Ambiguities and symmetry relations associated with fermionic tensor densities

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We consider the consistent evaluation of perturbative (divergent) Green functions associated with fermionic tensor densities and the derivation of symmetry relations for them. We show that, in spite of current algebra methods being not applicable, it is possible to derive symmetry properties analogous to the Ward identities of vector and axial-vector densities. The proposed method, which is applicable to any previously chosen order of perturbative calculation, gives the same results as those of current algebra when such a tool is applicable. By using a very general calculational strategy, concerning the manipulations and calculations involving divergent Feynman integrals, we evaluate the purely fermionic two-point functions containing tensor vertices and derive their symmetry properties. The present investigation is the first step in the study and characterization of possible anomalies involving fermionic tensor densities, particularly in purely fermionic three-point functions.

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I. INTRODUCTION

Fermionic densities play the central role in the construction of phenomenological quantum field theories (QFT's). They are, in the last instance, the quantities which couple the constituent matter fields with the intermediary gauge fields. In the context of perturbative solutions in QFT's, purely fermionic Green functions are crucial for the renormalization. Their symmetry relations or Ward identities must be explicitly verified case by case and, only in the situations where such properties can be maintained, in the explicitly evaluated expressions for the perturbative amplitudes, a theory has a chance for the renormalizability. This is far from being a trivial verification due to many reasons. First, as it is well known, the perturbative Green functions may involve divergences and, consequently, some kind of regularization scheme or equivalent philosophy must be adopted in order to verify the symmetry relations. However, frequently the regularized Green functions acquire ambiguous terms which are always related to violations of symmetry relations and/or other fundamental symmetries. So, at the same time that the regularizations are tools to make the manipulations and calculations, required to check if the symmetry properties can be maintained in the calculated Green functions, they can be the agents that lead to violations. One can attempt to define a consistent regularization as the scheme which eliminates all the ambiguities and violating terms in all situations, but this cannot be true. There are violations of symmetry relations in perturbative Green functions which are fundamental properties and cannot be avoided by any scheme. We are talking about anomalies. A deep understanding of a quantum field theory requires the clear identification of eventual anomalies [1–4]. This last sentence addresses us to a second difficulty related to the study of fermionic densities, or more specifically the fermionic tensor densities. Different from the nontensor densities (vector, axial-vector, scalar and pseudoscalar), whose Green functions can be related through current algebra methods [5,6], generating the desirable Ward identities [7,8] for the tensor densities such relations cannot be directly produced. So, if one wants to study the perturbative representations for the fermionic Green functions involving tensor vertices, searching for eventual anomalies, for example, it is necessary first to state the relations which will play the role of Ward identities. Only if such relations can be stated, in a second step we will be able to explicitly evaluate the involved Green functions, within the framework of a previously chosen prescription, and then verify if the expected "symmetry properties" can be maintained by the calculated expression, or if we will have unavoidable violations analogous to the well-known anomalies.

The study of fermionic tensor densities, on the other hand, can be important in many contexts of interest. In the context of the Nambu-Jona-Lasinio model, a purely fermionic model where the amplitudes, which are responsible for the predictions are, in the last instance, n-point purely fermionic Green functions, the tensor densities have been recently used in the description of vector mesons phenomenology [9]. In the context of QED, if one wants to extend the usual theory in order to include new terms, the lowest order, in the power counting, coupling between the electromagnetic field and the fermion field allowed by the gauge invariance is the one constructed through the tensor density. The resulting theory will be nonrenormalizable in the usual sense but predictions could be made in the context of effective QFT's as clearly pointed by Weinberg [10]. The fermionic tensor densities also appear in quantum theories for gravitation.

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The amplitudes which emerge from the linearized version of the coupling between a Weyl fermion and the *zweibein* field lead to the evaluation of Green functions that are similar mathematical structures to the fermionic tensor Green functions [11]. In particular, for the two-dimensional gravitation the Green functions involving fermionic tensor densities have been calculated in the framework of dispersion relations [12]. Fermionic tensor densities also play a crucial role in certain simple Abelian gauge models that include new kinds of matter fields [13,14].

In all the above cited situations where the evaluation of fermionic tensor Green functions becomes necessary, it would be extremely important that such an evaluation of the involved mathematical structures was made in a consistent way. By consistent way we understand that the amplitudes are obtained free from ambiguities and symmetry preserving, where they must be, and exhibiting the correct unavoidable violation, in the case of eventual anomalous amplitudes. However, before the evaluation of the Green functions, we have first to establish the constraints we must impose on the mathematical structures involved. The present contribution is an attempt to develop relations among the Green functions involving fermionic tensor densities with other nontensor densities or among themselves, which would play the role of Ward identities in furnishing constraints to the consistent evaluation of such structures. In order to achieve this goal, we consider an alternative strategy to generate relations among Green functions of the perturbative calculations that gives the same results of the current algebra methods, in situations where these theoretical tools can be applied. Exact relations are obtained to the tensor amplitudes and to their contractions with the external momenta involving nontensor Green functions. The adopted strategy, whose application is restricted to two-point functions in the present contribution, can be easily applied for n-point functions. In particular, its application to the three-point functions can reveal the existence of anomalies similar to those occurring in the AVV and AAA triangle amplitudes [15].

This work is organized as follows. In Sec. II we introduce some definitions and specify the notation which will be used. In Sec. we state the equivalence between the current algebra methods and a method based on identities relating the Green function of the perturbative calculations to generate Ward identities. In Sec. IV we discuss the impossibilities of current algebra methods to treat fermionic tensor densities and the difficulties relative to the derivation of symmetry properties for Green functions having tensor vertices. In Sec. V we derive relations among Green functions involving all two-point functions. In Sec. VI we present the calculational strategy we will use to evaluate the divergent Green functions considered, which we make in Sec. VII. In Sec. VIII we verify the relation among Green functions in the presence of ambiguous terms. In Sec. IX we adopt a specific point of view for the divergences which leads to the consistency in perturbative calculations and specifies the explicit form of the "consistent regularized amplitudes" and the symmetry relations for the two-point Green functions having tensor vertices. Finally, we present our final remarks in Sec. X.

II. DEFINITIONS AND NOTATION

We start by introducing the notation to be used and defining the quantities we will be concerned with for the rest of the work. For our present purposes consider a spin $1/2$, mass m free fermion model. There will be therefore a massive field which obeys Dirac's equation and with which we can construct currents $j_i(x)$ defined by

$$j_i(x) = \bar{\psi}(x)\Gamma_i\psi(x), \tag{1}$$

where Γ_i are the Dirac's matrices responsible for the transformation properties of the currents $(\hat{1};\gamma_5;\gamma_\mu;\gamma_\mu\gamma_5;\sigma_{\mu\nu})$, characterizing the scalar $S(x)$, pseudoscalar $P(x)$, vector $V_\mu(x)$, axial-vector $A_\mu(x)$, and tensor $T_{\mu\nu}(x)$ densities, respectively. Here we have introduced the definition $\sigma_{\mu\nu} = \frac{1}{2}[\gamma_\mu, \gamma_\nu]$. With the above definitions and the fermionic propagator, $iS_F(p) = i(\not{p} - m)^{-1}$, it is possible to construct n-point functions, which we define in the same way as in Ref. [4] as follows.

(I) One-point functions:

$$T^i(k_1, m) = \int \frac{d^4k}{(2\pi)^4}\mathrm{Tr}\Big\{\Gamma_i\frac{1}{[(\not{k} + \not{k}_1) - m]}\Big\}. \tag{2}$$

(II) Two-point functions:

$$T^{ij}(k_1, m; k_2, m) = \int \frac{d^4k}{(2\pi)^4}\mathrm{Tr}\Big\{\Gamma_i\frac{1}{[(\not{k} + \not{k}_1) - m]} \times\Gamma_j\frac{1}{[(\not{k} + \not{k}_2) - m]}\Big\}. \tag{3}$$

(III) Three-point functions:

$$T^{lij}(k_1, m; k_2, m; k_3, m) = \int \frac{d^4k}{(2\pi)^4}\mathrm{Tr}\Big\{\Gamma_l\frac{1}{[(\not{k} + \not{k}_3) - m]} \times\Gamma_i\frac{1}{[(\not{k} + \not{k}_1) - m]} \times\Gamma_j\frac{1}{[(\not{k} + \not{k}_2) - m]}\Big\}, \tag{4}$$

and so on. Here k_1, k_2, and k_3 represent the arbitrary choices for the internal momenta of the loop. Energy momentum conservation requires only that these quantities must be related to the external momenta, e.g., in the three-point functions we have $k_3 - k_1 = p$, $k_1 - k_2 = p'$, and $k_3 - k_2 = q$. In the definition for the two-point function above, on the other hand, the difference $k_1 - k_2$ represents the external momentum.

In our notation, for three-point functions, the vertex to the left is assumed to be connected with the "initial state" having external momentum q and the vertex operator Γ_l. The other two vertices correspond to "final states" with external momenta p and p' and the vertex operators Γ_i and Γ_j, respectively. Besides, the upper indices are associated, in the order they appear, with the respective Lorentz indices in the same order, whenever is the case. As an example, if $\Gamma_l = \gamma_\lambda$, $\Gamma_i = \gamma_\mu \gamma_5$ and $\Gamma_j = \gamma_5$ we will have $T^{VAP}_{\lambda\mu}(k_1, k_2, k_3)$. In particular, this means that if we have one particle in the initial state and two in the final one, a symmetrization in the final states will be required. For example, in the process $S \to VV$ we define the corresponding amplitude as

$$T^{S\to VV}_{\mu\nu} = T^{SVV}_{\mu\nu}(k_1, k_2, k_3) + T^{SVV}_{\nu\mu}(l_1, l_2, l_3). \qquad (5)$$

The first term represents the direct channel, and the second the crossed channel where l_1, l_2, and l_3 are the arbitrary choices for the corresponding internal momenta.

There are integral representations for the functions defined above with the Fourier transform of the currents [4]

$$\langle j_1(q) j_2(-q)\rangle \equiv \int e^{-ipx} d^4x \langle 0|T[j_1(x) j_2(0)]|0\rangle, \qquad (6)$$

in the case of two-point functions, and

$$\langle j_1(p) j_2(p') j_3(q)\rangle \equiv \int e^{-ipx} e^{-ip'y}\, d^4x\, d^4y \times \langle 0|T[j_1(x) j_2(y) j_3(0)]|0\rangle, \qquad (7)$$

for the three-point functions. With these elements, one can establish relations among the n-point functions, i.e., Ward identities, which we will consider next.

III. RELATIONS AMONG GREEN FUNCTIONS AND WARD IDENTITIES

It is a well-known fact that the Green functions of the perturbative solution of QFT's are related among them. Such relations are deeply associated with the symmetry content of the theories. There are many equivalent ways to state the referred symmetry relations. In what follows we will consider two equivalent ways which can be applied to generate Ward identities relating amplitudes of the perturbative calculation when fermionic tensor densities are not involved.

A. Current algebra methods and Ward identities

Perhaps the most popular way to generate symmetry relations is to use the standard methods of the current algebra [5,6]. In this case, the first step is to specify the fourdivergences of the currents. Such properties are intermediate in the vector and axial-vector currents. They are

$$\partial^\mu V_\mu(x) = 0, \qquad (8)$$

$$\partial^\mu A_\mu(x) = 2mP(x), \qquad (9)$$

which are direct consequence of the fact that the spin-$\frac{1}{2}$ fermion field obeys the Dirac equation. The tensor current we will consider in a moment. To derive the Ward identities we will also need the following commutation relations for the vector and axial-vector currents, at equal time:

$$[V_0(\vec{x}, t), V_\mu(\vec{y}, t)] = 0, \qquad (10)$$

$$[V_0(\vec{x}, t), A_\mu(\vec{y}, t)] = 0, \qquad (11)$$

$$[A_0(\vec{x}, t), V_\mu(\vec{y}, t)] = 0. \qquad (12)$$

The above properties are consequences of the canonical commutation relations for the fields $\psi(x)$ and $\overline{\psi}(x)$. In addition to these ingredients, we have also to consider the derivative of a time ordered product of field operators which means

$$\begin{aligned} &\partial_\mu \langle 0|T[J_\alpha(x) O^1(y_1)\cdots O^n(y_n)]|0\rangle \\ &\quad = \langle 0|T\{[\partial_\mu J_\alpha(x)] O^1(y_1)\cdots O^n(y_n)\}|0\rangle \\ &\qquad + \sum_{i=1}^{n} \langle 0|T[J_0(x), O^i(y_i)]\delta(x^0 - y_i^0) O^1 \\ &\qquad \times (y_1)\cdots O^{i-1}(y_{i-1}) O^{i+1}(y_{i+1})\cdots O^n(y_n)|0\rangle. \end{aligned} \qquad (13)$$

It is instructive, especially for future purposes, to consider an explicit example. Let us take the AVV Green function, which, according to definition (7), can be written as

$$\langle A_\alpha(p) V_\beta(p') V_\gamma(q)\rangle = \int d^4x\, d^4y\, e^{-ipx} e^{-ip'y} \times \langle 0|T[A_\alpha(x) V_\beta(y) V_\gamma(0)]|0\rangle. \qquad (14)$$

In order to obtain a Ward identity, we contract the above expression with the external momentum at their respective vertex. Taking the contraction at the axial vertex, we have

$$p^\alpha \langle A_\alpha(p) V_\beta(p') V_\gamma(q)\rangle = \int d^4x\, d^4y\, e^{-ipx} e^{-ip'y} \partial^\alpha \times \langle 0|T[A_\alpha(x) V_\beta(y) V_\gamma(0)]|0\rangle.$$

The derivative inside the time ordered product can be performed with the help of Eq. (13). Then we get

$$p^{\alpha}\langle A_{\alpha}(p)V_{\beta}(p')V_{\gamma}(q)\rangle = \int d^4x\, d^4y\, e^{-ipx}e^{-ip'y} \times\{\langle 0|T[\partial^{\alpha}A_{\alpha}(x)V_{\beta}(y)V_{\gamma}(0)]|0\rangle + \langle 0|T([A_0(x), V_{\beta}(y)]V_{\gamma}(0))|0\rangle \times\delta(x^0-y^0) + \langle 0|T([A_0(x), \times V_{\gamma}(0)]V_{\beta}(y))|0\rangle\delta(x^0)\}. \quad (15)$$

In the first term the proportionality can be used between the fourdivergence of the axial current and the pseudoscalar one, Eq. (9). The two remaining terms must vanish as a consequence of the commutation relation at equal time for the involved currents, Eqs. (10)–(12). So, we get

$$p^{\alpha}\langle A_{\alpha}(p)V_{\beta}(p')V_{\gamma}(q)\rangle = 2m\langle P(p)V_{\beta}(p')V_{\gamma}(q)\rangle.$$

On the other hand, contracting the AVV function at a vector vertex, say p'_{β}, we obtain

$$p'^{\beta}\langle A_{\alpha}(p)V_{\beta}(p')V_{\gamma}(q)\rangle = \int d^4x\, d^4y\, e^{-ipx}e^{-ip'y}\{\langle 0|T(A_{\alpha}(x)\partial^{\beta}V_{\beta}(y)V_{\gamma}(0))|0\rangle + \langle 0|T([V_0(y), A_{\alpha}(x)]V_{\gamma}(0))|0\rangle\delta(y^0-x^0) + \langle 0|T([V_0(y), V_{\gamma}(0)]A_{\alpha}(x))|0\rangle\delta(y^0)\}. \quad (16)$$

The property (8) of the vector current states the Ward identities

$$p'^{\beta}\langle A_{\alpha}(p)V_{\beta}(p')V_{\gamma}(q)\rangle = 0, \quad (17)$$

$$q^{\gamma}\langle A_{\alpha}(p)V_{\beta}(p')V_{\gamma}(q)\rangle = 0. \quad (18)$$

The obtained Ward identities are very well known and must be obeyed in any order of perturbative solution. The main implication, for practical purposes, is the fact that the above obtained relations represent constraints to be imposed on any explicit evaluation of the involved amplitudes in any order of the perturbative calculations. Let us now consider an alternative approach.

B. Relations among Green functions

The standard methods of the current algebra state properties or constraints which must be valid in any order of a perturbative calculation. In practical situations, however, the amplitudes are evaluated in a specific order so that it is possible to state a one by one correspondence between the currents and their perturbative representations or n-point Green functions. This means that the symmetry relations can be viewed, in a certain order of the perturbative calculations, as nothing more than identities relating the corresponding Green functions, which implies that Ward identities can be directly derived from the expressions of the perturbative amplitudes. In fact, the Ward identities can be derived in any previously chosen order by noting the existence of identities relating the mathematical objects used to define the n-point Green functions. Such identities are only consequences of the Dirac gamma matrices algebra. To make this point clear, consider the identity

$$\begin{aligned}(k_3-k_2)^{\alpha}&\left\{\gamma_{\nu}\frac{1}{(\not k+\not k_2)-m}\gamma_{\alpha}\gamma_5\frac{1}{(\not k+\not k_3)-m}\gamma_{\mu}\frac{1}{(\not k+\not k_1)-m}\right\}\\ &= -2m\left\{\gamma_{\nu}\frac{1}{(\not k+\not k_2)-m}\gamma_5\frac{1}{(\not k+\not k_3)-m}\gamma_{\mu}\frac{1}{(\not k+\not k_1)-m}\right\}+\left\{\gamma_{\mu}\gamma_5\frac{1}{(\not k+\not k_1)-m}\gamma_{\nu}\frac{1}{(\not k+\not k_2)-m}\right\}\\ &\quad-\left\{\gamma_{\nu}\gamma_5\frac{1}{(\not k+\not k_3)-m}\gamma_{\mu}\frac{1}{(\not k+\not k_1)-m}\right\}.\end{aligned} \quad (19)$$

If we take traces and next integrate in both sides, we get a relation among perturbative Green functions,

$$\begin{aligned}(k_3-k_2)^{\alpha}&\int\frac{d^4l}{(2\pi)^4}\mathrm{tr}\left\{\gamma_{\nu}\frac{1}{(\not k+\not k_2)-m}\gamma_{\alpha}\gamma_5\frac{1}{(\not k+\not k_3)-m}\gamma_{\mu}\frac{1}{(\not k+\not k_1)-m}\right\}\\ &= -2m\int\frac{d^4l}{(2\pi)^4}\mathrm{tr}\left\{\gamma_{\nu}\frac{1}{(\not k+\not k_2)-m}\gamma_5\frac{1}{(\not k+\not k_3)-m}\gamma_{\mu}\frac{1}{(\not k+\not k_1)-m}\right\}\\ &\quad-\int\frac{d^4l}{(2\pi)^4}\mathrm{tr}\left\{\gamma_{\nu}\gamma_5\frac{1}{(\not k+\not k_3)-m}\gamma_{\mu}\frac{1}{(\not k+\not k_1)-m}\right\}+\int\frac{d^4l}{(2\pi)^4}\mathrm{tr}\left\{\gamma_{\mu}\gamma_5\frac{1}{(\not k+\not k_1)-m}\gamma_{\nu}\frac{1}{(\not k+\not k_2)-m}\right\},\end{aligned}$$

which, in the adopted notation, can be written as

$$(k_3-k_2)^{\alpha}T^{AVV}_{\alpha\mu\nu} = -2mT^{PVV}_{\mu\nu} + T^{AV}_{\mu\nu}(k_1,k_2) + T^{AV}_{\mu\nu}(k_3,k_1). \quad (20)$$

This is only a relation between Green functions and not yet a Ward identity. The summation of this result with the corresponding crossed channel, which, according to the adopted notation, is written as

$$q^{\alpha}T^{A\rightarrow VV}_{\alpha\mu\nu} = -2mT^{P\rightarrow VV}_{\mu\nu} + T^{AV}_{\mu\nu}(k_1, k_2) + T^{AV}_{\mu\nu}(k_3, k_1) + T^{AV}_{\mu\nu}(l_1, l_2) + T^{AV}_{\mu\nu}(l_3, l_1), \qquad (21)$$

became the corresponding Ward identity. For the sake of completeness, we also consider the identity

$$(k_3 - k_1)^{\mu} \times \left\{\gamma_{\nu}\frac{1}{(k + k_2) - m}\gamma_{\alpha}\gamma_5\frac{1}{(k + k_3) - m}\gamma_{\mu}\frac{1}{(k + k_1) - m}\right\} = \gamma_{\alpha}\gamma_5\frac{1}{(k + k_1) - m}\gamma_{\nu}\frac{1}{(k + k_2) - m} - \gamma_{\alpha}\gamma_5\frac{1}{(\not{k} + \not{k}_3) - m}\gamma_{\nu}\frac{1}{(\not{k} + \not{k}_2) - m}, \qquad (22)$$

which implies

$$(k_3 - k_1)^{\mu}T^{AVV}_{\alpha\mu\nu} = T^{AV}_{\alpha\nu}(k_1, k_2) - T^{AV}_{\alpha\nu}(k_3, k_2). \qquad (23)$$

This means that

$$p^{\mu}T^{A\rightarrow VV}_{\alpha\mu\nu} = T^{AV}_{\alpha\nu}(k_1, k_2) - T^{AV}_{\alpha\nu}(k_3, k_2) + T^{AV}_{\alpha\nu}(l_3, l_2) - T^{AV}_{\alpha\nu}(l_3, l_1), \qquad (24)$$

and, in a completely similar way

$$p'^{\nu}T^{A\rightarrow VV}_{\alpha\mu\nu} = T^{AV}_{\alpha\mu}(k_3, k_2) - T^{AV}_{\alpha\mu}(k_3, k_1) + T^{AV}_{\alpha\mu}(l_1, l_2) - T^{AV}_{\alpha\mu}(l_3, l_2). \qquad (25)$$

Actually, the same expressions we obtained above can be produced by an equivalent procedure, the use of convenient identities in the interior of the traces. Such a procedure can be found in almost all quantum field theory textbooks and in traditional papers, especially those related to the subject of triangle anomalies [4,16]. In order to rewrite the left-hand side of the equation, after introducing the contraction inside the traces, the following identities are used:

$$(\not{l}_i - \not{l}_j)\gamma_5 = -\gamma_5[(\not{l} + \not{l}_i) - m] - [(\not{l} + \not{l}_j) - m]\gamma_5 - 2m\gamma_5, \qquad (26)$$

$$(\not{l}_i - \not{l}_j) = [(\not{l} + \not{l}_i) - m] - [(\not{l} + \not{l}_j) - m]. \qquad (27)$$

Following the above described procedure, it is possible to state an uncountable number of relations among perturbative Green functions. For the nontensorial two-point functions defined in Eq. (3), we get

$$(l_1 - l_2)^{\mu}T^{VV}_{\mu\nu}(l_1, l_2) = T^{V}_{\nu}(l_2) - T^{V}_{\nu}(l_1), \qquad (28)$$

$$(l_1 - l_2)^{\mu}T^{AA}_{\mu\nu}(l_1, l_2) = -2mT^{PA}_{\nu}(l_1, l_2) + T^{V}_{\nu}(l_2) - T^{V}_{\nu}(l_1), \qquad (29)$$

$$(l_1 - l_2)^{\nu}T^{AP}_{\nu}(l_1, l_2) = -2mT^{PP}(l_1, l_2) - T^{S}(l_2) - T^{S}(l_1), \qquad (30)$$

$$(l_1 - l_2)^{\mu}T^{AV}_{\mu\nu}(l_1, l_2) = -2mT^{PV}_{\nu}(l_1, l_2) + T^{A}_{\nu}(l_2) - T^{A}_{\nu}(l_1), \qquad (31)$$

$$(l_1 - l_2)^{\nu}T^{AV}_{\mu\nu}(l_1, l_2) = T^{A}_{\nu}(l_2) - T^{A}_{\nu}(l_1). \qquad (32)$$

Additional relations can be identified after the Dirac traces are evaluated. For example,

$$T^{AV}_{\mu\nu}(l_i, l_j) = \frac{i}{2m}\varepsilon_{\mu\nu\lambda\xi}(l_i - l_j)^{\lambda}(T^{SV})^{\xi}(l_i, l_j), \qquad (33)$$

$$T^{PA}_{\mu}(l_i, l_j) = \frac{1}{2m}(l_1 - l_2)_{\mu}[T^{SS}(l_i, l_j) + T^{PP}(l_i, l_j)], \qquad (34)$$

$$T^{VV}_{\mu\nu}(l_i, l_j) - T^{AA}_{\mu\nu}(l_i, l_j) = g_{\mu\nu}[T^{SS}(l_i, l_j) + T^{PP}(l_i, l_j)]. \qquad (35)$$

At this point, it is important to emphasize that the relations among Green functions do not correspond to Ward identities in general. The correct symmetry properties, or the Ward identities themselves, will emerge only if the evaluation of all the involved amplitudes are made in a consistent way which requires the adoption of a regularization scheme or an equivalent philosophy given the divergent character. There are structures in the relations among Green functions which vanish as a consequence of the equal time commutation relation among currents, in the methods of the current algebra, which need to be obtained as automatically zero when they are directly evaluated by an explicit calculation. This means that the regularization must attribute properties to the regularized divergent integrals which will be, in the last instance, responsible for the identically zero value for the referred structures. In different words, there are two steps in the calculations. The first involves some type of manipulations and calculations in divergent amplitudes without assuming any specific properties for the divergent integrals. At this stage, the relations among Green functions must be satisfied even if undefined pieces are still present. In a second stage, we will assume some properties for the regularization which must lead to the expected symmetry properties for the amplitudes. This conceptual point of view for the problem is very important for the purposes of the present contribution. We will return to this aspect later.

IV. TENSOR DENSITIES: CURRENT ALGEBRA AND RELATIONS AMONG GREEN FUNCTIONS

Let us now consider the case of tensor densities. It is important to state in a clear way which are the difficulties involved in the derivation of symmetry properties for amplitudes having tensor densities.

A. The current algebra methods

If one wants to use current algebra methods, the first ingredient that must be stated is the fourdivergence of the antisymmetric tensor current, which is given by

$$\partial^\mu T_{\mu\nu}(x) = \partial^\mu[\bar{\psi}(x)\sigma_{\mu\nu}\psi(x)]. \qquad (36)$$

The above expression cannot be reduced to a combination of other fermionic currents. In order to make this statement clear, let us develop the expression (36) further

$$\begin{aligned}\partial^\mu T_{\mu\nu}(x) &= \bar{\psi}(x)\overleftarrow{\partial}^\mu \sigma_{\mu\nu}\psi(x) + \bar{\psi}(x)\sigma_{\mu\nu}\partial^\mu\psi(x)\\ &= \bar{\psi}(x)\overleftarrow{\partial}^\mu \tfrac{1}{2}(\gamma_\mu\gamma_\nu - \gamma_\nu\gamma_\mu)\psi(x)\\ &\quad + \bar{\psi}(x)\tfrac{1}{2}(\gamma_\mu\gamma_\nu - \gamma_\nu\gamma_\mu)\partial^\mu\psi(x). \qquad (37)\end{aligned}$$

Using then the anticommutation of the γ matrices, we can write

$$\begin{aligned}\partial^\mu T_{\mu\nu}(x) &= \bar{\psi}(x)\overleftarrow{\partial}^\mu(\gamma_\mu\gamma_\nu - g_{\mu\nu})\psi(x)\\ &\quad + \bar{\psi}(x)(-\gamma_\nu\gamma_\mu + g_{\mu\nu})\partial^\mu\psi(x)\\ &= \bar{\psi}(x)\overleftarrow{\partial}^\mu\gamma_\mu\gamma_\nu\psi(x) - \bar{\psi}(x)\gamma_\nu\gamma_\mu\partial^\mu\psi(x)\\ &\quad - \bar{\psi}(x)\overleftarrow{\partial}_\nu\psi(x) + \bar{\psi}(x)\partial_\nu\psi(x)\\ &= 2mV_\nu(x) - \bar{\psi}(x)\overleftarrow{\partial}_\nu\psi(x) + \bar{\psi}(x)\partial_\nu\psi(x). \qquad (38)\end{aligned}$$

The first term on the right-hand side states a proportionality between the fourdivergence of the tensor current and the vector current, similar to those which appear in the case of axial and pseudoscalar ones. However, it is now immediately noted that the last two terms, having derivatives of the fermionic fields $\psi(x)$ and $\bar{\psi}(x)$, cannot be written in terms of the densities defined in Eq. (1). Without this ingredient, we are prevented from using the standard methods of current algebra in order to derive Ward identities for the Green functions associated with the fermionic tensor densities. Let us try to consider then the alternative option discussed in the preceding section.

B. Relations among Green functions

Given the impossibility of using the methods of current algebra in which case a general result, valid for any order of the perturbative calculations, may be obtained, we have no option than to adopt the procedure described previously, which can generate constraints for the amplitudes in a specific order. This limitation does not represent a trouble due to the fact that, in practical situations, we invariably have to consider a perturbative evaluation of the amplitudes, when the methods are completely equivalent, as we have shown. The calculations, as expected, are not so immediate as in the case of those situations involving vector or axial-vector currents as we shall see. In order to make the difficulties involved clear, let us consider a relatively simple three-point function belonging to the class of tensor Green functions, the $T^{SST}_{\mu\nu}$, which is defined as

$$\begin{aligned}T^{SST}_{\mu\nu} &= \int \frac{d^4l}{(2\pi)^4}\mathrm{Tr}\Big\{\hat{1}\frac{1}{[(\not{l}+\not{l}_3)-m]}\\ &\quad\times\hat{1}\frac{1}{[(\not{l}+\not{l}_1)-m]}\sigma_{\mu\nu}\frac{1}{[(\not{l}+\not{l}_2)-m]}\Big\}. \qquad (39)\end{aligned}$$

The external momentum associated with the vertex carrying the Lorentz indexes is $l_1 - l_2$. Taking the contraction with the above expression, we get

$$\begin{aligned}(l_1-l_2)^\mu T^{SST}_{\mu\nu} &= \int \frac{d^4l}{(2\pi)^4}\mathrm{Tr}\Big\{\hat{1}\frac{1}{[(\not{l}+\not{l}_3)-m]}\\ &\quad\times\hat{1}\frac{1}{[(\not{l}+\not{l}_1)-m]}(l_1-l_2)^\mu\sigma_{\mu\nu}\\ &\quad\times\frac{1}{[(\not{l}+\not{l}_2)-m]}\Big\}. \qquad (40)\end{aligned}$$

The result can be written in a more convenient form by using the identity

$$\begin{aligned}(l_1-l_2)^\mu\sigma_{\mu\nu} &= [(\not{l}+\not{l}_1)-m]\gamma_\nu + \gamma_\nu[(\not{l}+\not{l}_2)-m]\\ &\quad + 2m\gamma_\nu - [(l+l_1)_\nu + (l+l_2)_\nu], \qquad (41)\end{aligned}$$

which is

$$(l_1-l_2)^\mu T^{SST}_{\mu\nu} = T^{SV}_\nu(l_3,l_2) + T^{VS}_\nu(l_3,l_1) + 2mT^{SSV}_\nu - G^{SST}_\nu, \qquad (42)$$

where we have defined

$$\begin{aligned}G^{SST}_\nu &= \int\frac{d^4l}{(2\pi)^4}[(l+l_1)_\nu + (l+l_2)_\nu]\\ &\quad\times\mathrm{Tr}\Big\{\hat{1}\frac{1}{[(\not{l}+\not{l}_3)-m]}\hat{1}\frac{1}{[(\not{l}+\not{l}_1)-m]}\\ &\quad\times\hat{1}\frac{1}{[(\not{l}+\not{l}_2)-m]}\Big\}. \qquad (43)\end{aligned}$$

Clearly, G^{SST}_ν cannot be immediately identified with simple Green functions due to the presence of the term $(l + l_1)_\nu + (l + l_2)_\nu$. Actually it is not surprising to find such types of difficulties also in this approach due to the fact that the fourdivergence of the tensor current is not a simple reduction to other currents. In order to attempt to circumvent this trouble, we need to take the traces over the Dirac matrices, writing the above term as a combination of Feynman integrals to perhaps put the result so obtained as a combination of simple Green functions of the perturbative calculation or their external momenta contracted expressions.

V. RELATIONS AMONG GREEN FUNCTIONS HAVING TENSOR VERTEX OPERATORS

In the preceding discussions, only the difficulties related to the derivation of symmetry properties for amplitudes having tensor densities have been considered. From now on, we will be concerned with the main motivation of the present contribution, the derivation of the constraints which will play an analogous role to the one played by the Ward identities for conventional fermionic densities.

For this purpose, it is convenient to introduce a set of definitions which will be useful in next steps. First, consider the object

$$t^{ij}_{\mu\nu\pm} = 4[(l+l_i)_\mu(l+l_j)_\nu \pm (l+l_i)_\nu(l+l_j)_\mu], \quad (44)$$

where l_i are the adopted arbitrary internal lines momenta of the corresponding loops. In terms of the above-defined structures, we define other objects which will appear after the traces evaluation in the calculations of two-point functions. They are

$$(T_2)^{ij}_{\mu\nu\pm} = \int\frac{d^4l}{(2\pi)^4}\frac{t^{ij}_{\mu\nu\pm}}{[(l+l_i)^2-m^2][(l+l_j)^2-m^2]}. \quad (45)$$

Note that the tensor $(T_2)^{ij}_{\mu\nu+}$ $[(T_2)^{ij}_{\mu\nu-}]$ is symmetric (antisymmetric) by the interchanging in the indexes i, j as well as in the Lorentz indexes μ, ν. In addition, the tensor above is a specific combination of two propagator Feynman integrals as are all two-point functions. This observation indicates that the defined structures can be related to nontensor two-point functions. This means that if we find relations among the above-defined tensors and nontensor two-point functions we are constructing relations among the tensor amplitudes and the conventional ones which are very well known concerning their symmetry properties. This is precisely what we are searching for due to the fact that, if it is possible to write the tensor amplitudes and their external momenta contracted expressions in terms of nontensorial amplitudes, the next steps will be under control since the constraints on the evaluation of such amplitudes are perfectly well known, allowing us to evaluate the tensor amplitudes in a consistent way. Having this in mind, let us now consider the reduction of the tensors (45) and after this their external momenta contracted expressions to amplitudes. The tensors $(T_2)^{ij}_{\mu\nu\pm}$ have, in fact, simple reductions through identities to nontensor amplitudes. It is immediate to identify that, after the traces evaluation the following relations emerge:

$$(T_2)^{ij}_{\mu\nu+} = \tfrac{1}{2}[T^{VV}_{\mu\nu}(l_i,l_j) + T^{AA}_{\mu\nu}(l_i,l_j)] + \tfrac{1}{2}g_{\mu\nu}[T^{SS}(l_i,l_j) - T^{PP}(l_i,l_j)], \quad (46)$$

$$(T_2)^{ij}_{\mu\nu-} = -\frac{i}{2}\varepsilon_{\mu\nu\lambda\xi}(T^{AV})^{\lambda\xi}(l_i,l_j). \quad (47)$$

The identities (33)–(35), among others, can be used to relate the tensors on the left-hand side to other amplitudes.

Now we consider the reductions for the external momenta contracted expressions since they are crucial in the construction of relations among Green functions that we want to identify. For this purpose, consider first the contraction of the two-point function external momentum $l_i - l_j$ with the tensor $(T_2)^{ij}_{\mu\nu\pm}$, which gives

$$(l_i-l_j)^\mu(T_2)^{ij}_{\mu\nu+} = T^V_\nu(l_j) - T^V_\nu(l_i) - (l_i-l_j)_\nu T^{PP}(l_i,l_j). \quad (48)$$

On the other hand, we can also state that

$$(l_i-l_j)^\mu(T_2)^{ij}_{\mu\nu-} = -\frac{i}{2}\varepsilon_{\alpha\nu\lambda\xi}(l_i-l_j)^\alpha(T^{AV})^{\lambda\xi}(l_i,l_j). \quad (49)$$

Again the expressions above can be replaced with alternative, but equivalent, forms by using the identities (33)–(35).

Let us now state the external momenta contracted expressions for the tensor two-point functions. We first consider the TT two-point function. The contraction of the function $T^{TT}_{\mu\nu\alpha\beta}(l_1,l_2)$ with the momentum $(l_1-l_2)_\mu$, written initially as

$$(l_1-l_2)^\mu T^{TT}_{\mu\nu\alpha\beta}(l_1,l_2) = \int\frac{d^4l}{(2\pi)^4}\mathrm{Tr}\left\{(l_1-l_2)^\mu\sigma_{\mu\nu}\times\frac{1}{\not l+\not l_1-m}\sigma_{\alpha\beta}\frac{1}{\not l+\not l_2-m}\right\}, \quad (50)$$

can be modified if the identity

$$(l_1-l_2)^\mu\sigma_{\mu\nu} = -\gamma_\nu[\not l+\not l_1-m] - [\not l+\not l_2-m]\gamma_\nu - 2m\gamma_\nu + [(l+l_1)_\nu + (l+l_2)_\nu], \quad (51)$$

is considered. We then obtain

$$(l_1-l_2)^\mu T^{TT}_{\mu\nu\alpha\beta}(l_1,l_2) = -\int\frac{d^4l}{(2\pi)^4}\mathrm{Tr}\left\{\gamma_\nu\sigma_{\alpha\beta}\frac{1}{\not l+\not l_2-m}\right\} - \int\frac{d^4l}{(2\pi)^4}\mathrm{Tr}\left\{\frac{1}{\not l+\not l_1-m}\sigma_{\alpha\beta}\gamma_\nu\right\} - \int\frac{d^4l}{(2\pi)^4}2m\,\mathrm{Tr}\left\{\gamma_\nu\frac{1}{\not l+\not l_1-m}\sigma_{\alpha\beta}\frac{1}{\not l+\not l_2-m}\right\} + \int\frac{d^4l}{(2\pi)^4}[(l+l_1)_\nu + (l+l_2)_\nu]\mathrm{Tr}\left\{\frac{1}{\not l+\not l_1-m}\sigma_{\alpha\beta}\frac{1}{\not l+\not l_2-m}\right\}. \quad (52)$$

An additional and convenient change is allowed through the use of the properties

$$\sigma_{\alpha\beta}\gamma_\nu = -g_{\alpha\nu}\gamma_\nu + g_{\beta\nu}\gamma_\alpha + i\varepsilon_{\alpha\beta\nu\sigma}\gamma_5\gamma_\sigma,$$
$$\gamma_\nu\sigma_{\alpha\beta} = g_{\alpha\nu}\gamma_\nu - g_{\beta\nu}\gamma_\alpha + i\varepsilon_{\alpha\beta\nu\sigma}\gamma_5\gamma_\sigma.$$

Thus, it is possible to write

$$(l_1 - l_2)^\mu T^{TT}_{\mu\nu\alpha\beta}(l_1, l_2) = -g_{\alpha\nu}[T^V_\beta(l_2) - T^V_\beta(l_1)] + g_{\beta\nu}[T^V_\alpha(l_2) - T^V_\alpha(l_1)] - 2mT^{VT}_{\nu\alpha\beta}(l_1, l_2) + G^{TT}_{\nu\alpha\beta}(l_1, l_2), \quad (53)$$

where we have introduced the definition

$$G^{TT}_{\nu\alpha\beta}(l_1, l_2) = \int \frac{d^4l}{(2\pi)^4}[(l + l_1)_\nu + (l + l_2)_\nu] \times \mathrm{Tr}\left\{\frac{1}{\not{l} + \not{l}_1 - m}\sigma_{\alpha\beta}\frac{1}{\not{l} + \not{l}_2 - m}\right\}. \quad (54)$$

It is immediate to see that $G^{TT}_{\nu\alpha\beta}$ does not correspond, at least not directly, to any two-point function defined in (3). This situation reflects in our procedure the problem which we have found when the fourdivergence of the antisymmetric fermionic current $T(x) = \overline{\Psi}(x)\sigma_{\mu\nu}\Psi(x)$ has been considered. In order to circumvent this trouble, it becomes necessary to take an additional step, the evaluation of Dirac traces present in the expression for $G^{TT}_{\nu\alpha\beta}$. In the corresponding results it is possible to identify the tensors $(T_2)^{ij}_{\mu\nu\pm}$. The result for the involved traces can be written as

$$\mathrm{Tr}\left\{\frac{1}{\not{l} + \not{l}_1 - m}\sigma_{\alpha\beta}\frac{1}{\not{l} + \not{l}_2 - m}\right\} = (t_2)^{12}_{\alpha\beta-}\{[(l + l_1)^2 - m^2][(l + l_2)^2 - m^2]\}^{-1}. \quad (55)$$

Now we consider the identities

$$(l + l_1)_\nu(t_2)^{12}_{\alpha\beta-} = (l + l_1)_\nu[(l_1 - l_2)_\alpha(l + l_2)_\beta - (l_1 - l_2)_\beta(l + l_2)_\alpha], \quad (56)$$

$$(l + l_2)_\nu(t_2)^{12}_{\alpha\beta-} = (l + l_2)_\nu[(l_1 - l_2)_\alpha(l + l_1)_\beta - (l_1 - l_2)_\beta(l + l_1)_\alpha], \quad (57)$$

and substitute them in expression (54) to get

$$G^{TT}_{\nu\alpha\beta} = (l_1 - l_2)_\alpha(T_2)^{12}_{\nu\beta+} - (l_1 - l_2)_\beta(T_2)^{12}_{\nu\alpha+}. \quad (58)$$

Since the tensors $(T_2)^{ij}_{\mu\nu\pm}$ can be written in terms of amplitudes, according to expressions (46) and (47), we can say that the relation among Green functions which we were looking for is already constructed. We write it in the form

$$(l_1 - l_2)^\mu T^{TT}_{\mu\nu\alpha\beta}(l_1, l_2) = -g_{\alpha\nu}[T^V_\beta(l_2) - T^V_\beta(l_1) + (l_1 - l_2)_\beta T^{SS}(l_1, l_2)] + g_{\beta\nu}[T^V_\alpha(l_2) - T^V_\alpha(l_1) + (l_1 - l_2)_\alpha T^{SS}(l_1, l_2)] + (l_1 - l_2)_\alpha T^{AA}_{\nu\beta}(l_1, l_2) - (l_1 - l_2)_\beta T^{AA}_{\nu\alpha}(l_1, l_2) - 2mT^{VT}_{\nu\alpha\beta}(l_1, l_2). \quad (59)$$

An alternative form can be obtained by using the relation (35)

$$(l_1 - l_2)^\mu T^{TT}_{\mu\nu\alpha\beta}(l_1, l_2) = -g_{\alpha\nu}[T^V_\beta(l_2) - T^V_\beta(l_1) - (l_1 - l_2)_\beta T^{PP}(l_1, l_2)] + g_{\beta\nu}[T^V_\alpha(l_2) - T^V_\alpha(l_1) - (l_1 - l_2)_\alpha T^{PP}(l_1, l_2)] + (l_1 - l_2)_\alpha T^{VV}_{\nu\beta}(l_1, l_2) - (l_1 - l_2)_\beta T^{VV}_{\nu\alpha}(l_1, l_2) - 2mT^{VT}_{\nu\alpha\beta}(l_1, l_2). \quad (60)$$

All the steps performed involve only identities at the traces level and therefore no relevant role has been played by the divergent character of the amplitudes. At this point we can ask ourselves: What does the relation above mean? Only one interpretation can be given. When the TT two-point function is explicitly evaluated, as well as all the one and two-point functions present on the right-hand side of the equation, and the contraction with the external momentum is taken, it must be possible to identify in the obtained expression the combination of amplitudes which appeared on the right-hand side, in spite of the divergences involved. This means that identity (59) must be preserved by the calculated expressions through a consistent regularization strategy adopted to handle the divergences.

Given the fact that another three Lorentz indexes are present in the TT amplitude, it is also possible to state constraints on successive contractions with the external momentum. It is immediate to note that when these contractions involve both Lorentz indexes of a tensor operator the result must vanish identically

$$(l_1 - l_2)^\alpha(l_1 - l_2)^\beta T^{TT}_{\mu\nu\alpha\beta}(l_1, l_2) = 0. \quad (61)$$

This is due to the property $(l_1 - l_2)^\alpha(l_1 - l_2)^\beta\sigma_{\alpha\beta} = 0$. This property states additional constraints on the consistent evaluation of TT two-point functions. Although the requirements given by Eqs. (59)–(61) seem to be obvious at this point, the divergent character fo the amplitudes makes this property far from being trivial.

Following strictly the same procedure, the properties for the remaining tensor two-point functions can be stated. For the TS amplitude,

$$(l_1 - l_2)^\mu T^{TS}_{\mu\nu}(l_1, l_2) = \frac{1}{2m}\{(l_1 - l_2)^2 T^{VS}_\nu(l_1, l_2) - (l_1 - l_2)_\nu[T^S(l_2) - T^S(l_1)]\}, \quad (62)$$

or, equivalently, given the identity (33),

$$(l_1 - l_2)^\mu T^{TS}_{\mu\nu}(l_1, l_2) = -\frac{i}{2}\varepsilon_{\alpha\nu\lambda\xi}(l_1 - l_2)^\alpha (T^{AV})^{\lambda\xi}(l_1, l_2). \tag{63}$$

Now, for the TP Green function we get

$$(l_1 - l_2)^\mu T^{TP}_{\mu\nu}(l_1, l_2) = -2mT^{VP}_\nu(l_1, l_2). \tag{64}$$

In the case of TV amplitude, we have two types of relations. For the vector index we get

$$(l_1 - l_2)^\alpha T^{TV}_{\mu\nu\alpha}(l_1, l_2) = 0, \tag{65}$$

as it should be expected. On the other hand, we obtain

$$\begin{aligned}(l_1 - l_2)^\mu T^{TV}_{\mu\nu\alpha}(l_1, l_2) = \frac{1}{2m}(l_1 - l_2)^2[T^{VV}_{\alpha\nu}(l_1, l_2) \\ -T^{AA}_{\alpha\nu}(l_1, l_2)] \\ -(l_1 - l_2)_\nu T^{PA}_\alpha(l_1, l_2),\end{aligned} \tag{66}$$

due to the identity (35). Finally, we consider the TA function. For the axial index we get

$$(l_1 - l_2)^\alpha T^{TA}_{\mu\nu\alpha}(l_1, l_2) = 2mT^{TP}_{\mu\nu}(l_1, l_2). \tag{67}$$

On the other hand, we get

$$(l_1 - l_2)^\mu T^{TA}_{\mu\nu\alpha}(l_1, l_2) = -2mT^{VA}_{\nu\alpha}(l_1, l_2), \tag{68}$$

which is, due to (33), equivalent to

$$(l_1 - l_2)^\mu T^{TA}_{\mu\nu\alpha}(l_1, l_2) = (-i)\varepsilon_{\nu\alpha\lambda\xi}(l_1 - l_2)^\lambda (T^{SV})^\xi(l_1, l_2), \tag{69}$$

for the tensor index.

At this point some comments are in order. The obtained results for the amplitudes as well as for the external momentum contracted expressions are very general. Only identities have been used to identify the relations stated. We have made at this point no calculation for any divergent integral involved. This means that the results are not compromised with a regularization or similar. It is also interesting to note that in the contracted expressions we have included those with the vector and axial-vector indexes. The expected properties, the identically zero value and the proportionality with the pseudoscalar, respectively, are evident, showing the consistency of the procedure. The next step in our investigation is the explicit evaluation of the amplitudes and after this verify if it is possible to preserve in the calculated expressions the constraints which we have derived. For this purpose, it becomes necessary to adopt a prescription to handle the divergences in Feynman integrals. We will adopt a very general strategy which is briefly described in the next section.

VI. THE CALCULATIONAL METHOD TO HANDLE DIVERGENT INTEGRALS

To explicitly evaluate the involved divergent integrals, we adopt an alternative strategy to handle the divergences [17]. Rather than the specification of some regularization, to justify all the necessary manipulations, we will assume the presence of a regulating distribution only in an implicit way. Schematically,

$$\begin{aligned}\int \frac{d^4k}{(2\pi)^4} f(k) \to \int \frac{d^4k}{(2\pi)^4} f(k)\left\{\lim_{\Lambda_i^2\to\infty} G_{\Lambda_i}(k^2, \Lambda_i^2)\right\} \\ = \int_\Lambda \frac{d^4k}{(2\pi)^4} f(k).\end{aligned} \tag{70}$$

Here $\Lambda_i' s$ are parameters of the generic distribution $G(k^2, \Lambda_i^2)$ that, in addition to the obvious finiteness character of the modified integral, should have two other very general properties. It must be even in the integrating momentum k, due to Lorentz invariance maintenance, as well as a well-defined connection limit must exist; i.e.,

$$\lim_{\Lambda_i^2\to\infty} G_{\Lambda_i}(k^2, \Lambda_i^2) = 1. \tag{71}$$

The first property implies that all odd integrals vanish. The second one guarantees, in particular, that the value of finite integrals in the amplitudes will not be modified. Having this in mind, we manipulate the integrand of the divergent integrals to generate a mathematical expression where all the divergences are located in internal momenta independent structures. This goal can be achieved by using an adequate identity such as

$$\begin{aligned}\frac{1}{[(k + k_i)^2 - m^2]} = \sum_{j=0}^{N} \frac{(-1)^j(k_i^2 + 2k_i\cdot k)^j}{(k^2 - m^2)^{j+1}} \\ + \frac{(-1)^{N+1}(k_i^2 + 2k_i\cdot k)^{N+1}}{(k^2 - m^2)^{N+1}[(k + k_i)^2 - m^2]},\end{aligned} \tag{72}$$

where k_i is (in principle) an arbitrary routing to a loop internal line momentum. The value for N should be adequately chosen. The minor value should be the one that leads the last term in the above expression to be present in a finite integral, and therefore, by virtue of the well-defined connection limit assumptions, the corresponding integration can be performed without restrictions and free from the specific effects of an eventual regularization. All the remaining structures become independent of the internal lines momenta. We then eliminate all the integrals with odd integrand, as a trivial consequence of the even character of the implicit regulating distribution. In the divergent structures obtained this way no additional assumptions are made. They are organized in five objects, namely,

$$\Box_{\alpha\beta\mu\nu} = \int_\Lambda \frac{d^4k}{(2\pi)^4} \frac{24k_\mu k_\nu k_\alpha k_\beta}{(k^2-m^2)^4} - g_{\alpha\beta}\int_\Lambda \frac{d^4k}{(2\pi)^4} \times \frac{4k_\mu k_\nu}{(k^2-m^2)^3} - g_{\alpha\nu}\int_\Lambda \frac{d^4k}{(2\pi)^4}\frac{4k_\beta k_\mu}{(k^2-m^2)^3} - g_{\alpha\mu}\int_\Lambda \frac{d^4k}{(2\pi)^4}\frac{4k_\beta k_\nu}{(k^2-m^2)^3}, \tag{73}$$

$$\Delta_{\mu\nu} = \int_\Lambda \frac{d^4k}{(2\pi)^4}\frac{4k_\mu k_\nu}{(k^2-m^2)^3} - \int_\Lambda \frac{d^4k}{(2\pi)^4}\frac{g_{\mu\nu}}{(k^2-m^2)^2}, \tag{74}$$

$$\nabla_{\mu\nu} = \int_\Lambda \frac{d^4k}{(2\pi)^4}\frac{2k_\nu k_\mu}{(k^2-m^2)^2} - \int_\Lambda \frac{d^4k}{(2\pi)^4}\frac{g_{\mu\nu}}{(k^2-m^2)}, \tag{75}$$

$$I_{\log}(m^2) = \int_\Lambda \frac{d^4k}{(2\pi)^4}\frac{1}{(k^2-m^2)^2}, \tag{76}$$

$$I_{\rm quad}(m^2) = \int_\Lambda \frac{d^4k}{(2\pi)^4}\frac{1}{(k^2-m^2)}. \tag{77}$$

This systematization is sufficient for discussions in fundamental theories at the one-loop level. In nonrenormalizable ones, new objects can be defined following this philosophy. In the two (or more) loop levels of calculations, new basic divergent structures can be equally defined in a completely analogous way. The main point is to avoid the explicit evaluation of such divergent structures, in which case a regulating distribution needs to be specified.

The divergent integrals which are necessary for the present calculations are

$$(I_1; I_1^\mu) = \int \frac{d^4k}{(2\pi)^4}\frac{(1; k^\mu)}{[(k+k_1)^2-m^2]}, \tag{78}$$

$$(I_2; I_2^\mu; I_2^{\mu\nu}) = \int \frac{d^4k}{(2\pi)^4} \times \frac{(1; k^\mu; k^\mu k^\nu)}{[(k+k_1)^2-m^2][(k+k_2)^2-m^2]}. \tag{79}$$

The results are

$$(I_1) = [I_{\rm quad}(m^2)] + k_1^\alpha k_1^\beta[\Delta_{\alpha\beta}], \tag{80}$$

$$(I_1)_\mu = -k_{1\mu}[I_{\rm quad}(m^2)] - k_1^\beta[\nabla_{\beta\mu}] - \tfrac{1}{3}k_1^\beta k_1^\alpha k_1^\nu[\Box_{\alpha\beta\mu\nu}] - \tfrac{1}{3}k_{1\mu}k_1^\alpha k_1^\beta[\Delta_{\alpha\beta}] + \tfrac{1}{3}k_1^2 k_1^\alpha[\Delta_{\alpha\mu}], \tag{81}$$

$$(I_2) = [I_{\log}(m^2)] - \left(\frac{i}{(4\pi)^2}\right)\{Z_0[(k_1-k_2)^2; m^2]\}, \tag{82}$$

$$(I_2)_\mu = -\tfrac{1}{2}(k_1+k_2)^\alpha[\Delta_{\alpha\mu}] - \tfrac{1}{2}(k_1+k_2)_\mu(I_2), \tag{83}$$

$$\begin{aligned}(I_2)_{\mu\nu} =& \frac{1}{2}[\nabla_{\mu\nu}] + \frac{1}{6}(k_2^\alpha k_2^\beta + k_1^\alpha k_2^\beta + k_1^\alpha k_1^\beta)[\Box_{\alpha\beta\mu\nu}] - \frac{1}{12}(k_1-k_2)^2[\Delta_{\mu\nu}] + \frac{1}{6}(k_{2\nu}k_2^\beta + k_{1\nu}k_2^\beta + k_{1\nu}k_1^\beta)[\Delta_{\beta\mu}] \\ &+ \frac{1}{6}(k_{2\mu}k_2^\beta + k_{1\mu}k_2^\beta + k_{1\mu}k_1^\beta)[\Delta_{\beta\nu}] + \frac{1}{2}g_{\mu\nu}[I_{\rm quad}(m^2)] - \frac{1}{12}g_{\mu\nu}(k_1-k_2)^2[I_{\log}(m^2)] + \frac{1}{6}(2k_{2\nu}k_{2\mu} + k_{1\nu}k_{2\mu} \\ &+ k_{1\mu}k_{2\nu} + 2k_{1\nu}k_{1\mu})[I_{\log}(m^2)] + \left(\frac{i}{(4\pi)^2}\right)[(k_1-k_2)_\mu(k_1-k_2)_\nu - g_{\mu\nu}(k_1-k_2)^2] \times \Big(-Z_2[(k_1-k_2)^2, m^2] \\ &+ \frac{1}{4}Z_0[(k_1-k_2)^2, m^2]\Big) - \frac{1}{4}(k_1+k_2)_\mu(k_1+k_2)_\nu\left(\frac{i}{(4\pi)^2}\right)\{Z_0[(k_1-k_2)^2, m^2]\}.\end{aligned} \tag{84}$$

We have introduced the two-point function structures [17]

$$Z_k(p^2; m^2) = \int_0^1 dz\, z^k \ln\left(\frac{p^2 z(1-z) - m^2}{-m^2}\right). \tag{85}$$

The integration could be easily performed, but for our present purposes this is not necessary.

At this point it is important to emphasize the general aspects of this method. No shifts have been performed and, in fact, no divergent integrals have been calculated. All final results produced by this approach can be mapped into those of any specific technique. The finite parts are the same as they should be by physical reasons. The divergent parts can be easily obtained. All we need is to evaluate the remaining divergent structures. By virtue of this general character, the present strategy can be used simply to systematize the procedures, even if one wants to use traditional techniques. Those parts that depend on a specific regularization method are naturally separated allowing us to analyze such dependence in a particular problem, which is very interesting. Let us now use the above obtained result to calculate physical amplitudes.

VII. EXPLICIT EVALUATION OF GREEN FUNCTIONS

In Sec. V, we obtained relations to be satisfied by the calculated expressions for the tensor two-point functions. In the preceding section, we described a strategy to handle the divergent Feynman integrals. We now use the results for the divergent integrals presented in the previous section to solve explicitly the considered Green functions. It is interesting to start with the expressions for the nontensor Green functions. Since all the tensor amplitudes have been written in terms of the conventional ones, the evaluation of such structures is enough also to complete the evaluation of tensor Green functions. The conventional Green functions have been evaluated in detail within the point of view of the adopted strategy in Ref. [18]. We only quote their expressions.

(I) One-point functions:

$$T^S(l_1) = 4m\{[I_{\rm quad}(m^2)] + l_1^\beta l_1^\alpha[\Delta_{\beta\alpha}]\}, \quad (86)$$

$$T^V_\mu(l_1) = 4\{-l_1^\beta[\nabla_{\beta\mu}] - \tfrac{1}{3}l_1^\beta l_1^\alpha l_1^\nu[\Box_{\alpha\beta\mu\nu}] + \tfrac{1}{3}l_1^2 l_1^\nu[\Delta_{\mu\nu}] + \tfrac{2}{3}l_{1\mu}l_1^\alpha l_1^\beta[\Delta_{\alpha\beta}]\}. \quad (87)$$

(II) Two-point functions:

$$T^{SS}(l_1,l_2) = 4\bigg\{[I_{\rm quad}(m^2)] + \frac{1}{2}[4m^2-(l_1-l_2)^2] \times[I_{\rm log}(m^2)] - \frac{1}{2}[4m^2-(l_1-l_2)^2]\bigg(\frac{i}{(4\pi)^2}\bigg) \times\{Z_0[(l_1-l_2)^2,m^2]\}\bigg\} + (l_1-l_2)^\alpha(l_1-l_2)^\beta \times[\Delta_{\alpha\beta}] + (l_1+l_2)^\alpha(l_1+l_2)^\beta[\Delta_{\alpha\beta}], \quad (88)$$

$$T^{PP}(l_1,l_2) = 4\bigg\{-[I_{\rm quad}(m^2)] + \frac{1}{2}(l_1-l_2)^2[I_{\rm log}(m^2)] - \frac{1}{2}(l_1-l_2)^2\bigg(\frac{i}{(4\pi)^2}\bigg)\{Z_0[(l_1-l_2)^2,m^2]\}\bigg\} -(l_1-l_2)^\alpha(l_1-l_2)^\beta[\Delta_{\alpha\beta}] -(l_1+l_2)^\alpha(l_1+l_2)^\beta[\Delta_{\alpha\beta}], \quad (89)$$

$$T^{PA}_\mu(l_1,l_2) = 4m(l_1-l_2)_\mu\bigg\{[I_{\rm log}(m^2)] -\bigg(\frac{i}{(4\pi)^2}\bigg)\{Z_0[(l_1-l_2)^2,m^2]\}\bigg\}, \quad (90)$$

$$T^{VS}_\mu(l_1,l_2) = -4m(l_1+l_2)^\xi[\Delta_{\xi\mu}], \quad (91)$$

$$T^{AV}_{\mu\nu}(l_1,l_2) = -2i\varepsilon_{\mu\nu\alpha\beta}(l_2-l_1)^\beta(l_1+l_2)^\xi[\Delta^\alpha_\xi], \quad (92)$$

$$T^{VV}_{\mu\nu}(l_1,l_2) = \frac{4}{3}[(l_1-l_2)^2 g_{\mu\nu} - (l_1-l_2)_\mu(l_1-l_2)_\nu] \times\bigg\{[I_{\rm log}(m^2)] - \bigg(\frac{i}{(4\pi)^2}\bigg) \times\bigg[\frac{1}{3} + \frac{[2m^2+(l_1-l_2)^2]}{(l_1-l_2)^2} \times\{Z_0[(l_1-l_2)^2,m^2]\}\bigg]\bigg\} + A_{\mu\nu}, \quad (93)$$

$$T^{AA}_{\mu\nu}(l_1,l_2) = \frac{4}{3}[(l_1-l_2)^2 g_{\mu\nu} - (l_1-l_2)_\mu(l_1-l_2)_\nu] \times\bigg\{[I_{\rm log}(m^2)] - \bigg(\frac{i}{(4\pi)^2}\bigg)\bigg[\frac{1}{3} + \frac{[2m^2+(l_1-l_2)^2]}{(l_1-l_2)^2} \times\{Z_0[(l_1-l_2)^2,m^2]\}\bigg]\bigg\} - 8m^2 g_{\mu\nu} \times\bigg\{[I_{\rm log}(m^2)] - \bigg(\frac{i}{(4\pi)^2}\bigg)\{Z_0[(l_1-l_2)^2,m^2]\}\bigg\} +A_{\mu\nu}. \quad (94)$$

where, in the last two expressions we have defined

$$A_{\mu\nu} = 4[\nabla_{\mu\nu}] + (l_1-l_2)^\alpha(l_1-l_2)^\beta[\tfrac{1}{3}\Box_{\alpha\beta\mu\nu} + \tfrac{1}{3}g_{\alpha\nu}\Delta_{\mu\beta} + g_{\alpha\mu}\Delta_{\beta\nu} - g_{\mu\nu}\Delta_{\alpha\beta} - \tfrac{2}{3}g_{\alpha\beta}\Delta_{\mu\nu}] +[(l_1-l_2)^\alpha(l_1+l_2)^\beta - (l_1+l_2)^\alpha(l_1-l_2)^\beta] \times[\tfrac{1}{3}\Box_{\alpha\beta\mu\nu} + \tfrac{1}{3}g_{\nu\alpha}\Delta_{\mu\beta} + \tfrac{1}{3}g_{\alpha\mu}\Delta_{\beta\nu}] + (l_1+l_2)^\alpha \times(l_1+l_2)^\beta[\Box_{\alpha\beta\mu\nu} - g_{\mu\beta}\Delta_{\nu\alpha} - g_{\alpha\mu}\Delta_{\beta\nu} -3g_{\mu\nu}\Delta_{\alpha\beta}]. \quad (95)$$

Since the tensor amplitudes have been written in terms of the tensors $(T_2)^{ij}_{\mu\nu\pm}$ it is interesting to write first the expression for these objects. For this purpose it is only necessary to substitute the expressions for the Feynman integrals (80)–(84). The result is

$$(T_2)^{ij}_{\mu\nu+} = \frac{4}{3}[(l_i-l_j)^2 g_{\mu\nu} - (l_i-l_j)_\mu(l_i-l_j)_\nu] \times\bigg\{I_{\rm log}(m^2) - \bigg(\frac{i}{(4\pi)^2}\bigg)\bigg[\frac{1}{3} + \frac{2m^2+(l_i-l_j)^2}{(l_i-l_j)^2} \times\{Z_0[m^2,(l_i-l_j)^2]\}\bigg]\bigg\} - g_{\mu\nu}T^{PP}(l_1,l_2) + A_{\mu\nu}, \quad (96)$$

$$(T_2)^{ij}_{\mu\nu-} = -2\{(l_i-l_j)_\mu(l_i+l_j)^\xi[\Delta_{\xi\nu}] -(l_i-l_j)_\nu(l_i+l_j)^\xi[\Delta_{\xi\mu}]\}, \quad (97)$$

Now we are at the position to consider the evaluation of the tensor Green functions. First we consider the *ST* two-point function, which means, in definition (3), take $\Gamma_i = 1$ and $\Gamma_j = \sigma_{\mu\nu}$,

$$T^{ST}_{\mu\nu}(l_1, l_2) = \int \frac{d^4 l}{(2\pi)^4} \mathrm{Tr}\left\{\hat{1} \frac{1}{\not{l} + \not{l}_1 - m} \sigma_{\mu\nu} \frac{1}{\not{l} + \not{l}_2 - m}\right\} = (T_2)^{12}_{\mu\nu -} = -2\{(l_1 - l_2)_\mu (l_1 + l_2)^\xi [\Delta_{\xi\nu}] - (l_1 - l_2)_\nu (l_1 + l_2)^\xi [\Delta_{\xi\mu}]\}. \quad (98)$$

Next, we calculate the PT amplitude

$$T^{PT}_{\mu\nu}(l_1, l_2) = \int \frac{d^4 l}{(2\pi)^4} \mathrm{Tr}\left\{\gamma_5 \frac{1}{\not{l} + \not{l}_1 - m} \sigma_{\mu\nu} \frac{1}{\not{l} + \not{l}_2 - m}\right\} = \frac{i}{2} \varepsilon_{\mu\nu\alpha\beta} (T_2)^{12}_{\alpha\beta -} = (-i) 2 \varepsilon_{\mu\nu\alpha\beta} (l_1 - l_2)^\alpha (l_1 + l_2)^\xi [\Delta^\beta_\xi], \quad (99)$$

where Eq. (97) has been used also. On the other hand, we get for the AT function

$$T^{AT}_{\alpha\mu\nu}(l_1, l_2) = \int \frac{d^4 l}{(2\pi)^4} \mathrm{Tr}\left\{\gamma_\alpha \gamma_5 \frac{1}{\not{l} + \not{l}_1 - m} \sigma_{\mu\nu} \frac{1}{\not{l} + \not{l}_2 - m}\right\} = 4im \varepsilon_{\mu\nu\alpha\beta} (l_1 + l_2)^\xi [\Delta^\beta_\xi]. \quad (100)$$

In a completely similar way, we get

$$T^{VT}_{\alpha\mu\nu}(l_1, l_2) = \int \frac{d^4 l}{(2\pi)^4} \mathrm{Tr}\left\{\gamma_\alpha \frac{1}{\not{l} + \not{l}_1 - m} \sigma_{\mu\nu} \frac{1}{\not{l} + \not{l}_2 - m}\right\} = 4m (g_{\alpha\nu} g_{\mu\lambda} - g_{\alpha\mu} g_{\nu\lambda})(l_1 - l_2)^\lambda \left\{[I_{\log}(m^2)] - \left(\frac{i}{(4\pi)^2}\right)\{Z_0[(l_1 - l_2)^2, m^2]\}\right\}, \quad (101)$$

and also

$$T^{TT}_{\alpha\beta\mu\nu}(l_1, l_2) = \int \frac{d^4 l}{(2\pi)^4} \mathrm{Tr}\left\{\sigma_{\alpha\beta} \frac{1}{\not{l} + \not{l}_1 - m} \sigma_{\mu\nu} \frac{1}{\not{l} + \not{l}_2 - m}\right\} = g_{\alpha\mu} (T_2)^{12}_{\beta\nu +} - g_{\alpha\nu} (T_2)^{12}_{\beta\mu +} + g_{\beta\nu} (T_2)^{12}_{\alpha\mu +} - g_{\beta\mu} (T_2)^{12}_{\alpha\nu +} + (g_{\alpha\nu} g_{\beta\mu} - g_{\alpha\mu} g_{\beta\nu}) \times T^{SS}(l_1, l_2). \quad (102)$$

The last equation can be written in an explicit form by using Eqs. (96) and (88)

$$T^{TT}_{\alpha\beta\mu\nu}(l_1, l_2) = (g_{\alpha\mu} g_{\beta\xi} g_{\nu\lambda} - g_{\alpha\nu} g_{\beta\xi} g_{\mu\lambda} + g_{\beta\nu} g_{\alpha\xi} g_{\mu\lambda} - g_{\beta\mu} g_{\alpha\xi} g_{\nu\lambda}) \frac{4}{3}[(l_1 - l_2)^2 g^{\xi\lambda} - (l_1 - l_2)^\xi (l_1 - l_2)^\lambda] \times \left\{I_{\log}(m^2) - \left(\frac{i}{(4\pi)^2}\right)\left[\frac{1}{3} + \frac{2m^2 + (l_1 - l_2)^2}{(l_1 - l_2)^2}\{Z_0[m^2, (l_1 - l_2)^2]\}\right]\right\} + 4(g_{\alpha\nu} g_{\beta\mu} - g_{\alpha\mu} g_{\beta\nu}) \times \left\{-[I_{\rm quad}(m^2)] + \frac{1}{2}[4m^2 + (l_1 - l_2)^2][I_{\log}(m^2)] - \frac{1}{2}[4m^2 + (l_1 - l_2)^2]\left(\frac{i}{(4\pi)^2}\right)\{Z_0[(l_1 - l_2)^2, m^2]\} - \frac{1}{4}(l_1 - l_2)^\lambda (l_1 - l_2)^\xi [\Delta_{\lambda\xi}] - \frac{1}{4}(l_1 + l_2)^\lambda (l_1 + l_2)^\xi [\Delta_{\lambda\xi}]\right\} + g_{\alpha\mu} A_{\beta\nu} - g_{\alpha\nu} A_{\beta\mu} + g_{\beta\nu} A_{\alpha\mu} - g_{\beta\mu} A_{\alpha\nu}. \quad (103)$$

which completes the calculations.

VIII. ARBITRARINESS AND RELATION AMONG GREEN FUNCTIONS

In Sec. V we stated relations between the external momentum contracted expressions of tensor amplitudes and themselves and conventional ones. The referred relations are at the one-loop level deeply related to the "Ward identities" for tensor amplitudes. The identities are constructed only at the level of the integrand which means that no specific aspects related to the divergent character have been used. If a certain method is adopted to evaluate all the structures involved, all the derived relations must be preserved simultaneously. On the other hand, in the previous section we evaluated all the one- and two-point functions defined in Eqs. (2) and (3). In the corresponding expressions all the intrinsic arbitrariness, which is the choice of the internal lines momenta and the choice for the regularization, are still preserved. In order to show the general consistency of the employed method for the manipulations and calculations performed, we will verify if the relations among Green functions are preserved before any assumption about the arbitrariness involved.

Let us start by the VT amplitude. First note that the expression (101) can be organized as

$$T^{VT}_{\alpha\mu\nu}(l_1, l_2) = \frac{1}{2m}(l_1 - l_2)_\mu [T^{VV}_{\alpha\nu}(l_1, l_2) - T^{AA}_{\alpha\nu}(l_1, l_2)] - \frac{1}{2m}(l_1 - l_2)_\nu [T^{VV}_{\alpha\mu}(l_1, l_2) - T^{AA}_{\alpha\mu}(l_1, l_2)], \quad (104)$$

where Eqs. (93) and (94) have been used. It is now clear conditions (65) and (66) are satisfied. Next we consider the TT Green function, given by expression (103). After some algebraic effort we note that this expression can be put into the form

$$T^{TT}_{\alpha\beta\mu\nu}(l_1, l_2) = g_{\alpha\mu} T^{AA}_{\beta\nu}(l_1, l_2) - g_{\alpha\nu} T^{AA}_{\beta\mu}(l_1, l_2) + g_{\beta\nu} T^{AA}_{\alpha\mu}(l_1, l_2) - g_{\beta\mu} T^{AA}_{\alpha\nu}(l_1, l_2) + (g_{\alpha\mu} g_{\beta\nu} - g_{\alpha\nu} g_{\beta\mu}) T^{SS}(l_1, l_2), \quad (105)$$

where Eqs. (88) and (94) have been used. It is evident that the contraction with the external momentum gives us Eq. (59). Taking now the ST amplitude, Eq. (98), we first note that, by using (91) and (47), it can be identified as

$$T^{ST}_{\mu\nu}(l_1, l_2) = \frac{1}{2m}[(l_1 - l_2)_\mu T^{SV}_\nu(l_1, l_2) - (l_1 - l_2)_\nu T^{SV}_\mu(l_1, l_2)] \tag{106}$$

$$= -\frac{i}{2}\varepsilon_{\mu\nu\lambda\xi}(T^{AV})^{\lambda\xi}(l_1, l_2). \tag{107}$$

The constraint identified for the contraction with the external momentum

$$(l_1 - l_2)^\mu T^{TS}_{\mu\nu}(l_1, l_2) = \frac{1}{2m}\{(l_1 - l_2)^2 T^{SV}_\nu(l_1, l_2) - (l_1 - l_2)_\nu[T^S(l_2) - T^S(l_1)]\} = -\frac{i}{2}\varepsilon_{\mu\nu\lambda\xi}(l_1 - l_2)^\mu(T^{AV})^{\lambda\xi}(l_1, l_2), \tag{108}$$

is automatically preserved. Now, for the TP Green function we get first the identification

$$T^{PT}_{\mu\nu}(l_1, l_2) = T^{VA}_{\mu\nu}(l_1, l_2), \tag{109}$$

which means that

$$(l_1 - l_2)^\mu T^{TP}_{\mu\nu}(l_1, l_2) = -2m T^{VP}_\nu(l_1, l_2), \tag{110}$$

as it should be. For the AT amplitude, given by expression (100), we first note that the identity

$$T^{AT}_{\alpha\mu\nu}(l_1, l_2) = -i\varepsilon_{\mu\nu\alpha\beta}(T^{SV})^\beta(l_1, l_2) \tag{111}$$

can be identified. It becomes obvious that

$$(l_1 - l_2)^\alpha T^{TA}_{\mu\nu\alpha}(l_1, l_2) = 2m T^{TP}_{\mu\nu}(l_1, l_2), \tag{112}$$

$$(l_1 - l_2)^\mu T^{TA}_{\mu\nu\alpha}(l_1, l_2) = -2m T^{VA}_{\nu\alpha}(l_1, l_2) = (-i)\varepsilon_{\nu\alpha\lambda\xi}(l_1 - l_2)^\lambda(T^{SV})^\xi(l_1, l_2). \tag{113}$$

For the sake of completeness, it is interesting to note that in the functions having the "dual" vertex $\tilde{T} = \sigma_{\mu\nu}\gamma_5$, after the explicit evaluation of amplitudes it is possible to identify relations with other amplitudes similar to those presented above. We can write, for example,

$$T^{\tilde{T}\tilde{T}}_{\mu\nu\alpha\beta}(l_1, l_2) = -g_{\alpha\mu}T^{VV}_{\nu\beta}(l_1, l_2) + g_{\alpha\nu}T^{VV}_{\mu\beta}(l_1, l_2) - g_{\beta\nu}T^{VV}_{\mu\alpha}(l_1, l_2) + g_{\beta\mu}T^{VV}_{\nu\alpha}(l_1, l_2) + (g_{\alpha\mu}g_{\beta\nu} - g_{\alpha\nu}g_{\beta\mu})T^{PP}(l_1, l_2), \tag{114}$$

$$T^{\tilde{T}S}_{\mu\nu}(l_1, l_2) = T^{AV}_{\mu\nu}(l_1, l_2), \tag{115}$$

$$T^{\tilde{T}A}_{\mu\nu\alpha}(l_1, l_2) = -g_{\nu\alpha}T^{SV}_\mu(l_1, l_2) + g_{\mu\alpha}T^{SV}_\nu(l_1, l_2), \tag{116}$$

$$T^{P\tilde{T}}_{\mu\nu}(l_1, l_2) = T^{TS}_{\mu\nu}(l_1, l_2), \tag{117}$$

$$T^{\tilde{T}V}_{\mu\nu\alpha}(l_1, l_2) = i\varepsilon_{\mu\nu\alpha\beta}(T^{AP})^\beta(l_1, l_2). \tag{118}$$

These relations simplify the analyses of the correspondent symmetry relations.

The analysis presented in this section is useful to show the consistency of the strategy we have adopted. All the identities among the considered Green functions are preserved before any assumption about the internal lines momenta as well as about a regularization. We must now consider the requirements we have to impose in order to get the desirable consistency with symmetry requirements like gauge invariance.

IX. AMBIGUITIES AND CONSISTENCY IN REGULARIZATION

Note that, in the preceding expressions for the one- and two-point functions, we have not assumed any specific consequences of a regularization. All the obtained expressions, at this stage, must obey all the corresponding relations among Green functions. In fact, it is a simple matter to verify that this is true. However, the results written above are certainly not consistent with the required symmetry properties for the amplitudes which are directly stated by the current algebra methods. The vector currents are not conserved at this stage and the proportionalities between the axial and the pseudoscalar current are not those expected. We have to find a consistent regularization. This search must be guided by physical reasons or symmetry impositions on some amplitudes based on general grounds. As an example of such constraints, take the vector one-point function. The Furry's theorem states that this amplitude must be identically zero. This result can be achieved only if the adopted regularization, which until this point is maintained only implicitly, possesses the properties

$$\Box^{\rm reg}_{\alpha\beta\mu\nu} = \nabla^{\rm reg}_{\mu\nu} = \Delta^{\rm reg}_{\mu\nu} = 0, \tag{119}$$

which we call consistency conditions. Note that the above conditions can be adopted as a definition for a regularization. They are automatically satisfied in the dimensional regularization [19]. Given this assumption, the Ward identities for the two-point functions are all automatically fulfilled. It is possible to verify that, as a consequence of our strategy to handle the divergences of perturbative amplitudes, all the relations among nontensorial Green functions are preserved except the AVV and AAA triangle [20]; i.e., as a consequence of the consistency conditions, all the Ward identities are preserved except the ones involved in the triangle anomalies as it should be. After these important discussions we are ready to consider the main point involved in this contribution.

Assuming then the validity of the consistency conditions, we can state the expressions for the amplitudes as well as for the contracted expressions. As an immediate consequence of these requirements we get, for the nontensorial one and two-point functions,

$$T^{S}(l_1) = 4m[I_{\mathrm{quad}}(m^2)], \tag{120}$$

$$T^{V}_{\mu}(l_1) = 0, \tag{121}$$

$$T^{SS}(l_1, l_2) = 4\left\{[I_{\mathrm{quad}}(m^2)] + \frac{1}{2}[4m^2 - (l_1 - l_2)^2] \times [I_{\log}(m^2)] - \frac{1}{2}[4m^2 - (l_1 - l_2)^2]\left(\frac{i}{(4\pi)^2}\right) \times \{Z_0[(l_1 - l_2)^2, m^2]\}\right\}, \tag{122}$$

$$T^{PP}(l_1, l_2) = 4\left\{-[I_{\mathrm{quad}}(m^2)] + \frac{1}{2}(l_1 - l_2)^2[I_{\log}(m^2)] - \frac{1}{2}(l_1 - l_2)^2\left(\frac{i}{(4\pi)^2}\right)\{Z_0[(l_1 - l_2)^2, m^2]\}\right\}, \tag{123}$$

$$T^{PA}_{\mu}(l_1, l_2) = 4m(l_1 - l_2)_{\mu}\left\{[I_{\log}(m^2)] - \left(\frac{i}{(4\pi)^2}\right)\{Z_0](l_1 - l_2)^2, m^2]\}\right\}, \tag{124}$$

$$T^{VS}_{\mu}(l_1, l_2) = 0, \tag{125}$$

$$T^{AV}_{\mu\nu}(l_1, l_2) = 0, \tag{126}$$

$$T^{VV}_{\mu\nu}(l_1, l_2) = \frac{4}{3}[(l_1 - l_2)^2 g_{\mu\nu} - (l_1 - l_2)_{\mu}(l_1 - l_2)_{\nu}] \times \left\{[I_{\log}(m^2)] - \left(\frac{i}{(4\pi)^2}\right) \times \left[\frac{1}{3} + \frac{[2m^2 + (l_1 - l_2)^2]}{(l_1 - l_2)^2} \times \{Z_0[(l_1 - l_2)^2, m^2]\}\right]\right\}, \tag{127}$$

$$T^{AA}_{\mu\nu}(l_1, l_2) = \frac{4}{3}[(l_1 - l_2)^2 g_{\mu\nu} - (l_1 - l_2)_{\mu}(l_1 - l_2)_{\nu}] \times \left\{[I_{\log}(m^2)] - \left(\frac{i}{(4\pi)^2}\right)\left[\frac{1}{3} + \frac{[2m^2 + (l_1 - l_2)^2]}{(l_1 - l_2)^2} \times \{Z_0[(l_1 - l_2)^2, m^2]\}\right]\right\} - 8m^2 g_{\mu\nu}\left\{[I_{\log}(m^2)] - \left(\frac{i}{(4\pi)^2}\right)\{Z_0[(l_1 - l_2)^2, m^2]\}\right\}. \tag{128}$$

On the other hand, the tensor amplitudes become

$$T^{ST}_{\mu\nu}(l_i, l_j) = T^{PT}_{\mu\nu}(l_i, l_j) = T^{AT}_{\alpha\mu\nu}(l_i, l_j) = 0, \tag{129}$$

$$T^{VT}_{\alpha\mu\nu}(l_1, l_2) = 4m(g_{\alpha\nu}g_{\mu\lambda} - g_{\alpha\mu}g_{\nu\lambda})(l_1 - l_2)^{\lambda}\left\{[I_{\log}(m^2)] - \left(\frac{i}{(4\pi)^2}\right)\{Z_0[(l_1 - l_2)^2, m^2]\}\right\}, \tag{130}$$

$$T^{TT}_{\alpha\beta\mu\nu}(l_1, l_2) = (g_{\alpha\mu}g_{\beta\xi}g_{\nu\lambda} - g_{\alpha\nu}g_{\beta\xi}g_{\mu\lambda} + g_{\beta\nu}g_{\alpha\xi}g_{\mu\lambda} - g_{\beta\mu}g_{\alpha\xi}g_{\nu\lambda})\frac{4}{3}[(l_1 - l_2)^2 g^{\xi\lambda} - (l_1 - l_2)^{\xi}(l_1 - l_2)^{\lambda}] \times \left\{I_{\log}(m^2) - \left(\frac{i}{(4\pi)^2}\right)\left[\frac{1}{3} + \frac{2m^2 + (l_1 - l_2)^2}{(l_1 - l_2)^2}\{Z_0[m^2, (l_1 - l_2)^2]\}\right]\right\} + 4(g_{\alpha\nu}g_{\beta\mu} - g_{\alpha\mu}g_{\beta\nu})\left\{-[I_{\mathrm{quad}}(m^2)] + \frac{1}{2}[4m^2 + (l_1 - l_2)^2][I_{\log}(m^2)] - \frac{1}{2}[4m^2 + (l_1 - l_2)^2]\left(\frac{i}{(4\pi)^2}\right)\{Z_0[(l_1 - l_2)^2, m^2]\}\right\}. \tag{131}$$

The above listed expressions can be referred as the "consistent regularized amplitudes" (CRA's).

Now, given the expressions for the CRA's in the nontensorial amplitudes, the constraints which we have derived for tensor amplitudes, which play the role of symmetry relations or Ward identities, become then

$$(l_1 - l_2)^{\alpha} T^{TV}_{\mu\nu\alpha}(l_1, l_2) = 0, \tag{132}$$

$$(l_1 - l_2)^{\mu} T^{TS}_{\mu\nu}(l_1, l_2) = 0, \tag{133}$$

$$(l_1 - l_2)^{\mu} T^{TP}_{\mu\nu}(l_1, l_2) = 0, \tag{134}$$

$$(l_1 - l_2)^\mu T^{TA}_{\mu\nu\alpha}(l_1, l_2) = 0, \tag{135}$$

$$(l_1 - l_2)^\alpha T^{TA}_{\mu\nu\alpha}(l_1, l_2) = 0, \tag{136}$$

and

$$(l_1 - l_2)^\mu T^{TV}_{\mu\nu\alpha}(l_1, l_2) = \frac{1}{2m}(l_1 - l_2)^2[T^{VV}_{\alpha\nu}(l_1, l_2) - T^{AA}_{\alpha\nu}(l_1, l_2)] - (l_1 - l_2)_\nu T^{PA}_\alpha(l_1, l_2), \tag{137}$$

$$(l_1 - l_2)^\mu T^{TT}_{\mu\nu\alpha\beta}(l_1, l_2) = -2m T^{VT}_{\nu\alpha\beta}(l_1, l_2) + (l_1 - l_2)_\alpha \times [T^{AA}_{\nu\beta}(l_1, l_2) + g_{\nu\beta} T^{SS}(l_1, l_2)] - (l_1 - l_2)_\beta [T^{AA}_{\nu\alpha}(l_1, l_2) + g_{\nu\alpha} T^{SS}(l_1, l_2)] \tag{138}$$

$$= -2m T^{VT}_{\nu\alpha\beta}(l_1, l_2) - (l_1 - l_2)_\beta [T^{VV}_{\nu\alpha}(l_1, l_2) + g_{\alpha\nu} T^{PP}(l_1, l_2)] + (l_1 - l_2)_\alpha [T^{VV}_{\nu\beta}(l_1, l_2) + g_{\nu\beta} T^{PP}(l_1, l_2)]. \tag{139}$$

Where all the amplitudes appearing in the last two equations are the CRA's ones. Note that with the adopted definition for CRA's, all the arbitrariness associated with the internal momenta choices are automatically removed. Let us now go to the final remarks.

X. FINAL REMARKS

In the present work, we have considered the questions relative to the consistent evaluation of Green functions having tensor operators and their symmetry relations. Given the occurrence of divergences, such types of calculations are plagued by the very well-known arbitrariness and their associated ambiguities. So, even if the evaluation of these structures does not represent a trouble from the point of view of traditional regularization methods, it remains the question of consistency in such eventual calculations, because the referred strategies may not be automatically consistent. It becomes necessary to identify all the possible constraints to be satisfied by the calculated amplitudes in order to verify the consistency of the obtained results. In order to get such constraints for the conventional densities, we have at our disposal the Ward identities but for the tensor densities there are no such identities. Having this in mind, we used a procedure to evaluate the amplitudes and to generate constraints to be satisfied by the explicit expressions of tensor Green functions. The procedure produces results which are completely equivalent to those produced through the standard methods of current algebra in the case of conventional fermionic currents. Given the inadequacy of the current algebra, we proposed to use alternative methods for the tensor Green functions in order to derive constraints or symmetry properties. When such constraints are stated to vector or axial-vector Lorentz indexes, carried by a Green function possessing a tensor vertex also, it is immediate to verify that the obtained constraints are precisely those which are obtained by using current algebra methods. All the results for the tensor amplitudes and for their corresponding external momentum contracted expressions have been written as a combination of Green functions involving the conventional operators. Thus, all the constraints which are imposed on these amplitudes can be used as a guide to construct the corresponding consistent results for the tensor structures. Having this in mind, we used conclusions stated in previous investigations concerning the consistency in the evaluation of physical amplitudes having divergences. In the referred analyses, all the arbitrariness is maintained in the evaluation of divergent structures. The values for the undefined pieces are fixed by the requirements that are imposed on the amplitudes by very general symmetry grounds. Such requirements are put in terms of a set of conditions for specific combinations of purely divergent integrals having the same degree of divergence. They have been denominated as consistency conditions and revealed enough to guarantee the desirable consistency in perturbative calculations. The amplitudes which are obtained after the imposition of such conditions are referred as "consistent regularized amplitudes." They are the evaluated amplitudes having the structures $\Box_{\alpha\beta\mu\nu}, \nabla_{\mu\nu}$, and $\Delta_{\mu\nu}$ removed from them. The expressions thus resulting for all tensor amplitudes are free from ambiguities and simultaneously "symmetry preserving" since they automatically satisfy the constraints derived for them. We can consider that the initial purposes of the present contribution have been adequately achieved. We believe that the derived amplitudes and their corresponding symmetry properties are adequate to be used in phenomenological predictions. The most important aspect involved in this work, however, is the perspective opened by the present analysis which is the study of three-point functions and their eventual anomalies. It is possible to show that anomalies in Ward identities analogous to those occurring in the *AVV* and *AAA* triangle amplitudes will appear in triangle amplitudes associated with tensor densities. A work along this line is in preparation.

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