PHYSICAL REVIEW D, VOLUME 70, 065003

Topological-charge anomalies in supersymmetric theories with domain walls

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(Received 24 April 2004; published 20 September 2004)

Domain walls in $1 + 2$ dimensions are studied to clarify some general features of topological-charge anomalies in supersymmetric theories, by extensive use of a superfield supercurrent. For domain walls quantum modifications of the supercharge algebra arise not only from the short-distance anomaly but also from another source of long-distance origin, induced spin in the domain-wall background, and the latter dominates in the sum. A close look into the supersymmetric trace identity, which naturally accommodates the central-charge anomaly and its superpartners, shows an interesting consequence of the improvement of the supercurrent: Via an improvement the anomaly in the central-charge can be transferred from induced spin in the fermion sector to an induced potential in the boson sector. This fact reveals a dual character, both fermionic and bosonic, of the central-charge anomaly, which reflects the underlying supersymmetry. The one-loop superfield effective action is also constructed to verify the anomaly and Bogomol'nyi-Prasad-Sommerfield (BPS) saturation of the domain-wall spectrum.

DOI: 10.1103/PhysRevD.70.065003 PACS numbers: 11.10.Kk, 11.30.Pb

I. INTRODUCTION

There is an interesting interplay between supersymmetry and topological excitations. As Witten and Olive [1] pointed out, in the presence of topological excitations the supercharge algebra is modified to include central charges and in certain supersymmetric theories the spectrum of topological excitations which saturate the Bogomol'nyi-Prasad-Sommerfield (BPS) bound [2] classically is determined exactly through the central-charge. An argument based on multiplet shortening for BPS-saturated excitations shows that saturation persists beyond the classical level in many cases [1,3].

It, however, remained somewhat obscure whether and how BPS saturation could continue at the quantum level in some simple supersymmetric theories where the excitation spectrum is affected by quantum corrections and renormalization. In this connection, solitons (or kinks) in two-dimensional theories with $N = 1$ supersymmetry [4] had long been examined by a number of authors [5–16]. It was eventually shown by Shifman, Vainshtein and Voloshin [14] that the central-charge acquires a quantum anomaly so that, together with the quantum correction to the kink mass, BPS saturation is maintained at the quantum level. Their analysis revealed the importance of enforcing supersymmetry although actual calculations were made with component fields.

Fujikawa and van Nieuwenhuizen [17] developed a superspace approach to this problem and derived the central-charge anomaly by making a local supersymmetry transformation on the superfield. Subsequently a superfield formulation of the central-charge anomaly was presented [18] by making extensive use of a superfield supercurrent that places the supercurrent, energy-momentum tensor and topological current in a supermultiplet.

The purpose of this paper is to present a further study of the central-charge anomaly, especially its origins and character, for domain walls in $1 + 2$ dimensions, for which nontrivial BPS saturation of the quantum spectrum has been reported [19,20]. For solitons in two dimensions the central-charge anomaly derives entirely from the superconformal anomaly (of short-distance origin). For domain walls in three dimensions, in contrast, quantum modifications of the supercharge algebra come not only from the short-distance anomaly but also from another source of long-distance origin, induced spin in the domain-wall background, and the latter dominates in the sum. We point out some interesting consequences of the "improvement" of the superfield supercurrent: Via an improvement the central-charge operator changes its form while its (physical) expectation value remains unchanged. One can thereby transform induced spin in the fermion sector into an induced potential in the boson sector. This fact reveals a dual character, both fermionic and bosonic, of the central-charge anomaly, which reflects the underlying supersymmetry.

In Sec. II we review some basic features of supersymmetric theories with topological charges. In Sec. III we calculate the one-loop effective action in superspace and identify a possible anomaly in the central-charge. In Sec. IV we introduce a superfield supercurrent, examine its conservation law at the quantum level and determine the central-charge anomaly and its superpartners. In Sec. V we consider the improvement of the superfield supercurrent and its effect on the supersymmetric trace identities. In Sec. VI we study physical origins of the central-charge anomaly and examine what happens in the case of extended supersymmetry. Section VII is devoted to a summary and discussion.

II. $N = I$ SUPERSYMMETRY IN THREE DIMENSIONS

Let us first review some basic features of supersymmetric theories with topological excitations in three (or two) dimensions. Consider the Wess-Zumino model [21] consisting of a real scalar field ϕ and a real (Majorana) spinor field $\psi_\alpha = (\psi_1, \psi_2)$, along with a real auxiliary field F, described by the action $S = \int d^3x \mathcal{L}$ and

$$\mathcal{L} = \frac{1}{2}\{\bar{\psi} i \not{\partial} \psi + (\partial_\mu \phi)^2 + F^2\} + FW'(\phi) - \frac{1}{2} W''(\phi)\bar{\psi}\psi, \tag{2.1}$$

with the Dirac matrices (in a Majorana representation)

$$\gamma^0 = \sigma_2, \qquad \gamma^1 = i\sigma_3, \qquad \gamma^2 = i\sigma_1. \tag{2.2}$$

Here $\bar{\psi} \equiv \psi\gamma^0 = i(\psi_2, -\psi_1)$ and $W'(\phi) = dW(\phi)/d\phi$, etc., Eliminating the auxiliary field F from $\mathcal{L}$, i.e., setting $F \to -W'(\phi)$ yields the potential term $-\frac{1}{2}[W'(\phi)]^2$. We suppose that the superpotential $W(\phi)$ has more than one extrema with $W'(\phi) = 0$ so that the model supports topologically stable excitations. A simple choice [4]

$$W(\phi) = \frac{m^2}{4\lambda}\phi - \frac{\lambda}{3}\phi^3 \tag{2.3}$$

supports a classical static domain-wall solution

$$\phi_{\rm DW}(x) = v \tanh(mx^1/2) \tag{2.4}$$

with $v = m/(2\lambda)$, uniform in x^2 and interpolating between the two distinct vacua with $\langle\phi\rangle_{\rm vac} = \pm v$ at spatial infinities $x^1 = \pm\infty$. The domain-wall has a finite energy density (or surface tension)

$$M^{\rm cl}_{\rm DW}/L_y = m^3/(6\lambda^2), \tag{2.5}$$

where $L_y = \int dy$ denotes the length in the $x^2 \equiv y$ direction. In two dimensions the same solution (2.4) describes a static kink [4] with energy $M^{\rm cl}_{\rm kink} = m^3/(6\lambda^2)$. The super-sine-Gordon model with $W(\Phi) = mv^2 \sin(\Phi/v)$ also supports analogous domain walls and solitons.

The action $S = \int d^3x \mathcal{L}$ is invariant under supersymmetry transformations

$$\begin{aligned} \delta\phi(x) &= \bar{\xi}\psi(x), \\ \delta\psi_\alpha(x) &= -i(\gamma^\mu\xi)_\alpha \partial_\mu\phi(x) + \xi_\alpha F(x), \\ \delta F(x) &= -i\bar{\xi}\gamma^\mu\partial_\mu\psi(x), \end{aligned} \tag{2.6}$$

where $\xi_\alpha = (\xi_1, \xi_2)$ is a two-component Grassmann number; $\bar{\xi} = \xi\gamma^0$ and $\bar{\xi}\psi = \bar{\xi}_\alpha\psi_\alpha$. The associated Noether supercurrent is written as

$$J^\mu_\alpha = (\partial_\nu\phi)(\gamma^\nu\gamma^\mu\psi)_\alpha - iF(\gamma^\mu\psi)_\alpha. \tag{2.7}$$

The conserved supercharges

$$Q_\alpha = \int d^2x J^0_\alpha \tag{2.8}$$

generate, within the canonical formalism, the transformation law of the supercurrent

$$i[\bar{\xi}_\beta Q_\beta, J^\mu_\alpha] = -2i(\gamma_\lambda\xi)_\alpha(T^{\mu\lambda} + \epsilon^{\mu\lambda\nu}F\partial_\nu\phi), \tag{2.9}$$

with the canonical energy-momentum tensor

$$T^{\mu\lambda} = \frac{i}{2}\bar{\psi}\gamma^\mu\partial^\lambda\psi + \partial^\mu\phi\partial^\lambda\phi - \frac{1}{2}g^{\mu\lambda}\{(\partial_\nu\phi)^2 - F^2\} \tag{2.10}$$

and the topological current

$$\epsilon^{\mu\lambda\nu}F\partial_\nu\phi = -\epsilon^{\mu\lambda\nu}\partial_\nu W(\phi). \tag{2.11}$$

Here F stands for $-W'(\phi)$ owing to the equation of motion $\delta S/\delta F = F + W'(\phi) \to 0$. In deriving Eq. (2.9), use has been made of the matrix identity specific to $1 + 2$ dimensions,

$$\gamma^\mu\gamma^\nu = g^{\mu\nu} - i\epsilon^{\mu\nu\lambda}\gamma_\lambda \tag{2.12}$$

with $\epsilon^{012} = 1$.

Let us note that the energy-momentum tensor $T^{\mu\lambda}$ has a portion antisymmetric in (μ, λ),

$$T^{\mu\lambda}_{\rm asym} = -\frac{1}{8}\epsilon^{\mu\lambda\nu}\{\partial_\nu(\bar{\psi}\psi) + 2X_\nu\}. \tag{2.13}$$

[In two dimensions $T^{\mu\lambda}_{\rm asym} \propto \epsilon^{\mu\lambda}\bar{\psi}\gamma^0\gamma^1(\delta S/\delta\bar{\psi})$ vanishes (at the quantum level [18]).] Here

$$X^\nu = -\bar{\psi}\gamma^\nu\gamma^\rho\partial_\rho\psi = i\bar{\psi}\gamma^\nu(\delta S/\delta\bar{\psi}) \tag{2.14}$$

is proportional to the equation of motion $\delta S/\delta\bar{\psi} = i\gamma^\mu\partial_\mu\psi - W''\psi \to 0$ and vanishes. It is thus natural to isolate the symmetric part $\Theta^{\mu\lambda} \equiv T^{\mu\lambda}_{\rm sym}$ and to regard $T^{\mu\lambda}_{\rm asym}$ as part of the (canonical) topological current

$$\zeta^{\mu\lambda}_{\rm can} = \epsilon^{\mu\lambda\nu}\{F\partial_\nu\phi - (1/8)\partial_\nu(\bar{\psi}\psi)\}, \tag{2.15}$$

which may be rewritten as $-\epsilon^{\mu\lambda\nu}\partial_\nu\{W(\phi) + (1/8)\bar{\psi}\psi\}$.

From Eq. (2.9) follows the supercharge algebra

$$\{Q_\alpha, \bar{Q}_\beta\} = 2(\gamma_\lambda)_{\alpha\beta}(P^\lambda + Z^\lambda) \tag{2.16}$$

where $P^\lambda = \int d^2\mathbf{x}\Theta^{0\lambda}$ is the total energy and momentum. The central-charge

$$Z^\lambda = \int d^2\mathbf{x}\zeta^{0\lambda}_{\rm can}, \tag{2.17}$$

for the classical domain-wall configuration $\phi_{DW}(x) \to \pm v$ and $\psi_\alpha(x) \to 0$ as $x^1 \to \pm\infty$, reads $Z^1 = 0$ and

$$Z^2/L_y = [W(\phi)]^{x^1=\infty}_{x^1=-\infty} = m^3/(6\lambda^2). \tag{2.18}$$

The $N = 1$ superalgebra thus gets centrally extended in the presence of domain walls (as well as solitons in two dimensions) [1].

The Wess-Zumino model (2.1) is neatly rephrased using the superfield formalism [22]. The structure of $N = 1$ supersymmetry is formally the same for two and three dimensions, with points $z = (x^\mu, \theta_\alpha)$ in $N = 1$ superspace labeled by spacetime coordinates x^μ and two Majorana coordinates $\theta_\alpha = (\theta_1, \theta_2)$. The supermultiplet nature of the fields (ϕ, ψ_α, F) is encoded in a real superfield,

$$\Phi(z) \equiv \Phi(x, \theta) = \phi(x) + \bar{\theta}\psi(x) + \frac{1}{2}\bar{\theta}\theta F(x), \quad (2.19)$$

where $\bar{\theta} \equiv \theta\gamma^0 = i(\theta_2, -\theta_1)$ and $\bar{\theta}\theta = \bar{\theta}_\alpha\theta_\alpha = -2i\theta_1\theta_2$. Under translations $x^\mu \to x^\mu - i\bar{\theta}\gamma^\mu\xi$ and $\theta_\alpha \to \theta_\alpha + \xi_\alpha$ in superspace, the component fields undergo the supersymmetry transformations (2.6).

The action $S = \int d^3x\mathcal{L}$ is cast in a superfield form [4]

$$S[\Phi] = \int d^5z\left\{\frac{1}{4}(\bar{D}_\alpha\Phi)D_\alpha\Phi + W(\Phi)\right\} \quad (2.20)$$

with $d^5z = d^3xd^2\theta$ and $\int d^2\theta\frac{1}{2}\bar{\theta}\theta = 1$. Here the spinor derivatives

$$D_\alpha = \partial/\partial\bar{\theta}_\alpha - (\not{p}\theta)_\alpha \quad (2.21)$$

and $\bar{D}_\alpha \equiv D_\beta(\gamma^0)_{\beta\alpha}$, with $\not{p} \equiv \gamma^\mu p_\mu$ and $p_\mu = i\partial_\mu$, obey

$$\{D_\alpha, D_\beta\} = 2(p\gamma^0)_{\alpha\beta}; \quad (2.22)$$

see Ref. [18] for some formulas involving D_α.

The superalgebra (2.16) has an important consequence. For the supercharge Q_2, in particular, it reads

$$(Q_2)^2 = P^0 - P^2 - Z^2. \quad (2.23)$$

The classical domain-wall solution (2.4) (giving $P^2 = 0$) obeys the first-order equation $(\partial/\partial x^1)\phi = W'(\phi)$, and is BPS-saturated [1] in the sense that it is inert under the action of Q_2,

$$Q_2|\mathrm{DW}\rangle = (P^0 - Z^2)|\mathrm{DW}\rangle = 0. \quad (2.24)$$

The supercharge Q_1, on the other hand, acts nontrivially. The BPS saturation (2.24) thus implies that the domain-wall tension $\langle P^0\rangle$ is given by the central-charge $\langle Z^2\rangle$ exactly; here $\langle\cdots\rangle$ stands for the expectation value $\langle\mathrm{DW}|\cdots|\mathrm{DW}\rangle$ for short. It is clear from Eqs. (2.5) and (2.18) that this equality holds at the classical level.

The BPS saturation $Q_2|\mathrm{DW}\rangle = 0$ and the resulting equality $\langle P^0\rangle = \langle Z^2\rangle$, once established classically, generally persist at the quantum level. This follows from multiplet shortening for BPS-saturated excitations in many cases [1]. Solitons in two dimensions and the domain-wall under consideration also belong to short one-dimensional representations, preserving only half of the original supersymmetry [14,16].

Some remarks are in order here. To validate the formal reasoning based on the superalgebra, one has to preserve supersymmetry in actual calculations. This is most naturally achieved by use of superfields, as we shall verify later. As a result, the supercharge algebra (2.16) holds as it is at the quantum level [although the charges $(Q_\alpha, P^\lambda, Z^\lambda)$ may deviate from their classical expressions; see, e.g., Eq. (4.11)]. The BPS saturation (2.24) implies that the domain-wall superfield has the form [14]

$$\Phi_{\mathrm{DW}}(z) = \phi_{\mathrm{DW}}\left(x^1 - \frac{1}{2}\bar{\theta}\theta\right), \quad (2.25)$$

which thus relates the domain-wall background-field $\phi_{\mathrm{DW}}(x^1)$ and the associated F component so that

$$\partial_1\phi_{\mathrm{DW}}(x^1) = -F_{\mathrm{DW}}(x^1). \quad (2.26)$$

Note that the action of Q_2 (or supertranslations with ξ_1) preserves the interval $x^1 - \frac{1}{2}\bar{\theta}\theta$ and hence $\Phi_{\mathrm{DW}}(z)$ as well.

To fix the functional form of $\phi_{\mathrm{DW}}(x^1)$ one may use the effective action in superspace. Suppose we have calculated the effective action [23] $\Gamma[\Phi_c]$ as a functional of the classical field $\Phi_c(z)$ [$= \langle\Phi(z)\rangle$ in the presence of a classical source $J_\Phi(z)$] in a loopwise expansion. It is a sum of the classical action (2.20) and loop corrections, $\Gamma[\Phi_c] = S[\Phi_c] + \Gamma_{\mathrm{loop}}[\Phi_c]$, and the associated classical equation of motion $\delta\Gamma[\Phi_c]/\delta\Phi_c(z) = 0$ governs the quantum dynamics of $\Phi_c(z)$. The key fact [18] is that this superfield equation $\delta\Gamma[\Phi_c]/\delta\Phi_c(z) = 0$ directly turns into the BPS equation for $\Phi_{\mathrm{DW}}(z)$, on substitution $\Phi_c(z) \to \Phi_{\mathrm{DW}}(z)$ (and on noting that $D_1\Phi_{\mathrm{DW}} = 0$ and $\bar{D}D\Phi_{\mathrm{DW}} = 2\partial_1\Phi_{\mathrm{DW}}$). The superspace effective action thus accommodates BPS-saturated excitations quite naturally.

III. SUPERSPACE EFFECTIVE ACTION

In this section we calculate the effective action and identify a possible anomaly in the central-charge. We use the background-field method [23] and expand Φ around the classical field Φ_c, $\Phi(z) = \Phi_c(z) + \chi(z)$. The quantum fluctuation χ at the one-loop level is governed by the action $\int d^5z\frac{1}{2}\chi\mathcal{D}\chi$ with the superspace operator

$$\mathcal{D} = -\frac{1}{2}\bar{D}D + W''(\Phi_c). \quad (3.1)$$

The associated χ propagator is given by $i/\mathcal{D}$, which we regularize in a supersymmetric way as

$$\langle\chi(z)\chi(z')\rangle^{\mathrm{reg}} = \langle z|\frac{i}{\mathcal{D}}e^{\tau\mathcal{D}^2}|z'\rangle, \quad (3.2)$$

with $\tau \to 0_+$ in the ultraviolet (UV) regulator $e^{\tau\mathcal{D}^2}$.

One can evaluate the χ propagator by expanding it in powers of D_α acting on $M \equiv W''(\Phi_c)$. The calculation is essentially the same as in the two-dimensional (2d) kink case; one may simply evaluate Eq. (C2) of Ref. [18] in three dimensions. To $O(D^2)$ the result is

$$\langle \chi(z)\chi(z)\rangle = 2\kappa - \frac{|M|}{4\pi} - \frac{\bar{D}DM}{16\pi|M|} + \frac{(\bar{D}M)DM}{32\pi M|M|}, \quad (3.3)$$

where $\kappa \equiv 1/(8\pi\sqrt{\pi\tau})$ is UV-cutoff dependent. Integrating this with respect to Φ_c, as done earlier [18], then yields the one-loop effective action to $O(D^2)$,

$$\Gamma_1[\Phi_c] = \int d^5z \left[\kappa M - \frac{M|M|}{16\pi} + \frac{(\bar{D}_\alpha M)D_\alpha M}{64\pi|M|} \right]. \quad (3.4)$$

The $O(D^0)$ terms in the total one-loop effective action $\Gamma[\Phi_c] = S[\Phi_c] + \Gamma_1[\Phi_c]$ now read

$$U_{\rm eff}(\Phi_c) = W(\Phi_c) + \kappa W_c'' - \frac{1}{16\pi} W_c''|W_c''|, \quad (3.5)$$

where $W_c'' \equiv M = W''(\Phi_c)$. Rewriting it in favor of the expectation value $\langle W(\Phi)\rangle = W(\Phi_c) + \frac{1}{2} W_c''\langle \chi\chi\rangle + \cdots$ yields

$$U_{\rm eff}(\Phi_c) = \langle W(\Phi)\rangle + \frac{1}{16\pi} W_c''|W_c''|. \quad (3.6)$$

This shows that the superpotential deviates from the classical superpotential (operator) $W(\Phi)$ by $(1/16\pi)W''(\Phi)|W''(\Phi)|$ at the one-loop level, suggesting a quantum anomaly in the central-charge. [The identification (3.6) is meant to $O(D^0)$. Interestingly, its right-hand side agrees with $\Gamma_1[\Phi_c]$ to $O(D^2)$, since the difference $\sim(1/32\pi)\bar{D}D|W_c''|$ vanishes under $\int d^5z$.]

The UV-divergent term $\kappa W_c''$ in $U_{\rm eff}(\Phi_c)$ can be eliminated by mass renormalization. To this end let m_r be a finite mass scale and set $m^2 = m_r^2 + \delta m^2$ in $U_{\rm eff}(\Phi_c)$. A convenient choice for the mass counterterm is

$$\delta m^2 = 8\lambda^2\kappa, \quad (3.7)$$

the net effect of which is to set $m \to m_r$ and $\kappa \to 0$ in $U_{\rm eff}(\Phi_c)$.

The effective action $\Gamma[\Phi_c]$ to $O(D^2)$ governs the asymptotic ($\mathbf{x} \to \pm\infty$) characteristics of the domain-wall state, which are sufficient for determining the central-charge and for verifying BPS saturation. Retaining only the bosonic components $\phi_c(x)$ and $F_c(x)$ of $\Phi_c(z)$ in $\Gamma[\Phi_c]$, one obtains the Lagrangian for the static wall,

$$\mathcal{L}_{\rm stat} = -\frac{1}{2}\left\{\sqrt{\alpha}\partial_1\phi_c \mp \frac{1}{\sqrt{\alpha}} U_{\rm eff}'(\phi_c)\right\}^2 \mp \partial_1 U_{\rm eff}(\phi_c) - \frac{1}{2}\alpha(\partial_2\phi_c)^2, \quad (3.8)$$

with $\alpha(\phi_c) = 1 + (1/16\pi)\{W'''(\phi_c)\}^2/|W''(\phi_c)|$. This leads to the BPS equation for the domain-wall,

$$\partial_1\phi_c = -F_c = (1/\alpha)U_{\rm eff}'(\phi_c), \qquad \partial_2\phi_c = 0, \quad (3.9)$$

with the asymptotic values of ϕ_c at $x^1 = \pm\infty$ now determined from $U_{\rm eff}'(\phi_c) \equiv dU_{\rm eff}(\phi_c)/d\phi_c = 0$. [The superfield equation $\delta\Gamma[\Phi_c]/\delta\Phi_c(z) = 0$ also leads to the same BPS equation.] The central-charge $Z_c = \int d^2\mathbf{x}\,\partial_1 U_{\rm eff}(\phi_c)$ then gives the surface tension

$$M_{\rm DW}/L_y = Z_c/L_y = \frac{m_r^3}{6\lambda^2} - \frac{m_r^2}{8\pi} \quad (3.10)$$

at $\phi_c(x^1 = \infty) = m_r/(2\lambda) - \lambda/(4\pi)$, in agreement with earlier results [19]. Here the quantum correction $m_r^2/(8\pi)$ derives from $(1/16\pi)W_c''|W_c''|$ in $U_{\rm eff}(\Phi_c)$. Note that it is the potential $U_{\rm eff}(\Phi_c)$ that should be minimized, rather than the operator potential $W(\Phi) + (1/16\pi)W''(\Phi)\times|W''(\Phi)|$ which, upon minimization, leads to a (divergent) unrenormalized expression with a quantum correction of the wrong sign.

The direct calculation (3.4) reveals that the super-sine-Gordon model with $W(\Phi) = mv^2\sin(\Phi/v)$, though non-renormalizable by power counting, is renormalizable at the one-loop level. It leads to the domain-wall surface tension

$$M_{\rm DW}/L_y = 2m_r v^2 - (m_r^2/8\pi), \quad (3.11)$$

upon setting $m = m_r + \delta m$ and $\delta m = \kappa m_r/v^2$.

IV. SUPERFIELD SUPERCURRENT

In this section we study possible quantum modifications of the supercharge algebra. The first step is to make a proper choice of conserved symmetry currents. This is not an easy step if one notes that there is some arbitrariness in defining currents (such as J_α^μ, $\Theta^{\mu\lambda}$, etc.) in supersymmetric theories: One may use either the auxiliary field F or its (classical) equivalent $-W'(\phi)$ to form currents but such possible choices are not necessarily the same at the quantum level, as we shall see soon.

Fortunately, in the present case one may simply adopt a superfield supercurrent used in the 2d kink case [18], which (now adapted to $1+2$ dimensions) reads

$$\mathcal{V}_\alpha^\mu = -i(D_\alpha\bar{D}_\lambda\Phi)(\gamma^\mu)_{\lambda\beta}D_\beta\Phi. \quad (4.1)$$

This current is a real spinor-vector superfield and places the supercurrent J_α^μ, energy-momentum tensor $T^{\mu\lambda}$ and topological current in a supermultiplet, as seen from the component expression

$$\mathcal{V}_\alpha^\mu = J_\alpha^\mu - 2i(\gamma_\lambda\theta)_\alpha(T^{\mu\lambda} + \epsilon^{\mu\lambda\nu}F\partial_\nu\phi) + \theta_\alpha X^\mu + \frac{1}{2}\bar{\theta}\theta f_\alpha^\mu. \quad (4.2)$$

Here J_α^μ and $T^{\mu\lambda}$ are defined by Eqs. (2.7) and (2.10) [with the auxiliary field F not identified with $-W'(\phi)$], respectively; $X^\mu = i\bar{\psi}\gamma^\mu(\delta S/\delta\bar{\psi})$ as defined in Eq. (2.14). We refer to one more current f_α^μ somewhat later. This current $\mathcal{V}_\alpha^\mu$ obeys a conservation law of the form

$$\partial_\mu\mathcal{V}_\alpha^\mu = (D_\alpha\bar{D}_\beta\Phi)D_\beta\frac{\delta S}{\delta\Phi} - \left(D_\alpha\bar{D}_\beta\frac{\delta S}{\delta\Phi}\right)D_\beta\Phi, \quad (4.3)$$

where

$$\frac{\delta S}{\delta \Phi} = -\frac{1}{2}\bar{D}D\Phi + W'(\Phi) \tag{4.4}$$

is an identity. One would think that current conservation $\partial_\mu \mathcal{V}^\mu_\alpha = 0$ simply follows from the equation of motion $\delta S/\delta\Phi = 0$. Care is required here, however. In general, while the equations of motion hold by themselves, operator products of the form (equations of motion) $\times$ (fields) are potentially singular and, when properly regulated, may not vanish. Indeed, in Fujikawa's method [24,25] all known anomalies arise from regularized Jacobian factors which take precisely such form. One therefore has to keep track of such potentially anomalous products to determine the conservation laws at the quantum level; see Ref. [26] for an early study of the superconformal anomaly along this line.

Here we quote only some general features of the potentially anomalous products, studied earlier [18]. Consider a product of the form

$$\{\Omega\Phi(z)\}\Xi\frac{\delta S}{\delta\Phi(z)}, \tag{4.5}$$

where Ω and Ξ may involve operators D_α and ∂_μ. One can evaluate it using the regularized propagator (3.2) at the one-loop level. The key result is that the regularized products enjoy the reciprocal property

$$(\Omega\Phi)\Xi\frac{\delta S}{\delta\Phi} = \pm(\Xi\Phi)\Omega\frac{\delta S}{\delta\Phi}, \tag{4.6}$$

where the minus sign applies only when both Ω and Ξ are Grassmann-odd. An immediate consequence of this formula and Eq. (4.3) is the conservation of the supercurrent $\partial_\mu \mathcal{V}^\mu_\alpha = 0$ at the quantum level. The simplest anomalous product we shall use later is

$$\Phi\frac{\delta S}{\delta\Phi} = -2\kappa W''(\Phi) \tag{4.7}$$

at the one-loop level, with $\kappa \equiv 1/(8\pi\sqrt{\pi\tau})$ as before; for a derivation one may evaluate Eq. (B2) of Ref. [18] in three dimensions.

The highest component in $\mathcal{V}^\mu_\alpha$ is written as

$$f^\mu_\alpha = -2\epsilon^{\mu\lambda\nu}\partial_\lambda(iF\gamma_\nu\psi - \phi\partial_\nu\psi) + r^\mu_\alpha, \tag{4.8}$$

where r^μ_α collectively stands for potentially anomalous products which take essentially the same form as in the 2d kink case. One can show quite generally, using the formula (4.6), that $X^\mu = r^\mu = 0$ at the quantum level [18]. Correspondingly, the associated spinor charge vanishes

$$\int_{-\infty}^{\infty} d^2\mathbf{x} f^0_\alpha = 0, \tag{4.9}$$

as long as the spinor field $\psi_\alpha \to 0$ for $x^1 \to \pm\infty$ while all the fields, like the classical domain-wall configuration $\phi_{DW}(x)$, are uniform for $x^2 \to \pm\infty$ [so that $\phi_i|_{x^2=\infty} = \phi_i|_{x^2=-\infty}$ with $\phi_i = (\phi, \psi_\alpha, F)$]. Hence only Q_α, P^λ and Z^λ form an irreducible supermultiplet, yielding a conserved-charge superfield

$$\int_{-\infty}^{\infty} d^2\mathbf{x}\mathcal{V}^0_\alpha = Q_\alpha - 2i(\gamma_\lambda\theta)_\alpha(P^\lambda + Z^\lambda), \tag{4.10}$$

which, upon supertranslations, correctly reproduces the supercharge algebra (2.16).

While anomalous products have left the conservation law $\partial_\mu \mathcal{V}^\mu_\alpha = 0$ intact, they cause some changes in the component currents of $\mathcal{V}^\mu_\alpha$. Consider, e.g., the topological current $\zeta^{\mu\lambda}_{\rm can}$ defined by Eq. (2.15) [with $X_\nu = 0$] and rewrite the first term as $F\partial_\nu\phi = -\partial_\nu W + (\delta S/\delta F)\partial_\nu\phi$, using $\delta S/\delta F = F + W'$. The key formula (4.6) then implies that $(\delta S/\delta F)\partial_\nu\phi = (1/2)\partial_\nu \times (\phi\delta S/\delta F)$, with the anomalous product $\phi(\delta S/\delta F) = -2\kappa W''(\phi)$ read [27] from the superfield product (4.7). In effect, F multiplied with $\partial\phi$ acts like $-(W' + \kappa W''')$ at the quantum level; the auxiliary field F thus changes its role in composite operators. As a result $\zeta^{\mu\lambda}_{\rm can}$ deviates from the classical expression (2.15) $\sim O(\hbar^{-1})$ by $\kappa W'' \sim O(\hbar^0)$,

$$\zeta^{\mu\lambda}_{\rm can} = -\epsilon^{\mu\lambda\nu}\partial_\nu\left\{W(\phi) + \frac{1}{8}\bar{\psi}\psi + \kappa W''(\phi)\right\}. \tag{4.11}$$

Note, however, that $\zeta^{\mu\lambda}_{\rm can}$, on eliminating F, is only apparently modified; its very definition (2.15) with F is left intact. Analogous apparent modifications take place in J^μ_α and $\Theta^{\mu\nu}$ as well. This is the general manner how the symmetry currents in supersymmetric theories accommodate quantum anomalies while leaving their supermultiplet structure and conservation laws untouched, as observed earlier in the 2d kink case [18].

The central-charge $\langle Z^\lambda\rangle = \int d^2\mathbf{x}\langle\zeta^{0\lambda}_{\rm can}\rangle$ is now related to the operators $W(\phi)$, $W''(\phi)$ and $\bar{\psi}\psi$ at spatial infinities $\mathbf{x} \to \pm\infty$. The composite operators $\bar{\psi}\psi$ and ϕ^2, in general, become nonvanishing in the presence of a classical field $\phi_{\rm c}(x)$, as seen from $\langle\chi\chi\rangle$ in Eq. (3.3). Actually, using the relation $(1/2)\bar{D}D\Phi^2 = \Phi\bar{D}D\Phi + (\bar{D}\Phi)D\Phi$, one can relate $\langle\bar{\psi}\psi\rangle \sim \langle\bar{D}\chi(z)D\chi(z)\rangle$ to $\langle\chi\chi\rangle$,

$$\begin{aligned}\langle\bar{D}\chi D\chi\rangle &= \left\{\frac{1}{2}\bar{D}D - 2W''_{\rm c}\right\}\langle\chi\chi\rangle + 2\langle\Phi\delta S/\delta\Phi\rangle \\ &= -8\kappa W''_{\rm c} + \frac{1}{2\pi}W''_{\rm c}|W''_{\rm c}| + O(D^2).\end{aligned} \tag{4.12}$$

In forming $\langle Z^\lambda\rangle$, the divergent term in $\langle\bar{\psi}\psi\rangle$ and the short-distance anomaly $\kappa W''$ combine to cancel so that

$$\frac{1}{8}\langle\bar{\psi}\psi\rangle + \kappa\langle W''(\phi)\rangle = \frac{1}{16\pi}W''_{\rm c}|W''_{\rm c}| + \cdots, \tag{4.13}$$

in confirmation of $U_{\rm eff}(\Phi_{\rm c})$ in Eq. (3.6) and hence the surface tension (3.10).

V. IMPROVEMENT AND TRACE IDENTITIES

In this section we examine the central-charge anomaly in the light of superconformal symmetry. The supercurrent $\mathcal{V}^{\mu}_{\alpha}$ is composed of super-Poincare currents and is also used [28] to construct the superconformal currents. As seen from the conservation law $\partial_{\mu}(\not{x}\mathcal{V}^{\mu}) = \gamma_{\mu}\mathcal{V}^{\mu}$ of the first-moment current, in particular, explicit breaking to superconformal symmetry is characterized by the quantity $\gamma_{\mu}\mathcal{V}^{\mu}$. Writing $i(\gamma_{\mu}\mathcal{V}^{\mu})_{\alpha} = 2(\bar{D}D\Phi)D_{\alpha}\Phi - (D_{\alpha}\bar{D}_{\beta}\Phi)D_{\beta}\Phi$ and isolating a term involving the equation of motion one can cast it in the form

$$i(\gamma_{\mu}\mathcal{V}^{\mu})_{\alpha} = -4D_{\alpha}W_{\rm eff}(\Phi), \tag{5.1}$$

$$W_{\rm eff}(\Phi) = W(\Phi) + \frac{1}{8}(\bar{D}\Phi)D\Phi - \frac{1}{2}\Phi\frac{\delta S}{\delta\Phi}, \tag{5.2}$$

with $-(1/2)\Phi(\delta S/\delta\Phi) = \kappa W''(\Phi)$ as quoted in Eq. (4.7). One may equally well write $W_{\rm eff}(\Phi)$ as

$$W_{\rm eff}(\Phi) = \frac{1}{16}\bar{D}D\Phi^2 + \tilde{W}_{\rm eff}(\Phi), \tag{5.3}$$

$$\tilde{W}_{\rm eff}(\Phi) = W(\Phi) - \frac{1}{4}\Phi W'(\Phi) - \frac{1}{4}\Phi\frac{\delta S}{\delta\Phi}. \tag{5.4}$$

Equation (5.1) is a supersymmetric version of the trace identity [29], as seen from the component expression

$$i(\gamma_{\mu}\mathcal{V}^{\mu})_{\alpha} = i(\gamma_{\mu}J^{\mu})_{\alpha} + 2\theta_{\alpha}\Theta^{\mu}_{\mu} - 2i(\gamma^{\nu}\theta)_{\alpha}\epsilon_{\nu\mu\lambda}\zeta^{\mu\lambda}_{\rm can} + \frac{1}{2}\bar{\theta}\theta(i\gamma_{\mu}f^{\mu})_{\alpha}. \tag{5.5}$$

The $(\gamma^{\nu}\theta)_{\alpha}$ component of Eq. (5.1), in particular, agrees with Eq. (4.11). This shows that the quantum modification of the topological current $\zeta^{\mu\lambda}_{\rm can}$ derives from the superconformal anomaly $-(1/2)\Phi(\delta S/\delta\Phi) = \kappa W''(\Phi)$ in $W_{\rm eff}(\Phi)$, as in the 2d kink case.

Let us here note that for $W(\Phi) = 0$ both $\phi(x)$ and $\psi_{\alpha}(x)$ are free and massless, and the model has exact conformal symmetry. Accordingly the $(1/8)\bar{D}\Phi D\Phi$ term in $W_{\rm eff}(\Phi)$ is not a genuine breaking term, and it can be removed by an appropriate redefinition, i.e., the so-called improvement, of the symmetry currents. As for the improvement [29] one may recall that one is free to modify a conserved current j^{μ} by adding a divergence of an antisymmetric tensor $\propto \partial_{\nu}j^{\mu\nu}$ without changing the conservation law $\partial_{\mu}j^{\mu} = 0$ and the charge $\int d^2\mathbf{x}j^0$.

For $\mathcal{V}^{\mu}_{\alpha}$ let us try an antisymmetric spinor-tensor superfield $I^{\nu\mu} = \Phi[\gamma^{\nu},\gamma^{\mu}]D\Phi = -i\epsilon^{\nu\mu\lambda}\gamma_{\lambda}D\Phi^2$ which obeys $\partial_{\nu}(\gamma_{\mu}I^{\nu\mu}) = -2\partial_{\nu}(\gamma^{\nu}D\Phi^2) = iD_{\alpha}\bar{D}D\Phi^2$. We define the improved supercurrent by $\mathcal{V}^{\mu}_{\alpha} - (1/4)\partial_{\nu}I^{\nu\mu}_{\alpha}$ or

$$\tilde{\mathcal{V}}^{\mu}_{\alpha} = \mathcal{V}^{\mu}_{\alpha} - \frac{i}{4}\epsilon^{\mu\nu\rho}\partial_{\nu}(\gamma_{\rho}D\Phi^2)_{\alpha}, \tag{5.6}$$

which then satisfies the "improved" trace identity

$$i(\gamma_{\mu}\tilde{\mathcal{V}}^{\mu})_{\alpha} = -4D_{\alpha}\tilde{W}_{\rm eff}(\Phi) \tag{5.7}$$

with the superpotential $\tilde{W}_{\rm eff}(\Phi)$ defined in Eq. (5.4).

Passing from $\mathcal{V}^{\mu}_{\alpha}$ to $\tilde{\mathcal{V}}^{\mu}_{\alpha}$ yields the following supermultiplet of improved symmetry currents

$$\begin{aligned}
\tilde{J}^{\mu}_{\alpha} &= J^{\mu}_{\alpha} - \frac{i}{2}\epsilon^{\mu\nu\rho}\partial_{\nu}\{(\gamma_{\rho}\psi)_{\alpha}\phi\},\\
\tilde{\Theta}^{\mu\lambda} &= \Theta^{\mu\lambda} + \frac{1}{8}(g^{\mu\lambda}\partial^2 - \partial^{\mu}\partial^{\lambda})\phi^2,\\
\tilde{\zeta}^{\mu\lambda} &= \epsilon^{\mu\lambda\nu}\Big\{F\partial_{\nu}\phi - \frac{1}{4}\partial_{\nu}(F\phi)\Big\},\\
\tilde{f}^{\mu}_{\alpha} &= f^{\mu}_{\alpha} + \frac{i}{2}(g^{\mu\nu}\partial^2 - \partial^{\mu}\partial^{\nu})(\gamma_{\nu}\psi\phi)_{\alpha}.
\end{aligned} \tag{5.8}$$

Interestingly, the present improvement has removed the antisymmetric component $T^{\mu\lambda}_{\rm asym} \propto \epsilon^{\mu\lambda\nu}\partial_{\nu}(\bar{\psi}\psi)$ from $T^{\mu\lambda}$, yielding the symmetric energy-momentum tensor $\tilde{\Theta}^{\mu\lambda}$ with a well-known improvement term [29] and the topological current $\tilde{\zeta}^{\mu\lambda}$ involving no fermion field. From Eq. (5.7) follow the improved trace identity and its superpartners:

$$\begin{aligned}
i\gamma_{\mu}\tilde{J}^{\mu} &= -4\psi\tilde{W}'_{\rm eff}(\phi),\\
\tilde{\Theta}^{\mu}_{\mu} &= -2F\tilde{W}'_{\rm eff}(\phi) + \tilde{W}''_{\rm eff}(\phi)\bar{\psi}\psi,\\
\tilde{\zeta}^{\mu\lambda} &= -\epsilon^{\mu\lambda\nu}\partial_{\nu}\tilde{W}_{\rm eff}(\phi),\\
i\gamma_{\mu}\tilde{f}^{\mu} &= -4i\partial_{\mu}[\gamma^{\mu}\psi\tilde{W}'_{\rm eff}(\phi)].
\end{aligned} \tag{5.9}$$

With $\tilde{\mathcal{V}}^{\mu}_{\alpha}$ one again finds a conserved-charge superfield

$$\int_{-\infty}^{\infty} d^2x\,\tilde{\mathcal{V}}^{0}_{\alpha} = \tilde{Q}_{\alpha} - 2i(\gamma_{\lambda}\theta)_{\alpha}(\tilde{P}^{\lambda} + \tilde{Z}^{\lambda}), \tag{5.10}$$

which shows that the improved charges $\tilde{Q}_{\alpha}$, $\tilde{P}^{\lambda}$ and $\tilde{Z}^{\lambda}$ obey the same supercharge algebra as in Eq. (2.16). Note that $\tilde{Q}_{\alpha}$ and $\tilde{P}^{\lambda}$ are essentially the same as the original charges Q_{α} and P^{λ}, under the same asymptotic ($\mathbf{x} \to \pm\infty$) condition on the fields as discussed for the fermionic charge $\int d^2\mathbf{x}f^0_{\alpha}$ in Eq. (4.9). This in turn implies that the central charges $Z^{\lambda} = \int d^2\mathbf{x}\zeta^{0\lambda}_{\rm can}$ and $\tilde{Z}^{\lambda} = \int d^2\mathbf{x}\tilde{\zeta}^{0\lambda}$, though different in form, are physically equivalent.

It is enlightening to verify this equivalence. It is a simple task to evaluate, using Eqs. (3.3), (4.7), and (4.12), the expectation values of the effective superpotentials $W_{\rm eff}(\Phi)$ and $\tilde{W}_{\rm eff}(\Phi)$ to one-loop or $O(\hbar^0)$:

$$\langle W_{\rm eff}(\Phi)\rangle = U_{\rm eff}(\Phi_{\rm c}) + \frac{1}{8}\bar{D}\Phi_{\rm c}D\Phi_{\rm c}, \tag{5.11}$$

$$\langle \tilde{W}_{\rm eff}(\Phi)\rangle = U_{\rm eff}(\Phi_{\rm c}) - \frac{1}{4}\Phi_{\rm c}U'_{\rm eff}(\Phi_{\rm c}), \tag{5.12}$$

apart from terms of $O(D^2)$, with $U_{\rm eff}(\Phi_{\rm c})$ defined in Eq. (3.5). Here we see that $\langle W_{\rm eff}(\Phi)\rangle$ and $\langle\tilde{W}_{\rm eff}(\Phi)\rangle$ precisely agree with $U_{\rm eff}(\Phi_{\rm c})$ at spatial infinities $x^1 \to \pm\infty$ where $\psi_{\rm c} \to 0$ and $\phi_{\rm c} \to \pm v$ with v determined from $U'_{\rm eff}(v) = 0$. The resulting central charges

$$\langle Z^{\lambda}_{\rm can}\rangle/L_y = \langle \tilde{Z}^{\lambda}\rangle/L_y = \delta^{\lambda 2}2U_{\rm eff}(v) \tag{5.13}$$

are thus in agreement with Eq. (3.10) obtained from the effective action.

VI. INDUCED SPIN AND EXTENDED SUPERSYMMETRY

In this section we explore physical origins of the central-charge anomaly. In Eq. (4.13) we have seen that the main quantum correction to the central-charge comes from the $\frac{1}{8}\langle\bar{\psi}\psi\rangle$ portion of $\langle Z^2\rangle$. In three dimensions, with the spatial-rotation matrix $\sigma^{12} \equiv (i/2)[\gamma^1, \gamma^2] = \sigma_2 = \gamma^0$, the fermion mass term is nothing but the spin density $\psi^{\dagger}\frac{1}{2}\sigma^{12}\psi = \frac{1}{2}\bar{\psi}\psi$. Accordingly, through the "Zeemann coupling" $W''\bar{\psi}\psi$, the domain-wall configuration works to align the fermion-spin oppositely in the two domains;, e.g., for $x^1 < 0$, $W'' > 0$ so that $\langle\bar{\psi}\psi\rangle < 0$ is preferred. The quantum central-charge $\sim\langle\bar{\psi}\psi\rangle$ is therefore ascribed to induced spin in the domain-wall background.

Actually, it is possible to evaluate the induced polarization reliably with free fermions if one notes that, except for the vicinity of the wall, the effective fermion mass is almost constant $W'' \approx \mp m$ in each domain. Consider the relativistic expression $\langle\bar{\psi}(x)\psi(x)\rangle = -{\rm tr}\langle x|i/(\not{p} - m)|x\rangle$ and integrate over p^0 first. The result clarifies the meaning of $\langle\bar{\psi}\psi\rangle$: It is written as a sum over (twice) aligned spins of negative-energy fermions filling the Dirac sea, the fermionic vacuum:

$$\langle\bar{\psi}\psi\rangle = -\sum_{\mathbf{p}}\frac{m}{|m|} = -\frac{m}{2\pi}(\sqrt{\Lambda^2 + m^2} - |m|), \tag{6.1}$$

where $\sum_{\mathbf{p}} \equiv \int\{d^2\mathbf{p}/(2\pi)^2\}|m|/\epsilon = |m|\int d\epsilon/(2\pi)$ is the phase-space sum and $\epsilon = \sqrt{m^2 + \mathbf{p}^2}$. The divergent contribution involving the momentum cutoff Λ^2 is associated with the infinite depth or infinite phase-space of the Dirac sea [30]. The conformal anomaly $\sim\kappa W''(\phi)$ works to cancel this infinite intrinsic spin of the fermionic vacuum, leaving finite induced spin $m^2/(2\pi)$ for the central-charge, as we have seen in Eq. (4.13).

Note here that, upon improvement (5.6), the induced fermion-spin $\sim\frac{1}{8}\langle\bar{\psi}\psi\rangle$ in the topological current $\zeta^{\mu\lambda}_{\rm can}$ is transferred into an induced (bosonic) superpotential $-\frac{1}{4}\langle\phi W'(\phi)\rangle + \frac{1}{2}\kappa\langle W''(\phi)\rangle$ in the improved current $\tilde{\zeta}^{\mu\lambda}$; see Eq. (5.9). This reveals a dual (fermionic/bosonic) character of the central-charge anomaly. This dualism is unexpected but is quite natural since in supersymmetric theories fermionic and bosonic quantum fluctuations are intimately related, as seen from Eq. (4.12).

The dual character of the anomaly can be verified for the 2d kink case [18] as well. There the trace identity is governed by the superpotential

$$W^{\rm kink}_{\rm eff}(\Phi) = W(\Phi) - \frac{1}{2}\Phi\frac{\delta S}{\delta\Phi} \tag{6.2}$$

and the central-charge anomaly comes from the superconformal anomaly $-\frac{1}{2}\Phi(\delta S/\delta\Phi) = W''(\Phi)/(4\pi)$. We try the following improvement. Let $I_2^{\nu\mu} = [\gamma^{\nu}, \gamma^{\mu}]D\Phi^2$ and consider the improved supercurrent $\tilde{\mathcal{V}}^{\mu}_{\alpha} = \mathcal{V}^{\mu}_{\alpha} - (1/4)\partial_{\nu}(I_2)^{\nu\mu}_{\alpha}$. It satisfies the improved trace identity

$$i(\gamma_{\mu}\tilde{\mathcal{V}}^{\mu})_{\alpha} = -2D_{\alpha}\tilde{W}^{\rm kink}_{\rm eff}(\Phi) \tag{6.3}$$

with the new effective superpotential

$$\tilde{W}^{\rm kink}_{\rm eff}(\Phi) = W(\Phi) - \frac{1}{2}\Phi W'(\Phi) - \frac{1}{4}(\bar{D}\Phi)D\Phi. \tag{6.4}$$

Here the improvement has been made to remove the anomaly from $W^{\rm kink}_{\rm eff}(\Phi)$. One can then verify that the central-charge anomaly entirely resides in the fermion sector $-\frac{1}{4}\langle\bar{D}\Phi D\Phi\rangle = W''(\Phi_{\rm c})/(4\pi) + C$ while $-\frac{1}{2}\times\langle\Phi W'(\Phi)\rangle = -\frac{1}{2}\Phi_{\rm c}U'_{\rm eff}(\Phi_{\rm c}) - C$ works to remove the divergent piece $C \sim (W''_{\rm c}/8\pi)\log[\Lambda^2/(W''_{\rm c})^2]$ from the former. Thus in this case one would interpret the central-charge anomaly as due to induced quantum number $\langle\bar{\psi}\psi\rangle$ in the kink background.

Finally, as for the presence or absence of anomalies in the central-charge it is instructive to look into the case of $N = 2$ supersymmetry, for which, in $1 + 1$ dimensions, the central-charge anomaly is known to be absent [12,14]. The relevant $N = 2$ model in (two or) three dimensions is obtained from the 4d Wess-Zumino model via dimensional reduction. In terms of two $N = 1$ real superfields $\Phi_1(z)$ and $\Phi_2(z)$ the superspace action is expressed in the form (2.20) with the kinetic term $\bar{D}\Phi D\Phi \to \sum_i\bar{D}\Phi_i D\Phi_i$ and the superpotential [14]

$$W(\Phi_1, \Phi_2) = \frac{m^2}{4\lambda}\Phi_1 - \frac{\lambda}{3}\Phi_1^3 + \lambda\Phi_1\Phi_2^2, \tag{6.5}$$

which is harmonic, $\sum_i W_{ii} = 0$ with $W_{ij} \equiv \partial^2 W/\partial\Phi_i\partial\Phi_j$, a property characteristic of extended supersymmetry. The classical domain-wall configuration is realized with $\phi_1(x) \to \phi_{\rm DW}(x)$ and $\phi_2(x) \to 0$.

The conserved supercurrent is written as a sum $\mathcal{V}^{\mu}_{\alpha} = \mathcal{V}^{\mu}_{\alpha}[\Phi_1] + \mathcal{V}^{\mu}_{\alpha}[\Phi_2]$ of $\mathcal{V}^{\mu}_{\alpha}$ in Eq. (4.1) and the associated trace identity is written as $i(\gamma_{\mu}\mathcal{V}^{\mu})_{\alpha} = -4D_{\alpha}W_{\rm eff}$ with

$$W_{\rm eff} = W(\Phi_1, \Phi_2) + \frac{1}{8}(\bar{D}\Phi_i)D\Phi_i - \frac{1}{2}\Phi_i(\delta S/\delta\Phi_i). \tag{6.6}$$

The central-charge is now read from this $W_{\rm eff}$. Note that in each domain the two species of fermions $D\Phi_{1,2}$ have effective masses opposite in sign, $W_{11}(\phi_i) = -W_{22}(\phi_i) = -2\lambda\phi_1 \sim -m$. The induced spin $\langle\bar{D}\Phi_i D\Phi_i\rangle$ from each species therefore differs in sign, yielding no net polarization in each domain. No infinite polarization or no intrinsic short-distance anomaly thereby remains with the sum $\sum_i\Phi_i(\delta S/\delta\Phi_i) \sim \sum_i W_{ii}$ vanishing, leaving no anomaly in the central-charge.

The absence of induced spin and that of the central-charge anomaly are thus correlated. This is not a coincidence. It is a consequence of the nonrenormalization theorem [22,31] which states that there is no quantum correction to chiral superpotentials. The chiral structure inherent in $N = 1$ supersymmetry in four dimensions is responsible for the absence of the central-charge anomaly in the present dimensionally-reduced model with $N = 2$ supersymmetry.

VII. SUMMARY AND DISCUSSION

In the present paper we have studied some general features of the central-charge anomaly for domain walls in three-dimensional supersymmetric theories. The way the anomaly arises in the supercharge algebra critically depends on the dimension of spacetime. For kinks in two dimensions the central-charge anomaly arises as part of the superconformal anomaly. For domain walls in three dimensions the central-charge has, besides the superpotential $W(\phi)$, a fermion-spin term $\sim\bar{\psi}\psi$ at the classical level. The quantum modifications of the supercharge algebra therefore come not only from the short-distance anomaly but also from quantum induced spin $\sim\langle\bar{\psi}\psi\rangle$, and the latter dominates in the sum. For domain walls the central-charge anomaly is thus ascribed to quantum induced spin of long-distance origin.

The best place to explore the central-charge anomaly is the supersymmetric trace identity, in view of the fact that the topological current lies in a supermultiplet together with the energy-momentum tensor and supercurrent. This naturally has led us to consider the improvement of the superfield supercurrent (since one normally has to improve the canonical energy-momentum tensor to arrive at the well-behaved conformal currents [29]). We have thereby seen that the anomaly in the central-charge, upon improvement, can be transferred from induced spin in the fermion sector to an induced potential in the boson sector, or vice versa. This has revealed an unexpected dual character, both fermionic and bosonic, of the central-charge anomaly. This (boson/fermion) dualism has a further consequence for kinks in two dimensions. There one can make an improvement so that the short-distance anomaly is transformed into induced fermion quantum number $\sim\langle\bar{\psi}\psi\rangle$; the central-charge anomaly thus has a dual character of either short- or long-distance origin as well.

We have also examined the case of extended supersymmetry and noted that the absence or presence of the short-distance anomaly and that of induced spin are correlated. This coincidence is quite natural in the light of the dual character of the central-charge anomaly, which itself is a reflection of the underlying supersymmetry.

Finally it would be worth remarking that the superfield formalism (+ regularization) provides a natural means of preserving supersymmetry at the quantum level, best suited for the analysis of anomalies. Extensive use of the superfield supercurrent has made manifest the supermultiplet nature of various symmetry currents, conservation laws and the associated anomalies. Use of superfields has also helped us systematize the process of improvement of the supercurrent, which, if done separately for each component current, could have been a laborious task.

ACKNOWLEDGMENTS

This work was supported in part by a Grant-in-Aid for Scientific Research from the Ministry of Education of Japan, Science and Culture (Grant No. 14540261).

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