b_1 \sim \exp(-m_\pi)$, to describe $T_c(m_\pi)$ in the above mass range. Physically, this is a consequence of the flat potential (large correlation length) for the Polyakov loop in the three-color pure gauge theory at T_c. We quantitatively estimate the end point of the line of first-order deconfining phase transitions: $m_\pi \simeq 4.2\sqrt{\sigma} \simeq 1.8$ GeV and $T_c \simeq 240$ MeV for three flavors and three colors.

I. INTRODUCTION

Lattice QCD calculations at finite temperature and with dynamical fermions are presently performed for quark masses exceeding their physical values; for a recent review see [1]. To date, pion masses as low as ≈ 400 MeV are feasible [2], about 3 times the physical pion mass. When comparing effective theories to first-principles numerical data obtained on the lattice it is therefore important to fix the parameters (coupling constants, vacuum expectation values, and so on) such as to match the values of physical observables, e.g., of m_π, to those of the lattice calculations. For example, the QCD equation of state in the confined phase appears to be described reasonably well by that of a hadron resonance gas model, *after extrapolating* the physical spectrum of hadrons and resonances to that from the lattice [3]. Thus, lattice data on the dependence of various observables on the quark (or pion) mass constrain effective theories for the QCD phase transition at finite temperature and could provide relevant information on the driving degrees of freedom.

In this paper, we analyze the dependence of the *chiral* symmetry restoration temperature on the vacuum mass of the pion using a linear sigma model in Sec. III. The linear sigma model provides an effective Lagrangian approach to low-energy QCD near the chiral limit [4,5]. It incorporates the global flavor symmetry, assuming that "color" can be integrated out. For example, it allows one to discuss the order of the $N_f = 2 + 1$ chiral phase transition as a function of the quark masses [4–7].

Instead of working up from zero quark mass, one could start with the quark masses taken to infinity, that is, with a pure gauge theory. Then, one can discuss the *deconfinement* transition at finite temperature within an effective Lagrangian for the Polyakov loop with global $Z(N_c)$

symmetry [8–15] (N_c is the number of colors). For finite pion mass, the symmetry is broken explicitly, and the phase transition (or crossover) temperature is shifted, relative to the pure gauge theory where pions are infinitely heavy. In Sec. IV, we determine the end point of the line of first-order transitions for three colors, and extract the magnitude of the explicit $Z(3)$ breaking from lattice data on ΔT_c.

II. PION MASS AND DECAY CONSTANT IN VACUUM

The Lagrangian of QCD with the quark-mass matrices set to zero is invariant under independent rotations of the N_f right-handed and left-handed quark fields. It exhibits a global $\mathrm{SU}(N_f)_R \times \mathrm{SU}(N_f)_L$ symmetry, leading to $2(N_f^2 - 1)$ conserved currents. Those are $N_f^2 - 1$ vector currents, $V_i^\mu = \bar{q}\gamma^\mu \lambda_i q/2$, and $N_f^2 - 1$ axial currents, $A_i^\mu = \bar{q}\gamma^\mu \gamma_5 \lambda_i q/2$, with λ_i the generators of $\mathrm{SU}(N_f)$, normalized according to $\mathrm{tr}\lambda_i \lambda_j = 2\delta_{ij}$. The $\mathrm{SU}(N_f)_V$ subgroup of vector transformations is preserved in the vacuum [16], while the $\mathrm{SU}(N_f)_A$ is broken spontaneously by a nonvanishing chiral condensate $\langle \bar{q}_R q_L \rangle \neq 0$, leading to nonconservation of the axial currents.

In reality, of course, even $\mathrm{SU}(N_f)_V$ is broken explicitly by the nonvanishing quark-mass matrix. Nevertheless, since at least m_u and m_d are very small in the physical limit, the $\mathrm{SU}(2)_V$ symmetry is almost exact in QCD. The small explicit breaking of $\mathrm{SU}(2)_V$ is responsible for the nonvanishing pion mass, as given by the Gell-Mann, Oakes, Renner relation

$$m_\pi^2 = \frac{1}{f_\pi^2} m_q \langle \bar{q}q \rangle. \tag{1}$$

We neglect isospin breaking effects here and so assume that $m_u = m_d \equiv m_q$. $\langle \bar{q}q \rangle$ denotes the sum of the vacuum expectation values of the operators $\bar{u}_R u_L$ and $\bar{d}_R d_L$, and their complex conjugates. The proportionality constant f_π

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is the pion decay constant. It should be noted that (1) is valid only at tree level, and that loop effects induce an implicit dependence of both f_π and $\langle \bar{q}q \rangle$ on m_q. For small m_q, this dependence can be computed in chiral perturbation theory [17]. For example, at next-to-leading order,

$$m_\pi^2 = M^2 \left[1 - \frac{1}{2} \left(\frac{M}{4\pi F} \right)^2 \log \frac{\Lambda_3^2}{M^2} \right], \qquad (2)$$

$$f_\pi = F \left[1 + \left(\frac{M}{4\pi F} \right)^2 \log \frac{\Lambda_4^2}{M^2} \right], \qquad (3)$$

where M and F are the couplings of the effective theory (equivalent to $\langle \bar{q}q \rangle$ and m_q), and Λ_3 and Λ_4 are two renormalization-group invariant scales. These relations link the behavior of f_π to that of m_π, the mass of a physical state. (In what follows, we use m_π to vary the strength of explicit symmetry breaking rather than using directly the scale dependent quark masses.)

More accurate results than Eqs. (2) and (3) can perhaps be obtained by computing quark propagators for various quark masses on the lattice. Reference [18] analyzed the propagators for gauge field configurations generated with the standard Wilson gauge action ("quenched QCD"), using overlap fermions with exact chiral symmetry. They obtained a parametrization of both m_π and f_π in terms of the mass m_q of u and d quarks (see Sec. 2 in [18]) which allows us to express f_π as a function of m_π. Their data cover an interval of $0.4 \lesssim m_\pi \lesssim 1$ GeV, and $0.15 \lesssim \sqrt{2} f_\pi \lesssim 0.22$ GeV.

III. LINEAR SIGMA MODEL AT FINITE TEMPERATURE

In this section, we discuss chiral symmetry restoration at finite temperature, and, in particular, the dependence of the symmetry restoration temperature on the pion mass. For simplicity, we restrict ourselves here to the two-flavor case. Our emphasis is not on the order of the transition as the strange quark mass is varied but rather on how the temperature at which the transition occurs (be it either a true phase transition or just a crossover) depends on the pion mass. Such dependence arises from two effects: first, of course, due to explicit symmetry breaking occurring when $m_\pi > 0$. Second, due to the "indirect" dependence of spontaneous symmetry breaking, i.e., of the condensate $\langle \bar{q}q \rangle$ respectively, f_π, on the pion mass (through pion loops, see previous section). The tree-level potential of the linear sigma model with $SU(2)_V \times SU(2)_A \cong O(4)$ symmetry is given by

$$V(\sigma, \vec{\pi}) = \frac{1}{2} m^2 \phi^2 + \frac{\lambda}{4} \phi^4 - H\sigma, \qquad (4)$$

with $\phi_a = (\sigma, \vec{\pi})$. For $m^2 < 0$ the $O(4)$ symmetry of the vacuum state is broken spontaneously to $O(3)$, leading to a nonvanishing scalar condensate $\langle \sigma \rangle = f_\pi$. The explicit

symmetry breaking term $\sim H$ provides a mass to the pions. At tree level, the masses are given by

$$m_\sigma^2 = m^2 + 3\lambda\sigma^2, \qquad m_\pi^2 = m^2 + \lambda\sigma^2. \qquad (5)$$

Below, we employ the Hartree approximation to investigate the dependence of the transition temperature on the pion mass. This approximation scheme is known to exhibit problems in the chiral limit in that the Goldstone theorem is violated and the phase transition is incorrectly predicted to be of first order (for $N_f = 2$). However, here we are interested only in the model with explicit symmetry breaking, where the theory exhibits a crossover. Specifically, we consider the region of pion masses covered by the lattice data in [2], $m_\pi \gtrsim 0.4$ GeV.

In [19] it was shown that such truncated nonperturbative resummation schemes can be renormalized with local counter terms obtained in the vacuum (see also [20] for $\lambda\phi^4$ theory). These ideas were applied in [21] to theories with global symmetries, and a Bogoliubov-Parasiuk-Hepp-Zimmerman–like renormalization scheme was introduced for the $O(4)$ linear sigma model in Hartree approximation without explicit symmetry breaking. The scheme can be straightforwardly extended to the case $H > 0$; see Eqs. (9)–(11) below. Those renormalized gap equations coincide with those introduced first by Lenaghan and Rischke in Ref. [22].

In this renormalization scheme a mass renormalization scale μ is introduced and the couplings then depend on that scale (cf., e.g., [22]). However, choosing

$$\mu^2 = \exp \left[\frac{m_\sigma^2 (\ln m_\sigma^2 - 1) - m_\pi^2 (\ln m_\pi^2 - 1)}{m_\sigma^2 - m_\pi^2} \right], \qquad (6)$$

the four-point coupling $\lambda(\mu) = \lambda_{\text{tree}}$ retains its tree-level (classical) value [22]. In other words, this renormalization prescription evolves the renormalization scale μ in such a way as to keep λ constant.

Explicitly, this leads to the following expressions for the couplings [22]:

$$\lambda = \frac{1}{2} \frac{m_\sigma^2 - m_\pi^2}{f_\pi^2}, \qquad H = f_\pi(m_\sigma^2 - 2\lambda f_\pi^2),$$
$$m^2 = -\frac{1}{2}(m_\sigma^2 - 3m_\pi^2) - 6\lambda Q_\mu(m_\pi), \qquad (7)$$

where

$$Q_\mu(M) \equiv \frac{1}{(4\pi)^2} \left[M^2 \ln \frac{M^2}{\mu^2} - M^2 + \mu^2 \right]. \qquad (8)$$

These equations determine the couplings in vacuum in terms of m_π, f_π, and m_σ. The dependence of f_π on m_π is taken from the data of Ref. [18] (cf. their Figs. 1 and 2 and the corresponding fits therein), as mentioned above. Roughly, for $m_\pi : 0.4 \to 1$ GeV, f_π increases by about 50%, leading to an increase of the explicit symmetry breaking term H by a factor of 10. We also require the

dependence of m_σ on m_π, which we take from a recent computation with standard Wilson fermions [23]. Those authors find that m_σ is essentially a linear function of m_π^2. We checked how our results in Fig. 1 depend on this assumption by using, alternatively, a linear dependence $m_\sigma = m_\pi + \text{const}$, with $m_\sigma = 0.6$ GeV for $m_\pi = 0.14$ GeV. We found essentially the same dependence of T_c on m_π.

At finite temperature, we use the effective potential for composite operators [24] to determine the masses and the scalar condensate in the Hartree approximation. We follow the derivation outlined in [22,25]. The resulting gap equations are

$$H = \langle\sigma\rangle[m_\sigma^2 - 2\lambda\langle\sigma\rangle^2], \tag{9}$$

$$m_\sigma^2 = m^2 + 3\lambda\{\langle\sigma\rangle^2 + [Q_T(m_\sigma) + Q_\mu(m_\sigma)] + [Q_T(m_\pi) + Q_\mu(m_\pi)]\}, \tag{10}$$

$$m_\pi^2 = m^2 + \lambda\{\langle\sigma\rangle^2 + [Q_T(m_\sigma) + Q_\mu(m_\sigma)] + 5[Q_T(m_\pi) + Q_\mu(m_\pi)]\}, \tag{11}$$

where the finite-temperature contribution of the tadpole diagram is given by

$$Q_T(m) \equiv \int \frac{d^3k}{(2\pi)^3} \frac{1}{\epsilon_k(m)} \frac{1}{\exp[\epsilon_k(m)/T] - 1},$$
$$\epsilon_k(m) \equiv \sqrt{\vec{k}^2 + m^2}. \tag{12}$$

Here, m_σ, m_π, and $\langle\sigma\rangle$ denote the effective masses and the scalar condensate at finite temperature, respectively. The self-consistent solution of the above gap equations for a given vacuum pion mass determines the temperature dependence of the scalar condensate as the order parameter of chiral symmetry restoration. For explicitly broken chiral symmetry, $H > 0$, the transition in this approach is a crossover. We define the crossover temperature T_c by the peak of $\partial\langle\sigma\rangle/\partial T$.

The dependence of T_c on m_π is depicted in Fig. 1 (left panel), where we have also shown lattice results obtained with two and three degenerate quark flavors, respectively [2] (the $N_f = 2$ data with standard action, the $N_f = 3$ data with improved $p4$ action). Driven by the increase of both f_π and H with m_π, the linear sigma model predicts a rather rapid rise of T_c with the pion mass, as compared to the data which is nearly flat on the scale of the figure. While lattice data indicate a rather weak dependence of T_c on the quark mass (see also Ref. [26]), models with spontaneous symmetry breaking in the vacuum naturally predict a rather steep rise of T_c with the vacuum expectation value $\langle\sigma\rangle_{\text{vac}} = f_\pi$, which itself increases with the quark (or pion) mass. Our findings here are in qualitative agreement with those from Ref. [27] who employed nonperturbative flow equations to compute the effective potential for two-flavor QCD within the linear sigma model. They also find a steeper slope of $T_c(m_\pi)$ than indicated by the lattice, even though their analysis appears to predict a somewhat weaker increase of f_π with m_π than the data of [18], which we employ here.

Figure 1 also shows the temperature dependence of the σ condensate (right panel). With m_σ a linear function of m_π^2 [23], the width of the crossover is approximately independent of the pion mass for $0.4 \lesssim m_\pi \lesssim 1$ GeV, while we found considerable broadening when m_σ is linear in m_π (not shown). The chiral susceptibility $\partial\langle\sigma\rangle/\partial T$ at its maximum is ≈ 0.25, i.e., the crossover is in fact quite broad for the range of m_π considered. Since this is at variance with lattice data on QCD thermodynamics (pressure and energy density as functions of temperature, see, e.g., the review in [1]), one might argue that the crossover is in fact not driven by the order parameter field but by heavier degrees of freedom [3,17]. Such degrees of freedom could reduce the pion-mass dependence of the transition substantially: using three-loop chiral perturbation theory (i.e., the nonlinear model), Gerber and Leutwyler find [17] that T_c increases rapidly from
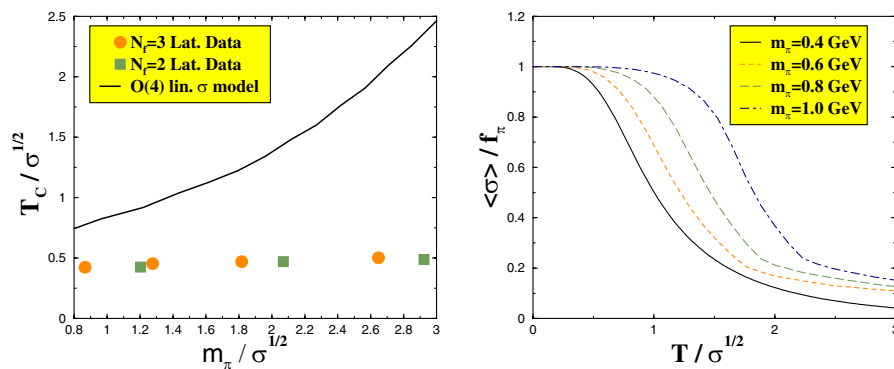

FIG. 1 (color online).   Left panel: The crossover temperature T_c as a function of the (vacuum) pion mass as obtained from the linear sigma model with $O(4)$ symmetry in comparison to lattice data [2] for two and three flavors. The scale for both T_c and m_π is set by the zero-temperature string tension in the pure gauge theory, $\sqrt{\sigma} \simeq 0.425$ GeV. Right panel: The scalar condensate, $\langle\sigma\rangle$, as a function of temperature for various pion masses.

≈ 190 MeV in the chiral limit (using their set of couplings) to ≈ 240 MeV for physical pion mass. However, when heavy states are included (in the dilute gas approximation), then T_c increases less rapidly, from ≈ 170 MeV in the chiral limit to ≈ 190 MeV for physical pion mass.

Hence, perhaps the transition is not primarily driven by an order parameter field and infrared dynamics. Another possible approach is discussed in the next section.

IV. EFFECTIVE LAGRANGIAN FOR THE POLYAKOV LOOP AT FINITE TEMPERATURE

In the limit $m_q \to \infty$ the quarks decouple and drop out of the theory. The (gauge-invariant) order parameter for the deconfining phase transition in such a pure gauge theory with N_c colors is the Polyakov loop

$$\ell = \frac{1}{N_c} \mathrm{tr} \mathcal{P} \exp\left[ig \int_0^{1/T} A_0(\vec{x}, \tau) d\tau \right]. \qquad (13)$$

A_0 denotes the temporal component of the gauge field in the fundamental representation, g is the gauge coupling, and path ordering is with respect to imaginary time τ. Its expectation value, $\ell_0(T)$, vanishes when $T < T_c$ and is nonzero above T_c. Indeed, by asymptotic freedom, $\ell_0 \to 1$ as $T \to \infty$. The simplest guess for a potential for the Polyakov loop is

$$V(\ell) = -\frac{b_2}{2}|\ell|^2 + \frac{1}{4}(|\ell|^2)^2 \qquad (N_c = 2). \qquad (14)$$

The Polyakov loop model [9–12] is a mean-field theory for ℓ. In a mean-field analysis all coupling constants are taken as constant with temperature, except for the mass term, $\sim b_2|\ell|^2$. About the transition, condensation of ℓ is driven by changing the sign of the two-point coupling: $b_2 > 0$ above T_c [$b_2(T) \to 1$ for $T \to \infty$], and < 0 below T_c.

For two colors, (14) is a mean-field theory for a second order deconfining transition [28]. The ℓ field is real, and so the potential defines a mass: $(m_\ell/T)^2 = (1/Z_s)\partial^2 V/\partial\ell^2$, where Z_s is the wave function normalization constant for ℓ [29]. The mass is measured from the two-point function of Polyakov loops in coordinate space, $\propto (1/r) \times \exp(-m_\ell r)$ as $r \to \infty$.

For three colors, ℓ is a complex valued field, and a term cubic in ℓ appears in $V(\ell)$,

$$V(\ell) = -\frac{b_2}{2}|\ell|^2 - \frac{b_3}{3}\frac{\ell^3 + \ell^{*3}}{2} + \frac{1}{4}(|\ell|^2)^2 \qquad (N_c = 3). \qquad (15)$$

At very high temperature, the favored vacuum is perturbative, with $\ell_0 \approx 1$, times $Z(3)$ rotations. We then choose $b_3 > 0$ so that in the $Z(3)$ model, there is always one vacuum with a real, positive expectation value for ℓ_0. This produces a first-order deconfining transition, where ℓ_0 jumps from 0 at T_c^- to $\ell_c = 2b_3/3$ at T_c^+ [10,11]; T_c is given by $b_2(T_c) = -2b_3^2/9$. The ℓ field has two masses,

from its real (m_ℓ) and imaginary ($\tilde{m}_\ell$) parts. At T_c^+, $\sqrt{Z_s}m_\ell/T = \ell_c$. The mass for the imaginary part of ℓ is $\sqrt{Z_s}\tilde{m}_\ell(T)/T \propto \sqrt{b_3}\ell$; at T_c^+, $\tilde{m}_\ell/m_\ell = 3$, twice the value expected from a perturbative analysis of the loop-loop correlation function, obtained by expanding ℓ from Eq. (13) to order A_0^3 [30]. This mass ratio receives corrections if five-point and six-point couplings are included in the effective Lagrangian [30] but those are not crucial for the present discussion. We note that, in principle, all of the above coupling constants could be determined on the lattice. The lattice regularization requires nonperturbative renormalization of the Polyakov loop in order to define the proper continuum limit of ℓ [31,32].

Within the above mean-field theory, dynamical quarks act like a "background magnetic field" which breaks the $Z(3)$ symmetry explicitly, and a term linear in ℓ also appears in $V(\ell)$ [33–36]:

$$V(\ell) = -b_1\frac{\ell + \ell^*}{2} - \frac{b_2}{2}|\ell|^2 - \frac{b_3}{3}\frac{\ell^3 + \ell^{*3}}{2} + \frac{1}{4}(|\ell|^2)^2$$
$$(N_c = 3, m_\pi < \infty). \qquad (16)$$

Hence, as m_π decreases from infinity, $b_1(m_\pi)$ turns on. The normalization of $b_2(T)$ for $T \to \infty$ is such that $\ell_0 \to 1$, i.e., $b_2(T = \infty) = 1 - b_1 - b_3$.

We first consider the case where b_1 is very small and take the term linear in ℓ as a perturbation; then the weakly first-order phase transition of the pure gauge theory persists (in what follows, the critical temperature in the pure gauge theory with $b_1 = 0$ will be denoted T_c^*). The critical temperature is determined from

$$b_2(T_c) = -\frac{2}{9}b_3^2\left(1 + \frac{27}{2}\frac{b_1}{b_3^3}\right) + \mathcal{O}(b_1^2). \qquad (17)$$

The order parameter jumps at T_c, from

$$\ell_0(T_c^-) = \frac{9}{2}\frac{b_1}{b_3^2} + \mathcal{O}(b_1^2), \qquad (18)$$

to

$$\ell_0(T_c^+) = \frac{2}{3}b_3 - \frac{9}{2}\frac{b_1}{b_3^2} + \mathcal{O}(b_1^2). \qquad (19)$$

Note that numerically $\ell_0(T_c^-)$ could be much larger than b_1 if the phase transition in the pure gauge theory is weak and so the correlation length $\xi = 1/m_\ell$ near T_c is large (i.e., if b_3 is small), as indeed appears to be the case for $N_c = 3$ colors [37]. In other words, it could be that on the lattice ℓ quickly develops a nonvanishing expectation value at T_c^- already for rather large quark (or pion) masses, but this does not automatically imply a large explicit symmetry breaking (see also Fig. 2).

From Eq. (17) we can estimate the shift of T_c induced by letting $m_\pi < \infty$. Writing the argument of b_2 in that equation as $T_c^* + \Delta T_c$ and expanding to first order in ΔT_c we obtain
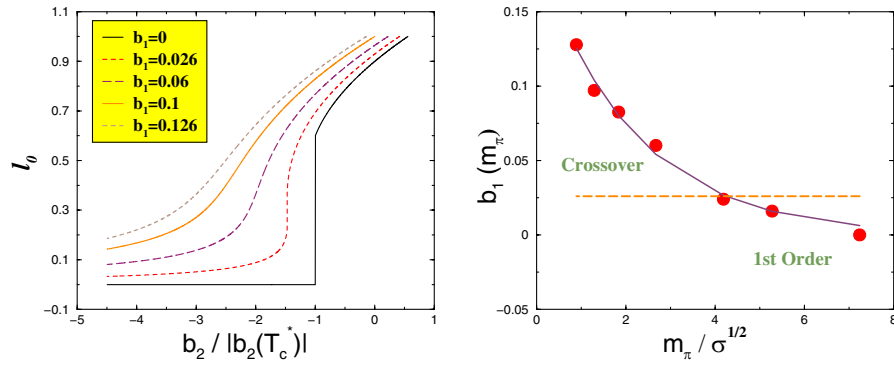
FIG. 2 (color online).   Left panel: The expectation value for the Polyakov loop, $\ell_0[b_2(T)]$, for various values of the explicit symmetry breaking coupling, b_1. All curves terminate at $\ell_0 = 1 \Leftrightarrow T = \infty$. Right panel: b_1 as a function of m_π, obtained by matching to three flavor lattice data for $T_c(m_\pi)$. The solid line corresponds to an exponential increase of b_1 with decreasing m_π; see text. The dashed horizontal line displays the end point of the line of first-order phase transitions in terms of b_1; the intersection with the $b_1(m_\pi)$ curve then gives the corresponding pion mass.

$$\frac{\Delta T_c}{T_c^*} = -3\frac{b_1}{b_3}\left(T\frac{\partial b_2}{\partial T}\right)^{-1}_{T=T_c^*} + \mathcal{O}(b_1^2)$$

$$= -\frac{2}{3}\ell_0(T_c^-)b_3\left(T\frac{\partial b_2}{\partial T}\right)^{-1}_{T=T_c^*} + \mathcal{O}(b_1^2). \qquad (20)$$

The shift in T_c with decreasing pion mass is proportional to the expectation value of the Polyakov loop just below T_c; all other factors on the right-hand side of Eq. (20) do not depend on b_1 or m_π. Numerical values for b_3 and for $b_2(T)$ were obtained in [10,12,38] by fitting the effective potential (15) to the pressure and energy density of the pure gauge theory with three colors: those are $b_3 \approx 0.9$ and $b_3^2/[T_c^*\partial b_2(T_c^*)/\partial T] \approx 1$, to within 10%. We therefore expect that numerically $\Delta T_c/T_c^*$ is roughly equal to $\ell_0(T_c^-)$.

Equations (18) and (19) seem to indicate that the discontinuity of ℓ_0 at T_c vanishes, i.e., that the phase transition turns into a crossover, at a pion mass such that $b_1(m_\pi) = 2b_3^3/27$. However, we cannot really extend our $\mathcal{O}(b_1)$ estimates to the end point of the line of first-order transitions because it applies, near T_c, only if $-4b_2(T_c) \ll b_3^2$, which translates into $b_1 \ll b_3^3/108$; see Eq. (17). To find the end point of the line of first-order transitions we solve numerically for the global minimum of (16) as a function of b_2, for given b_1; see Fig. 2 (left panel). The numerical solution is "exact" and does not assume that b_1 is small. We employ $b_3 = 0.9$ to properly account for the small latent heat of the pure gauge theory [10–12,30,38]. Also, for $b_1 = 0$, this b_3 corresponds to $\ell_c = 0.6$, which is close to the expectation value of the renormalized (fundamental) loop for the $N_c = 3$ pure gauge theory [31,32].

Clearly, for very small b_1 the order parameter ℓ_0 jumps at some $b_2^c \equiv b_2(T_c)$, i.e., the first-order phase transition persists. [The abscissa is normalized by $|b_2(T_c^*)| = 2b_3^2/9$.] We find that the discontinuity vanishes at $b_1^c = 0.026(1)$, so there is no true phase transition for $b_1 > b_1^c$.

Nevertheless, we define b_2^c even in the crossover regime via the peak of $\partial\ell_0(b_2)/\partial b_2$. The shift of b_2^c with increasing b_1 can now be converted into the shift of T_c itself by expanding about T_c^*:

$$\frac{\Delta T_c}{T_c^*} = \Delta b_2^c\left(T\frac{\partial b_2}{\partial T}\right)^{-1}_{T=T_c^*}, \qquad (21)$$

as already discussed above. We also note that from Fig. 2 (left panel), the susceptibility for the Polyakov loop at its maximum is $\partial\ell_0/\partial b_2 \simeq 3.5$, 2, and 1.5 for $b_1 = 0.06$, 0.1, and 0.126, respectively. That is, the crossover is rather sharp for the values of b_1 shown in the figure.

Explicit breaking of the $Z(3)$ symmetry of the gauge theory has previously been studied in [34–36] and has been identified as the essential factor in determining the end point of deconfining phase transitions. Moreover, while the term $\sim b_1$ quickly washes out the transition, those studies showed that along the line of first-order transitions the shift of T_c (or, alternatively, of the critical coupling β_c) is moderate, which agrees with our findings. However, the numerical values for the critical "external field" at the end point obtained in [35,36] from actual Monte Carlo simulations cannot be compared directly to our estimate for b_1^c because we work here with the renormalized (continuum-limit) loop, not the bare loop.

Reference [2] studied finite-temperature QCD with $N_f = 3$ flavors and various quark masses on the lattice (with improved $p4$ action) and determined the critical (or crossover) temperature as a function of the pion mass. Using Eq. (21) we can match our $\Delta T_c/T_c^*$ to the data from [2] to determine $b_1(m_\pi)$. In other words, we extract the function $b_1(m_\pi)$ required to match the effective Lagrangian (16) to $T_c(m_\pi)$ found on the lattice. The result is shown in Fig. 2 on the right. (Again, the pion mass is normalized to the zero-temperature string tension in the pure gauge theory, $\sqrt{\sigma} \simeq 0.425$ GeV.)

Evidently, the $\approx 33\%$ reduction of T_c from $m_\pi = \infty$ (pure gauge theory) to $m_\pi/\sqrt{\sigma} \approx 1$ requires only small explicit breaking of the $Z(3)$ symmetry for the Polyakov loop ℓ: we find that $b_1 < 0.15$ even for the smallest pion masses available on the lattice. This is due to the rather weak first-order phase transition of the pure gauge theory with $N_c = 3$ colors, reflected by the strong dip of the string tension in the confined phase near T_c^- and of the Polyakov loop screening mass m_ℓ in the deconfined phase near T_c^+ [37]; cf. also the discussion in [9–11,30].

Moreover, $b_1(m_\pi)$ appears to follow the expected behavior $\sim \exp(-m_\pi)$. The exponential fit shown by the solid line corresponds to $b_1(m_\pi) = a \exp(-b m_\pi/\sqrt{\sigma})$, with $a = 0.19$ and $b = 0.47$. Surprisingly, by naive extrapolation one obtains a pretty small explicit symmetry breaking even in the chiral limit, $b_1 \approx 0.2$.

Finally, the end point of the line of first-order transitions at $b_1^c = 0.026$ (indicated by the dashed horizontal line) intersects the curve $b_1(m_\pi)$ at $m_\pi/\sqrt{\sigma} \approx 4.2$. For heavier pions the theory exhibits a first-order deconfining phase transition, which then turns into a crossover for $m_\pi \lesssim 4.2\sqrt{\sigma} \approx 1.8$ GeV. According to our estimate, the end point of the line of first-order transitions occurs at $\Delta T_c/T_c^* \approx 12\%$, which is slightly less than a previous (qualitative) estimate of 26% from Ref. [14].

Of course, so far our analysis is restricted to pion masses $m_\pi/\sqrt{\sigma} \gtrsim 1$. On the other hand, one might cross a chiral critical point for some pion mass $m_\pi/\sqrt{\sigma} < 1$ [7]. Attempting a fit with the model (16) beyond that point would then lead to deviations from $b_1 \sim \exp(-m_\pi)$.

V. DISCUSSION

Three-color QCD exhibits a (weakly) first-order deconfining phase transition at a temperature $T_c/\sqrt{\sigma} \approx 0.63$ in the limit of infinitely heavy quarks ($\sqrt{\sigma} \approx 0.425$ GeV denotes the string tension at $T = 0$ in this theory). Near T_c, the screening mass for the fundamental Polyakov loop ℓ drops substantially [37], and so one might hope to capture the physics of the phase transition using some effective Lagrangian for ℓ [8–15].

For finite quark masses, a term linear in ℓ appears which breaks the $Z(3)$ center symmetry explicitly. This reduces the deconfinement temperature, with $\Delta T_c/T_c^*$ on the order of the expectation value of the Polyakov loop at T_c^-; cf. Eq. (20).

At some point then, the line of first-order deconfinement phase transitions ends [7,34–36,39]; see Fig. 3. We have provided a quantitative estimate of this point, $m_\pi \simeq 4.2\sqrt{\sigma} \simeq 1.8$ GeV and $T_c \simeq 240$ MeV for $N_f = 3$ degenerate flavors, by matching our effective Lagrangian for the Polyakov loop to lattice data on $T_c(m_\pi)$ [2]. Assuming that $b_1 \propto N_f$ [36] shifts "D" to $m_\pi \simeq 1.4$ GeV for $N_f = 2$ and to 0.8 GeV for $N_f = 1$.

Going to even smaller quark (or pion) masses leaves a crossover between the low-temperature and high-
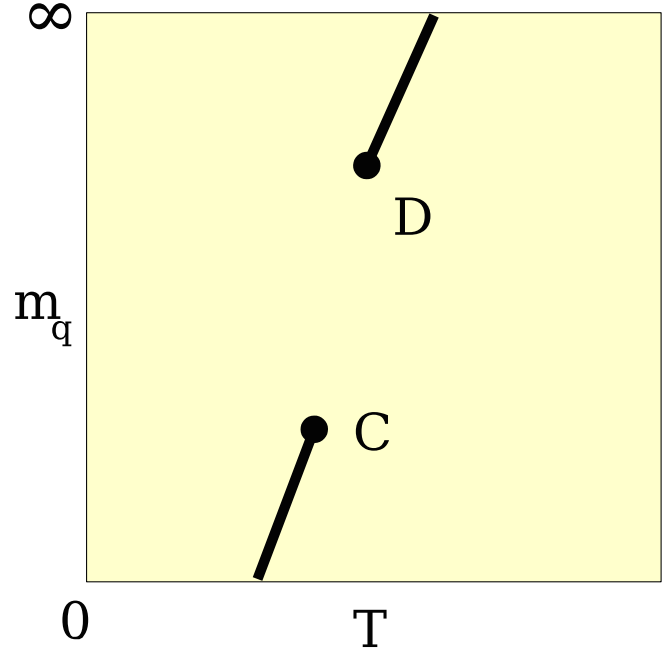

FIG. 3 (color online). Schematic phase diagram in the temperature vs quark-mass plane [7]. C is the chiral critical point, D the deconfining critical point.

temperature regimes of QCD. The dependence of the crossover temperature T_c on the pion mass appears to be well described by a small explicit breaking of the $Z(3)$ center symmetry, $b_1 \sim \exp(-m_\pi)$, down to $m_\pi/\sqrt{\sigma} \simeq 1$, which is the smallest pion mass covered by the lattice data of Ref. [2]. On the other hand, a linear sigma model leads to a stronger dependence of T_c on m_π than seen in the data.

In turn, in the chiral limit, and for $N_f = 3$ flavors, one expects a first-order chiral phase transition [4,7,39]. The linear sigma model should then be an appropriate effective Lagrangian for low-energy QCD [4–7]. The first-order chiral phase transition ends in a critical point "C" if either the mass of the strange quark or those of all three quark flavors increase. Given that the explicit symmetry breaking term for the Polyakov loop remains rather small when extrapolated to $m_\pi \to 0$, that is $b_1 \to 0.2$, we speculate that C might be rather close to the chiral limit. Indeed, recent lattice estimates for $N_f = 3$ place C at $m_\pi \simeq 290$ MeV [40] for standard staggered fermion action and $N_t = 4$ lattices; improved ($p4$) actions predict values as low as $m_\pi \simeq 67$ MeV [41].

Of course, the question arises why, for pion masses down to $\simeq 400$ MeV, the QCD crossover is described rather naturally by a slight "perturbation" of the $m_\pi = \infty$ limit, in the form of an explicit breaking of the global $Z(N_c)$ symmetry for the Polyakov loop. Physically, the reason is the flatness of the potential for ℓ in the pure gauge theory at T_c, see, e.g., the figures in [10,38], which causes the sharp drop of the screening mass for ℓ near T_c^+

[37]. This is natural, given that finite-temperature expectation values of Polyakov loops at $N_c = 3$ are close to those at $N_c = \infty$ [32], where the potential at T_c becomes entirely flat [32,42]. Hence, a rather small "tilt" of the potential (due to explicit symmetry breaking) quickly washes out the deconfining phase transition of the pure gauge theory and causes a significant shift ΔT_c of the crossover temperature already for small b_1. If so, then for $N_c \rightarrow \infty$, at the Gross-Witten point, the end point D should be located at $b_1 = 0$; the discontinuity for the Polyakov loop at T_c, which in a mean-field model for the pure gauge theory is $1/2$ at $N_c = \infty$ [32,42] then vanishes for arbitrarily small explicit symmetry breaking. This has previously been noted by Green and Karsch [34] within a mean-field model. If confirmed by lattice Monte Carlo studies, we might improve our understanding of the degrees of freedom driving the QCD crossover for pion masses above the chiral critical point C.

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