

Dynamical chiral symmetry breaking in unquenched QED₃

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We investigate dynamical chiral symmetry breaking in unquenched QED₃ using the coupled set of Dyson-Schwinger equations for the fermion and photon propagators. For the fermion-photon interaction we employ an ansatz which satisfies its Ward-Green-Takahashi identity. We present self-consistent analytical solutions in the infrared as well as numerical results for all momenta. In Landau gauge, we find a phase transition at a critical number of flavors of $N_f^{\text{crit}} \approx 4$. In the chirally symmetric phase the infrared behavior of the propagators is described by power laws with interrelated exponents. For $N_f = 1$ and $N_f = 2$ we find small values for the chiral condensate in accordance with bounds from recent lattice calculations. We investigate the Dyson-Schwinger equations in other linear covariant gauges as well. A comparison of their solutions to the accordingly transformed Landau gauge solutions shows that the quenched solutions are approximately gauge covariant, but reveals a significant amount of violation of gauge covariance for the unquenched solutions.

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I. INTRODUCTION

Over the years, quantum electrodynamics in $(2 + 1)$ dimensions (QED₃) has been studied for a variety of reasons. On the one hand it served as a laboratory for investigating nonperturbative phenomena such as dynamical mass generation or confinement in a comparatively simple framework devoid of the technical complications of non-Abelian gauge theories (for reviews see Refs. [1–3]). On the other hand QED₃ has regained recent interest due to possible applications in condensed matter systems. High- T_c cuprate superconductors possess an unconventional d -wave symmetry of the pairing condensate. Such a pairing gap has nodes at the electronic Fermi surface at which the low energy dispersion becomes linear and thus can be described as massless fermions. Since the electronic motion is mainly confined to the two dimensional copper-oxygen planes in these systems an effective low energy description of the cuprates in terms of a quantum electrodynamics in two spatial dimensions with two massless fermion flavors has been suggested [4–6]. In this picture the antiferromagnetically ordered insulating state of the cuprates would correspond to a state of broken chiral symmetry. For this reason there would be considerable interest in a study of the chiral phase transition as well as the infrared spectral properties of the fermion propagator in both the chirally symmetric and in the ordered phase of QED₃.

QED in $(2 + 1)$ dimensions is a super-renormalizable theory and has an intrinsic mass scale given by the dimensionful coupling constant $\alpha = N_f e^2/8$. With

the help of the photon polarization $\Pi(p)$ a dimensionless running coupling $\bar{\alpha} = \alpha/(p[1 + \Pi(p)]) = N_f e^2/(8p[1 + \Pi(p)])$ can be defined which separates the nonperturbative infrared momentum regime from the perturbative ultraviolet behavior [7]. A nonzero fermion mass would provide a second mass scale. Various studies of the Dyson-Schwinger equation (DSE) of the fermion propagator suggest that, in the chiral limit, the interactions generate a dynamical fermion mass $M(p^2)$ (at least for a small number of fermion flavors), and that this generated mass scale $M(p^2 = 0)$ is considerably smaller than the scale defined by the coupling constant α [7–15].

It is the smallness of this generated mass scale that poses problems in lattice Monte-Carlo simulations of QED₃ [16–19]. Finite volume effects are large and the relevant signal to determine the chiral phase transition, the dimensionless chiral condensate, is very small. Furthermore the presence of an infrared cutoff as such has been shown to reduce the value of the critical number of flavors, N_f^{crit} [20]. Thus recent studies for the number of flavors $N_f = 2$ [18] and $N_f = 4$ [18,19] determined bounds on the chiral condensate, but no definite value for N_f^{crit} could be extracted. A definite signal for chiral symmetry breaking was obtained only for $N_f = 1$ [19]. Given these problems it seems evident that a continuum method is needed to shed light on the infrared properties of QED₃.

The DSEs of the propagators of QED₃ have long been investigated employing various levels of approximation.

Early investigations of the fermion DSE based on a large N_f expansion indicated chiral symmetry to be broken only if the number of flavors N_f is smaller than a critical value of $N_f^{\text{crit}} = 32/\pi^2 \approx 3.2$ to leading order in Landau gauge [8] and $N_f^{\text{crit}} = 4/3(32/\pi^2)$ including next to leading order corrections in a nonlinear gauge [21]. These results have been questioned in Refs. [9–11], where it was argued that the $1/N_f$ -expansion is not an appropriate tool to address these nonperturbative phenomena. Using a slightly different truncation of the fermion DSE, it was found that chiral symmetry is broken for all values of N_f , although the generated mass scale is exponentially decreasing for increasing N_f [11]. Subsequent work on the *coupled* DSEs for the fermion and the photon propagator, however, again found chiral symmetry restoration for $N_f > N_f^{\text{crit}}$, with a value of N_f^{crit} between three and four [13,15]. All investigations so far are either quenched or employ a fermion-photon interaction which manifestly violates gauge symmetry.

Certainly, gauge invariance is a key property of a local quantum field theory and has to play a vital role in these investigations. Reliable results from DSEs can only be expected if the fermion-photon vertex respects local gauge symmetry. A necessary (though not sufficient) condition in this respect is given by the Ward-Green-Takahashi identity (WGTI) [22], which determines the “longitudinal” part of the fermion-photon interaction uniquely in terms of the propagator functions [23]. The remaining transverse part of the vertex is unrestricted by the WGTI and has to be determined either directly from the vertex DSE or modeled by a suitable ansatz. As the vertex DSE is considerably more complicated to solve than the ones for the propagators all work up to now concentrated on the latter strategy. Constraints on the structure of the transverse part of the vertex have been derived from gauge covariance [24,25] and multiplicative renormalizability [26]. Also perturbative vertex corrections put constraints on the transverse fermion-photon vertex [27–32] and proposals for nonperturbative generalizations of these structures have been made [28–31].

A further important nonperturbative tool to assess the gauge transformation properties of a given vertex ansatz are the Landau-Khalatnikov-Fradkin transformations (LKFT) [33]. These transformation laws leave the DSEs and the WGTI form invariant and in principle allow one to test whether a given ansatz for a vertex is gauge covariant. Such an investigation, however, is hampered by the fact that the transformation law for the vertex is quite complicated. Therefore an indirect strategy has been applied: One calculates the propagator with a given vertex ansatz in the fermion and photon DSE in various gauges and compares with the corresponding results from the LKFT of the propagator [24,31,34,35]. The success of this strategy has been limited by the problem that the LKFT is formulated in coordinate space and the necessary

Fourier-transform can be carried out analytically only for very special cases.

Our aim in this paper is to make progress in the direction of a gauge covariant solution for the propagators of QED₃. We investigate the coupled set of DSEs for the fermion and photon propagator employing a fermion-photon interaction which respects the WGTI. In addition, it contains a transverse part that has been shown to respect the LKFT properties in the quenched massless case and is thus a good starting point for unquenched QED₃. With the absence of the fermion mass scale in the symmetric phase, the assumption of a power law behavior of the dressing functions in the infrared seems natural. For the same reasons similar power laws have been found for the ghost and gluon propagators in quenched and unquenched QCD₄ [36–39]. We will see how far we can get with this assumption here, given the limitations of the chosen truncation scheme.

The paper is organized as follows: In Sec. II we discuss the DSEs for the fermion and photon propagator and define our truncation for the fermion-photon interaction. Furthermore, we recall the ultraviolet behavior of the propagators as known from methods such as the $1/N_f$ -expansion, the loop expansion, and the operator product expansion. In Sec. III we present an analytical determination of the infrared behavior of the coupled system of fermion and photon equations in the symmetric phase. We show that, within the limits of our truncation scheme, the infrared behavior of the propagators in Landau gauge is given by simple power laws, confirming a long-standing conjecture from perturbative arguments [10,11,40]. We also calculate the critical number of flavors N_f^{crit} for chiral symmetry breaking using these analytic solutions in the infrared, and determine the behavior of the fermion scalar self-energy close to N_f^{crit} for a simplified version of our fermion-photon vertex. Furthermore we investigate the gauge dependence of these power law solutions.

Numerical results in the broken and symmetric phases are presented in Sec. IV. In passing we reanalyze the quenched fermion and photon DSEs and demonstrate that the Curtis-Pennington vertex resolves an inconsistency in determining the chiral condensate from the fermion propagator, noted in Ref. [12]. We then proceed to solve the unquenched system of photon and fermion DSEs employing various ansätze for the fermion-photon vertex. No further approximations are made. Our results in Landau gauge nicely reproduce the analytical results in the ultraviolet as well as in the infrared momentum regime. With our most elaborate vertex ansatz the critical number of flavors is $N_f^{\text{crit}} \approx 4$. The order parameter of the phase transition, the dimensionless chiral condensate $(-\langle\bar{\Psi}\Psi\rangle)/e^4$ is very small, of the order of 10^{-3} even in the quenched limit, and decreases exponentially as one approaches the phase transition. For $N_f = 1$ and $N_f = 2$

we find values of the condensate in agreement with the lattice bounds [18,19]. We conclude with a discussion of our results in Sec. V.

II. THE DYSON-SCHWINGER EQUATIONS IN QED₃

We consider QED₃ with a four-component spinor representation for the Dirac algebra and N_f fermions. This allows a definition of chiral symmetry similar to the cases of QED₄ and QCD₄. With massless fermions, the Lagrangian has a $U(2N_f)$ ‘‘chiral’’ symmetry, which is broken to $SU(N_f) \times SU(N_f) \times U(1) \times U(1)$ if the fermions become massive.¹ The order parameter for this symmetry breaking is the chiral condensate. The question is: Is this chiral symmetry broken dynamically? We use the set of DSEs to investigate this question.

A. The fermion and photon propagators

The DSEs for the photon and fermion propagators in Euclidean space are given by

$$D_{\mu\nu}^{-1}(p) = D_{0,\mu\nu}^{-1}(p) - Z_1 N_f e^2 \int \frac{d^3 q}{(2\pi)^3} \text{Tr}[\gamma_\mu S(q) \Gamma_\nu(q, k) S(k)], \quad (1)$$

$$S^{-1}(p) = S_0^{-1}(p) + Z_1 e^2 \int \frac{d^3 q}{(2\pi)^3} \gamma_\mu S(q) \Gamma_\nu(q, p) D_{\mu\nu}(k), \quad (2)$$

with the momentum routing $k_\mu = q_\mu - p_\mu$. A diagrammatic notation of these equations is given in Fig. 1.

The general form of the dressed fermion propagator $S(p, \xi)$ and the photon propagator $D_{\mu\nu}(p, \xi)$ is given by

$$S(p, \xi) = \frac{i\not{p}A(p^2, \xi) + B(p^2, \xi)}{p^2 A^2(p^2, \xi) + B^2(p^2, \xi)}, \quad (3)$$

$$D_{\mu\nu}(p, \xi) = \left(\delta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2} \right) \frac{1}{p^2 [1 + \Pi(p^2)]} + \xi \frac{p_\mu p_\nu}{p^4}. \quad (4)$$

Here ξ is the gauge parameter in linear covariant gauges, with $\xi = 0$ denoting Landau gauge. The fermion functions A , B , and $M = B/A$ depend on the gauge parameter ξ . On the other hand, the vacuum polarization Π is independent of ξ . Physical quantities such as the fermion pole mass and the chiral condensate are also independent of the gauge parameter. In order to keep the notation as

¹Note that in this formalism a fermion mass term is even under parity. It is also possible to formulate QED₃ with two-component spinors; however, in such a formulation there is no chiral symmetry, and a fermion mass term breaks parity [41].

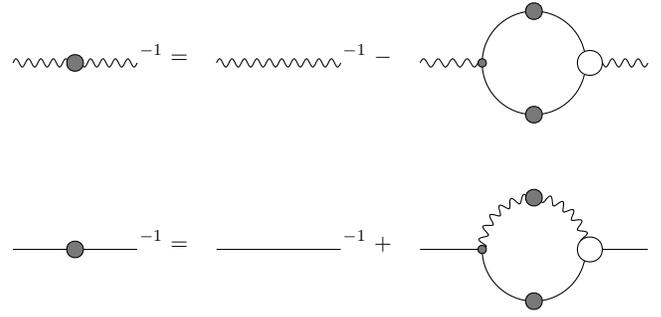


FIG. 1. The Dyson-Schwinger equations of the photon and fermion propagators in diagrammatic notation.

clear as possible, we will treat all dependence of the dressing functions on ξ implicitly from now on.

The vertex normalization constant Z_1 is related to the fermion wave function normalization Z_2 by a WGTI, $Z_1 = Z_2$. Since QED₃ is free of ultraviolet divergences, there is no need for any renormalization, though finite renormalizations of the fermion and photon fields are possible and leave the physical content of the theory invariant. In our numerical procedure we set $A(\mu^2) = 1$ at a large normalization point μ^2 and determine Z_2 self-consistently.

B. The fermion-photon vertex

In general there are several possible strategies to choose an appropriate approximation for the fermion-photon vertex $\Gamma_\nu(q, p)$ in Eqs. (1) and (2). The simplest option would be to replace the dressed vertex by the bare vertex γ_ν . However, this violates, among other things, gauge invariance and the renormalization properties of the theory. If one wants to preserve these symmetries, one has to use a suitably dressed vertex $\Gamma_\nu(q, p)$.

One way to dress the vertex would be to solve its corresponding DSE. However, the vertex DSE contains an unknown four-point function, the fermion-antifermion scattering kernel; one has to truncate the infinite set of DSEs somewhere in order to obtain a tractable set of equations, and this would only shift the problem up the hierarchy. Furthermore, one faces the technical difficulties involved in solving an integral equation in two independent momenta, i.e., in three independent variables.

A different strategy, which we adopt in this paper, is to employ an ansatz for the vertex, which has to satisfy at least two requirements:

- it must approach the perturbative form of the vertex for large momenta;
- it must satisfy the WGTI

$$i(q - p)_\nu \Gamma_\nu(p, q) = S^{-1}(p) - S^{-1}(q). \quad (5)$$

Condition (a) reflects the fact that QED₃ is an asymptotically free theory as explained in the introduction. This condition furthermore implicitly specifies the symmetry properties of the vertex, i.e., its behavior under charge conjugation and Lorenz transformations. Condition (b) is dictated by gauge invariance and determines the longitudinal part of the vertex. Furthermore, it uniquely fixes the vertex when the two fermion momenta are equal

$$\Gamma_\nu(p, p) = i \frac{\partial S^{-1}(p)}{\partial p_\nu}. \quad (6)$$

The two conditions (a) and (b) are necessary but not sufficient to ensure gauge covariance of the propagators. We will come back to this point frequently later on.

Any vertex satisfying condition (b) leads to the following interesting property of the fermion equation: With the explicit form, see Eq. (4), of the photon propagator in the fermion DSE, the integral on the right-hand side can be split into two pieces,

$$\begin{aligned} S^{-1}(p) &= S_0^{-1}(p) + Z_1 e^2 \int \frac{d^3 q}{(2\pi)^3} \gamma_\mu S(q) \frac{1}{k^2 [1 + \Pi(k^2)]} \\ &\quad \times \left(\delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2} \right) \Gamma_\nu(q, p) \\ &\quad + Z_1 e^2 \xi \int \frac{d^3 q}{(2\pi)^3} \gamma_\mu S(q) \frac{k_\mu k_\nu}{k^4} \Gamma_\nu(q, p), \end{aligned} \quad (7)$$

with the momentum convention $k_\mu = q_\mu - p_\mu$ for the photon momentum. Because of the appearance of the longitudinal projection $k_\nu \Gamma_\nu(q, p)$ in the second line of

this equation, each vertex truncation satisfying the WGTI treats this piece exactly. This will be important later on in our infrared analysis.

A suitable basis to construct a vertex ansatz satisfying the requirements (a) and (b) has been given in Ref. [23]. It consists of 12 tensor structures, which can be split up in a set of four components, Γ_ν^{BC} , completely determined by the WGTI and eight transverse components, Γ_ν^T

$$\Gamma_\nu(p, q) = \Gamma_\nu^{\text{BC}}(p, q) + \Gamma_\nu^T(p, q). \quad (8)$$

The WGTI is solved by the Ball-Chiu (BC) construction

$$\begin{aligned} \Gamma_\nu^{\text{BC}}(p, q) &= \frac{A(p^2) + A(q^2)}{2} \gamma_\nu + i \frac{B(p^2) - B(q^2)}{p^2 - q^2} (p + q)_\nu \\ &\quad + \frac{A(p^2) - A(q^2)}{2(p^2 - q^2)} (\not{p} + \not{q})(p + q)_\nu. \end{aligned} \quad (9)$$

The WGTI furthermore constrains the $\sigma_{\mu\nu}(p_\mu + q_\mu)$ -component of the vertex to be zero. The eight transverse components satisfy

$$k_\nu \Gamma_\nu^T(p, q) = 0, \quad \Gamma_\nu^T(p, p) = 0, \quad (10)$$

and are otherwise constrained by condition (a). Much work has been invested to determine $\Gamma_\nu^T(q, p)$ in the perturbative region [26–32,42] to constrain possible non-perturbative ansätze, but so far without conclusive results.

A minimal ansatz for $\Gamma_\nu^T(q, p)$ ensuring multiplicative renormalizability in four dimensional quenched QED has been given by Curtis and Pennington [26]

$$\Gamma_\nu^{T,\text{CP}}(p, q) = \frac{A(p^2) - A(q^2)}{2} \frac{[(p^2 - q^2)\gamma_\nu - (\not{p} - \not{q})(p + q)_\nu](p^2 + q^2)}{(p^2 - q^2)^2 + [M^2(p^2) + M^2(q^2)]^2}. \quad (11)$$

Burden and Roberts [24] discovered another favorable property of the Curtis-Pennington (CP) vertex $\Gamma_\nu^{\text{CP}}(q, p) = \Gamma_\nu^{\text{BC}}(q, p) + \Gamma_\nu^{T,\text{CP}}(q, p)$, which holds in *quenched massless* QED in both three and four dimensions: With the help of the LKFT [33] for the propagators they showed that the CP-vertex indeed preserves gauge covariance in these special cases. More sophisticated ansätze for the transverse parts of the vertex have been given in Refs. [25,28–31]. Contrary to early expectations, it has been noted [30] that $\Gamma_\nu^T(p, q)$ does not vanish in Landau gauge and contains terms that have to be explicitly dependent on the gauge parameter ξ .

C. The truncated Dyson-Schwinger equations

A numerical investigation of all these ansätze, though highly desirable, is a formidable task. Up to now the quenched fermion DSE and the photon DSE of QED₃ have been studied employing a bare fermion-photon vertex as well as the BC construction [12,24] and the CP-

vertex [43]. Partly unquenched calculations can be found in Refs. [7–11,44] where the photon propagator has been approximated by its $1/N_f$ -expression. Fully unquenched dynamical solutions employing the bare as well as the first part of the BC-vertex have been reported in Refs. [13,15]. In our work we will extend these investigations and solve the unquenched equations employing the CP-construction in the fermion DSE and the BC-vertex in the photon equation. The reason for this hybrid choice is the following: The transverse term in the CP-vertex has been constructed for quenched QED, i.e., its structure is adapted to the kinematical situation in the fermion DSE and it is believed to approximate some parts of the real fermion-photon vertex that are important in the fermion DSE. In the photon DSE, however, a different kinematical region of the vertex is probed. The $\Gamma_\nu^{T,\text{CP}}$ -term leads to divergences here [45] and is thus not a good approximation to the important parts of the real vertex in the photon DSE.

Substituting the CP-vertex into the fermion DSE and taking appropriate traces we arrive at

$$\begin{aligned}
B(p^2) = & Z_2 e^2 \int \frac{d^3 q}{(2\pi)^3} \frac{1}{k^2 [q^2 A^2(q^2) + B^2(q^2)]} \left(\frac{1}{1 + \Pi(k^2)} \left\{ \frac{A(p^2) + A(q^2)}{2} 2B(q^2) + [A(p^2) - A(q^2)] B(q^2) \Omega(p^2, q^2) \right. \right. \\
& + [\Delta AB(q^2) - \Delta BA(q^2)] \left[-\frac{k^2}{2} + (p^2 + q^2) - \frac{(p^2 - q^2)^2}{2k^2} \right] \left. \left. + \xi \left[\frac{A(p^2) + A(q^2)}{2} B(q^2) + \Delta AB(q^2) \frac{(p^2 - q^2)^2}{2k^2} \right. \right. \right. \\
& \left. \left. \left. - \Delta BA(q^2) \left(\frac{(p^2 - q^2)^2}{2k^2} + \frac{q^2 - p^2}{2} \right) \right] \right) \right) \quad (12)
\end{aligned}$$

$$\begin{aligned}
A(p^2) = & Z_2 + Z_2 e^2 \int \frac{d^3 q}{(2\pi)^3} \frac{A(q^2)}{p^2 k^2 [q^2 A^2(q^2) + B^2(q^2)]} \left(\frac{1}{1 + \Pi(k^2)} \left\{ \frac{A(p^2) + A(q^2)}{2} \left(-\frac{k^2}{2} + \frac{(p^2 - q^2)^2}{2k^2} \right) + [A(p^2) \right. \right. \\
& \left. \left. - A(q^2)] \Omega(p^2, q^2) \left(\frac{p^2 + q^2 - k^2}{2} \right) - \left[\Delta A \frac{p^2 + q^2}{2} + \Delta BM(q^2) \right] \left[-\frac{k^2}{2} + (p^2 + q^2) - \frac{(p^2 - q^2)^2}{2k^2} \right] \right\} \right. \\
& \left. + \xi \left[\frac{A(p^2) + A(q^2)}{2} \left(\frac{p^2 + q^2}{2} - \frac{(p^2 - q^2)^2}{2k^2} \right) + \Delta A \frac{(p^2 - q^2)}{2} \left(\frac{p^2 - q^2}{2} + \frac{q^4 - p^4}{2k^2} \right) - \Delta BM(q^2) \left(\frac{(p^2 - q^2)}{2} \right. \right. \right. \\
& \left. \left. \left. + \frac{(q^2 - p^2)^2}{2k^2} \right) \right] \right) \quad (13)
\end{aligned}$$

Here we used the abbreviations

$$\begin{aligned}
\Delta A &= \frac{A(p^2) - A(q^2)}{p^2 - q^2}, & \Delta B &= \frac{B(p^2) - B(q^2)}{p^2 - q^2}, \\
\Omega(p^2, q^2) &= \frac{p^4 - q^4}{(p^2 - q^2)^2 + [M^2(p^2) + M^2(q^2)]^2}.
\end{aligned}$$

Furthermore we have used the WGTI $Z_1 = Z_2$ for the vertex and wave function renormalization constants.

In the photon equation we contract the Lorenz indices with the general tensor [46,47]

$$\mathcal{P}_{\mu\nu}^{(\zeta)}(p) = \delta_{\mu\nu} - \zeta \frac{P_\mu P_\nu}{p^2}. \quad (14)$$

Inserting the BC-vertex in the general expression for the vacuum polarization, we obtain

$$\begin{aligned}
\Pi(p^2) = & -Z_2 e^2 N_f \int \frac{d^3 q}{(2\pi)^3} \frac{1}{q^2 A^2(q^2) + B^2(q^2)} \frac{1}{k^2 A^2(k^2) + B^2(k^2)} \left\{ \frac{A(q^2) + A(k^2)}{2} [W_1(p^2, q^2, k^2) A(q^2) A(k^2) \right. \\
& + W_2(p^2, q^2, k^2) B(q^2) B(k^2)] + \frac{A(q^2) - A(k^2)}{2(q^2 - k^2)} [W_3(p^2, q^2, k^2) A(q^2) A(k^2) + W_4(p^2, q^2, k^2) B(q^2) B(k^2)] \\
& \left. + \frac{B(q^2) - B(k^2)}{q^2 - k^2} [W_5(p^2, q^2, k^2) A(q^2) B(k^2) + W_6(p^2, q^2, k^2) B(q^2) A(k^2)] \right\}. \quad (15)
\end{aligned}$$

The general form of the kernels W_i as well as a discussion of the (weak) dependence of the photon equation on the parameter ζ of the projector (14) can be found in Appendix A. Here we give the kernels for the special value $\zeta = 3$, suggested in Ref. [46] to avoid spurious divergences

$$W_1(p^2, q^2, k^2) = \frac{3k^4}{p^4} - k^2 \left(\frac{2}{p^2} + \frac{6q^2}{p^4} \right) - 1 - \frac{2q^2}{p^2} + \frac{3q^4}{p^4}, \quad (16)$$

$$W_2(p^2, q^2, k^2) = 0, \quad (17)$$

$$\begin{aligned}
W_3(p^2, q^2, k^2) &= \frac{3k^6}{p^4} - k^4 \left(\frac{4}{p^2} + \frac{3q^2}{p^4} \right) + k^2 \left(1 - \frac{3q^4}{p^4} \right) \\
&+ q^2 - \frac{4q^4}{p^2} + \frac{3q^6}{p^4}, \quad (18)
\end{aligned}$$

$$\begin{aligned}
W_4(p^2, q^2, k^2) &= \frac{-6k^4}{p^4} + k^2 \left(\frac{4}{p^2} + \frac{12q^2}{p^4} \right) - 2 + \frac{4q^2}{p^2} \\
&- \frac{6q^4}{p^4}, \quad (19)
\end{aligned}$$

$$W_5(p^2, q^2, k^2) = \frac{3k^4}{p^4} - k^2 \left(\frac{4}{p^2} + \frac{6q^2}{p^4} \right) + 1 + \frac{3q^4}{p^4}, \quad (20)$$

$$W_6(p^2, q^2, k^2) = \frac{3k^4}{p^4} - k^2 \left(\frac{6q^2}{p^4} \right) + 1 - \frac{4q^2}{p^2} + \frac{3q^4}{p^4}. \quad (21)$$

D. $1/N_f$ -expansion and asymptotic behavior

Several previous studies of QED₃ have used the $1/N_f$ -expansion² to justify truncations or to calculate the asymptotic behavior of the dressing functions. Our approach does not rely on this expansion, but it is interesting to compare our results with the ones obtained in this way. We therefore shortly summarize the anticipated behavior of the dressing functions based on the $1/N_f$ -expansion.

For the vacuum polarization, one finds for N_f massless fermion flavors to leading order in a $1/N_f$ -expansion [7]

$$\Pi(p^2) = \frac{N_f e^2}{8p} = \frac{\alpha}{p}, \quad (22)$$

independently of the value for the gauge parameter ξ . In the full theory, this expression remains valid in the ultraviolet asymptotic limit, as has been demonstrated in quenched approximation employing a BC-vertex in the fermion loop [48]. Our numerical results show that this is also the case in the unquenched case. In the infrared we will find a modified power law for the photon.

The asymptotic behavior of the vector self-energy to two-loop order in *quenched approximation* is given by [29]

$$\begin{aligned} A(p^2 \rightarrow \infty) &= 1 + \frac{\xi e^2}{16p} + \frac{e^4 \xi^2}{64\pi^2 p^2} + \frac{3e^4}{64\pi^2 p^2} \left(\frac{\pi^2}{4} - \frac{7}{3} \right) \\ &+ O(\alpha^3) \end{aligned} \quad (23)$$

Note that to this order the vector dressing function A receives positive corrections in all gauges, i.e., $A(p^2) \downarrow 1$ for $p^2 \rightarrow \infty$. Furthermore, these corrections to its asymptotic value $A = 1$ behave like inverse powers of the momentum.

On the other hand, from an *unquenched* $1/N_f$ -expansion in Landau gauge, employing a bare vertex and the $1/N_f$ photon propagator given in Eq. (22), one obtains for $p < \alpha$

$$A(p^2) = 1 + \frac{8}{3N_f \pi^2} \ln(p/\alpha), \quad (24)$$

²This is equivalent to a perturbative expansion for small e^2 while keeping $\alpha = N_f e^2/8$ fixed. As QED₃ is an asymptotically free theory this expansion will provide correct answers in the ultraviolet.

up to terms that are regular for $p \rightarrow 0$. Certainly this expression cannot be valid in the ultraviolet region [as $A(p^2) \rightarrow 1$ for large momenta], nor in the (far) infrared region, reflecting the inconsistency of ignoring vertex corrections. Nevertheless, it has been argued [10,11,40] that it could be the first term in the build up of an anomalous dimension

$$A(p^2) = \left(\frac{p^2}{\alpha^2} \right)^\eta, \quad (25)$$

with

$$\eta = \frac{4}{3\pi^2 N_f} \approx \frac{0.135}{N_f}, \quad (26)$$

in the infrared region. We will come back to this possibility in our infrared analysis in the next section.

Finally, the analysis of the asymptotic behavior of the scalar dressing function $B(p^2)$ in the chirally broken phase of massless QED₃ is outlined in [12]. Since we are interested in dynamical chiral symmetry breaking, which is a purely nonperturbative phenomenon, we cannot rely on the $1/N_f$ -expansion to obtain an expression for B . Using the operator product expansion however, one finds that asymptotically

$$B(p^2 \rightarrow \infty) = \frac{2 + \xi}{4} \frac{\langle \bar{\Psi}\Psi \rangle}{p^2}, \quad (27)$$

with $A(p^2 \rightarrow \infty) \rightarrow 1$. Thus the chiral condensate $\langle \bar{\Psi}\Psi \rangle$ can be obtained from the fermion propagator in two ways: On the one hand it can be read off from the asymptotic behavior of the B-function and on the other hand it is given by the trace of the propagator in coordinate space. In Ref. [12] slight deviations between these two methods have been found. We will demonstrate that these deviations do not occur in our truncation. The asymptotic form, Eq. (27), is reproduced to very good accuracy for a range of values of the gauge parameter ξ in our numerical analysis.

III. INFRARED ASYMPTOTIC BEHAVIOR

Unlike the situation in a perturbative analysis, where one has a definite starting point to work out results order by order, an analysis of the nonperturbative momentum regime of QED₃ is based solely on self-consistency and relies therefore on a physically motivated ansatz to start with. Inspired by the conjecture of the previous subsection and guided by our numerical analysis, our working hypothesis will be that at least in Landau gauge the infrared behavior of QED₃ in the symmetric phase is given by power laws. We will investigate this assumption employing different vertex truncations and see how far we can get. Finally we will investigate whether our results are gauge covariant.

A. Infrared analysis in Landau gauge

1. The symmetric phase

For the following analysis we will not use the full CP- or BC-vertex constructions but only the term proportional to γ_μ , denoted 1BC,

$$\Gamma_\mu^{1BC}(p, q, k) = \frac{A(p^2) + A(q^2)}{2} \gamma_\mu. \quad (28)$$

This choice has the advantage that the equations are simplified significantly and it already contains all qualitative features of the solution employing the full CP/BC-vertex in the infrared region, as will be demonstrated by our numerical calculations given in Sec. IV. Furthermore, it has the merit that we can solve the DSEs in the infrared analytically. This vertex has been considered before in Ref. [15], though no attempt was made there to solve the DSEs analytically.

The starting point of our investigation is a power law ansatz for the vector dressing function

$$A(p^2) = c p^{2\kappa}, \quad (29)$$

with the constant c and the power κ to be determined self-consistently. We expect this ansatz to be valid in the infrared, i.e., in the momentum region $p \ll \alpha$; for $p > \alpha$ the function $A(p^2)$ rapidly approaches its free form, $A(p^2) = 1$. The integrals on the right-hand side of the DSEs are dominated by the infrared contributions, coming from the region $p < \alpha$. Thus one can safely substitute the power law for $A(p^2)$ and cut off the integrals at $p = \alpha$. Alternatively, one can substitute the power law over the entire momentum range in the integrals, provided one keeps track of possible ultraviolet divergences. After integration, the resulting power behavior on the right-hand side of the equations then has to match the power law on the left-hand side.

Given the ansatz, Eq. (29), we first have to derive the corresponding power law of the photon polarization. After substituting the ansatz (29) into the right-hand side of the photon equation, Eq. (15), the integral can be carried out with the help of Eq. (B6). We arrive at

$$\begin{aligned} \Pi_{1BC}(p^2) &= Z_2 \frac{\alpha}{c} \frac{4}{\pi} \frac{\Gamma(3/2 - \kappa)\Gamma(1/2 + \kappa)}{\Gamma(3 - \kappa)\Gamma(1 + \kappa)} p^{-1-2\kappa}, \\ \Pi_{1BC}(p^2) &=: Z_2 \frac{\alpha}{c} w_{1BC} p^{-1-2\kappa}, \end{aligned} \quad (30)$$

where we have introduced the dimensionless function $w_{1BC}(\kappa)$

$$w_{1BC}(\kappa) = : \frac{4}{\pi} \frac{\Gamma(3/2 - \kappa)\Gamma(1/2 + \kappa)}{\Gamma(3 - \kappa)\Gamma(1 + \kappa)}. \quad (31)$$

For the analysis of the fermion DSE, Eq. (13), we assume that $\kappa > -1/2$, as $\kappa \leq -1/2$ only admits the trivial solution $A \equiv 1$ cf. Sec. III B below. With $p \ll \alpha$ the photon dressing is given by

$$\frac{1}{1 + \Pi_{1BC}(p^2)} \approx \frac{c}{\alpha w_{1BC}} p^{1+2\kappa}. \quad (32)$$

Together with the power law Eq. (29) we then obtain

$$\begin{aligned} c p^{2\kappa} &= Z_2 + \frac{c}{w_{1BC} N_f \pi^3} \int d^3 q \left\{ \frac{k^{-1+2\kappa}}{p^2 q^{2+2\kappa}} \frac{p^{2\kappa} + q^{2\kappa}}{2} \right. \\ &\quad \left. \times \left(-\frac{k^2}{2} + \frac{(p^2 - q^2)^2}{2k^2} \right) \right\}. \end{aligned} \quad (33)$$

The treatment of this type of equation has been discussed in detail in Refs. [49,50] for the system of ghost and gluon DSEs in QCD₄. To proceed one has to distinguish two cases, $\kappa < 0$ and $\kappa > 0$. In the first case the left-hand side of the equation becomes singular for $p^2 \rightarrow 0$ and has to be matched by a corresponding singularity in the integral on the right-hand side. In this case the constant term, Z_2 , stemming from the bare propagator is suppressed and can simply be discarded. (This case is analogous to the gluon equation in QCD₄). On the other hand, if $\kappa > 0$ the left-hand side goes to zero. The renormalization constant Z_2 thus has to be canceled by a constant term generated by the integral. A straightforward way to deal with this situation is to discard the constant term Z_2 and at the same time to eliminate the constant term hiding in the integral by employing dimensional regularization. Furthermore we have to eliminate a spurious divergence introduced by employing the power law ansatz over the whole momentum range. We are then left with

$$\begin{aligned} p^{2\kappa} &= \frac{p^{2\kappa}}{w_{1BC} N_f \pi^2} \left(\frac{1}{2\kappa(1 - 2\kappa)} + \frac{\pi}{(3 + 2\kappa)} \right. \\ &\quad \left. \times \frac{\Gamma(\kappa)\Gamma(1 - \kappa)}{\Gamma(3/2 - \kappa)\Gamma(1/2 + \kappa)} \right). \end{aligned} \quad (34)$$

Note that the normalization factor Z_2 as well as the coefficient c of the power law has been dropped out of the equation as expected. The powers of momentum match on both sides of the equation thus confirming that the power law is indeed a self-consistent solution of the DSEs in the chirally symmetric phase. Eqs. (31) and (34) together determine the exponent κ and therefore completely describe the behavior of the photon and fermion propagators in the infrared in the given truncation scheme.

From our analysis we find a possible explanation, why the authors of Refs. [10,11,44] did not find a phase transition in their truncation scheme: As the feedback from the function A onto the vacuum polarization is not taken into account in their approach, i.e., $\Pi(p^2) \sim 1/p$, the right-hand side of the DSE for A is proportional to p^0 , which only matches the left-hand side if the A -function becomes a (trivial) constant in the infrared. Thus there is no self-consistent power law solution in this truncation scheme. This feedback was first considered in Refs. [13–15].

An explicit numerical solution of the Eqs. (31) and (34) is shown in Fig. 2. For the sake of comparison we also display the solution for the case of a bare fermion-photon vertex, which can be obtained from a similar analysis. Both results are very well fitted by a series of powers of $1/N_f$:

$$\kappa_{\text{bare}} = \frac{0.135}{N_f} + \frac{0.090}{N_f^2} + O(1/N_f^3), \quad (35)$$

$$\kappa_{\text{IBC}} = \frac{0.115}{N_f} + \frac{0.044}{N_f^2} + O(1/N_f^3), \quad (36)$$

which suggests a connection to the $1/N_f$ -expansion. Comparing the first term of our result for the bare vertex with Eq. (25) we find that the $1/N_f$ -result is indeed the first term in the build up of an anomalous dimension. The additional $1/N_f^2$ -term in our fit indicates that also loop corrections to the next order in a $1/N_f$ -expansion sum up and contribute to the anomalous dimension. However, one should keep in mind that our calculation is not a $1/N_f$ expansion of the DSEs. It is therefore not surprising that our result for the order $1/N_f^2$ contribution to the anomalous dimension deviates from that obtained in a $1/N_f$ expansion [51]

$$\kappa_{1/N_f} = \frac{4}{3\pi^2 N_f} - \frac{8(32 - 3\pi^2)}{9\pi^4 N_f^2} + O(1/N_f^3) \quad (37)$$

$$\approx \frac{0.135}{N_f} - \frac{0.022}{N_f^2} + O(1/N_f^3). \quad (38)$$

Furthermore, the vertex dressing of the IBC-vertex modifies $\kappa(N_f)$ to quite some extent.

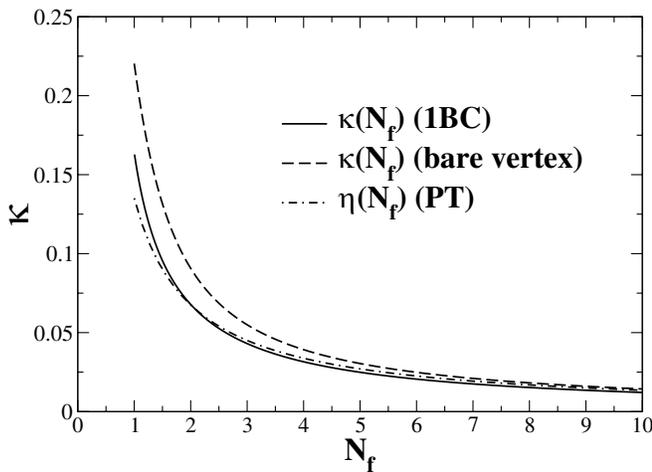


FIG. 2. Here we display the anomalous dimension of the fermion vector dressing function obtained from our infrared analysis, $\kappa(N_f)$, compared with the conjecture from perturbation theory, $\eta(N_f)$.

The infrared powers presented in Fig. 2 are not the only solutions of the Eqs. (31) and (34). In the range $-1/2 < \kappa < 1$ we found a second solution which is excellently fitted by

$$\kappa_{\text{IBC}} = 0.5 - \frac{0.050}{N_f} - \frac{0.006}{N_f^2} - \frac{0.028}{N_f^3}. \quad (39)$$

However, contrary to the solution (36), this solution does depend very heavily on the projection method in the photon equation, i.e., it depends on the parameter ζ introduced in Eq. (14). Furthermore, it does not connect to the ultraviolet behavior of the dressing functions, i.e., we do not find numerical solutions of the DSEs interpolating between the infrared behavior, Eq. (39), and the ultraviolet asymptotic behavior given in subsection II D. We therefore discard the solution (39) in the following.

We have shown so far that in the symmetric phase the power law (29) leads to a self-consistent solution of the fermion and photon DSEs, assuming that the vertex is dominated by its γ_μ -part. The crucial question is, of course, whether the power law survives when additional structure of the vertex is taken into account. That this is indeed the case for our choice of the fermion-photon vertex can be shown by a simple dimensional analysis. Plugging the power law into the CP-vertex, Eqs. (9) and (11), we find that with a vanishing B-function all terms in the vertex depend on combinations of momenta which are of the same order $p^{2\kappa}$ as the leading term. Thus after integration the CP-vertex will only change the coefficients of the right-hand sides of the fermion and photon DSEs, Eqs. (30) and (34), but not the general power law behavior. We thus expect a modified function $\kappa_{CP}(N_f)$ as compared to $\kappa_{\text{IBC}}(N_f)$ and $\kappa_{\text{bare}}(N_f)$. That these modifications are small will be confirmed by our numerical analysis in Sec. IV below. Further modifications have to be expected from including other transverse parts of the fermion-photon vertex and it is by no means excluded that κ finally becomes negative. We will further discuss this possibility later on in Sec. III B 2.

2. The chirally broken phase close to N_f^{crit}

Next we investigate the chirally broken phase close to the critical value N_f^{crit} of the phase transition (assuming for now that there is a chirally broken phase, and a critical value of N_f). In this region the dynamically generated fermion mass will be extremely small compared to α . The momentum range $B(0) \ll p \ll \alpha$ will dominate the integral on the right-hand side of the DSE for the B-function, Eq. (12), and therefore the chirally symmetric solutions for the photon polarization and the dressing function A

$$A(p^2) = cp^{2\kappa} \quad (40)$$

$$\Pi(p^2) = Z_2 \frac{\alpha}{c} w(\kappa) p^{-1-2\kappa}, \quad (41)$$

determined in the last subsection, can be substituted in the integral on the right-hand side for all momenta. Choosing again the case of the 1BC-vertex for simplicity we obtain

$$B(p^2) = Z_2 \frac{\alpha}{N_f \pi^3} \int d^3 q \frac{1}{k^2 + Z_2 \frac{\alpha w_{1BC}(\kappa)}{c} k^{1-2\kappa}} \times \frac{2B(q)}{c^2 q^{2+4\kappa} + B(q)^2} \frac{c(p^{2\kappa} + q^{2\kappa})}{2}. \quad (42)$$

The angular integrals are easily performed, and we obtain

$$B(p^2) = \frac{2Z_2 c \alpha}{p N_f \pi^2} \frac{1}{1 + 2\kappa} \int_0^\infty dq q \frac{B(q)(p^{2\kappa} + q^{2\kappa})}{c^2 q^{2+4\kappa} + B^2(q)} \times \ln \left(\frac{Z_2 \frac{\alpha w_{1BC}(\kappa)}{c} + (p+q)^{1+2\kappa}}{Z_2 \frac{\alpha w_{1BC}(\kappa)}{c} + |p-q|^{1+2\kappa}} \right). \quad (43)$$

As we are interested in the momenta $p, q \ll \alpha$ the scale α cuts off the integral and the logarithm can safely be expanded. Furthermore close to the phase transition the

interesting region is the one with $B^2(q) \ll c^2 q^{2+4\kappa}$, therefore the equation can be linearized. Thus we arrive at

$$B(p) = \frac{4}{w_{1BC}(\kappa) N_f \pi^2} \int_0^\alpha dq \frac{B(q)(p^{2\kappa} + q^{2\kappa})}{q^{4\kappa}} \times [\max(p, q)]^{2\kappa-1}. \quad (44)$$

The analysis of this type of equation is well-known [8,52]. The integral equation can be solved directly by substituting the power law $B(p) \sim p^b$ and comparing coefficients on both sides. On the other hand, by converting the (nonlinear) integral equation into a differential equation one obtains the boundary condition

$$\left[p \frac{dB(p)}{dp} + B(p) \right]_{p=\alpha} = 0, \quad (45)$$

which has to be satisfied by the power law solution. Therefore a nontrivial exponent b has to be either complex or $b = -1$.

From the integral equation, Eq. (44), we obtain both the chirally symmetric solution $B(p) \equiv 0$ and the nontrivial solution

$$b = -\frac{1}{2} + 2\kappa \pm \frac{1}{2} \sqrt{(1 - 4\kappa + 8\kappa^2) - \frac{16(1 - 2\kappa)}{\omega(\kappa)}} - 4 \sqrt{\kappa^2(1 - 2\kappa)^2 - \frac{16\kappa^2(1 - 2\kappa)}{\omega(\kappa)} + \frac{16(1 - 2\kappa)^2}{[\omega(\kappa)]^2}}, \quad (46)$$

where we have used the abbreviation $\omega(\kappa) := w_{1BC}(\kappa) \pi^2 N_f$. Above a critical value N_f^{crit} both solutions of the exponent b are in the interval $-1 < b < 0$ and not compatible with the boundary condition Eq. (45): The system is in the symmetric phase. Setting the discriminant of the outer root in Eq. (46) equal to zero and using Eq. (30) we find a critical number of flavors of

$$N_f^{\text{crit}, 1BC} \approx 3.56, \quad (47)$$

for the case of the 1BC-vertex. For the sake of comparison we also give the result in the bare vertex truncation

$$N_f^{\text{crit}, \text{bare}} \approx 3.96. \quad (48)$$

Note that for $\kappa = 0$ we have $A = 1$, $\Pi(p^2) = \alpha/p$ and $w_{1BC} \equiv 1$, and consequently recover the well-known result from the $1/N_f$ -expansion [8]

$$b = -\frac{1}{2} \pm \frac{1}{2} \sqrt{1 - \frac{32}{\pi^2 N_f}}, \quad (49)$$

which gives

$$N_f^{\text{crit}, 1/N_f} = 32/\pi^2 \approx 3.24. \quad (50)$$

[This limit also serves to determine the correct sign in front of the inner root of Eq. (46).]

Although the critical number of flavors is not too far away from the old $1/N_f$ result there is a clear qualitative difference between solutions from the full (coupled) set of DSEs and the one from a $1/N_f$ -expansion: The nonperturbative nature of the DSEs manifests itself in the power law solution of the vector dressing function $A(p^2)$, i.e., in $\kappa \neq 0$. Such a behavior can never be obtained in a perturbative expansion and is in marked contrast to the assumption $A = 1$ employed in the leading order $1/N_f$ -expansion. Similar criticism has been raised in Refs. [9–11,44], and is at the origin of a long-standing controversy.

Having determined the location of the phase transition we now investigate the behavior of the B-function for $N_f \rightarrow N_f^{\text{crit}}$ from below. Abbreviating

$$b = -\frac{1}{2} + 2\kappa \pm \frac{1}{2} f_{1BC}(N_f, \kappa), \quad (51)$$

the oscillatory solution of the linearized equation, Eq. (44), can be written in the form

$$B(p) = p^{-1/2+2\kappa} \sin \left[\frac{1}{2} f_{1BC}(N_f, \kappa) (\ln[p/B(0)] + \delta) \right] \quad (52)$$

with a phase δ and the relevant scale for mass generation,

$B(0)$, in the logarithm. Plugging this solution into the boundary condition Eq. (45) we take the limit $N_f \rightarrow N_f^{\text{crit}}$ and arrive at the condition

$$\begin{aligned} & \frac{1}{2} f_{\text{IBC}}(N_f, \kappa) (\ln[p/B(0)] + \delta) \\ &= n\pi - \frac{1}{1+4\kappa} f_{\text{IBC}}(N_f, \kappa). \end{aligned} \quad (53)$$

It can be shown [7,8] that the value $n = 1$ gives the lowest vacuum energy. Therefore

$$B(0) = \alpha e^{[2/(1+4\kappa)+\delta]} \exp\left[\frac{-2\pi}{f_{\text{IBC}}(N_f, \kappa)}\right]. \quad (54)$$

We find an exponentially decreasing mass $M(0) = B(0)/A(0)$ at zero momentum close to the chiral phase transition. This is in agreement with the numerical findings of Ref. [15], where the same IBC-vertex was used in both the fermion and the photon DSE. A similar expression describes the B-function in the bare vertex truncation and in the $1/N_f$ -expansion [7,8,21]. Our numerical study in Sec. IV will demonstrate that an exponential decrease of $B(0)$ with a modified function $f(N_f, \kappa)$ will also emerge when the CP-vertex is employed. This type of exponential behavior near N_f^{crit} , which is different from the usual first or second order phase transition, is reminiscent of a conformal phase transition [53]. Strictly speaking however, it is not a conformal phase transition, because QED₃ is a super-renormalizable theory with a dimensionful coupling constant, and the conformal symmetry is broken in both the chirally symmetric and the broken phase.

B. Infrared analysis in general linear covariant gauges

Having determined the infrared behavior of the photon polarization and the fermion dressing functions in Landau gauge we now turn to general linear covariant gauges. In the following we will investigate whether our ansatz for the fermion-photon interaction is sophisticated enough to generate gauge covariance of the photon polarization and the fermion propagator in the symmetric phase. To this end we will follow the strategy to first reanalyze the DSEs in general linear covariant gauges and then compare our findings with the corresponding ones from performing a LKFT of our Landau gauge solutions. We will investigate whether our truncation allows for power law solutions in general linear covariant gauges and whether the LKFT is consistent with such a scenario.

1. The photon and fermion DSEs

In QED the photon polarization is a gauge invariant object. This is evident from its LKFT, given below in Eq. (63). Thus in general a subtle interplay of the fermion propagators and the fermion-photon interaction has to

guarantee the invariance of the photon polarization in its DSE. On the perturbative one-loop level, it has been shown recently [30], that at least in the symmetric phase there have to be terms in the transverse part of the vertex that are explicitly dependent on the gauge parameter ξ . These terms are not constrained by the WGTI, and are missing in the vertex truncation investigated in this work. For general gauges these terms will be important in the photon DSE and we therefore cannot expect the photon polarization to be gauge invariant at our level of truncation.

Assuming a ξ -dependent power law for the vector dressing function of the fermions

$$A(p^2) = c(\xi) p^{2\kappa(\xi)}, \quad (55)$$

and employing the BC-vertex in the photon DSE we end up with the same expression for $\Pi(p^2)$ as in Landau gauge,

$$\Pi(p^2) = :Z_2 \frac{\alpha}{c(\xi)} (w_{\text{IBC}}[\kappa(\xi)] + w_{\text{2BC}}[\kappa(\xi)]) p^{-1-2\kappa(\xi)}. \quad (56)$$

where w_{IBC} has been given in Eq. (31) and w_{2BC} abbreviates contributions from the remaining term of the BC-vertex. The detailed form of w_{2BC} can be calculated analytically, but will not be given here as it is not important in the following. We note that on this level of truncation gauge invariance of the photon polarization in the infrared requires the exponent κ to be independent of the gauge parameter ξ . It is somewhat surprising that as a result, also the infrared behavior of the vector fermion dressing function A is gauge independent, since it is governed by the same exponent κ . This may or may not be an inconsistency in our truncation as this function A is in general a gauge dependent object. At this stage of the investigation one may hope that the dependence of κ on ξ is weak, leading to qualitatively similar results at least in the vicinity of Landau gauge.

Next we analyze the fermion DSE, Eq. (7), with the CP-vertex. In the ξ -part of the equation the vertex is replaced by the inverse fermion propagator according to the WGTI. We then obtain

$$\begin{aligned} A(p^2) = & Z_2 + \frac{Z_2 \alpha}{N_f \pi^3} \int d^3 q \left\{ \frac{1}{1 + \Pi(k^2)} \frac{1}{p^2 k^2 q^2 A(q^2)} \right. \\ & \times \left(-\frac{k^2}{2} + \frac{(p^2 - q^2)^2}{2k^2} \right) \left(\frac{A(p^2) + A(q^2)}{2} \right. \\ & \left. \left. + \frac{A(p^2) - A(q^2)}{p^2 - q^2} \frac{p^2 + q^2}{2} \right) + \frac{\xi}{p^2 k^2 q^2} \left[\frac{A(p^2)}{A(q^2)} \right. \right. \\ & \left. \left. \times \left(\frac{p^2 q^2}{2k^2} - \frac{p^4}{2k^2} + \frac{p^2}{2} \right) + \left(\frac{p^2 q^2}{2k^2} - \frac{q^4}{2k^2} + \frac{q^2}{2} \right) \right] \right\}. \end{aligned} \quad (57)$$

As has been noted in Ref. [24] the CP-vertex leads to

gauge covariant DSEs in the quenched massless case. Indeed in this limit, $\Pi(k^2) = 0$, and we have the gauge covariant solution

$$\frac{1}{A(p^2)} = 1 - \frac{e^2 \xi}{8\pi p} \arctan\left(\frac{8\pi p}{e^2 \xi}\right), \quad (58)$$

of Eq. (57). In Landau gauge, this solution reduces to the trivial solution $A(p^2) = 1$. The question is: Is there a self-consistent power law solution for the unquenched case?

Substituting the power laws, Eqs. (55) and (56), in Eq. (57), we have to distinguish several cases:

(a) $\kappa > -1/2$:

In this case the photon propagator contributes as

$$\frac{1}{1 + \Pi_{\text{IBC}}(k^2)} \approx \frac{c}{Z_2 \alpha w_{\text{IBC}}} k^{1+2\kappa}, \quad (59)$$

and with substituted power laws the integration of the right-hand side of Eq. (57) leads to

$$c p^{2\kappa} = Z_2 \frac{1}{N_f \pi^2} \left\{ \frac{c p^{2\kappa}}{Z_2 (w_{\text{IBC}} + w_{\text{BC}})} h(\kappa) + \frac{1}{p} \xi \frac{\Gamma(3/2 - \kappa) \Gamma(1/2 + \kappa)}{\Gamma(1 + \kappa) \Gamma(1 - \kappa)} \right\}, \quad (60)$$

where $h(\kappa)$ abbreviates a combination of Γ -functions and hypergeometric functions, which need not be specified here. The longitudinal $1/p$ -term dominates the right-hand side for all gauges except Landau gauge and again we do not find a self-consistent power law to this level of truncation. The only way to obtain such a solution with $\kappa > -1/2$ is the presence of a ξ -dependent transverse term in the vertex canceling the ξ -dependent longitudinal piece.³ Then the precise value of κ is again determined by the coefficients of the remaining terms.

(b) $\kappa \leq -1/2$, $\kappa \neq -1$:

In this case the vacuum polarization vanishes for $k \rightarrow 0$ and the photon propagator is proportional to $1/k^2$ in the infrared region. Integrating the right-hand side of Eq. (57), the ξ -independent part vanishes, and we obtain

$$c p^{2\kappa} = \frac{1}{p} \xi \frac{Z_2 \alpha}{N_f \pi} \frac{\Gamma(3/2 - \kappa) \Gamma(1/2 + \kappa)}{\Gamma(1 + \kappa) \Gamma(1 - \kappa)}. \quad (61)$$

Note that this remaining ξ -dependent expression is exact [cf. the comments below Eq. (7)]. Matching the coefficients (naively) gives $\kappa = -1/2$, but due to the divergence of the coefficient on the right-hand side this is not a solution. There might exist a

³Based on a less rigorous analysis of the infrared a similar conjecture has already been made in [44].

potential solution with $\kappa = -1$, which we will consider separately below. Here we conclude that there is no nontrivial self-consistent power law solution with $-1 < \kappa \leq -1/2$ within our truncation. Any possible nontrivial solutions in this range have to be generated by other transverse parts of the vertex. For example, a term proportional to $p^{2\kappa}$ after integration with appropriate coefficients would be more singular in the infrared than the ξ -piece and lead to a self-consistent nontrivial solution.

(c) $\kappa = -1$:

As in case (b), the vacuum polarization vanishes for $k \rightarrow 0$ and thus Eq. (57) becomes effectively quenched. In this case there is at least one solution, namely, Eq. (58). Certainly, this trivial solution will not survive when further parts of the transverse vertex will be taken into account, but it serves well in the following to illustrate an important point: Interestingly, the infrared behavior of this solution is

$$A(p^2) = 3 \left(\frac{e^2 \xi}{8\pi} \right)^2 \frac{1}{p^2} + O(p^4), \quad (62)$$

for $p \ll e^2 \xi / (8\pi)$ and thus $\kappa = -1$. Nevertheless the corresponding pure power law is not a self-consistent solution, as can be seen from Eq. (61): The right-hand side vanishes for $\kappa = -1$. The reason for this behavior is to be found in the appearance of the new scale $e^2 \xi / (8\pi)$ introduced by the gauge transformation from Landau gauge to general linear covariant gauges. For small values of the gauge parameter this new scale divides the momentum range $0 < p < \alpha$ into two regions and in general we cannot expect the pure power law to describe the physics in this whole region.

In all cases considered so far we do not find a self-consistent power law solution in non-Landau gauges. Two possible reasons for this behavior have been identified. Missing transverse parts of the fermion-photon vertex could play an important role in these gauges. Furthermore the appearance of the new scale $e^2 \xi / (8\pi)$ may invalidate a simple power law ansatz in the infrared region. In the next subsection we will investigate, whether one can derive additional information from the LKFT.

2. Landau-Khalatnikov-Fradkin transformation of the Landau gauge solution

Assuming for the moment that the power law for the vector dressing function is qualitatively correct in Landau gauge, we will now use the LKFT to determine the corresponding solutions in other gauges. The transformation laws for the photon and fermion propagators are most easily specified in coordinate space and we give

the transformation rules for the propagators in Euclidean space. The photon propagator $D_{\mu\nu}(x, \Delta)$ in general gauges can be obtained from its transverse Landau gauge form $D_{\mu\nu}(x, 0)$ by the transformation law

$$D_{\mu\nu}(x, \Delta) = D_{\mu\nu}(x, 0) + \partial_\mu \partial_\nu \Delta(x), \quad (63)$$

with the arbitrary function Δ . The corresponding transformation law for the fermion propagator is

$$S(x, \Delta) = S(x, 0)e^{[\Delta(x) - \Delta(0)]e^2}. \quad (64)$$

These transformation laws leave the DSE and the WGTI form invariant. In linear covariant gauges and in general dimension d the function $\Delta(x)$ is given by

$$\Delta(x) = -\xi \int \frac{d^d q}{(2\pi)^d} \frac{e^{-iq \cdot x}}{q^4}, \quad (65)$$

which leads to the familiar form of the photon propagator in linear covariant gauges

$$D_{\mu\nu}(p, \xi) = \left(\delta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2} \right) \frac{1}{p^2 [1 + \Pi(p^2)]} + \xi \frac{p_\mu p_\nu}{p^4}, \quad (66)$$

in momentum space with the gauge invariant photon polarization $\Pi(p^2)$. Furthermore for QED₃ one obtains the transformation law

$$S(x, \xi) = S(x, 0)e^{-x\xi e^2/(8\pi)}, \quad (67)$$

for the fermion propagator in coordinate space.

In the symmetric phase of Landau gauge QED₃ we found the power law solution

$$S(p, 0) = \frac{i\not{p}}{p^2} \frac{1}{c p^{2\kappa}}, \quad (68)$$

which leads to the corresponding expression

$$S(x, 0) = \int \frac{d^3 p}{(2\pi)^3} e^{-ip \cdot x} \frac{i\not{p}}{p^2} \frac{1}{c p^{2\kappa}}, \quad (69)$$

$$S(x, 0) = \frac{\Gamma(1 - 2\kappa) \sin(\kappa\pi)}{c 4\pi^2} \frac{1 - 2\kappa}{\kappa} \frac{\gamma_i x_i}{x^{3-2\kappa}},$$

in coordinate space (all integration formulas used in this subsection are given in Appendix B). Applying the transformation (67) we transform the propagator to general gauges and perform the inverse Fourier-transform

$$S(p, \xi) = \frac{\Gamma(1 - 2\kappa) \sin(\kappa\pi)}{c 4\pi^2} \frac{1 - 2\kappa}{\kappa} \times \int d^3 x e^{ip \cdot x} \frac{\gamma_i x_i}{x^{3-2\kappa}} e^{-x\xi e^2/(8\pi)} \quad (70)$$

$$S(p, \xi) = \frac{i\not{p}}{p^2} \frac{1}{c \cos(\kappa\pi)} \frac{2\kappa - 1}{2\kappa} \times \left[\frac{\cos\{2\kappa \arctan[p 8\pi/(\xi e^2)]\}}{\{p^2 + [\xi e^2/(8\pi)]^2\}^\kappa} - \frac{\sin\{(2\kappa - 1) \arctan[p 8\pi/(\xi e^2)]\}}{\{p^2 + [\xi e^2/(8\pi)]^2\}^{\kappa-1/2} p(2\kappa - 1)} \right]. \quad (71)$$

Note that in performing the LKFT with the infrared power law alone we have implicitly assumed that contributions from $p > \alpha = N_f e^2/8$ have no significant influence on the behavior of the transformed propagator for $p \ll \alpha$. Furthermore note that the LKFT has introduced a new scale, $\xi e^2/(8\pi)$. In order to be consistent with the previous assumption we have to restrict the gauge parameter to small values, i.e., $0 \ll \xi e^2/(8\pi) \ll \alpha$. We then obtain two momentum regions of interest where we can expand our solution:

$$A(p, \xi) = \begin{cases} c(p^2)^{-1} \cos(\kappa\pi) \frac{3}{1-4\kappa^2} \left(\frac{\xi e^2}{8\pi}\right)^{2\kappa+2} & \text{for } p \ll \frac{\xi e^2}{8\pi} \\ c(p^2)^\kappa & \text{for } p \gg \frac{\xi e^2}{8\pi} \end{cases} \quad (72)$$

As expected this expression smoothly connects to the Landau gauge power law when $\xi \rightarrow 0$. In all other linear covariant gauges we obtain the Landau gauge power law for momenta $p \gg \frac{\xi e^2}{8\pi}$. Below this scale we find surprising agreement with the LKFT of the trivial solution, Eq. (61), i.e., a power $\kappa = -1$.

3. Self-consistent power law solutions in covariant gauges

If we take the LKFT result based on the power law solution in Landau gauge, Eq. (72), at face value and combine it with the information we extracted from the infrared analysis of the coupled fermion and photon DSEs, Eqs. (56) (which implicitly defines w_{INC} and $w_{2\text{BC}}$), (60), and (61), two consistent scenarios of massless QED₃ with power law behavior in the infrared are possible:

- (I) Landau gauge behaves differently in the extreme infrared than any other linear covariant gauge. In Landau gauge we have a power law with small positive or negative values of the exponent κ (dependent on the details of the vertex truncation). In other gauges we essentially obtain the free solution with $\kappa = -1$ for $p \ll \frac{\xi e^2}{8\pi}$ and the same power law as in the Landau gauge for momenta $\frac{\xi e^2}{8\pi} \ll p \ll \alpha$. As the region $p \ll \frac{\xi e^2}{8\pi}$ can always be gauged away, all interesting infrared physics is already contained in the Landau gauge power law. In the DSE this scenario could be realized by a transverse term in the vertex which is proportional to the gauge parameter ξ and leads to a term of the

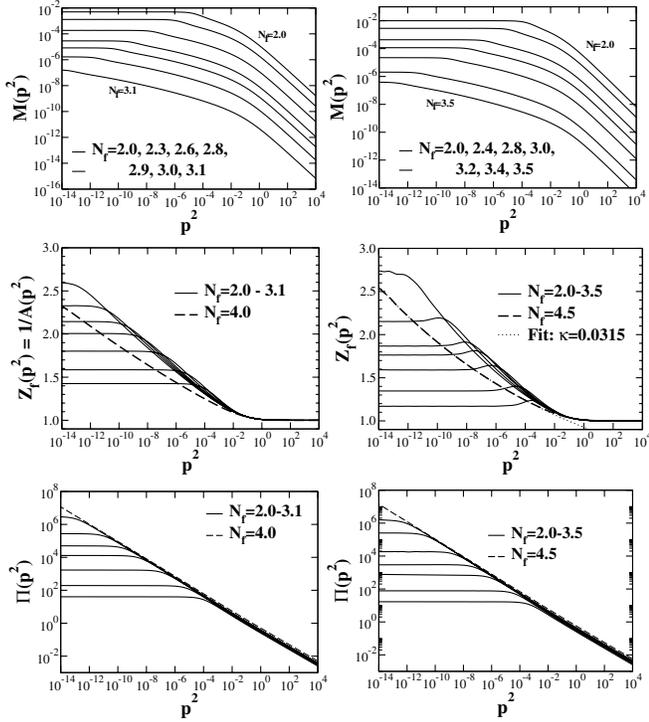


FIG. 3. Shown are the variation of the mass function, the wave function renormalization and the polarization with the number of flavors in Landau gauge, $\xi = 0$. On the left-hand side we employed the first term of the BC-vertex (1BC) in both the photon and the fermion DSEs. On the right-hand side we display the same functions calculated with the CP-vertex in the fermion DSE and the BC-vertex in the photon DSE. The scale is set by choosing $e^2 = 1$.

order $1/p^2$ in the infrared after integration of the fermion DSE. A subtle cancellation of terms in the photon equation has to guarantee a gauge invariant photon polarization.

- (II) Landau gauge behaves similar to other linear covariant gauges. This entails that the solutions with small positive values for κ found in Sec. III A are artifacts of the truncation scheme, and the true solution is a power law in the infrared with an exponent $\kappa = -1$ for all values of the gauge parameter ξ including Landau gauge. Then the gauge dependent scale $\frac{\xi e^2}{8\pi}$ does not distinguish two momentum regions with different behavior of the propagator. In the DSE this scenario requires a

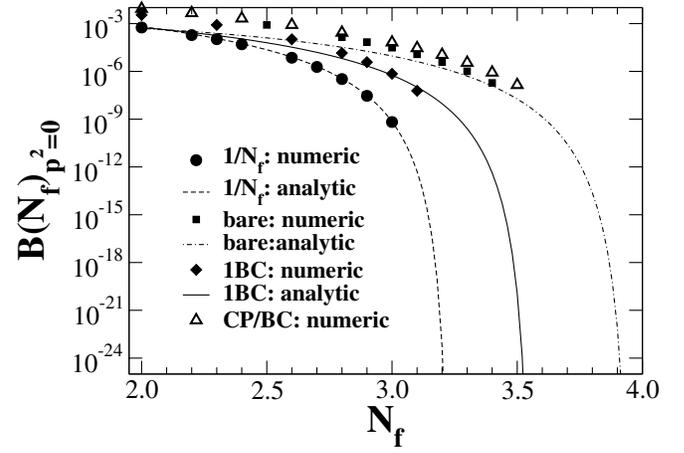


FIG. 4. The phase transition: Analytical vs numerical results for the scalar fermion dressing function, $B(p^2 = 0)$, as function of N_f . Shown are results for four different truncation schemes: The $1/N_f$ -expansion of Ref. [8], and three different vertex truncations of the coupled photon and fermion DSEs. The scale is set by choosing $e^2 = 1$.

transverse term in the vertex which does *not* explicitly contain the gauge parameter ξ and leads to a term of the order $1/p^2$ in the infrared after integration. Then the dressing function A and subsequently the photon polarization would be gauge invariant in the infrared. The fermion and photon DSEs would effectively decouple for momenta $p \ll \alpha$.

Both possibilities are consistent with our findings from the LKFT and only a detailed analysis of the transverse parts of the fermion-photon vertex can decide which one is realized in QED₃.

IV. NUMERICAL RESULTS

In the previous section we performed in some detail an analytical determination of the infrared behavior of QED₃ close to and above the phase transition assuming a power law behavior of the fermion vector dressing function. Here we present our numerical solutions of the unquenched system of DSEs in both the massive and the massless phase, deferring the presentation of our results in quenched QED₃ to Appendix C.

TABLE I. The chiral condensate [calculated via Eq. (73)] obtained in the 1BC-vertex model and employing the CP-vertex in the fermion DSE and the BC-vertex in the photon DSE.

N_f	$-\langle \bar{\Psi}\Psi \rangle / e^4$								
	1.0	2.0	2.8	3.0	3.1	3.3	3.4	3.5	
1BC-vertex	$7 \cdot 10^{-4}$	$3.5 \cdot 10^{-5}$	$2.5 \cdot 10^{-8}$	$3.9 \cdot 10^{-10}$	$1.5 \cdot 10^{-11}$				
CP-vertex	$1.2 \cdot 10^{-3}$	$1.3 \cdot 10^{-4}$	$1.7 \cdot 10^{-6}$	$2.6 \cdot 10^{-7}$	$8.9 \cdot 10^{-8}$	$5.0 \cdot 10^{-9}$	$7.4 \cdot 10^{-10}$	$6.6 \cdot 10^{-11}$	

A. Unquenched results and phase transition in Landau gauge

Our results for the fermion mass function $M(p^2)$, the wave function renormalization $Z_f(p^2)$ and the photon polarization $\Pi(p^2)$ in unquenched Landau gauge are shown in Fig. 3. On the left panel we display results obtained with the first term of the BC-vertex (1BC) only. On the right panel we give the results obtained with the CP-vertex in the fermion DSE and the BC construction in the photon DSE. All results in this section are obtained with the Brown-Pennington projection $\zeta = 3$ in the photon equation, other choices lead only to minor modifications. As expected, in the ultraviolet all curves follow their respective asymptotic limits, given in Eqs. (27) and (22). In the infrared we find finite, nonzero dressing functions throughout the dynamically broken phase. Furthermore, we can clearly see two distinct mass scales: The fermion dressing functions have a kink near $p = e^2$ and a second kink at $p \sim M(0)$. Close to the phase transition, these scales are several orders of magnitude apart, which makes lattice simulations extremely difficult.

Above N_f^{crit} the functions turn into power laws thus justifying the basic assumption in our IR-analysis of Sec. III. The values of the exponents κ determined in the analytical calculation are reproduced on the 10% level by the numerical results. In the CP/BC-vertex case we find $\kappa = 0.0315$ at $N_f = 4.5$, which is much closer to the value obtained with a bare vertex, $\kappa_{\text{bare}} = 0.0343$, than to the corresponding value $\kappa_{\text{1BC}} = 0.0278$ obtained with the 1BC-vertex piece. Furthermore, notice that for these unquenched calculations, $Z_f(p^2) \geq 1$, at least for $N_f \geq 2$, in contrast to our findings in the quenched case (cf. Appendix C).

The dynamical mass generation close to the phase transition is studied in Fig. 4. Shown are results in four different truncation schemes: The leading order $1/N_f$ -expansion of Ref. [8] employs the perturbative expression, Eq. (22), for the photon polarization and chooses $A(p^2) \equiv 1$ together with a bare fermion-photon vertex. This is compared to our results from the fully unquenched system of DSEs with three different truncations for the vertex: bare, 1BC and the CP/BC-combination. As can be seen from the figure, the numerical results follow nicely the corresponding analytical results from Sec. III A, and are in qualitative agreement with the findings of Ref. [15]. Our most sophisticated vertex truncation, the CP/BC-combination, tends toward a critical value of the number of flavors of $N_f^{\text{crit}} \approx 4$.

Finally we list our results for the chiral condensate in two different truncation schemes for a range of values of N_f in Table I. For the CP-vertex, there is no discrepancy between the condensate as obtained from the trace of the

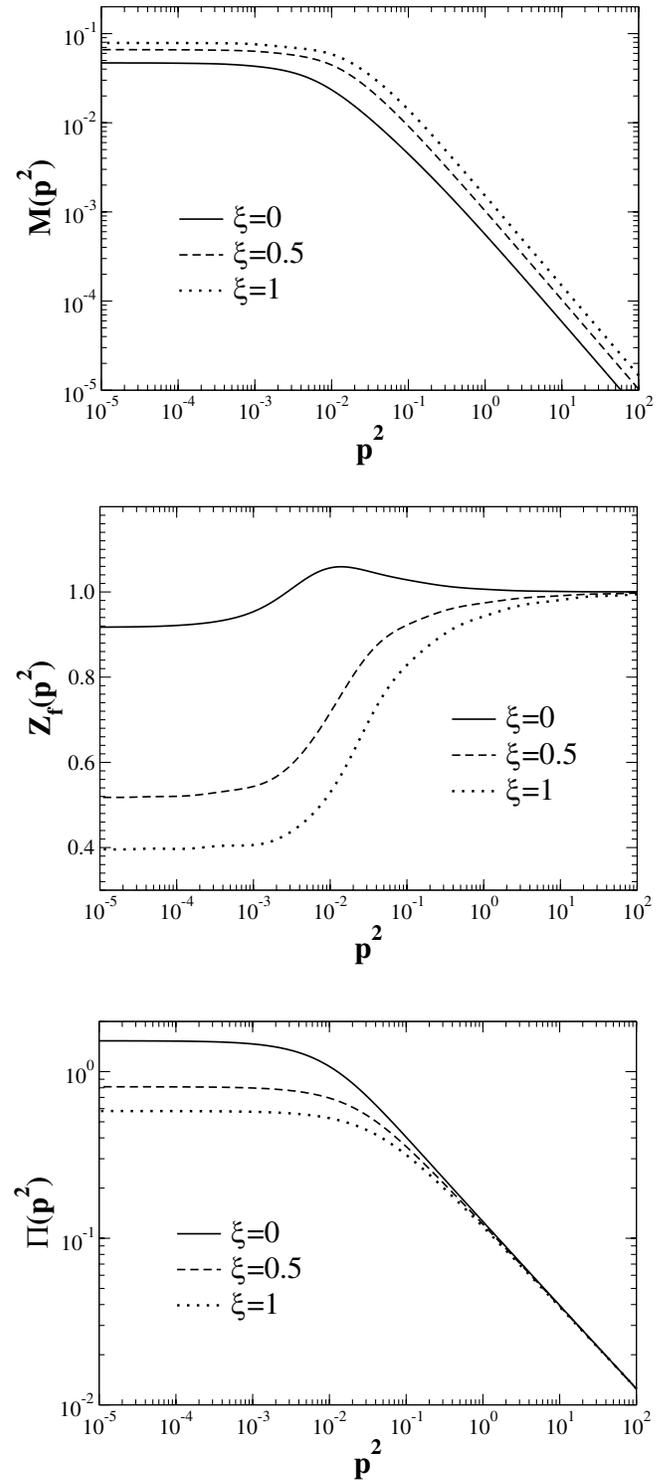


FIG. 5. Shown are our results for the dressing functions for three different values of the gauge parameter ξ and $N_f = 1$. The scale is set by choosing $e^2 = 1$.

fermion propagator

$$\langle \bar{\Psi}\Psi \rangle = -\text{Tr}[S(0)] = -4 \int \frac{d^3 q}{(2\pi)^3} \frac{B(q^2)}{q^2 A^2(q^2) + B^2(q^2)}, \quad (73)$$

and that extracted from the asymptotic behavior, Eq. (27); for the 1BC-vertex there is about 5% to 10% difference between the two methods. Listed are the condensates calculated from the trace of the fermion propagator. In accordance with the simple estimate given in Ref. [54], we find small condensates well below the phase transition. For $N_f = 1$ and $N_f = 2$ recent lattice simulations provide upper bounds of the $O(10^{-3})e^4$ and $O(10^{-4})e^4$, respectively [18,19]. These bounds are certainly consistent with our values. Thus the combined evidence of the DSEs and the lattice Monte-Carlo simulations indicate the presence of dynamical chiral symmetry breaking at $N_f = 1$ and $N_f = 2$.

B. Unquenched results in general linear covariant gauges

Here we present numerical solutions for unquenched QED₃ in linear covariant gauges in the CP/BC-vertex truncation scheme. According to our previous discussion in Sec. III B and the numerical results in the quenched approximation in Appendix C, we expect artifacts from violating gauge invariance. The results presented in Fig. 5 for the case of $N_f = 1$ flavors indicate that this is indeed the case. The photon polarization functions clearly depend on the gauge parameter ξ . On a quantitative level this can also be seen from the values of the chiral condensate displayed in Table II. Induced by the feedback of the photon propagator on the fermion, the variation of the condensate with the gauge parameter is much larger for the unquenched than for the quenched theory. Only the Landau gauge results are consistent with the bounds from lattice simulations.

As expected from the infrared analysis of Sec. III B there is no chirally symmetric self-consistent power law solution for $\xi \neq 0$. In fact we did not find any self-consistent solution in the symmetric phase for $\xi \neq 0$, and consequently the system stays in the chirally broken phase at least for values of N_f as large as 7, which is the largest N_f we have investigated. For larger values of N_f

the numerical analysis becomes increasingly tedious. The corresponding dressing functions in Feynman gauge are presented in Fig. 6. An interesting difference with the Landau gauge solutions is that now $0 < Z_f(p^2) \leq 1$ on the entire momentum range, in contrast to the unquenched Landau gauge solutions, for which $Z_f(p^2) \geq 1$ on part ($N_f = 1$) or all ($N_f \geq 2$) of the momentum range. Also note that now there appears to be only one scale at which the generated fermion mass function $M(p^2)$ has a kink.

V. SUMMARY AND CONCLUSIONS

In this work we have investigated the chiral phase transition of QED₃ in the Green's functions approach. Employing different ansätze for the fermion-photon vertex we have solved the coupled set of Dyson-Schwinger equations for the fermion and photon propagators. No other approximations have been made in our numerical calculations. In addition, the infrared behavior of the propagators close to the phase transition and in the symmetric phase has been investigated employing methods that have been successfully used previously in four dimensional QCD [36–39,49] and QED [52]. Special care has been taken to preserve gauge invariance as much as possible. The chosen vertex ansatz satisfies the Ward-Green-Takahashi identity. Furthermore it has the correct properties under Landau-Khalatnikov-Fradkin transformations in the special case of massless quenched QED. Nevertheless, our results indicate that this is not enough: The resulting vacuum polarization is not gauge invariant, nor is the chiral condensate. Clearly, further structure in the transverse part of the vertex is needed to properly ensure gauge covariance.

In Landau gauge we find a self-consistent power law solution for the photon polarization and the vector fermion dressing function in the infrared region in the symmetric phase. With a bare fermion-photon vertex the anomalous dimension κ is directly related to the coefficients of a well-known result from the $1/N_f$ -expansion. We thus confirmed a long-standing conjecture from the renormalization group [10,11,40]. We would like to emphasize, however, that such a power law solution is genuinely nonperturbative in nature and cannot be obtained to any finite order in perturbation theory or the $1/N_f$ -expansion. The dependence of κ on the number of

TABLE II. The chiral condensate [calculated via Eq. (73)] obtained in the CP/BC-vertex truncation in different gauges.

gauge parameter	$-\langle \bar{\Psi}\Psi \rangle / (10^{-5} e^4)$						
	$N_f = 0$	$N_f = 1$	$N_f = 2$	$N_f = 3$	$N_f = 4$	$N_f = 5$	$N_f = 6$
$\xi = 0$	333	121	13	0.026	?	0	0
$\xi = 0.5$	340	165	79	39	23	15	11
$\xi = 1$	351	202	108	74	55	37	29
$\xi = 2$	356	259	189	143	107	92	77

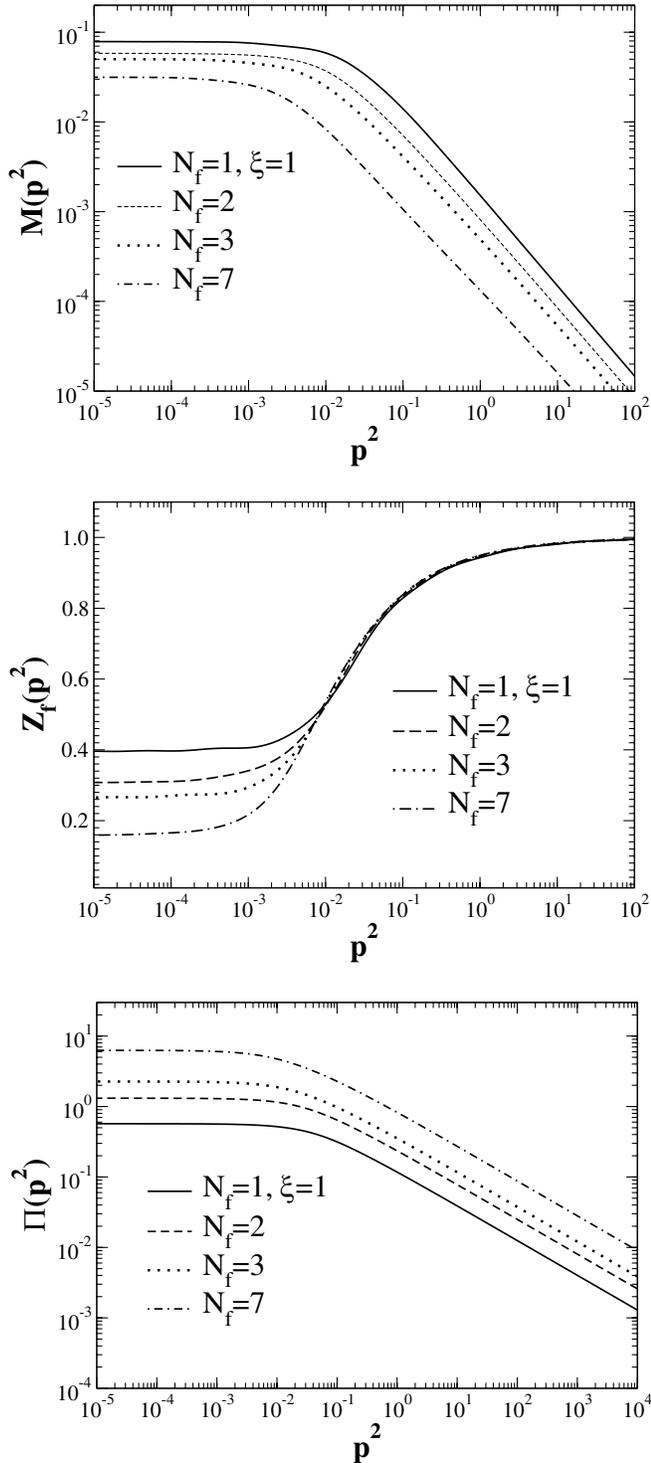


FIG. 6. Shown are the variation of the mass function, the wave function renormalization and the polarization with the number of flavors in Feynman gauge, $\xi = 1$. The scale is set by choosing $e^2 = 1$.

flavors is modified by nonperturbative contributions in the vertex dressing. We find small positive values of κ for all vertex dressings considered so far, indicating a vanishing propagator function $A(p^2)$ for $p^2 \rightarrow 0$. This is not

what one would expect on physical grounds [4]. It remains to be seen whether further contributions from the transverse part of the fermion-photon vertex are capable to drive κ to negative values.

In the chirally broken phase, the power law behavior gets modified by the (dynamically generated) fermion mass, which effectively acts as an infrared cutoff. We have determined the dependence of the chiral condensate and of the scalar fermion dressing function $B(p^2 = 0)$ on the number of flavors and found an exponential decrease close to the phase transition. If this behavior turns out to be correct in the full theory, one can hardly hope to be able to determine the critical number of flavors from lattice Monte-Carlo simulations. A sign pointing in this direction is found for small N_f : The chiral condensate is very small compared to the natural mass scale e^2 . Furthermore it agrees with the values recently determined on the lattice [18,19]. The qualitative behavior of the B-function and the condensate does not depend on our choice for the vertex ansatz, though there are quantitative differences, in particular, near the critical number of flavors, N_f^{crit} . The results with our most sophisticated vertex suggest a critical number of flavors $N_f^{\text{crit}} \approx 4$.

In other linear covariant gauges we find a completely different picture. No indications for a phase transition are seen in our numerical analysis. The value of the condensate becomes heavily dependent on the gauge parameter, and for $\xi \neq 0$ our results exceed the bounds set by lattice simulations. It appears as if no self-consistent power law solutions exist in the symmetric phase. A possible explanation for this fact can be found with the help of the Landau-Khalatnikov-Fradkin transformation laws. We find that, given a power law solution in Landau gauge with exponent κ , the gauge transformed propagator has the same anomalous dimension only for momenta $\frac{\xi e^2}{8\pi} \ll p \ll \alpha$, whereas below this scale it has an anomalous dimension $\kappa = -1$. Such a solution cannot be found from the Dyson-Schwinger equations with our vertex truncation. We thus conclude that further transverse structure in the vertex is mandatory to obtain gauge covariant solutions for the propagators of QED₃. These extra terms could allow for three possible scenarios in the symmetric phase of QED₃: (a) terms explicitly proportional to the gauge parameter ξ could lead to solutions in accordance with the Landau-Khalatnikov-Fradkin transformation of the Landau gauge solutions; or (b) they could invalidate all Landau gauge solutions found so far; or (c) they could allow for solutions in the symmetric phase that are not power laws in the infrared region. Although we do believe that the scenario (a) is most likely to be realized we cannot exclude the other possibilities so far. A thorough investigation of all existing proposals for the transverse structure of the fermion-photon vertex in this respect seems desirable and will be carried out in future work.

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APPENDIX A: REGULARIZATION OF THE PHOTON DSE

The aim here is to provide a formulation for the photon equation that can be used not only in combination with the Brown-Pennington projector, $\zeta = 3$ [cf. Equation (14)], but with general values of ζ . The dependence of the Landau gauge vacuum polarization on ζ then provides a measure of the violation of transversality in the photon equation.

To see the problems arising with $\zeta \neq 3$ we analyze the ultraviolet behavior of the photon DSE, Eq. (15). The kernels W_i are then given by

$$W_1(p^2, q^2, k^2) = \frac{\zeta k^4}{p^4} + k^2 \left(\frac{1-\zeta}{p^2} - \frac{2\zeta q^2}{p^4} \right) - 1 + \frac{(1-\zeta)q^2}{p^2} + \frac{\zeta q^4}{p^4}, \quad (\text{A1})$$

$$W_2(p^2, q^2, k^2) = \frac{2(3-\zeta)}{p^2}, \quad (\text{A2})$$

$$W_3(p^2, q^2, k^2) = \frac{\zeta k^6}{p^4} - k^4 \left(\frac{1+\zeta}{p^2} + \frac{\zeta q^2}{p^4} \right) + k^2 \left(1 + \frac{2(\zeta-3)q^2}{p^2} - \frac{\zeta q^4}{p^4} \right) + q^2 - \frac{(\zeta+1)q^4}{p^2} + \frac{\zeta q^6}{p^4}, \quad (\text{A3})$$

$$W_4(p^2, q^2, k^2) = \frac{-2\zeta k^4}{p^4} + k^2 \left(\frac{4}{p^2} + \frac{4\zeta q^2}{p^4} \right) - 2 + \frac{4q^2}{p^2} - \frac{2\zeta q^4}{p^4}, \quad (\text{A4})$$

$$W_5(p^2, q^2, k^2) = \frac{\zeta k^4}{p^4} - k^2 \left(\frac{1+\zeta}{p^2} + \frac{2\zeta q^2}{p^4} \right) + 1 + \frac{(\zeta-3)q^2}{p^2} + \frac{\zeta q^4}{p^4}, \quad (\text{A5})$$

$$W_6(p^2, q^2, k^2) = \frac{\zeta k^4}{p^4} - k^2 \left(\frac{3-\zeta}{p^2} + \frac{2\zeta q^2}{p^4} \right) + 1 + \frac{(-\zeta-1)q^2}{p^2} + \frac{\zeta q^4}{p^4}. \quad (\text{A6})$$

Let us first concentrate on the term proportional to the kernel W_1 . In the ultraviolet we can use the angular approximation $A(q^2) = A(k^2) = 1$, furthermore the momenta q^2 and k^2 are larger than all fermion masses. One angular integral yields a trivial factor 2π and the integral is dominated by the part $q^2 > p^2$

$$\begin{aligned} \Pi^{\text{UV1}}(p^2) &= -\frac{g^2 N_f}{(2\pi)^2} \int_p^\infty dq q^2 \int_0^\pi d\theta \sin(\theta) \frac{1}{q^2 k^2} \\ &\times \left[\frac{\zeta k^4}{p^4} + k^2 \left(\frac{1-\zeta}{p^2} - \frac{2\zeta q^2}{p^4} \right) - 1 + \frac{(1-\zeta)q^2}{p^2} + \frac{\zeta q^4}{p^4} \right]. \end{aligned} \quad (\text{A7})$$

Now we perform the angular integrals according to Eqs. (B1) through (B5) and expand the resulting logarithm for momenta $q^2 \gg p^2$. To leading order we obtain

$$\begin{aligned} \Pi^{\text{UV1}}(p^2) &= -\frac{g^2 N_f}{(2\pi)^2} \int_p^\infty dq \left\{ \frac{2(3-\zeta)}{3p^2} + \frac{(-10-2\zeta)}{15q^2} \right. \\ &\left. + O(p^2/q^4) \right\}, \end{aligned} \quad (\text{A8})$$

which is convergent if $\zeta = 3$ but contains a linear divergence for all other values. Treating all other terms in the fermion loop in the same way we arrive at the expression

$$\begin{aligned} \Pi^{\text{UV}}(p^2) &= -\frac{g^2 N_f}{(2\pi)^2} \int_p^\infty dq \left\{ \frac{2(3-\zeta)}{3p^2} + \frac{(-10-2\zeta)}{15q^2} + \dots + B(q^2)^2 \left(\frac{2(3-\zeta)}{p^2 q^2} + \frac{2(3-\zeta)}{3q^4} \right) + \frac{A'(q^2)}{2} \left[-\frac{4q^2(3-\zeta)}{3p^2} \right. \right. \\ &\left. \left. + \frac{5-\zeta}{15} + \dots + B(q^2)^2 \left(\frac{4(3-\zeta)}{3p^2} - \frac{5+3\zeta}{15q^2} + \dots \right) \right] + B'(q^2)A(q^2)B(q^2) \left[-\frac{8(3-\zeta)}{3p^2} + \frac{10+6\zeta}{15q^2} + \dots \right] \right\}. \end{aligned} \quad (\text{A9})$$

which contains linear divergences proportional to $(3 - \zeta)$ at various places. In order to eliminate these terms we subtract appropriate expressions from the kernels W_i given in Eqs. (16) through (21). This results in the modified kernels

$$\tilde{W}_1(p^2, q^2, k^2) = W_1(p^2, q^2, k^2) - \frac{2k^2(3 - \zeta)}{3p^2}, \quad (\text{A10})$$

$$\tilde{W}_2(p^2, q^2, k^2) = 0, \quad (\text{A11})$$

$$\tilde{W}_3(p^2, q^2, k^2) = W_3(p^2, q^2, k^2) + \frac{8q^2k^2(3 - \zeta)}{3p^2}, \quad (\text{A12})$$

$$\tilde{W}_4(p^2, q^2, k^2) = W_4(p^2, q^2, k^2) - \frac{8k^2(3 - \zeta)}{3p^2}, \quad (\text{A13})$$

$$\tilde{W}_5(p^2, q^2, k^2) = W_5(p^2, q^2, k^2) + \frac{4k^2(3 - \zeta)}{3p^2}, \quad (\text{A14})$$

$$\tilde{W}_6(p^2, q^2, k^2) = W_6(p^2, q^2, k^2) + \frac{4k^2(3 - \zeta)}{3p^2}. \quad (\text{A15})$$

Based on our analytical infrared calculus as well as on our numerical calculations we investigated the dependence of the solutions on the projection parameter ζ and

found very small effects not affecting any of our conclusions in the main body of this work.

APPENDIX B: ANGULAR AND RADIAL INTEGRALS

In $d = 3$ dimensions the following angular integrals are needed for the UV-analysis of the photon equation

$$\int_0^\pi d\theta \frac{\sin(\theta)}{k^4} = \frac{2}{(q^2 - p^2)^2}, \quad (\text{B1})$$

$$\int_0^\pi d\theta \frac{\sin(\theta)}{k^2} = \frac{1}{pq} \ln\left(\frac{p+q}{|p-q|}\right), \quad (\text{B2})$$

$$\int_0^\pi d\theta \sin(\theta) = 2, \quad (\text{B3})$$

$$\int_0^\pi d\theta \sin(\theta)k^2 = 2(p^2 + q^2), \quad (\text{B4})$$

$$\int_0^\pi d\theta \sin(\theta)k^4 = 2p^4 + 2q^4 + \frac{20}{3}p^2q^2, \quad (\text{B5})$$

where $k^2 = (q - p)^2 = p^2 + q^2 - 2pq \cos(\theta)$. For the IR-analysis of the coupled system we need the following integrals

$$\int d^d q \frac{1}{(q^2)^a (k^2)^b} = \pi^{d-d/2} (p^2)^{d/2-a-b} \frac{\Gamma(d/2 - a)\Gamma(d/2 - b)\Gamma(a + b - d/2)}{\Gamma(a)\Gamma(b)\Gamma(d - a - b)}, \quad (\text{B6})$$

and for the Fourier-transforms necessary for the LKFT we need

$$\int_0^\pi d\theta \sin\theta \cos\theta e^{\pm ipx \cos\theta} = \mp 2i \left(\frac{\cos(px)}{px} - \frac{\sin(px)}{(px)^2} \right), \quad (\text{B7})$$

$$\int_0^\infty dx x^b \sin(ax) = \frac{\Gamma(1+b)}{a^{1+b}} \sin\left[(1+b)\frac{\pi}{2}\right], \quad 0 < |b+1| < 1, \quad [3.761(4)] \quad (\text{B8})$$

$$\int_0^\infty dx x^b \cos(ax) = \frac{\Gamma(1+b)}{a^{1+b}} \cos\left[(1+b)\frac{\pi}{2}\right], \quad 0 < (b+1) < 1, \quad [3.761(9)] \quad (\text{B9})$$

$$\int_0^\infty dx x^b \sin(ax) e^{-cx} = \frac{\Gamma(1+b)}{(a^2 + c^2)^{(1+b)/2}} \times \sin\left[(1+b) \arctan\left(\frac{a}{c}\right)\right], \quad b > -2, \quad c > 0, \quad [3.944(5)] \quad (\text{B10})$$

TABLE III. The chiral condensate extracted from the asymptotics of the fermion mass function, see Eq. (27), and obtained by taking the trace of the propagator, for different values of the gauge parameter ξ , all in the quenched ($N_f = 0$) approximation. The units are $10^{-3}e^4$.

$(-\langle\bar{\Psi}\Psi\rangle)/(10^{-3}e^4)$	$\xi = 0$	$\xi = 0.5$	$\xi = 1$			
	asympt.	-Tr[S(0)]	asympt.	-Tr[S(0)]	asympt.	-Tr[S(0)]
BC-vertex	3.34	3.24	3.54	3.25	3.67	3.26
CP-vertex	3.29	3.29	3.34	3.35	3.46	3.46

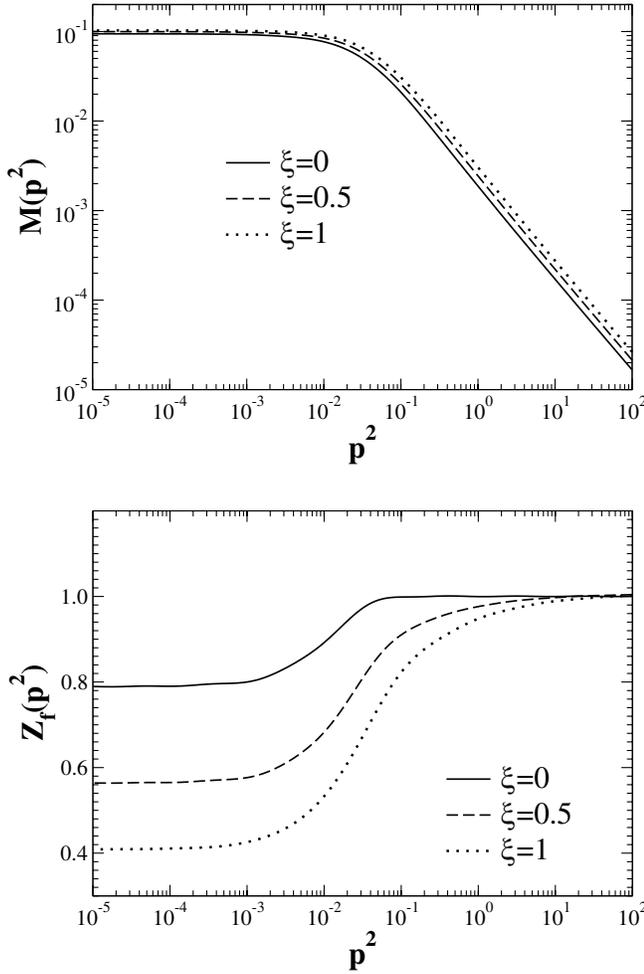


FIG. 7. Here we display the fermion mass function, $M(p^2)$, and the wave function renormalization, $Z_f(p^2) = 1/A(p^2)$ in quenched approximation for three different values of the gauge parameter ξ . The scale is set by choosing $e^2 = 1$.

$$\int_0^\infty dx x^b \cos(ax) e^{-cx} = \frac{\Gamma(1+b)}{(a^2 + c^2)^{(1+b)/2}} \times \cos\left[(1+b) \arctan\left(\frac{a}{c}\right)\right],$$

$$b > -1, \quad c > 0, \quad [3.944(6)] \quad (\text{B11})$$

where the numbers in square brackets refer to the corresponding equations in Ref. [55].

APPENDIX C: NUMERICAL RESULTS IN QUENCHED APPROXIMATION

QED₃ in quenched approximation employing the BC-vertex in the fermion and photon equation has been in-

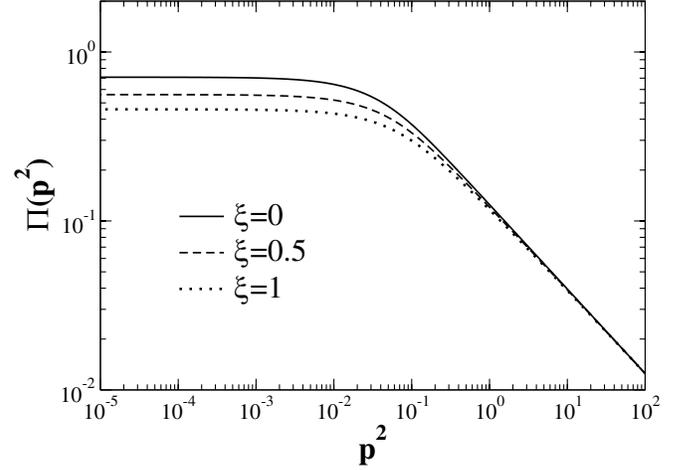


FIG. 8. Here we display the photon polarization $\Pi(p^2)$ as calculated from the photon DSE using $N_f = 1$ and the quenched fermion propagator functions without back-coupling for three different values of the gauge parameter ξ . The scale is set by choosing $e^2 = 1$.

vestigated in detail in Refs. [12,48] (for recent work see Ref. [35] and references therein). In the chirally broken phase, the gauge dependence of the photon polarization as well as the chiral condensate was found to be rather weak for the condensate and uncomfortably large for the photon polarization. For the condensate a discrepancy between the value extracted from the asymptotic form of the scalar fermion self-energy, see Eq. (27), and the value obtained from the trace of the fermion propagator, see Eq. (73), has been found. All we have to add to these investigations is an answer to this last problem. As can be seen from Table III, adding the CP term to the BC-vertex in the fermion DSE removes this discrepancy and slightly reduces the gauge dependence of the condensate.

The corresponding numerical solutions are displayed in Figs. 7 and 8. For the fermion propagator, no qualitative difference between Landau gauge and gauges with non-vanishing gauge parameter is found. Notice that $0 < Z_f(p^2) \leq 1$ on the entire momentum range, as one would expect from quenched perturbative theory, Eq. (23). For the photon polarization we find sizable violations of gauge invariance employing the BC-vertex in accordance with Ref. [48]. Adding the CP term also in the photon DSE does not help in this respect and furthermore introduces spurious divergences.

- [1] C. D. Roberts and A. G. Williams, *Prog. Part. Nucl. Phys.* **33**, 477 (1994).
- [2] R. Alkofer and L. von Smekal, *Phys. Rep.* **353**, 281 (2001).
- [3] N. E. Mavromatos and J. Papavassiliou, *cond-mat/0311421*.
- [4] M. Franz, Z. Tesanovic, and O. Vafek, *Phys. Rev. B* **66**, 054535 (2002); Z. Tesanovic, O. Vafek, and M. Franz, *Phys. Rev. B* **65**, 180511 (2002); M. Franz and Z. Tesanovic, *Phys. Rev. Lett.* **87**, 257003 (2001).
- [5] I. F. Herbut, *Phys. Rev. B* **66**, 094504 (2002); *Phys. Rev. Lett.* **88**, 047006 (2002).
- [6] A. A. Nersesyan and G. E. Vachnadze, *J. Low Temp. Phys.* **77**, 293 (1989).
- [7] T. W. Appelquist, M. J. Bowick, D. Karabali, and L. C. R. Wijewardhana, *Phys. Rev. D* **33**, 3704 (1986).
- [8] T. Appelquist, D. Nash, and L. C. R. Wijewardhana, *Phys. Rev. Lett.* **60**, 2575 (1988).
- [9] M. R. Pennington and S. P. Webb, BNL-40886 (unpublished).
- [10] D. Atkinson, P. W. Johnson, and P. Maris, *Phys. Rev. D* **42**, 602 (1990).
- [11] M. R. Pennington and D. Walsh, *Phys. Lett. B* **253**, 246 (1991); D. C. Curtis, M. R. Pennington, and D. Walsh, *Phys. Lett. B* **295**, 313 (1992).
- [12] C. J. Burden and C. D. Roberts, *Phys. Rev. D* **44**, 540 (1991).
- [13] P. Maris, *Phys. Rev. D* **52**, 6087 (1995).
- [14] V. P. Gusynin, A. H. Hams, and M. Reenders, *Phys. Rev. D* **53**, 2227 (1996).
- [15] P. Maris, *Phys. Rev. D* **54**, 4049 (1996).
- [16] E. Dagotto, J. B. Kogut, and A. Kocic, *Phys. Rev. Lett.* **62**, 1083 (1989); *Nucl. Phys.* **B334**, 279 (1990).
- [17] S. Hands and J. B. Kogut, *Nucl. Phys.* **B335**, 455 (1990).
- [18] S. J. Hands, J. B. Kogut, and C. G. Strouthos, *Nucl. Phys.* **B645**, 321 (2002).
- [19] S. J. Hands, J. B. Kogut, L. Scorzato, and C. G. Strouthos, *Phys. Rev. B* **70**, 104501 (2004).
- [20] V. P. Gusynin and M. Reenders, *Phys. Rev. D* **68**, 025017 (2003).
- [21] D. Nash, *Phys. Rev. Lett.* **62**, 3024 (1989).
- [22] J. C. Ward, *Phys. Rev.* **78**, 182 (1950); H. S. Green, *Proc. Phys. Soc. London, Sect. A* **66**, 873 (1953); Y. Takahashi, *Nuovo Cimento* **6**, 371 (1957).
- [23] J. S. Ball and T. W. Chiu, *Phys. Rev. D* **22**, 2542 (1980).
- [24] C. J. Burden and C. D. Roberts, *Phys. Rev. D* **47**, 5581 (1993).
- [25] Z. h. Dong, H. J. Munczek, and C. D. Roberts, *Phys. Lett. B* **333**, 536 (1994).
- [26] D. C. Curtis and M. R. Pennington, *Phys. Rev. D* **42**, 4165 (1990).
- [27] A. Kizilersu, M. Reenders, and M. R. Pennington, *Phys. Rev. D* **52**, 1242 (1995).
- [28] A. Bashir, A. Kizilersu, and M. R. Pennington, *Phys. Rev. D* **57**, 1242 (1998).
- [29] A. Bashir, A. Kizilersu, and M. R. Pennington, *Phys. Rev. D* **62**, 085002 (2000).
- [30] A. Bashir and A. Raya, *Phys. Rev. D* **64**, 105001 (2001).
- [31] A. Raya, Ph.D. thesis, University of Michoacana, Mexico, 2003, hep-th/0404138.
- [32] A. Bashir and R. Delbourgo, *J. Phys. A* **37**, 6587 (2004).
- [33] L. D. Landau and I. M. Khalatnikov, *Sov. Phys. JETP* **2**, 69 (1956); E. S. Fradkin, *Sov. Phys. JETP* **2**, 361 (1956); B. Zumino, *J. Math. Phys. (N.Y.)* **1**, 1 (1960).
- [34] A. Bashir and A. Raya, *Phys. Rev. D* **66**, 105005 (2002).
- [35] A. Bashir and A. Raya, hep-ph/0405142.
- [36] L. von Smekal, R. Alkofer, and A. Hauck, *Phys. Rev. Lett.* **79**, 3591 (1997); L. von Smekal, A. Hauck, and R. Alkofer, *Ann. Phys. (N.Y.)* **267**, 1 (1998).
- [37] D. Atkinson and J. C. Bloch, *Mod. Phys. Lett. A* **13**, 1055 (1998); *Phys. Rev. D* **58**, 094036 (1998).
- [38] C. S. Fischer and R. Alkofer, *Phys. Lett. B* **536**, 177 (2002).
- [39] C. S. Fischer and R. Alkofer, *Phys. Rev. D* **67**, 094020 (2003).
- [40] T. Appelquist and U. W. Heinz, *Phys. Rev. D* **24**, 2169 (1981).
- [41] T. Appelquist, M. J. Bowick, D. Karabali, and L. C. R. Wijewardhana, *Phys. Rev. D* **33**, 3774 (1986).
- [42] A. I. Davydychev, P. Osland, and L. Saks, *Phys. Rev. D* **63**, 014022 (2001).
- [43] A. Bashir, A. Huet, and A. Raya, *Phys. Rev. D* **66**, 025029 (2002).
- [44] D. Walsh, Ph.D. thesis, University of Durham, UK, 1990.
- [45] J. C. R. Bloch, Ph.D. thesis, University of Durham, UK, 1995 (hep-ph/0208074).
- [46] N. Brown and M. R. Pennington, *Phys. Rev. D* **39**, 2723 (1989).
- [47] C. S. Fischer, R. Alkofer, and H. Reinhardt, *Phys. Rev. D* **65**, 094008 (2002).
- [48] C. J. Burden, J. Praschifka, and C. D. Roberts, *Phys. Rev. D* **46**, 2695 (1992).
- [49] C. Lerche and L. von Smekal, *Phys. Rev. D* **65**, 125006 (2002).
- [50] D. Zwanziger, *Phys. Rev. D* **67**, 105001 (2003).
- [51] J. A. Gracey, *Nucl. Phys.* **B414**, 614 (1994).
- [52] R. Fukuda and T. Kugo, *Nucl. Phys.* **B117**, 250 (1976); C. N. Leung, S. T. Love, and W. A. Bardeen, *Nucl. Phys.* **B273**, 649 (1986).
- [53] V. A. Miransky and K. Yamawaki, *Phys. Rev. D* **55**, 5051 (1997); **56**, 3768 (1997).
- [54] T. Appelquist and L. C. R. Wijewardhana, hep-ph/0403250.
- [55] I. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals, Series, and Products* (Academic Press, London, 1994).