Vacuum fluctuations and Brownian motion of a charged test particle near a reflecting boundary

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We study the Brownian motion of a charged test particle coupled to electromagnetic vacuum fluctuations near a perfectly reflecting plane boundary. The presence of the boundary modifies the quantum fluctuations of the electric field, which in turn modifies the motion of the test particle. We calculate the resulting mean squared fluctuations in the velocity and position of the test particle. In the case of directions transverse to the boundary, the results are negative. This can be interpreted as reducing the quantum uncertainty which would otherwise be present.

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I. INTRODUCTION

In quantum electrodynamics, the effects of electromagnetic vacuum fluctuations upon an electron in empty space are usually regarded as unobservable. The divergent parts of the electron self-energy are absorbed by mass and wave function renormalizations, and the finite self-energy function can be taken to vanish for real (as opposed to virtual) electrons. However, *changes* in the vacuum fluctuations can produce observable effects. The Lamb shift and the Casimir effect are two examples of this.

In the present paper, we wish to discuss a very simple situation, the Brownian motion of a charged particle coupled to the quantized electromagnetic field. Just as a classical stochastic field will cause random motion of a test particle, one might also expect Brownian motion to be caused by quantum fluctuations. It is unclear whether this motion can be observable in the Minkowski vacuum state, although Gour and Sriramkumar [1] argue that it might be. Jaekel and Reynaud [2] have also discussed this issue in the context of mirrors coupled to vacuum fluctuations. Here we will be concerned with shifts due to the quantum state of the field being other than the Minkowski vacuum. One simple way to cause a nontrivial shift in the vacuum fluctuations is to introduce a reflecting boundary. In this paper, we will discuss the case of a single, perfectly reflecting plate, and calculate the effects of the modified electromagnetic vacuum fluctuations upon the motion of a charged test particle. The analogous calculations for the case of an uncharged, polarizable test particle were reported in Ref. [3]. The present problem bears some analogy to the problem of light cone fluctuations, where photons undergo Brownian motion due to modified quantum fluctuations of the quantized gravitational field [4–6].

II. THE LANGEVIN EQUATION AND ITS SOLUTIONS

We treat the particle as a point particle of mass m and electric charge e. In the limit of low velocities, the velocity $\mathbf{v}$ satisfies the nonrelativistic equation of motion with only an electric force term:

$$\frac{d\mathbf{v}}{dt} = \frac{e}{m}\mathbf{E}(\mathbf{x}, t). \tag{1}$$

We will restrict our attention to the case where the particle does not move significantly, so we can assume the position $\mathbf{x}$ to be constant. We also assume that dissipation can be ignored. If the particle starts at rest at time $t = 0$, then at time t the velocity is

$$\mathbf{v} = \frac{e}{m}\int_0^t \mathbf{E}(\mathbf{x}, t)dt, \tag{2}$$

and the mean squared speed in the i-direction is (no sum on i)

$$\langle \Delta v_i^2 \rangle = \frac{e^2}{m^2}\int_0^t\int_0^t [\langle E_i(\mathbf{x}, t_1)E_i(\mathbf{x}, t_2)\rangle - \langle E_i(\mathbf{x}, t_1)\rangle \times \langle E_i(\mathbf{x}, t_2)\rangle]dt_1 dt_2. \tag{3}$$

In general, there may be a classical, nonfluctuating field in addition to the fluctuating quantum field. However, in this case the electric field correlation function which appears in Eq. (3) is just the quantum field two-point function. Let the electric field be expressed as a sum of a classical and a quantum part: $\mathbf{E} = \mathbf{E}_c + \mathbf{E}_q$, where $\langle \mathbf{E} \rangle = \mathbf{E}_c$. Then

$$\langle \mathbf{E}(\mathbf{x}, t_1)\mathbf{E}(\mathbf{x}, t_2)\rangle - \langle \mathbf{E}(\mathbf{x}, t_1)\rangle\langle \mathbf{E}(\mathbf{x}, t_2)\rangle = \langle \mathbf{E}_q(\mathbf{x}, t_1)\mathbf{E}_q(\mathbf{x}, t_2)\rangle. \tag{4}$$

Thus Eq. (3) describes the velocity fluctuations around the

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mean trajectory caused by the classical field. Henceforth, we will drop the q subscript and understand $\langle \mathbf{E}(\mathbf{x}, t_1)\mathbf{E}(\mathbf{x}, t_2)\rangle$ to be the quantum two-point function.

In the presence of a boundary, this two-point function can be expressed as a sum of the Minkowski vacuum term and a correction term due to the boundary:

$$\langle \mathbf{E}(x)\mathbf{E}(x')\rangle = \langle \mathbf{E}(x)\mathbf{E}(x')\rangle_0 + \langle \mathbf{E}(x)\mathbf{E}(x')\rangle_R, \tag{5}$$

where the correction term is finite in the coincidence limit $x' = x$, so long as the point is not actually on the boundary. The Minkowski vacuum term would produce a formally divergent contribution to $\langle \Delta v^2\rangle$. However, as discussed above, this contribution is not expected to produce any observable consequences. Thus, we will keep only the boundary-dependent contribution, and write

$$\langle \Delta v_i^2\rangle = \frac{e^2}{m^2}\int_0^t\int_0^t \langle E_i(\mathbf{x}, t_1)E_i(\mathbf{x}, t_2)\rangle_R dt. \tag{6}$$

In the case of a single, perfectly reflecting plate, $\langle \mathbf{E}(x)\mathbf{E}(x')\rangle_R$ can be obtained by images [7]. Let the plate be located in the $z = 0$ plane. At a point a distance z from the plate, the components of $\langle \mathbf{E}(\mathbf{x}, t_1)\mathbf{E}(\mathbf{x}, t_2)\rangle_R$ are[1]

$$\langle E_x(\mathbf{x}, t')E_x(\mathbf{x}, t'')\rangle = \langle E_y(\mathbf{x}, t')E_y(\mathbf{x}, t'')\rangle = -\frac{\Delta t^2 + 4z^2}{\pi^2(\Delta t^2 - 4z^2)^3} \tag{7}$$

and

$$\langle E_z(\mathbf{x}, t')E_z(\mathbf{x}, t'')\rangle = \frac{1}{\pi^2(\Delta t^2 - 4z^2)^2}. \tag{8}$$

The velocity dispersion in the x-direction is given by

$$\begin{aligned}\langle \Delta v_x^2\rangle = \langle \Delta v_y^2\rangle &= \frac{e^2}{m^2}\int_0^t\int_0^t \langle E_x(\mathbf{x}, t')E_x(\mathbf{x}, t'')\rangle dt'dt'' \\ &= -\frac{e^2}{\pi^2 m^2}\int_0^t\int_0^t \frac{\Delta t^2 + 4z^2}{\pi^2(\Delta t^2 - 4z^2)^3}dt'dt'' \\ &= -\frac{e^2}{\pi^2 m^2}\int_0^t \frac{2(t-\tau)(\tau^2 + 4z^2)}{\pi^2(\tau^2 - 4z^2)^3}d\tau \\ &= \frac{e^2}{\pi^2 m^2}\left[\frac{t}{64z^3}\ln\left(\frac{2z + t}{2z - t}\right)^2 - \frac{t^2}{8z^2(t^2 - 4z^2)}\right].\end{aligned} \tag{9}$$

It should be pointed out that the above expression is singular at $t = 2z$. This corresponds to a time interval equal to the round-trip light travel time between the particle and the plane boundary. Presumably, this might be a result of our assumption of a rigid perfectly reflecting plane boundary, and would thus be smeared out in a more realistic treatment. For $t \gg z$, Eq. (9) becomes

$$\langle \Delta v_x^2\rangle = \langle \Delta v_y^2\rangle \approx -\frac{e^2}{3\pi^2 m^2}\frac{1}{t^2} - \frac{8e^2}{5\pi^2 m^2}\frac{z^2}{t^4}. \tag{10}$$

Therefore, the mean squared velocity fluctuation in the directions parallel to plane decreases to zero as time approaches to infinity.

The velocity dispersion in the z-direction is

$$\begin{aligned}\langle \Delta v_z^2\rangle &= \frac{e^2}{m^2}\int_0^t\int_0^t \langle E_z(\mathbf{x}, t')E_z(\mathbf{x}, t'')\rangle dt'dt'' \\ &= \frac{e^2}{\pi^2 m^2}\frac{t}{32z^3}\ln\left(\frac{2z + t}{2z - t}\right)^2.\end{aligned} \tag{11}$$

For $t \gg z$,

$$\langle \Delta v_z^2\rangle \approx \frac{e^2}{4\pi^2 m^2}\frac{1}{z^2} + \frac{e^2}{3\pi^2 m^2}\frac{1}{t^2}. \tag{12}$$

Unlike the velocity dispersion in the transverse directions, that in the direction perpendicular to the plate approaches a nonzero constant value at late times. The fact it does not continue to grow in time can be understood as a consequence of energy conservation. Unlike the case of Brownian motion due to thermal noise, here no dissipation is needed for $\langle \Delta v_i^2\rangle$ to be bounded at late times.

The mean squared position fluctuation in the x-direction can be calculated as follows

$$\begin{aligned}\langle \Delta x^2\rangle &= \int_0^t dt_1\int_0^{t_1} dt'\int_0^t dt_2\int_0^{t_2} dt''\langle E_x(\mathbf{x}, t')E_x(\mathbf{x}, t'')\rangle \\ &= \frac{e^2}{\pi^2 m^2}\left[\frac{t^3}{192z^3}\ln\left(\frac{t + 2z}{t - 2z}\right)^2 - \frac{t^2}{24z^2} - \frac{1}{6}\ln\left(\frac{t^2 - 4z^2}{4z^2}\right)\right].\end{aligned} \tag{13}$$

For $t \gg z$

$$\langle \Delta x^2\rangle = \langle \Delta y^2\rangle \approx -\frac{e^2}{3\pi^2 m^2}\ln(t/2z). \tag{14}$$

The corresponding position fluctuation in the z direction is

$$\langle \Delta z^2\rangle = \frac{e^2}{\pi^2 m^2}\left[\frac{t^2}{24z^2} + \frac{t^3}{96z^3}\ln\left(\frac{t + 2z}{t - 2z}\right)^2 + \frac{1}{6}\times\ln\left(\frac{t^2 - 4z^2}{4z^2}\right)\right], \tag{15}$$

and its limiting form for $t \gg z$ is

$$\langle \Delta z^2\rangle \approx \frac{e^2}{\pi^2 m^2}\left[\frac{t^2}{8z^2} + \frac{1}{3}\ln\left(\frac{t}{2z}\right) + \frac{1}{9} + O(z^2/t^2)\right]. \tag{16}$$

Recall that we have assumed that the particle does not significantly change its position, that is, $\langle \Delta z^2\rangle \ll z^2$. This condition will be fulfilled so long as

[1] Lorentz-Heaviside units with $c = \hbar = 1$ will be used here, except as otherwise noted.

$$t \ll \frac{2\sqrt{2}\pi}{e}(mz)z. \tag{17}$$

Note that mz is the ratio of the distance to the plate to the Compton wavelength of the particle, which is typically very large. Thus Eqs. (15) and (16) can be valid for times long compared to z, the light travel time to the plate.

Here we should also note that we have assumed no dissipation. In the case of a ground state, such as a Casimir vacuum, it would seem that there is no possibility of dissipation. However, we are dealing with a situation where the interaction between the charged particle and the quantized electromagnetic field is switched on suddenly, and a finite time is required for the system to settle into its steady state. During that time, dissipation of energy suppled by the act of switching is possible. The most likely source of dissipation here is radiation by the particle. This can be estimated using the Larmor formula, which gives the average power radiated by a nonrelativistic particle with acceleration a to be

$$P = \frac{e^2}{6\pi}a^2 = \frac{e^4}{6\pi m^2}\langle \mathbf{E}^2\rangle, \tag{18}$$

where in the second step we have used Eq. (1). After a time t, a particle radiating at this rate will change in its squared velocity by

$$\Delta v_{\rm rad}^2 = \frac{e^4 t}{3\pi m^3}\langle \mathbf{E}^2\rangle = \frac{e^4 t}{16\pi^3 z^4 m^3}, \tag{19}$$

where we used

$$\langle \mathbf{E}^2\rangle = \frac{3}{16\pi^2 z^4}. \tag{20}$$

The effects of radiation will be small compared the dispersion due to vacuum fluctuations so long as $\Delta v_{\rm rad}^2 \ll \langle \Delta v_z^2\rangle$, that is, so long as

$$t \ll \frac{4\pi}{e^2}(mz)z. \tag{21}$$

This condition will always be fulfilled for an electron so long as Eq. (17) holds.

III. INTERPRETATION OF THE RESULTS

A few comments are now in order for the above results. First, let us notice that the Brownian motion of a test charged particle subject to electromagnetic vacuum fluctuations will be anisotropic, since the behaviors of both the velocity and position dispersions are different in the longitudinal and transverse directions. The most dramatic feature is that $\langle \Delta v_x^2\rangle$ and $\langle \Delta x^2\rangle$ are both negative. This is counterintuitive and requires a physical explanation. A negative dispersion must imply a decrease in an uncertainty which would otherwise be present. One possibility is the usual uncertainty in position and velocity of a quantum particle. Quantum mechanically, a massive particle is described by a wave packet which must have a position and a momentum uncertainty. It is well established that the wave packet spreads out as time progresses, and so the position uncertainty will increase with time. Consequently, even if the particle is initially in a minimum uncertainty wave packet, at a later time, it will satisfy the uncertainty principle by a wider margin. If we recall that $\langle \Delta x^2\rangle$ is a difference between the case with the plane boundary and that without it, we can see that the negative sign of $\langle \Delta x^2\rangle$ can be understood as a reduction in the position spreading of the wave packet in the parallel directions as compared to what it would have been without the presence of a plane boundary. In a somewhat different context, it has been shown that dissipation can also suppress wave packet spreading [8]. Note that the reduction due to vacuum fluctuation is generally small as $\langle \Delta x^2\rangle$ is a logarithmic function of time. However, the corresponding position dispersion in the perpendicular direction, $\sqrt{\langle \Delta z^2\rangle}$, is positive and furthermore it grows linearly with time. Hence, the wave packet spreading in the z direction is reinforced by electromagnetic vacuum fluctuations and it will be larger than what it would have been without the boundary.

Let us now discuss in more detail the wave packet spreading due to the quantum nature of the particle and that due to electromagnetic vacuum fluctuations. Take, as an example, a Gaussian wave packet which represents a particle whose position and momentum are simultaneously determined, as closely as the uncertainty principle permits. We will use the subscript q to denote uncertainties due to the quantum nature of a particle, as opposed to those due to vacuum fluctuations. Assume the initial width of the wave packet is Δz_{q0}. It can be shown that the width of the packet at time t is

$$\Delta z_q = \sqrt{\Delta z_{q0}^2 + \frac{(\Delta p_z)^2 t^2}{m^2}}, \tag{22}$$

where Δp_z is the width of the wave packet in momentum space. Let $\Delta z_{q0}\Delta p_z = 1/2$, that is, choose the initial wave packet such that the uncertainty attains its theoretical minimum value. Then

$$\Delta z_q = \sqrt{\Delta z_{q0}^2 + \frac{t^2}{4\Delta z_{q0}^2 m^2}} \equiv \Delta z_{qm}. \tag{23}$$

The question we now want to ask is: how large the position fluctuation due to the vacuum fluctuations could be as compared to that due to the uncertainty principle and the wave packet spreading? For any fixed travel time t, we want to manipulate the initial size of the wave packet such as at time t, the width of the wave packet attains a minimum value. We find that this initial width is

$$\Delta z_{q0}^2 = \frac{t}{2m}, \tag{24}$$

and the corresponding minimum position in any direction uncertainty is

$$\Delta x_{qm} = \Delta z_{qm} = \sqrt{\frac{t}{m}}. \tag{25}$$

Let

$$\Delta x_f = \sqrt{|\langle \Delta x^2 \rangle|} \tag{26}$$

be the position uncertainty in the x-direction due to the effects of vacuum fluctuations, and Δz_f be the corresponding uncertainty in the z-direction. In the limit that $t \gg z$, we have, for the case that the charged particle is an electron,

$$\frac{\Delta x_f}{\Delta x_{qm}} = 2\sqrt{\frac{\alpha \ln(t/2z)}{3\pi t m}}, \tag{27}$$

where α is the fine structure constant. This ratio is always very small. The corresponding ratio for the z-direction is

$$\frac{\Delta z_f}{\Delta z_{qm}} = \sqrt{\frac{\alpha}{2\pi}}\sqrt{\frac{t}{m}}\frac{1}{z} = 3.4 \times 10^{-2}\frac{\Delta z_{qm}}{z}. \tag{28}$$

Since the initial size, Δz_{qm}, should be much less than z, in general this ratio is much less than one.

Note that dispersion in the transverse velocity, $\langle \Delta v_x^2 \rangle$, is essentially a transient effect which dies off rapidly in time. Although the dispersion in the transverse position, $\langle x^2 \rangle$, grows slowly in time, it can also be understood as consequence of the uncertainty in v_x at an earlier time, and hence also a transient effect. Such transient effects can be due to the way in which the system is prepared. Here we have assumed that the effect of the vacuum fluctuations begins at $t = 0$ without specifying the details of how the effects are switched on. One way this might be done is with electrons moving parallel to a finite plate and crossing the edge of the plate at $t = 0$. The effect of the switching can cause the electron to emit photons, which can in turn contribute to uncertainties in momentum and position. Thus, it may also be possible to interpret negative values of $\langle \Delta x^2 \rangle$ and $\langle \Delta v_x^2 \rangle$ as arising from a suppression of photon emission effects.

The increase in $\langle \Delta v_z^2 \rangle$ given in Eq. (12) can be associated with an effective temperature of

$$\begin{aligned} T_{\text{eff}} &= \frac{\alpha}{\pi}\frac{1}{k_B m z^2} = 1.7 \times 10^{-6}\left(\frac{1\mu m}{z}\right)^2, \\ K &= 1.7 \times 10^2\left(\frac{1\ \text{Å}}{z}\right)^2 K, \end{aligned} \tag{29}$$

where k_B is Boltzmann's constant. The approximation of a perfect reflector holds for metal plates at frequencies below the plasma frequency, which would require that $z \gtrsim 1\mu m$. The corresponding temperature, although small, is within a range that has been achieved experimentally. For $z \lesssim 1\mu m$, the plate is no longer a perfect reflector, but can be a partial reflector. Bragg scattering can produce significant reflection even at x-ray wavelengths.

Note that here we are concerned with the increase in the mean squared normal velocity due to the presence of the plate. Because the electron must be localized on a scale smaller than z, there is already a larger spread in v_z due to quantum uncertainty. However, in principle this could be canceled if one measured the change due to the boundary. A free electron near a conducting plate will also feel a classical image charge force, but this might be canceled by another classical force. It is of interest to compare Eq. (29) for an electron with the corresponding result for an atom [3], which is of order $0.1K$ if $z = 1$ Å, and falls as $1/z^8$ as z increases. Thus the case of an electron seems much closer to being experimentally accessible.

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