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Some boundary effects in quantum field theory

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We have constructed a quantum field theory in a finite box, with periodic boundary conditions, using the hypothesis that particles living in a finite box are created and/or annihilated by the creation and/or annihilation operators, respectively, of a quantum harmonic oscillator on a circle. An expression for the effective coupling constant is obtained, showing explicitly its dependence on the dimension of the box.

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I. INTRODUCTION

The fact that the energy eigenvalues of the quantum harmonic oscillator is given by $E_n = (n + 1/2)\hbar w$ allows us to interpret their successive energy levels as being obtained by the creation of a quantum particle of frequency w. This interpretation of the energy spectrum of the quantum harmonic oscillator was successfully used in the second quantization formalism. In short, one could say that Planck's hypothesis is realized in the second quantization formalism by the use of creation and annihilation operators of the quantum harmonic oscillator system [1]. This realization is obtained for the quantum harmonic oscillator defined on an infinite line.

Let us consider a situation in which we want to describe the interaction of quantum particles living in a finite box with boundary conditions, for example, using the second quantization formalism. In this context, it seems natural to assume the statement concerning the connection between Planck's hypothesis and the energy levels of a quantum harmonic oscillator in this finite space and therefore analyze the consequences of this assumption in the construction of a quantum field theory (QFT) in a compact manifold.

In Ref. [2] a discussion of a quantum harmonic oscillator in a circle and its associated Heisenberg algebra was presented. It was found that Mathieu's equation can satisfactorily describe the system and that the creation and annihilation operators of the system satisfy a sort of deformed Heisenberg algebra. In Ref. [3] a construction of a deformed scalar QFT based on q oscillator [4], which is a deformed Heisenberg algebra, was presented and in Ref. [5] a procedure to perform perturbative computation up to second order in the coupling constant was implemented. Subsequently, it was shown [6] that this deformed scalar quantum model is renormalizable up to second order in the coupling constant.

In this paper, we use the same procedure developed in Refs. [3,5] to perform a perturbative computation for a QFT in a box. To do this, we use the hypothesis, already mentioned, that in a compact space with periodic boundary conditions, particles are created and/or annihilated by the creation and/or annihilation operators of a Heisenberg algebra of the quantum harmonic oscillator defined on a circle. As a result, we find that the effective coupling constant which appears in the perturbation series depends on a dimensionless quantity related to the linear dimension of the box. This approach permits us to construct a field theory that creates, at any point of the space-time, particles described by a deformed Heisenberg algebra, which in the present case, the deformation parameter is inversely proportional to the dimension of the box. In this way we can investigate the interaction of point particles in compact spaces, showing how the boundary affects this interaction.

Finally, we have computed the variation of the effective coupling constant for two different values of the size of the box, namely, one corresponding to the time of nucleosynthesis of the standard cosmological model and the other to the present epoch. The choice of these values for the sizes of the box were done simply to perform a calculation and to show an example of the effect of the boundary on the effective coupling constant. This does not mean that our model has a connection with the standard cosmological model.

This paper is organized as follows: In Sec. II, we present a discussion of a quantum harmonic oscillator on a circle which is described by Mathieu's equation. The deformed Heisenberg algebra associated with Mathieu's equation is presented in Sec. III. In Sec. IV, we present a construction of a QFT in a box and perform some perturbative computation. In Sec. V, we present the bound for the variation of the coupling constant. Finally, in Sec. VI we conclude with some comments.

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II. THE QUANTUM HARMONIC OSCILLATOR ON A CIRCLE

In this section we discuss an equation defined on a finite interval of length L which reproduces the ordinary quantum harmonic oscillator in the limit $L \longrightarrow \infty$. For this proposal it is convenient to describe quantum mechanics on a periodic line, and to do this we follow Ohnuki-Kitakado's formalism [7]. According to this formalism, there are inequivalent quantum mechanics on S^1 (periodic line) depending on a parameter α ($0 \leq \alpha < 1$). The momentum operator G on S^1 in the coordinate representation is given in this formalism as [7,8]

$$G \longrightarrow \frac{1}{i}\frac{d}{d\theta} + \alpha, \qquad 0 \leq \alpha < 1, \tag{1}$$

and the coordinate operator is given in terms of the unitary operator W

$$W \longrightarrow e^{i\theta}. \tag{2}$$

Let us consider the following equation on S^1 [2]:

$$G^2\Psi + K[W + W^\dagger]\Psi = \epsilon\Psi, \tag{3}$$

where G and W were already defined.[1] In order to have the above equation in the coordinate representation, we substitute Eqs. (1) and (2) in Eq. (3) for $\alpha = 0$. Thus, we obtain

$$\frac{d^2\Psi(\theta)}{d\theta^2} + (\epsilon - 2K\cos\theta)\Psi(\theta) = 0, \tag{4}$$

with $\Psi(\theta = 0) = \Psi(\theta = 2\pi)$. This equation is the well known Mathieu's equation, which first appeared in 1868 in the study of the vibrations of a stretched membrane of elliptic cross section [9]. Mathieu's equation is an important equation in physics arising from the study of a variety of physical problems, from ordered crystals with the potential $\cos 2x$ [10] to the wave equation of scalar fields in the background of a D-brane metric [11]. Note that this is one possible equation on a periodic line since we chose for simplicity $\alpha = 0$ in Eq. (4). According to Ohnuki-Kitakado's formalism [7], there are inequivalent quantum mechanics on S^1 for each value of the parameter α ($0 \leq \alpha < 1$).

In order to consider the limit of Eq. (4) when the radius of the circle goes to infinity, we perform the change of variables

$$\theta = \frac{\pi}{L}y + \pi, \qquad -L \leq y \leq L. \tag{5}$$

Using Eq. (5), Eq. (4) becomes

$$\frac{d^2\Psi}{dy^2} + \left(E + \frac{2\pi^2}{L^2}K\cos\frac{\pi}{L}y\right)\Psi = 0, \tag{6}$$

where $E = \pi^2\epsilon/L^2$. Then, using a trivial trigonometric identity and calling $\lambda \equiv E + 2\pi^2 K/L^2$, we obtain

$$\frac{d^2\Psi}{dy^2} + \left[\lambda - \frac{\pi^4}{L^4}Ky^2\left(\frac{\sin \pi y/2L}{\pi y/2L}\right)^2\right]\Psi = 0. \tag{7}$$

The well known Schrödinger's equation for the quantum harmonic oscillator,

$$\frac{d^2\Psi}{dy^2} + (\lambda - y^2)\Psi = 0, \tag{8}$$

is obtained for $K = L^4/\pi^4$ in Eq. (7), if $L \longrightarrow \infty$. It is then reasonable to take Eq. (4) with $K = L^4/\pi^4$ as the Schrödinger equation for the quantum harmonic oscillator on the circle with energy eigenvalue given by $\lambda \equiv \pi^2\epsilon/L^2 + 2L^2/\pi^2$.

Suppose now we consider Mathieu's equation for $K = L^4/\pi^4$ and L asymptotic. In this case the first levels are concentrated in values $y \ll L$ and thus, according to previous discussion, these energy levels of Mathieu's equation, which we call ϵ_n^L, will provide the energy levels of the standard quantum harmonic oscillator when $L \longrightarrow \infty$. Now, analogously to the definition of a quantum particle through the ordinary quantum harmonic oscillator, we define n quantum particles on the circle of length L as having energy ϵ_n^L. By consistency, $\epsilon_n^{L\rightarrow\infty} - \epsilon_0^{L\rightarrow\infty} = n(\epsilon_1^{L\rightarrow\infty} - \epsilon_0^{L\rightarrow\infty})$.

In fact, there is a solution obtained by Ince and Goldstein [9,12,13] to Mathieu's equation, Eq. (4), for asymptotic values of K. Their expansion for ϵ, the characteristic value of the equation, in the present case, i.e., $K = L^4/\pi^4$, provides for λ the value

$$\lambda_n = \nu_n - (\nu_n^2 + 1)\frac{a^2}{2^6} - (5\nu_n^4 + 34\nu_n^2 + 9)\frac{a^3}{2^{16}} + \cdots, \tag{9}$$

where $\nu_n = 2n + 1$ and $a = \pi/L$. For $L \longrightarrow \infty$ ($a \longrightarrow 0$), we recognize the energy eigenvalues of the quantum harmonic oscillator. Thus, we see that the above asymptotic solution [9,12,13] of the characteristic values of Mathieu's equation is a deformation of the quantum harmonic oscillator with deformation parameter equal to $a = \pi/L$.

The above solution corresponds to the energy levels of Mathieu's equation when the parameter K appearing in Eq. (4) is large, i.e., when $a^4(2n + 1)^2/16$ is small [12,13]. Note that, even if L is large which leads to a localization of the solution, this is periodic with period $2L$.

Let us now consider dimensional variables. We call the dimensionless variables y and L as $y \equiv x/x_0$ and $L \equiv Z/x_0$, where x, Z are dimensional and x_0 is a scale dimensional parameter. In this case when the dimensionless variable y varies from $-L$ to L, the dimensional variable x varies as $-Z \leq x \leq Z$. Thus, Z is the dimen-

[1]It would also be possible to define an equation with quadratic powers of W and $W^\dagger$, but the above equation is the simplest one.

sional length of the one-dimensional space. Furthermore, as explained before, the well known Schrödinger's equation for the harmonic oscillator is obtained for $K = L^4/\pi^4$ in Eq. (7), when $L \to \infty$. In terms of dimensional quantities this limit is achieved when $Z \gg x_0$. Therefore, we could say that x_0 is a scale where deformed properties become relevant.

III. DEFORMED HEISENBERG ALGEBRA ASSOCIATED WITH MATHIEU'S EQUATION

The purpose of this section is to construct an algebra, like the Heisenberg algebra, for the Mathieu system described in the previous section. Like the standard algebra for the quantum harmonic oscillator, the algebra we are going to construct has creation and annihilation operators as part of its generators.

To this end let us consider an algebra generated by J_0, A, and $A^\dagger$ described by the relations [14]

$$J_0 A^\dagger = A^\dagger f(J_0), \tag{10}$$

$$A J_0 = f(J_0) A, \tag{11}$$

$$[A, A^\dagger] = f(J_0) - J_0, \tag{12}$$

where $\dagger$ is the Hermitian conjugate and, by hypothesis, $J_0^\dagger = J_0$ and $f(J_0)$ is a general analytic function of J_0.

Using the algebraic relations in Eqs. (10)–(12), we see that the operator

$$C = A^\dagger A - J_0 = A A^\dagger - f(J_0) \tag{13}$$

satisfies

$$[C, J_0] = [C, A] = [C, A^\dagger] = 0, \tag{14}$$

being thus a Casimir operator of the algebra.

We present now the representations of the algebra when the function $f(J_0)$ is a general analytic function of J_0. We assume that we have an n-dimensional irreducible representation of the algebra given by Eqs. (10)–(12) and also that there is a state $|0\rangle$ with the lowest eigenvalue of the Hermitian operator J_0

$$J_0|0\rangle = \alpha_0|0\rangle. \tag{15}$$

For each value of α_0 we have a different vacuum, and therefore a better notation for this state could be $|0\rangle_{\alpha_0}$. However, for simplicity, we shall omit subscript α_0.

Let $|m\rangle$ be a normalized eigenstate of J_0,

$$J_0|m\rangle = \alpha_m|m\rangle, \tag{16}$$

where

$$\alpha_m = f^m(\alpha_0) = f(\alpha_{m-1}), \tag{17}$$

and m denotes the number of iterations of α_0 through f.

As proved in Ref. [14], under the hypothesis stated previously,[2] for a general function f we obtain

$$J_0|m\rangle = f^m(\alpha_0)|m\rangle, \qquad m = 0, 1, 2, \cdots, \tag{18}$$

$$A^\dagger|m-1\rangle = N_{m-1}|m\rangle, \tag{19}$$

$$A|m\rangle = N_{m-1}|m-1\rangle, \tag{20}$$

where $N_{m-1}^2 = f^m(\alpha_0) - \alpha_0$. Note that for each function $f(x)$ the representations are constructed by the analysis of the above equations as done in Ref. [14] for the linear and quadratic $f(x)$.

When the functional $f(J_0)$ is linear in J_0, i.e., $f(J_0) = q^2 J_0 + s$, it was shown in Ref. [14] that the algebra in Eqs. (10)–(12) recovers the q-oscillator algebra for $\alpha_0 = 0$. Moreover, as shown in Ref. [14], where the representation theory was constructed in detail for the linear and quadratic functions $f(x)$, the essential tool to construct representations of the algebra in (10)–(12) for a general analytic function $f(x)$ is the analysis of the stability of the fixed points of $f(x)$ and their composed functions.

It was shown in Refs. [14,15] that there is a class of one-dimensional quantum systems described by these generalized Heisenberg algebras. This class is characterized by those quantum systems having energy eigenvalues given by

$$\varepsilon_{n+1} = f(\varepsilon_n), \tag{21}$$

where ε_{n+1} and ε_n are successive energy levels and $f(x)$ is a different function for each physical system. This function $f(x)$ is exactly the same function that appears in the construction of the algebra in Eqs. (10)–(12). In the algebraic description of this class of quantum systems, J_0 is the Hamiltonian operator of the system, and $A^\dagger$ and A are the creation and the annihilation operators, respectively. This Hamiltonian and the ladder operators are related by Eq. (13), where C is the Casimir operator of the representation associated to the quantum system under consideration.

Now let us show that the asymptotic solution to Mathieu's equation we presented in the last section belongs to the class of algebras discussed previously. In other words, we shall construct a Heisenberg-type algebra, an algebra with creation and annihilation operators, for the Ince-Goldstein solution [Eq. (9)] to the quantum harmonic oscillator on S^1 and we shall find the characteristic function $f(x)$ [see Eqs. (10)–(12)] for this algebra. Furthermore, we shall also propose a realization, as in the case of the standard quantum harmonic oscillator, of the ladder operators in terms of the physical operators of the system.

As described in Refs. [15,16], the first thing we have to do in order to describe the Heisenberg-type structure of a one-dimensional quantum system is to relate the energy of the system for two arbitrary successive levels [see

[2] J_0 is Hermitian and there exists a vacuum state.

Eq. (21)]. For the energy spectrum given in Eq. (9), i.e.,

$$\varepsilon_n^L = n + \frac{1}{2} - [(2n+1)^2 + 1]\frac{a^2}{2^7} - [5(2n+1)^4 + 34(2n+1)^2 + 9]\frac{a^3}{2^{17}} + \cdots, \tag{22}$$

we obtain

$$\varepsilon_{n+1}^L = \varepsilon_n^L + 1 - (n+1)\frac{a^2}{2^4} - (n+1) \times [10n(n+2) + 21]\frac{a^3}{2^{12}} + \cdots. \tag{23}$$

Thus, we have to invert Eq. (22) in order to obtain n in terms of ε_n^L. Taking n from Eq. (22), we get

$$\varepsilon_{n+1}^L \equiv f(\varepsilon_n^L) = \varepsilon_n^L + 1 - (2\varepsilon_n^L + 1)\frac{a^2}{2^5} - (2\varepsilon_n^L + 1) \times [20\varepsilon_n^L(\varepsilon_n^L + 1) + 27]\frac{a^3}{2^{14}} + \cdots. \tag{24}$$

According to Refs. [12,13], this solution is valid when $a^4(2n+1)^2/16$ is small. Thus, since $a = \pi/L$ is considered small, n cannot be very large.

Now if we assume that ε_n^L is the eigenvalue of operator J_0 on state $|n\rangle$, we identify $f(x)$ appearing in Eqs. (18)–(20) with that one in Eq. (24) for the quantum system under consideration. Then the algebraic structure describing the quantum system under consideration is obtained using $f(x)$ defined in Eq. (24) into Eqs. (10)–(12) and can be written as

$$[J_0, A^\dagger] = A^\dagger - A^\dagger(2J_0 + 1)\frac{a^2}{2^5} - A^\dagger(2J_0 + 1) \times [20J_0(J_0 + 1) + 27]\frac{a^3}{2^{14}} + \cdots, \tag{25}$$

$$[J_0, A] = -A + (2J_0 + 1)A\frac{a^2}{2^5} + (2J_0 + 1) \times [20J_0(J_0 + 1) + 27]A\frac{a^3}{2^{14}} + \cdots, \tag{26}$$

$$[A, A^\dagger] = 1 - (2J_0 + 1)\frac{a^2}{2^5} - (2J_0 + 1) \times [20J_0(J_0 + 1) + 27]\frac{a^3}{2^{14}} + \cdots, \tag{27}$$

where, according to Eqs. (18)–(20), A and $A^\dagger$ are the ladder operators for the system under consideration, i.e., $A^\dagger$ when applied to state $|m\rangle$, that has J_0 eigenvalue ε_m^L, gives, apart from a multiplicative factor depending on m, the state $|m+1\rangle$ has energy eigenvalue ε_{m+1}^L. A similar role is played by A.

Note that, when $a \to 0$ ($L \to \infty$), we reobtain the well known Heisenberg algebra, as it should be, since we showed in the previous section that Mathieu's equation, Eq. (4), for $K = L^4/\pi^4 = a^{-4}$ gives the well known Schrödinger's equation for the harmonic oscillator, Eq. (8), in this limit.

Now let us realize the operators A, $A^\dagger$, and J_0 in terms of physical operators as in the case of the one-dimensional quantum harmonic oscillator, following what was done in Refs. [15,16] for the square-well potential and q oscillators [3]. Let us consider a one-dimensional lattice in a momentum space where the momenta are allowed to take only discrete values, say p_0, $p_0 + a$, $p_0 + 2a$, $p_0 + 3a$, etc., with $a > 0$. The left and right discrete derivatives are given by

$$(\partial_p f)(p) = \frac{1}{a}[f(p+a) - f(p)], \tag{28}$$

$$(\bar{\partial}_p f)(p) = \frac{1}{a}[f(p) - f(p-a)], \tag{29}$$

which are the two possible definitions of derivatives on a lattice.

Let us now introduce the momentum shift operators

$$T = 1 + a\partial_p, \tag{30}$$

$$\bar{T} = 1 - a\bar{\partial}_p, \tag{31}$$

which shift the momentum value by a

$$(Tf)(p) = f(p+a), \tag{32}$$

$$(\bar{T}f)(p) = f(p-a), \tag{33}$$

and satisfies

$$T\bar{T} = \bar{T}T = \hat{1}, \tag{34}$$

where $\hat{1}$ means the identity on the algebra of functions of p.

Introducing the momentum operator P [17]

$$(Pf)(p) = pf(p), \tag{35}$$

we have

$$TP = (P + a)T, \tag{36}$$

$$\bar{T}P = (P - a)\bar{T}. \tag{37}$$

Now we go back to the realization of the deformed Heisenberg algebra Eqs. (25)–(27) in terms of physical operators. We can associate to the crystalline structure of Mathieu's equation discussed in the previous section the one-dimensional lattice we have just presented.

Observe that we can write J_0 for the asymptotic Ince-Goldstein solution to Mathieu's equation, Eq. (22), as

$$J_0 = \frac{P}{a} + \frac{1}{2} - \left[\left(2\frac{P}{a} + 1\right)^2 + 1\right]\frac{a^2}{2^7} - \left[5\left(2\frac{P}{a} + 1\right)^4 + 34\left(2\frac{P}{a} + 1\right)^2 + 9\right]\frac{a^3}{2^{17}} + \cdots, \tag{38}$$

where P is given in Eq. (35) and its application to the vector states $|m\rangle$ appearing in (18)–(20) gives

$$P|m\rangle = ma|m\rangle, \qquad m = 0, 1, \cdots, \tag{39}$$

and

$$\bar{T}|m\rangle = |m+1\rangle, \qquad m = 0, 1, \cdots, \tag{40}$$

where $\bar{T}$ and $T = \bar{T}^\dagger$ are defined in Eqs. (30)–(34). It is useful to note that from Eq. (39) it is possible to define the number operator N as $N \equiv P/a$.

With the definition of J_0 given in Eq. (38), we see that ε_n^L given in Eq. (23) is the J_0 eigenvalue of state $|n\rangle$ as desired. Let us now define

$$A^\dagger = S(P)\bar{T}, \tag{41}$$

$$A = TS(P), \tag{42}$$

where

$$S(P)^2 = \frac{P}{a} - \left[\left(2\frac{P}{a} + 1\right)^2 - 1\right]\frac{a^2}{2^7} - \left[5\left(2\frac{P}{a} + 1\right)^4 + 34\left(2\frac{P}{a} + 1\right)^2 - 39\right]\frac{a^3}{2^{17}} + \cdots \tag{43}$$

satisfies $S^2(P) = J_0 - \alpha_0$, where α_0, defined in Eq. (15), is ε_0^L. Following Ref. [2], one can show that $A^\dagger$, A, and J_0 given in Eqs. (41), (42), and (38), respectively, obey the algebra defined in Eqs. (25), (26), and (27).

Note that the realization we have found in Eqs. (41), (42), and (38) is qualitatively different from the realization of the standard harmonic oscillator. This is reasonable, since we have two physically different systems. Even if the standard quantum harmonic oscillator defined on $-\infty \leq x \leq \infty$ is a limiting case of the periodic one, it is not periodic and in this case there is no lattice associated to it. On the other hand, once L is finite, $-L \leq x \leq L$, the periodic structure is explicitly manifest and the realization in the finite case, given in Eqs. (41), (42), and (38), shows it clearly.

IV. A QUANTUM FIELD THEORY IN A BOX

In Sec. II we have presented a description of a quantum harmonic oscillator on a circle and in Sec. III, its associated Heisenberg-type algebra, i.e., an algebra having the Hamiltonian and the step operators as generators, corresponding to a quantum harmonic oscillator on a circle. This algebra is a deformed Heisenberg algebra which goes to the standard Heisenberg algebra when the radius of the circle goes to infinity.

In this section, using the hypothesis that the successive energy levels of the quantum harmonic oscillator on a circle are still obtained by the creation or/and annihilation of a quantum particle on a periodic structure, we are going to construct a quantum field theory in a compact space.

In the momentum space appropriated to the realization of the deformed Heisenberg algebra we discussed, besides the operator P defined in Eq. (35), one can define two self adjoint operators as

$$\chi \equiv -i[S(P)(1 - a\bar{\partial}_p) - (1 + a\partial_p)S(P)] = -i(A - A^\dagger), \tag{44}$$

$$Q \equiv S(P)(1 - a\bar{\partial}_p) + (1 + a\partial_p)S(P) = A + A^\dagger, \tag{45}$$

where ∂_p and $\bar{\partial}_p$ are the left and right discrete derivatives defined in Eqs. (28) and (29).

It can be verified that operators P, χ, and Q generate the following algebra on the momentum lattice:

$$[\chi, P] = iaQ, \tag{46}$$

$$[P, Q] = ia\chi, \tag{47}$$

$$[\chi, Q] = 2iS(P)[S(P + a) - S(P - a)]. \tag{48}$$

This algebra is the analog of the Heisenberg algebra in the deformed case.

Since the analog of the Heisenberg algebra for the deformed case has three generators, it is convenient to define three fields which we call $\phi(\vec{r}, t)$, $\Pi(\vec{r}, t)$, and $\wp(\vec{r}, t)$. In terms of Fourier series these fields are given as

$$\phi(\vec{r}, t) = \sum_{\vec{k}} \frac{1}{\sqrt{2\Omega\omega(\vec{k})}} Q_{\vec{k}}(t)e^{i\vec{k}.\vec{r}}, \tag{49}$$

$$\Pi(\vec{r}, t) = \sum_{\vec{k}} \frac{i\omega(\vec{k})}{\sqrt{2\Omega\omega(\vec{k})}} \chi_{-\vec{k}}(t)e^{i\vec{k}.\vec{r}}, \tag{50}$$

where $\omega(\vec{k}) = \sqrt{\vec{k}^2 + m^2}$, m is a real parameter, and Ω is the volume of a rectangular box and

$$\wp(\vec{r}, t) = \sum_{\vec{k}} \sqrt{\frac{\omega(\vec{k})}{2\Omega}} S_{\vec{k}} e^{i\vec{k}.\vec{r}}. \tag{51}$$

The time-dependent operators in the Hilbert space $Q_{\vec{k}}(t)$, $\chi_{\vec{k}}(t)$, and $S_{\vec{k}}$ will be defined in what follows, and the components of $\vec{k}$ are given by

$$k_i = \frac{2\pi l_i}{Z_i}, \qquad i = 1, 2, 3, \tag{52}$$

with $l_i = 0, \pm 1, \pm 2, \cdots$ and Z_i being the lengths of the three sides of a rectangular box Ω. We introduce for each point of this $\vec{k}$ space an independent deformed quantum harmonic oscillator constructed in the last two previous sections such that the deformed operators commute for different three-dimensional lattice points. We also introduce an independent copy of the one-dimensional momentum lattice defined in the previous section for each point of this $\vec{k}$ lattice so that $P^{\dagger}_{\vec{k}} = P_{\vec{k}}$ and $T_{\vec{k}}$, $\bar{T}_{\vec{k}}$, and $S_{\vec{k}}$ are defined by means of the previous definitions, Eqs. (30), (31), and (43), through the substitution $P \to P_{\vec{k}}$.

It is possible to show that

$$A^{\dagger}_{\vec{k}} = S_{\vec{k}}\bar{T}_{\vec{k}}, \tag{53}$$

$$A_{\vec{k}} = T_{\vec{k}}S_{\vec{k}}, \tag{54}$$

$$J_0(\vec{k}) = \frac{P_{\vec{k}}}{a} + \frac{1}{2} - \left[\left(2\frac{P_{\vec{k}}}{a}+1\right)^2 + 1\right]\frac{a^2}{2^7} - \left[5\left(2\frac{P_{\vec{k}}}{a}+1\right)^4 + 34\left(2\frac{P_{\vec{k}}}{a}+1\right)^2 + 9\right]\frac{a^3}{2^{17}} + \cdots, \tag{55}$$

where

$$S^2_{\vec{k}} = \frac{P_{\vec{k}}}{a} - \left[\left(2\frac{P_{\vec{k}}}{a}+1\right)^2 - 1\right]\frac{a^2}{2^7} - \left[5\left(2\frac{P_{\vec{k}}}{a}+1\right)^4 + 34\left(2\frac{P_{\vec{k}}}{a}+1\right)^2 - 39\right]\frac{a^3}{2^{17}} + \cdots, \tag{56}$$

satisfy the algebra in Eqs. (25)–(27) for each point of this $\vec{k}$ lattice and the operators $A^{\dagger}_{\vec{k}}$, $A_{\vec{k}}$, and $J_0(\vec{k})$ commute among them for different points of this $\vec{k}$ lattice.

Now we define operators χ and Q for each point of the three-dimensional lattice as

$$\chi_{\vec{k}} \equiv -i(T_{-\vec{k}}S_{-\vec{k}} - S_{\vec{k}}\bar{T}_{\vec{k}}) = -i(A_{-\vec{k}} - A^{\dagger}_{\vec{k}}), \tag{57}$$

$$Q_{\vec{k}} \equiv T_{\vec{k}}S_{\vec{k}} + S_{-\vec{k}}\bar{T}_{-\vec{k}} = A_{\vec{k}} + A^{\dagger}_{-\vec{k}}, \tag{58}$$

such that $\chi^{\dagger}_{\vec{k}} = \chi_{-\vec{k}}$ and $Q^{\dagger}_{\vec{k}} = Q_{-\vec{k}}$, exactly as it happens in the construction of a spin-0 field for the spin-0 quantum field theory [1]. These operators appear in the Fourier expansion of the fields given in Eqs. (49)–(51).

By a straightforward calculation, one can show that the Hamiltonian

$$H = \frac{1}{2}\int_{\Omega} d^3r[\Pi(\vec{r},t)^2 + \rho|\wp(\vec{r},t)|^2 + \phi(\vec{r},t)(-\vec{\nabla}^2 + m^2)\phi(\vec{r},t)] \tag{59}$$

can be written as

$$H = \frac{1}{2}\sum_{\vec{k}}\omega(\vec{k})[A^{\dagger}_{\vec{k}}A_{\vec{k}} + A_{\vec{k}}A^{\dagger}_{\vec{k}} + \rho S(N_{\vec{k}})^2] = \frac{1}{2}\sum_{\vec{k}}\omega(\vec{k})[S(N_{\vec{k}}+1)^2 + (1+\rho)S(N_{\vec{k}})^2], \tag{60}$$

where ρ is an arbitrary number and

$$S(N_{\vec{k}})^2 = N_{\vec{k}} - [(2N_{\vec{k}}+1)^2 - 1]\frac{a^2}{2^7} - [5(2N_{\vec{k}}+1)^4 + 34(2N_{\vec{k}}+1)^2 - 39]\frac{a^3}{2^{17}} + \cdots. \tag{61}$$

Since the term in the Hamiltonian (59) proportional to ρ is time independent, it seems that it cannot produce any relevant effect. Thus, for simplicity, we will take $\rho = 0$. In order that the energy of the vacuum state becomes zero, we replace H in Eq. (60) by

$$H = \frac{1}{2}\sum_{\vec{k}}\omega(\vec{k})[S(N_{\vec{k}}+1)^2 + S(N_{\vec{k}})^2 - N_0^2], \tag{62}$$

where

$$N_0^2 \equiv f(\alpha_0) - \alpha_0 = 1 - a^2/2^4 - 21a^3/2^{12} + \cdots. \tag{63}$$

Note that in the limit $L \to \infty$, the above Hamiltonian is proportional to the number operator.

The eigenvectors of H form a complete set and span the Hilbert space of this system. They are the following:

$$|0\rangle, A^{\dagger}_{\vec{k}}|0\rangle, A^{\dagger}_{\vec{k}}A^{\dagger}_{\vec{k}'}|0\rangle \quad \text{for } \vec{k} \neq \vec{k}', \qquad (A^{\dagger}_{\vec{k}})^2|0\rangle, \cdots, \tag{64}$$

where the state $|0\rangle$ satisfies as usual $A_{\vec{k}}|0\rangle = 0$ [see Eq. (12)] for all $\vec{k}$ and $A_{\vec{k}}$, $A^{\dagger}_{\vec{k}}$ for each $\vec{k}$ satisfying the deformed Heisenberg algebra Eqs. (25)–(27).

The time evolution of the fields can be studied by means of Heisenberg's equation for $A^{\dagger}_{\vec{k}}$, $A_{\vec{k}}$, and $S_{\vec{k}}[\equiv S(N_{\vec{k}})]$. Define

$$h(N_{\vec{k}}) \equiv \tfrac{1}{2}[S^2(N_{\vec{k}}+2) - S^2(N_{\vec{k}})]. \tag{65}$$

Thus, using Eqs. (60) or (62) and $[N, A^{\dagger}] = A^{\dagger}$, we obtain

$$[H, A^{\dagger}_{\vec{k}}] = \omega(\vec{k})A^{\dagger}_{\vec{k}}h(N_{\vec{k}}). \tag{66}$$

We can solve Heisenberg's equation for the deformed case and the result is

$$A^{\dagger}_{\vec{k}}(t) = A^{\dagger}_{\vec{k}}(0)e^{i\omega(\vec{k})h(N_{\vec{k}})t}. \tag{67}$$

Note that for $L \to \infty$ we have $h(N_{\vec{k}}) \to 1$ and Eq. (67) gives the correct result for this undeformed case. Furthermore, we easily see that operators $P_{\vec{k}}$ and $S_{\vec{k}}$ are time independent. We emphasize that the extra term $h(N_{\vec{k}})$ in the exponentials depends on the number operator, this being the main difference from the undeformed case. The Fourier transformation of Eq. (49) can then be written as

$$\phi(\vec{r},t) = \alpha(\vec{r},t) + \alpha(\vec{r},t)^{\dagger}, \tag{68}$$

where

$$\alpha(\vec{r},t) = \sum_{\vec{k}}\frac{1}{\sqrt{2\Omega\omega(\vec{k})}}e^{i\vec{k}.\vec{r} - i\omega(\vec{k})h(N_{\vec{k}})t}A_{\vec{k}}, \tag{69}$$

with $A_{\vec{k}}$ in Eq. (69) being time independent and $\alpha(\vec{r}, t)^\dagger$ is the Hermitian conjugate of $\alpha(\vec{r}, t)$.

The Dyson-Wick contraction between[3] $\phi(x_1)$ and $\phi(x_2)$ can be computed using Eqs. (68) and (69), which results in

$$D_F^N(x_1, x_2) = \sum_{\vec{k}} \frac{e^{i\vec{k}.\Delta\vec{r}_{12}}}{2\Omega\omega(\vec{k})}[S(N_{\vec{k}}+1)^2 e^{\mp i\omega(\vec{k})h(N_{\vec{k}})\Delta t_{12}} - S(N_{\vec{k}})^2 e^{\mp i\omega(\vec{k})h(N_{\vec{k}}-1)\Delta t_{12}}], \tag{70}$$

where $\Delta t_{12} = t_1 - t_2$, $\Delta\vec{r}_{12} = \vec{r}_1 - \vec{r}_2$. The minus sign in the exponent holds when $t_1 > t_2$ and the positive sign when $t_2 > t_1$. Note that when $L \to \infty$, $h(N_{\vec{k}}) \to 1$ and $S(N_{\vec{k}}+1)^2 - S(N_{\vec{k}})^2 \to 1$ recovering the standard result for the propagator.

We now present the result concerning the perturbative computation of the first order scattering process $1 + 2 \to 1' + 2'$ for $p_1 \neq p_2 \neq p_1' \neq p_2'$ with the initial state

$$|1, 2\rangle \equiv \frac{1}{N_0^2} A_{p_1}^\dagger A_{p_2}^\dagger |0\rangle \tag{71}$$

and the final state

$$|1', 2'\rangle \equiv \frac{1}{N_0^2} A_{p_1'}^\dagger A_{p_2'}^\dagger |0\rangle, \tag{72}$$

where A_{p_i} and $A_{p_i}^\dagger$ satisfy the algebraic relations in Eqs. (25)–(27). These particles are supposed to be described by the Hamiltonian given in Eq. (59) with an interaction given by $\lambda \int_{\Omega_t} :\phi(\vec{r}, t)^4: d^3r$, where $\Omega_t = \Omega \otimes t$ is the four volume of integration. To the lowest order in λ, we have (Γ means the standard S matrix)

$$\langle 1', 2'|\Gamma|1, 2\rangle_1 = -i\lambda \int_{\Omega_t} d^4x \langle 1', 2'|:\phi^4(x):|1, 2\rangle. \tag{73}$$

The first order computation follows, step by step, the computation of the first order scattering process given in Ref. [3] and gives us the following result:

$$\langle 1', 2'|\Gamma|1, 2\rangle_1 = \frac{-6i(2\pi)^4}{\Omega^2\sqrt{\omega_{\vec{p}_1}\omega_{\vec{p}_2}\omega_{\vec{p}'_1}\omega_{\vec{p}'_2}}} \frac{\lambda N_0^4}{h(0)} \delta^4(P_1 + P_2 - P_1' - P_2'), \tag{74}$$

where

$$P_i = (\vec{p}_i, \omega_{\vec{p}_i}), \qquad P_i' = (\vec{p}'_i, \omega_{\vec{p}'_i}), \tag{75}$$

and from Eq. (65)

$$h(0) = 1 - 3\frac{a^2}{2^5} - 123\frac{a^3}{2^{13}} + \dots. \tag{76}$$

Note that when $L \to \infty$ ($a \to 0$) we have $N_0 \to 1$, $h(0) \to 1$, the box Ω becomes an infinite box, and Eq. (74) becomes the standard undeformed result [1]. It is convenient to note at this point that we are, by hypothesis, identifying the linear dimensions of the box Ω, where we perform the spatial integration in Eq. (73) with the dimensional length Z ($L = Z/x_0$) of the circle where the harmonic oscillator is defined. This identification is not strictly necessary; it comes from our approach that everything happens inside the spatial box Ω. We suppose that in a universe approximated by a finite spatial box Ω the step operators of the quantum harmonic oscillator defined on a circle of length Z, where Z is a linear dimension of the box Ω, create and/or annihilate point particles. Thus, if the spatial box Ω increases, so does the length of the circle where the harmonic oscillator is defined.

To second order in λ the scattering process $1 + 2 \to 1' + 2'$ is given as

$$\langle 1', 2'|\Gamma|1, 2\rangle_2 = \frac{(-i)^2}{2}\lambda^2 \iint_{\Omega_t} d^4x d^4y \times \langle 1', 2'|T[:\phi^4(x)::\phi^4(y):]|1, 2\rangle, \tag{77}$$

where T denotes the time-ordered product. In order to convert the time-ordered product into a normal product, we use the Wick's expansion. The propagator in the present case [see Eq. (70)] is not a simple c number since it depends on the number operator N and this fact induces modifications in the standard Wick expansion. This subject was already discussed in Ref. [5] where the computation of a scattering process for a deformed QFT to second order in the coupling constant was presented.

Following Ref. [5] we find for the scattering process under consideration, up to second order in the coupling constant, is given by

$$\langle 1', 2'|\Gamma|1, 2\rangle_2 = \frac{1}{2\Omega^2\sqrt{\omega_{\vec{p}_1}\omega_{\vec{p}_2}\omega_{\vec{p}'_1}\omega_{\vec{p}'_2}}}\left(\frac{\lambda N_0^4}{h(0)}\right)^2 \delta^4(P_1 + P_2 - P_1' - P_2')(I + I' + I'' + I'''), \tag{78}$$

where

$$I = -\frac{(2\pi)^2}{4\Omega}\sum_{\vec{k}} \frac{1}{\sqrt{(\vec{k}^2 + m^2)[(\vec{s} - \vec{k})^2 + m^2]}}, \tag{79}$$

with $\vec{s} = \vec{p}_1 + \vec{p}_2$ and

$$I' = I(\vec{s} \to -\vec{s}), \tag{80}$$

$$I'' = I(\vec{s} \to \vec{t} \equiv \vec{p}_1 - \vec{p}'_1), \tag{81}$$

$$I''' = I(\vec{s} \to \vec{u} \equiv \vec{p}_1 - \vec{p}'_2). \tag{82}$$

In summary, the scattering process $1 + 2 \to 1' + 2'$ for $p_1 \neq p_2 \neq p_1' \neq p_2'$ with the initial and final states given in Eqs. (71) and (72), respectively, where A_{p_i}, $A_{p_i}^\dagger$ satisfy the algebraic relations in Eqs. (25)–(27) and the particles are supposed to be described by the Hamiltonian given in

[3] $x_i \equiv (\vec{r}_i, t_i)$

Eq. (59) with an interaction given by $\lambda \int_{\Omega_t} :\phi(\vec{r}, t)^4: d^3 r$ is given up to second order in the coupling constant λ as

$$\langle 1', 2' | \Gamma | 1, 2 \rangle = \frac{\lambda N_0^4}{h(0)} A_1 + \left(\frac{\lambda N_0^4}{h(0)} \right)^2 (A_2^s + A_2^t + A_2^u), \quad (83)$$

where A_1 is obtained from Eq. (74), A_2^s comes from I and I' in Eq. (78), and A_2^t and A_2^u come from I'' and I''', respectively.

Note that when $L \to \infty$ ($a \to 0$) we have $N_0 \to 1$, $h(0) \to 1$, the box Ω becomes an infinite box, Eq. (83) becomes the standard undeformed result with A_1, A_2^s, A_2^t, and A_2^u being the same contributions that we find in the *standard* $\lambda\ \phi^4$ (nondeformed) model corresponding to the tree level, the s, t, and u channels for one-loop level, respectively. Also, it is worth noticing that the perturbative expansion shows that the coupling constant which appears in the interacting Hamiltonian is modified as $\lambda \to \lambda N_0^4/h(0)$. This means that the effective coupling constant, $\lambda_{\text{eff}} \equiv \lambda N_0^4/h(0)$, in this framework is modified due to the presence of the deformation parameter $a = \pi/L$.

V. CONTRIBUTION FROM THE BOUNDARY FOR THE VARIATION OF THE COUPLING CONSTANT

The comments in the last paragraph of the previous section allow us to connect the effective coupling constant appearing in the perturbation expansion, which is given by $\lambda N_0^4/h(0)$, with the size of the box we are considering, i.e., the linear dimension Z of the box Ω. In this section, based on this connection we are going to compute the variation of the effective coupling constant for two different values of Z, namely, one corresponding to the time of nucleosynthesis of the standard cosmological model and the other to the present epoch. The choice of these values, as said before, was done just to perform a calculation. With this choice we are not assuming that the Universe is described by our model. In fact, we want just to have an idea of what would be the contribution from the boundary, in a compact space, to the variation of the coupling constant in the framework our approach.

In order to investigate the variation of the effective coupling constant, let us define $p \equiv N_0^4/h(0)$, which for two different values of L, namely, $L_\pm$, gives

$$p_\pm = \frac{(1 - a_\pm^2/2^4 - 21 a_\pm^3/2^{12} + \cdots)^2}{1 - 3a_\pm^2/2^5 - 123 a_\pm^3/2^{13} + \ldots}, \quad (84)$$

where we have used Eqs. (63) and (76) with $a_\pm = \pi/L_\pm$, $L_\pm = Z_\pm/x_0$, and $\lambda_{\text{eff}}^\pm = \lambda p_\pm$.

In what follows, let us compute the dimensionless quantity $\Delta\alpha/\alpha$, given by

$$\frac{\Delta\alpha}{\alpha} = \frac{(\lambda_{\text{eff}}^+)^2 - (\lambda_{\text{eff}}^-)^2}{(\lambda_{\text{eff}}^+)^2} = 1 - \frac{(\lambda_{\text{eff}}^-)^2}{(\lambda_{\text{eff}}^+)^2}, \quad (85)$$

where $\pm$ means the present time and the time at the moment of nucleosynthesis, respectively.

We have assumed that the creation and/or annihilation operators of the quantum harmonic oscillator on a periodic line create and/or annihilate a quantum particle. Along these lines we showed in Secs. II and III that there is a deformation parameter a which is connected to the linear size of the box where the second quantized formalism is constructed through $a = \pi/L$ with L given by $L = Z/x_0$. As discussed in Sec. II, x_0 is a scale where the deformation starts to become relevant. In what follows, we will assume that the value of x_0 is at least equal to the one corresponding to the scale of the electroweak phase transition just to have a reference size which will permit us to perform our calculation.

Now let us compute the dimensionless quantity $\Delta\alpha/\alpha$ for two different values of the dimensions of the box, namely, for $Z_+ \approx 10^{28}$ cm and $Z_- \approx 10^{19}$ cm in accordance with our choice . For these values of the size of the box, the deformation parameters are $a_+ = \pi/L_+ = \pi x_0/Z_+ \approx 10^{-15}$ and $a_- \approx 10^{-6}$. Because of the magnitude of the deformation parameters under consideration, it is sufficient to take the lowest order expansion for $p_\pm$, which is

$$p_\pm = 1 - \frac{a_\pm^2}{2^5} + \cdots. \quad (86)$$

Taking into account this expansion and the estimated value of a_-, we obtain for $\Delta\alpha/\alpha$ the following result:

$$\frac{\Delta\alpha}{\alpha} = \frac{a_-^2}{2^5} + \cdots < 10^{-12}. \quad (87)$$

VI. FINAL COMMENTS

In this paper we have constructed a QFT in a finite box. In order to construct this QFT, we used the hypothesis that particles living in a finite box with periodic boundary conditions are created and/or annihilated through the creation and/or annihilation operators, respectively, of a quantum harmonic oscillator on a circle.

The quantum harmonic oscillator we have used is described by Mathieu's equation and its associated creation and annihilation operators obey a deformed Heisenberg algebra. We have thus followed the treatment given in Refs. [3,5], which presents the construction of a deformed QFT based on q oscillators, in order to construct the present QFT in a finite box.

The perturbative series we have found shows an effective coupling constant given by $\lambda_{\text{eff}} = \lambda N_0^4/h(0)$, where N_0 and $h(0)$, see Eqs. (63) and (76), which depends on the dimensionless quantity $L = Z/x_0$, where Z is a linear dimension of the finite box Ω and x_0 is a scale where the modified description of the creation of particles starts to be relevant.

Even though our model is not a cosmological model, we considered an estimation of the bound for the variation of the effective coupling constant taking into account two values, Z, of the linear dimension of the finite box Ω, one corresponding to the time of the nucleosynthesis of the standard cosmological model and the other to the size of the Universe nowadays. The result obtained, $\Delta\alpha/\alpha < 10^{-12}$, is in the range of results on the constraints for the variation of the coupling constants for different epochs of the Universe [18]. Thus, we think that it would be interesting to analyze the approach we considered in this paper in the framework of a cosmological model in order to investigate the variation of the fundamental constants of nature.

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