

## Triply special relativity

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We describe an extension of special relativity characterized by *three* invariant scales, the speed of light  $c$ , a mass  $\kappa$ , and a length  $R$ . This is defined by a nonlinear extension of the Poincaré algebra  $\mathcal{A}$ , which we describe here. For  $R \rightarrow \infty$ ,  $\mathcal{A}$  becomes the Snyder presentation of the  $\kappa$ -Poincaré algebra, while for  $\kappa \rightarrow \infty$  it becomes the phase space algebra of a particle in de Sitter spacetime. We conjecture that the algebra is relevant for the low energy behavior of quantum gravity, with  $\kappa$  taken to be the Planck mass, for the case of a nonzero cosmological constant  $\Lambda = R^{-2}$ . We study the modifications of particle motion which follow if the algebra is taken to define the Poisson structure of the phase space of a relativistic particle.

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### I. INTRODUCTION

One of the most fascinating and central questions for contemporary physics is what is the symmetry of the low energy limit of quantum gravity. This question is especially interesting once it has been appreciated that Planck scale effects may be observable. This is because present and near future experiments are sensitive to corrections to the basic kinematical relations such as the energy-momentum relations

$$E^2 = p^2 + m^2 + a l_p E^3 + b l_p^2 E^4 + \dots \quad (1.1)$$

There may also be Planck scale corrections to the conservation laws for energy and momentum and to the transformation properties of particles under spacetime symmetries. Among the possible experimental windows to Planck scale effects are the spectrum of ultra high energy cosmic rays, and a possible Planck scale dependence of the speed of light with energy, observable in near future observations of gamma ray bursts.

Neglecting for a moment the role of the cosmological constant, there are three possibilities.

- **A.** *Poincaré invariance*, i. e., there is no residue of Planck scale physics in low energy phenomena.
- **B.** *Lorentz symmetry breaking*, so that probes sensitive to Planck scales discover that there really is a preferred reference frame.
- **C.** *Deformed or doubly special relativity (DSR)* [1], [2] which refers to the possibility that the principle of the relativity of inertial frames may be preserved, but in such a way that the Planck length or Planck energy becomes an observer independent threshold for new phenomena. The name comes from the fact that the symmetry algebra now preserves two observer independent invariant quantities, the speed of light  $c$ , and the Planck length  $l_p$ .

An example of this last possibility is  $\kappa$ -Poincaré symmetry [3], [4], [5] whose generators satisfy a nonlinear deformation of Poincaré invariance, governed by a dimensional parameter  $\kappa = l_p^{-1}$ . This can be understood as the symmetry algebra of a noncommutative deformation of Minkowski spacetime. Theories invariant under  $\kappa$ -Poincaré symmetry and other realizations of DSR have been constructed and studied in [6], [7], [8], [9], [10].

These three possibilities are distinguishable experimentally. The second is characterized by modified energy-momentum relations of the form of (1.1), but with ordinary conservation laws of energy and momentum, while possibility **C** is characterized by nonlinear corrections to both energy-momentum relations and conservation laws (see [11].)

We have previously conjectured that the third possibility is realized, both in nature and in the low energy limit of loop quantum gravity (LQG) [12], [13]. Support for the second conjecture comes from 2 + 1 gravity coupled to point particles, which is an exactly solvable model. A number of independent results show that the symmetry algebra which acts on observables of the theory is exactly  $\kappa$ -Poincaré symmetry. One reason to expect that the same thing will be true of the 3 + 1 theory is that modifications of energy-momentum relations of the form of (1.1) are seen in several calculations of the propagation of weakly coupled excitations of candidates for the ground state of loop quantum gravity. These describe matter fields or gravitons propagating on flat spacetime, but with modified dispersion relations. At the same time, it is unlikely that the low energy limit of a quantization of general relativity can have a preferred frame, as that is ruled out by diffeomorphism invariance, which is instituted by the requirement that the states are annihilated by the constraints that generate those transformations classically.

In this note we introduce an extension of doubly special relativity in which the Poincaré algebra is extended by a third invariant parameter, which we interpret as the cos-

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mological constant  $\Lambda$ . Since there are now three observer independent scales,  $c$ ,  $l_p$ , and  $R = \Lambda^{-1/2}$ , we refer to the resulting kinematical theory as *triple special relativity*. In the limit  $R \rightarrow 0$ , this new algebra reduces to the  $\kappa$ -Poincaré algebra, while in the limit  $l_p \rightarrow 0$  it reduces to the de Sitter (or anti-de Sitter) algebras that characterize the maximally symmetric solutions with cosmological constant.

We have both a physical and a theoretical motivation for extending the conjectured symmetry algebra of space-time in this way. The theoretical motivation begins with the observation that quantum gravity is unlikely to make sense unless the cosmological constant is a bare parameter of the theory. This comes from the expectation that  $\Lambda$  will be a relevant parameter that must be controlled to compute the low energy limit of any quantum theory of gravity. This is certainly true, perturbatively, and there is good evidence it is true nonperturbatively as well [14]. In addition, there is a beautiful argument that connects the symmetry of the low energy limit of quantum gravity with the symmetry in the presence of a nonzero cosmological constant [12]. This arises because it is known that in  $2 + 1$  and  $3 + 1$  dimensions the symmetry algebra is quantum deformed, with  $z = \ln(q)$

$$z \approx \sqrt{\Lambda} l_p \quad \text{for } d = 2 + 1 \quad (1.2)$$

[15–17],

$$z \approx \Lambda l_p^2 \quad \text{for } d = 3 + 1 \quad (1.3)$$

[18–20].

In the case of  $2 + 1$  gravity, the result that the symmetry algebra is quantum deformed when the cosmological constant is turned on is rigorous, a complete argument is given in [17]. For the case of  $3 + 1$  there is good evidence that the local gauge symmetry of the spacetime connection is quantum deformed from  $SU(2)$  to  $SU_q(2)$  [18–20]. In [21] an argument is given that this extends to the quantum deformation of the algebra of observables on the boundary of a spacetime with cosmological constant, so that the subgroup of the de Sitter algebra that generates the symmetries of the boundary is quantum deformed. This prompts the conjecture that the algebra of generators that preserve the ground state of  $3 + 1$  quantum gravity with nonzero  $\Lambda$  is quantum deformed.

In analogy to the standard Wigner-Inonu contraction that leads from de Sitter to Poincaré algebra in the  $\Lambda \rightarrow 0$  limit, we now consider taking the contraction of the quantum deformed symmetry algebra. The cosmological constant occurs both in the scaling of the translation generators and in either (1.2) or (1.3). As a result, the limit  $\Lambda \rightarrow 0$  may be no longer the Poincaré algebra. In the case of  $2 + 1$  gravity it is exactly the  $\kappa$ -Poincaré algebra [13]. Indeed this is exactly how the  $\kappa$ -Poincaré algebra was found in the first place [3], [4] (in any dimension).

In the case of  $3 + 1$  dimensions, one must take into account an additional renormalization of the energy and momentum generators. This is necessary because, unlike the case of  $2 + 1$  dimensions, there are local degrees of freedom, and these will induce a renormalization between the fundamental operators of the theory and the symmetry generators of the low energy limit. This will be proportional to a power of the ratio of the ultraviolet and infrared regulator. Since LQG is known to be ultraviolet finite, the former is the Planck scale. The latter is of course the cosmological constant itself. Thus we have

$$P_{a,\text{ren}} = \left( \frac{1}{\sqrt{\Lambda} l_p} \right)^r \sqrt{\Lambda} M_{5a}. \quad (1.4)$$

It turns out that for  $r < 1$  the contraction is the ordinary Poincaré algebra, while for  $r = 1$  it is again  $\kappa$ -Poincaré. (For  $r > 1$  the contraction does not exist.) This is supported as well by an explicit calculation [3], [4], [12].

The physical motivation stems from the observation that there appears to be a vacuum energy, which can be characterized, so far as all observations done to date, by a positive cosmological constant, whose value in Planck units is about

$$\lambda = G\Lambda\hbar \approx 10^{-120}. \quad (1.5)$$

It has, however, proved so far impossible to understand, from known physics, the value of the observed cosmological constant. This has remained true despite many attempts. One may then try a new approach to the problem of the cosmological constant by conjecturing that  $R \approx 10^{60} l_p$  constitutes a new scale in physics, at which novel, presently unknown laws, and principles come into effect. But if  $R$  is really a scale of new physics, then we would expect to see surprising phenomena in other cases in which the scale appears. Indeed, there are several such cases, including

- 1 The success of the modified Newtonian dynamics (MOND) formula, as a phenomenology of galaxy rotation curves. The situation may be summarized [22] by the statement that the need for either dark matter or a modification of Newtonian dynamics appears whenever the acceleration of a star falls below a critical value of the acceleration,  $a_0$  given roughly by

$$a_0 = 1.2 \times 10^{-8} \frac{\text{cm}}{\text{sec}^2} \approx c^2 \sqrt{\Lambda}. \quad (1.6)$$

Whether this indicates the need for a departure from standard physics, or instead, just is a phenomenological description of the effects of dark matter is clearly a pressing question, but in any case the phenomenology shows that the new phenomena is characterized by the scale of  $\Lambda$ . We may note that this means that the scale  $\Lambda$  can be read directly off the

date of galactic observations, and in more than one way. It can be read directly off the data for the Tully-Fisher relation, where it characterizes an observed relationship between luminous matter and the asymptotic velocity. It can also be read off of the discrepancy between observed accelerations of stars in galaxies and those predicted by Newtonian physics based on visible matter.

- 2 The Pioneer anomaly [23] consists of the observation of an additional, unexplained acceleration of all three satellites that have gone outside the solar system towards the sun, of a magnitude  $\approx 6 \times a_0$ .
- 3 There is a possible anomaly in cosmic microwave radiation (CMB) observations that can be interpreted as indicating that the fluctuations of modes with wavelengths greater than  $R$  are suppressed relative to the predictions of the Harrison-Zeldovich spectrum [24].
- 4 The possible observations of a time varying  $\alpha$  seen by [25] in quasar absorption line spectra can be interpreted as due to a variation of the speed of light of order,  $\dot{c} \approx 10^{-1} a_0$ .

It is perhaps fair to say that in every case in which we have observational evidence of phenomena characterized by the scale  $R$  there appears to be a divergence from theoretical expectations. Of course, some or all of these effects may turn out to be spurious or have simpler explanations. Still we may take these as hints suggesting we should look for modifications of physical principles at scales longer than  $R$ .

In the next section we take an algebraic approach by presenting an extension of the Poincaré algebra characterized by three invariant scales, which we may take to be  $c$ ,  $\kappa = m_p$ , and  $R = \Lambda^{-1/2}$ . In Section III we postulate that the algebra is the Poisson algebra for a relativistic particle. We study the resulting corrections to the equations of motion, particularly for the case of circular motion in a central potential. We find violations of the equivalence principle, and a new force that falls off as  $1/\text{distance}$ , as is the case for MOND. However, the new force is much too strong for the case of a star in orbit around a galaxy, because nonlinear effects coming from the fact that stars are very large in Planck units overwhelms the naive Newtonian limit.

## II. THE ALGEBRA

Let us begin with the Poincaré algebra. It has as a subalgebra, the Lorentz algebra

$$[M_{\mu\nu}, M_{\rho\sigma}] = g_{\mu\sigma}M_{\nu\rho} + g_{\nu\rho}M_{\mu\sigma} - g_{\mu\rho}M_{\nu\sigma} - g_{\nu\sigma}M_{\mu\rho}, \quad (2.1)$$

together with the translations  $P_\mu$ , which satisfy

$$[P_\mu, P_\nu] = 0, \quad (2.2)$$

to which we add the action of Lorentz transformations on translations

$$[M_{\mu\nu}, P_\rho] = -g_{\mu\rho}P_\nu + g_{\nu\rho}P_\mu. \quad (2.3)$$

This is easily extended to a phase space algebra, by which we mean the combination of the Poisson algebra for a free relativistic particle and the action of the symmetry generators acting on the position and momenta. We then take the commutators to indicate Poisson brackets so we have

$$[X_\mu, P_\nu] = g_{\mu\nu}. \quad (2.4)$$

The algebra is completed by the action of the Lorentz transformations on positions

$$[M_{\mu\nu}, X_\rho] = -g_{\mu\rho}X_\nu + g_{\nu\rho}X_\mu. \quad (2.5)$$

If we now turn on the cosmological constant  $\Lambda = R^{-2}$  the algebra is deformed to de Sitter algebra, which means replacing (2.2) with

$$[P_\mu, P_\nu] = \frac{1}{R^2}M_{\mu\nu}, \quad (2.6)$$

while the other relations remain unchanged.

It is useful for what we are about to do to observe that the curvature of position space is manifested by a non-commutativity of the conjugate variables. This of course is well-known from basic general relativity. But seeing it from a phase space point of view can lead one to ask whether one can do the reverse. That is, could one deform momentum space to a space of constant curvature? And would this be manifested by noncommutativity in the position observables?

Certainly one can do this. The result is an algebra given by the standard properties of the Lorentz transformations, (2.1), (2.3), (2.4), (2.5) together with

$$[X_\mu, X_\nu] = \frac{1}{\kappa^2}M_{\mu\nu}, \quad [P_\mu, P_\nu] = 0, \quad (2.7)$$

where we take  $l_p = \kappa^{-1}$  to be the Planck scale because it is a small scale deformation of standard physics.

Indeed, this is one way of writing the commutation relations that define  $\kappa$ -Poincaré symmetry and its action on  $\kappa$ -Minkowski spacetime. In this form it was first written down by Snyder [26]. Later this was shown [27] to be one basis for the  $\kappa$ -Poincaré algebra, which is now called the Snyder basis.

A confusing point is that the symmetry algebra generated by the  $M_{\mu\nu}$  and  $P_\mu$  appears to be a classical algebra. But it acts on a space of noncommutative coordinates which is otherwise flat. This is confusing because the classical Lie algebras are all symmetry algebras on classical manifolds (with commuting coordinates) of constant curvature. The point is that the relations (2.7) define a particular basis of a nontrivial Hopf algebra. If

one writes the remainder of the Hopf algebra relations one sees that the algebra is not a classical Lie algebra.

This indeed corresponds to the curvature of momentum space, as was shown in detail by one of us in [10], [27], [28]. It should also be mentioned that in the context of quantum groups, the duality between noncommutativity of the coordinates of the representation space and curvature in the space of generators was emphasized in the early work of Majid [29].

We can now ask if it is possible to do the trick twice. That is, can one make both the position and momentum spaces noncommutative? One wants then to realize the standard Lorentz transformation properties (2.1), (2.3), (2.5) and at the same time both<sup>1</sup>

$$[X_\mu, X_\nu] = \frac{1}{\kappa^2} M_{\mu\nu}, \quad [P_\mu, P_\nu] = \frac{1}{R^2} M_{\mu\nu}. \quad (2.8)$$

This can be done, but it requires deforming also the canonical commutation relation (2.4). One finds by explicit computation that the Jacobi identities are satisfied if one takes instead

$$[X_\mu, P_\nu] = g_{\mu\nu} - \frac{1}{\kappa^2} P_\mu P_\nu - \frac{1}{R^2} X_\mu X_\nu + \frac{1}{\kappa R} (X_\mu P_\nu + P_\mu X_\nu + M_{\mu\nu}). \quad (2.9)$$

Note that the subalgebras spanned by the pairs  $(M, X)$  and  $(M, P)$  are just the standard de Sitter algebras. Thus we can imagine the phase space as being composed of the product of two de Sitter spaces, with in addition a deformed Poisson bracket. Alternatively, the entire phase space is now a noncommutative space. We see that to do this gives us an algebra with three universal constants,  $c$ ,  $\kappa$ , and  $R$ .

It is also helpful to write the algebra we have found in terms of dimensionless variables

$$\tilde{X}^\mu = \kappa X^\mu, \quad \tilde{P}_\mu = R P_\mu. \quad (2.10)$$

The algebra involves only the dimensionless ratio

$$r = R\kappa. \quad (2.11)$$

Again, the standard Lorentz transformation properties are unchanged, while we now have

<sup>1</sup>In this presentation of the phase space algebra the variables  $X_\mu$  and  $P_\mu$  are not canonical. Expressing the Lorentz generators  $M_{\mu\nu} = X_\mu P_\nu - P_\mu X_\nu$  we find that the symplectic structure does not have the standard block off diagonal form. Darboux theorem guarantees however that locally one can always bring it to the canonical form (like Riemannian metric that is always locally Euclidean). Note that our symplectic structure becomes canonical in the limit  $\kappa, R \rightarrow \infty$ .

$$[\tilde{X}_\mu, \tilde{X}_\nu] = M_{\mu\nu}, \quad [\tilde{P}_\mu, \tilde{P}_\nu] = M_{\mu\nu}, \quad (2.12)$$

$$[\tilde{X}_\mu, \tilde{P}_\nu] = r g_{\mu\nu} + M_{\mu\nu} - \frac{1}{r} (\tilde{P}_\mu \tilde{P}_\nu + \tilde{X}_\mu \tilde{X}_\nu - \tilde{X}_\mu \tilde{P}_\nu - \tilde{P}_\mu \tilde{X}_\nu). \quad (2.13)$$

The algebra  $\mathcal{A}$  then is given by (2.12), (2.13) together with the standard (2.1), (2.5), (2.3). In the next section we will be considering it as defining the Poisson structure on the phase space of a relativistic particle. But it is also well defined as an operator algebra, with the orderings indicated. By extending slightly the construction of Snyder [26], one can find representation of  $\mathcal{A}$  in terms of operators acting on six-dimensional Minkowski space with coordinates  $\eta_A = (\eta_\mu, \eta_4, \eta_5) = (\eta_0, \dots, \eta_3, \eta_4, \eta_5)$  and the metric of signature  $(+, -, -, \dots, -)$ :

$$X^\mu = \frac{1}{\kappa} \left( \eta_4 \frac{\partial}{\partial \eta_\mu} - \eta^\mu \frac{\partial}{\partial \eta_4} \right) + \frac{R}{2} \frac{\eta^\mu}{\eta_5}, \quad (2.14)$$

$$P_\mu = -\frac{1}{R} \left( \eta_5 \frac{\partial}{\partial \eta^\mu} - \eta_\mu \frac{\partial}{\partial \eta_5} \right) + \frac{\kappa}{2} \frac{\eta_\mu}{\eta_4}. \quad (2.15)$$

### III. THE MOTION OF PARTICLES

Since the formalism we have developed involves the phase space, we can use it to describe the dynamics of particles. We then take  $\mathcal{A}$  as the definition of the Poisson brackets acting on the phase space  $\Gamma = \{\tilde{X}^\mu, \tilde{P}_\nu\}$ . Our goal in this section is to understand the physical meaning of the modifications coming from the deformations of the phase space algebra parameterized by  $l_p$  and  $R$ .

The dynamics on the phase space is specified by a reparametrization invariant action principle, with the Hamiltonian

$$H = N\mathcal{H}, \quad (3.1)$$

where  $N$  is the lapse and  $\mathcal{H}$  is the Hamiltonian constraint. The equation of motion for the lapse  $N$  yields the Hamiltonian constraint

$$\mathcal{H} = 0. \quad (3.2)$$

The equations of motion for positions and momenta are given by

$$\dot{\tilde{X}}^\mu = N[\tilde{X}^\mu, \mathcal{H}], \quad \dot{\tilde{P}}_\mu = N[\tilde{P}_\mu, \mathcal{H}], \quad (3.3)$$

subject to the initial data constraint (3.2).

For the free particle, the Hamiltonian constraint is given by the Casimir of the momentum sector of the phase space algebra, i.e., the Casimir of the  $(M, P)$  subalgebra. This Casimir reads

$$\mathcal{H}_0 = \tilde{P}_\mu \tilde{P}^\mu - M_{\mu\nu} M^{\mu\nu}. \quad (3.4)$$

It is easy to see that energy, momentum, and angular

momentum of particles are conserved because

$$[\tilde{P}_\mu, \mathcal{H}_0] = [M_{\mu\nu}, \mathcal{H}_0] = 0. \quad (3.5)$$

It is also not difficult to verify that, apart from a scaling and ordering, the standard definition of the Lorentz generators is unchanged,

$$M_{\mu\nu} = -\frac{1}{r}(\tilde{X}_\mu \tilde{P}_\nu - \tilde{P}_\mu \tilde{X}_\nu). \quad (3.6)$$

We want to study the question of whether the phenomenology of MOND can be recovered just from the modifications made so far to dynamics. To do this we add a static potential of the form  $\mathcal{U}(\tilde{\rho})$ , where, in the rest frame of the source,  $\tilde{\rho}^2 = \tilde{X}^i \tilde{X}^i$ . Here we have made a 3 + 1 split of spacetime, with  $\tilde{X}^\mu = (\tilde{X}^0, \tilde{X}^i)$ , with  $i = 1, 2, 3$ . Thus, the Hamiltonian constraint is now

$$\mathcal{H} = \tilde{P}_\mu \tilde{P}^\mu - M_{\mu\nu} M^{\mu\nu} + \mathcal{U}. \quad (3.7)$$

We now compute the equations of motion, using the Poisson brackets (2.13), (3.7). Using (3.6) to simplify the resulting expressions we find that

$$\begin{aligned} \dot{\tilde{X}}_\mu &= 2N\tilde{P}_\mu \left[ r + \frac{1}{r}(2\tilde{X} \cdot \tilde{P} - \tilde{X}^2 - \tilde{P}^2) + \frac{1}{2r}\tilde{X}^\lambda \frac{\partial \mathcal{U}}{\partial \tilde{X}^\lambda} \right] \\ &\quad - \frac{N}{r}\tilde{X}_\mu \tilde{P}^\lambda \frac{\partial \mathcal{U}}{\partial \tilde{X}^\lambda}, \end{aligned} \quad (3.8)$$

$$\dot{\tilde{P}}_\mu = -Nr \frac{\partial \mathcal{U}}{\partial \tilde{X}^\mu} + \frac{N}{r}[\tilde{X}_\mu(2\tilde{P}^\lambda - \tilde{X}^\lambda) - \tilde{P}_\mu \tilde{P}^\lambda] \frac{\partial \mathcal{U}}{\partial \tilde{X}^\lambda}. \quad (3.9)$$

We now impose conditions that single out circular motion. These are

$$\tilde{P}^\lambda \frac{\partial \mathcal{U}}{\partial \tilde{X}^\lambda} = 0, \quad \tilde{P} \cdot \tilde{X} = -\tilde{E}\tilde{t}. \quad (3.10)$$

We also posit that the potential is Newton's gravitational potential

$$\mathcal{U} = m^2 + c \frac{GMm}{\tilde{\rho}}, \quad (3.11)$$

where  $c$ , like  $N$  is to be determined by matching to the nonrelativistic Newton's laws and  $M$  is the mass of the central body.

These reduce the equations of motion for the spatial components to

$$\dot{\tilde{X}}_i = 2Nr\tilde{P}_i \left[ 1 - \frac{1}{r^2}(\tilde{X}^2 + \tilde{P}^2 - 2\tilde{E}\tilde{t}) + \frac{GcMm}{2r^2\tilde{\rho}} \right], \quad (3.12)$$

$$\dot{\tilde{P}}_i = -\frac{NGcMm\tilde{X}_i}{\tilde{\rho}} \left[ \frac{r}{\tilde{\rho}^2} + \frac{1}{r\tilde{\rho}} \right]. \quad (3.13)$$

We now choose  $N$  so that for the physical, dimensional variables

$$m\dot{X}_i = \tilde{P}_i. \quad (3.14)$$

This requires

$$N = \frac{1}{2mAR^2}, \quad (3.15)$$

where

$$A = 1 - \frac{1}{r^2}(\tilde{X}^2 + \tilde{P}^2 - 2\tilde{E}\tilde{t}) + \frac{GcMm}{2r^2\tilde{\rho}}. \quad (3.16)$$

Using (3.15),  $\dot{\tilde{P}}_i$  becomes

$$\dot{\tilde{P}}_i = -\frac{GcMl_p^2\tilde{X}_i}{2R^2A\tilde{\rho}} \left[ \frac{r}{\tilde{\rho}^2} + \frac{1}{r\tilde{\rho}} \right]. \quad (3.17)$$

Combining these we find the acceleration is

$$\ddot{\tilde{X}}_i = -\frac{GcM\tilde{X}_i}{2R^3Al_p\tilde{\rho}} \left[ \frac{r}{\tilde{\rho}^2} + \frac{1}{r\tilde{\rho}} \right]. \quad (3.18)$$

We now go back to dimensionless variables. We note that  $A = 1 + (\dots)/R^2$ , so that as  $R \rightarrow \infty$  with all other variables held fixed,  $A \rightarrow 1$ . Thus, it is natural to expect that  $A$  contains corrections which are unimportant except on cosmological scales. We therefore choose  $c$  so that the Newtonian limit is obtained as  $R \rightarrow \infty$ , so that

$$\ddot{X}_i = -\frac{GM\hat{X}_i}{A} \left[ \frac{1}{\rho^2} + \frac{l_p}{R\rho} \right]. \quad (3.19)$$

This fixes

$$c = \frac{2R^2m}{l_p}. \quad (3.20)$$

Assuming that  $A \approx 1$ , we do find an apparent MOND-like force, which is the term that falls off like  $1/\rho$ . However this is much too small, and it only becomes comparable to the Newtonian force for  $\rho \approx R^2/l_p$ . We also fail to see the emergence of a critical acceleration scale.

However, this is only the case if the masses are very small. For it is easy to see that the effect of the  $A$  term leads to drastic violations of the equivalence principle. Consider the term in  $A$  proportional to

$$z = \frac{GcMm}{2r^2\tilde{\rho}} = \frac{GMm^2l_p^2}{\rho} = \frac{GM}{\rho} \left( \frac{m}{m_p} \right)^2. \quad (3.21)$$

For a proton, in orbit around a galaxy,  $z \approx 10^{-40}$ . But the situation for a star around a galaxy is very different. In this case, the second factor overwhelms the first, so that  $z \approx 10^{72}$ . Thus, since  $A \approx 1 + z + \dots \approx z$ , the Newtonian limit is not obtained for the case of a star in circular orbit in a galaxy, instead we find an acceleration

$$\ddot{x}_i = -\frac{\hat{x}_i}{m^2 l_p^2 \rho}, \quad (3.22)$$

which is very far from the Newtonian limit.

The lesson is that due to the nonlinearities in the algebra, there are corrections to the dynamics that lead to massive violations of the equivalence principle. We may fix the constants so that Newton's laws are satisfied for masses much less than the Planck mass. This happens because the standard terms in the Poisson brackets dominate. But for stars in orbit around a galaxy, the new terms in the Poisson brackets such as the  $M_{\mu\nu}$  term in (2.9) are much more important than the conventional  $\eta_{\mu\nu}$  terms. The reason is that factors like  $(m/m_p)^2 \approx 10^{76}$  for a star can overcome suppressions of order  $l_p/R \approx 10^{60}$ .

Related to this is the observation that since the algebra is nonlinear, it is no longer true that the description of a composite system follows in a simple way from the action on the constituents. It is straightforward to show that if

$\mathcal{A}$  is posited as the Poisson algebra for elementary particles, it will not be satisfied for the total momentum and center of mass coordinates of a composite system, if they are given by the usual linear formulas of standard mechanics. We expect that this is related to similar issues that arise in the application of *DSR* to composite systems. These questions must be resolved before it can be determined whether the symmetry algebra described here may or may not be relevant for real physics.

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