

# Quasinormal modes in pure de Sitter spacetimes

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(Received 6 April 2004; published 16 September 2004)

We have studied scalar perturbations as well as fermion perturbations in pure de Sitter spacetimes. For scalar perturbations we have shown that well-defined quasinormal modes in  $d$ -dimensions can exist provided that the mass of scalar field  $m > \frac{d-1}{2l}$ . The quasinormal modes of fermion perturbations in three and four dimensional cases have also been investigated. We found that different from other dimensional cases, in the three dimensional pure de Sitter spacetime there is no quasinormal mode for the s-wave. This interesting difference caused by the spacial dimensions is true for both scalar and fermion perturbations.

DOI: 10.1103/PhysRevD.70.064024

PACS numbers: 04.30.-w, 04.62.+v

## I. INTRODUCTION

It is well known that the surrounding geometry of a black hole will experience damped oscillations under perturbations. The frequencies and damping times of the oscillations are entirely fixed by the black hole parameters and independent of the initial perturbations. These oscillations are called “quasinormal modes”(QNM), which is believed as a characteristic “sound” of black holes and would lead to the direct identification of the black hole existence through gravitational wave observation to be realized in the near future [1]. Because of the potential astrophysical interest, a great deal of effort has been devoted to the study of black holes’ QNMs. Most of these studies were concerned with black holes immersed in an asymptotically flat spacetime [2]. Considering the case when the black hole is immersed in an expanding universe, QNMs of black holes in de Sitter(dS) space have also attracted much attention [3].

Motivated by the discovery of the AdS/CFT correspondence, the investigation of QNM in antide Sitter(AdS) spacetimes became appealing in the past several years. It was argued that the QNMs of AdS black holes have direct interpretation in term of the dual conformal field theory (CFT) [4–9]. In dS space the relation between bulk dS spacetime and the corresponding CFT at the past boundary  $I^-$  and future boundary  $I^+$  in the framework of scalar perturbation spectrums has also been discussed [10]. A quantitative support of the dS/CFT correspondence was provided.

Recently QNMs in asymptotically flat spaces have acquired further attention, since the possible connection between the classical vibrations of a black hole spacetime

and various quantum aspects was proposed by relating the real part of the QNM frequencies to the Barbero-Immirzi (BI) parameter, a factor introduced by hand in order that loop quantum gravity reproduces correctly the black hole entropy [11]. In order to see whether this quantum connection is true in Schwarzschild dS (SdS) spacetime, a number of extensions have been made [12][13][14][15]. For the nearly extreme SdS black hole, like the Schwarzschild spacetime, the real part is found still proportional to the black hole surface gravity, but instead of the integer  $n$  labeling the modes, with proportional coefficients  $\sqrt{\ell(\ell+1) - \frac{1}{4}}$  for scalar and electromagnetic perturbations and  $\sqrt{(\ell+2)(\ell-1) - \frac{1}{4}}$  for gravitational perturbations. It is too early to confirm the recent conjecture by relating the  $\ell$ -dependent real part to the BI parameter[12]. The imaginary part of the QNM frequencies for the nearly extreme SdS black hole was found having an equally spacial structure with the level spacing depending on the surface gravity of the black hole, which agrees to the result in asymptotically flat spacetime and is independent of whether the perturbation is scalar, electromagnetic or gravitational[12]. This result was confirmed even for the very small SdS black holes [13] and later further supported by using the Born approximation [14].

The dependence of the surface gravity for both the real part and imaginary part of QNM frequencies suggests that there is a possible connection between the QNMs and thermodynamics of the black hole horizon. But what is the effect of the cosmological horizon here? It is believed that cosmological horizon has very similar thermodynamical behavior to that of the black hole horizon [16]. Is there a connection between the QNMs and surface gravity of the cosmological horizon? In order to answer this question, in this paper we are going to investigate the pure dS spacetime. It was argued that for a massless

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minimally coupled scalar field, there exists no QNMs in the pure dS space-times, however for a massive scalar field, there do exist well-defined QNMs[14]. We will examine this argument in different topological pure dS space-times by considering scalar perturbation and fermion perturbation. Different from the purpose to test the dS/CFT correspondence[10], we here concentrate on the region within the cosmological horizon.

The structure of the paper is as follows: In Sec. II, we will study the scalar perturbation in d-dimensional pure dS spacetime with a spherically symmetric space. The extension to topological pure dS space-times will be shown in Sec. III. In Sec. IV, we will present the discussion of the fermion perturbation. Our main results will be summarized in Sec. V.

## II. SCALAR PERTURBATION IN PURE DE SITTER SPACE

The static metric of a  $d$  dimensional dS space reads

$$ds^2 = -f(r)dt^2 + f^{-1}(r)dr^2 + r^2 d\Omega_{d-2}^2, \quad (1)$$

where  $f(r) = 1 - r^2/l^2$  ( $l$  is the minimal radius of dS space) and  $r^2 d\Omega_{d-2}^2$  represents the metric on the  $d-2$  dimensional sphere  $S^{d-2}$  of radius  $r$ .

We begin our discussion with a massive scalar field  $\Phi$ , satisfying the Klein-Gordon equation

$$\Phi_{; \nu}^{;\nu} = m^2 \Phi \quad (2)$$

This equation can be separated by  $\Phi = [u(r)/r^{(d-2)/2}]e^{-i\omega t}Y_\ell(\Omega_{d-2})$ . Here the spherical harmonic  $Y_\ell(\Omega_{d-2})$  is the eigen function of  $d-2$  dimensional Laplace-Beltrami operator  $\nabla_{d-2}^2$  with the eigenvalue  $-\ell(\ell + d - 3)$ . Using the tortoise coordinate,  $r_* = \int dr/f(r) = l \tanh^{-1}(r/l)$ , we can write the radial part into a Schrödingerlike equation

$$-\frac{d^2 u}{dr_*^2} + V(r_*)u = \omega^2 u \quad (3)$$

with effective potential

$$V(r_*) = -\frac{\mathcal{A}}{l^2 \cosh^2(r_*/l)} + \frac{\mathcal{B}}{l^2 \sinh^2(r_*/l)} \quad (4)$$

where  $\mathcal{A}$  and  $\mathcal{B}$  are defined by

$$\mathcal{A} = \frac{d-2}{2} \left( \frac{d-2}{2} + 1 \right) - m^2 l^2,$$

$$\mathcal{B} = \ell(\ell + d - 3) + \frac{d-2}{2} \left( \frac{d-2}{2} - 1 \right).$$

When  $\mathcal{B} > 0$ , the effective potential (4) diverges to a positive infinity at the origin ( $r = 0$ ) and vanishes at the cosmological horizon. On the other hand, if  $\mathcal{B} < 0$ , the potential falls down to negative infinity at the origin which indicates the instability of the perturbation. We will show it in detail later.

In the form of a new variable  $z = 1/\cosh^2(r_*/l)$ , Eq. (3) becomes

$$z(1-z)u'' + \left(1 - \frac{3}{2}z\right)u' + \frac{1}{4} \left[ \frac{\omega^2 l^2}{z} - \frac{\mathcal{B}}{1-z} + \mathcal{A} \right] u = 0. \quad (5)$$

Further using the ansatz  $u = z^\alpha(1-z)^\beta F(z)$ , we have

$$z(1-z)F'' + \left[ 1 + 2\alpha - (2\alpha + 2\beta + \frac{3}{2})z \right] F' + \left\{ \frac{1}{z} \left( \alpha^2 + \frac{\omega^2 l^2}{4} \right) + \frac{1}{1-z} \left( \beta^2 - \frac{1}{2}\beta - \frac{\mathcal{B}}{4} \right) - \left[ (\alpha + \beta)^2 + \frac{1}{2}(\alpha + \beta) - \frac{\mathcal{A}}{4} \right] \right\} F = 0 \quad (6)$$

If we properly select values of  $\alpha, \beta$  to make terms like  $\frac{1}{z}$  and  $\frac{1}{1-z}$  disappear, the solution to Eq. (6) is the standard hypergeometric function. It is not hard to find that  $z^\alpha \sim \exp(\pm i\omega r_*)$  when  $z \rightarrow 0$  (that is, approaching the horizon), so that the two independent solutions exactly correspond to the incoming and outgoing waves at the cosmological horizon. The general solution is

$$u(z) = u_1(z) + u_2(z) = C_1 z^{-\alpha} (1-z)^\beta {}_2F_1(a-c+1, b-c+1, 2-c, z) + C_2 z^\alpha (1-z)^\beta {}_2F_1(a, b, c, z) \quad (7)$$

where  $C_1, C_2$  are constants, and the parameters of the hypergeometric function are given by

$$c = 2\alpha + 1 \quad a = \alpha + \beta + \frac{1}{4}(1 + \sqrt{1 + 4\mathcal{A}})$$

$$b = \alpha + \beta + \frac{1}{4}(1 - \sqrt{1 + 4\mathcal{A}})$$

Note that the condition of writing the solution in the form of (7) is that  $c$  is not an integer.

We will restrict ourselves in the well-accepted definition of the quasinormal mode which is, in this special case, that these modes are purely outgoing waves at the cosmological horizon and vanish at  $r = 0$ . Indeed this boundary condition is determined by the behavior of the effective potential. Since the sign of  $\alpha$  is arbitrary, it is easy to check that choosing either  $u_1$  or  $u_2$  as the outgoing wave does not lead to any difference. Here we take  $C_1 = 0$  (so  $\alpha = -i\omega l/2$ ), hence the incoming wave is eliminated. According to the property of hypergeometric function[17], we can change the wave function into

$$\begin{aligned}
 u_2(z) &= C_2 z^\alpha (1-z)^\beta \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \\
 &\quad \times {}_2F_1(a, b, a+b-c+1, 1-z) \\
 &\quad + C_2 z^\alpha (1-z)^{(1/2)-\beta} \frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)} \\
 &\quad \times {}_2F_1(c-a, c-b, c-a-b+1, 1-z) \quad (8)
 \end{aligned}$$

To make it zero at the origin, we use the poles of Gamma functions. And to obtain the discrete poles which result in the level spacing frequencies we should assume that

$$\beta\left(\frac{1}{2} - \beta\right) = -\frac{\mathcal{B}}{4} \leq 0,$$

which is identical to our previous analysis of the effective potential. It is interesting to find that in the three dimensional case, this condition is equivalent to  $l^2 \geq 1/4$ , which implies that there exists no QNM for s-wave perturbation.

There are two sets of poles. (i) When  $\beta = \frac{d+2\ell}{4} - \frac{1}{2} \geq \frac{1}{2}$  (note that for the three dimensional case we have excluded the s-wave perturbation for the reason mentioned), the poles are  $a = -n$  or  $b = -n$  ( $n = 0, 1, \dots$ ). These poles indeed remove the divergent part of the wave function around the origin since both  $c$  and  $a+b-c = \frac{d+2\ell-3}{2}$  could not be nonpositive integers and the numerator remains regular. (ii) When  $\beta = -\frac{d+2\ell}{4} + 1 < 0$ , the poles are  $c-a = -n$  or  $c-b = -n$  ( $n = 0, 1, \dots$ ). Because both  $c$  and  $c-a-b = \frac{d+2\ell-3}{2}$  could not be nonpositive integers, these poles are well-defined.

The corresponding frequencies are

$$\omega = -\frac{i}{l}(2n + \ell + h_\pm) \quad (9)$$

or

$$\omega = -\frac{i}{l}(2n - \ell - d + 3 + h_\pm) \quad (10)$$

where  $h_\pm = \frac{d-1}{2} - \sqrt{(\frac{d-1}{2})^2 - m^2 l^2}$ .

The above discussion should be restricted to the condition that  $c$  is not an integer. Now let us see what will happen when this condition is violated. It only occurs in the massless case  $m = 0$ . We assume that  $c = k$ , where  $k = 0, \pm 1, \pm 2, \dots$ . When  $c = k \geq 1$ , we can obtain the solution satisfying the boundary condition[17]

$$u(z) \propto z^{-i\omega l/2} (1-z)^{(d+2l-2)/4} {}_2F_1(a, b, k, z) \quad (11)$$

and the corresponding quasinormal frequencies are

$$\omega = \frac{i}{l}(k-1) \quad (12)$$

When  $c = k \leq 0$ , the proper solution is then[17]

$$\begin{aligned}
 u(z) &\propto z^{-i\omega l/2} (1-z)^{(d+2l-2)/4} \\
 &\quad \times {}_2F_1(a-k+1, b-k+1, 2-k, z) \quad (13)
 \end{aligned}$$

and the quasinormal frequencies are

$$\omega = -\frac{i}{l}(k-1). \quad (14)$$

It is surprising that all these modes (9), (10), (12), and (14) are just some kinds of ‘‘distribution’’ along the radius. To show that, we compute its flux[7]

$$\mathcal{F} = \sqrt{|g|} \frac{1}{2i} (R^* \partial_r R - R \partial_r R^*) \quad (15)$$

where  $R = u_2(r)/r^{\frac{d-2}{2}}$  is the radial factor of the wave function. If frequencies  $\omega$  in (9), (10), (12), and (14) are all purely imaginary values,  $\alpha = -i\omega l/2$  is real and so are the parameters  $a, b$  and  $c$ . Then  $R(r)$  is proportional to a purely real function defined in the region  $0 \leq r \leq l$ , which results in the vanishing of the flux (15) everywhere as well as at the horizon! This is in contradiction with the definition of QNM.

However for the massive case, one can see from the scalar perturbation spectrum (9) and (10) that if  $m^2 l^2 > (\frac{d-1}{2})^2$ , the frequencies are not purely imaginary values which insure that the flux does not vanish. These modes are purely outgoing waves at the cosmological horizon and so are well-defined QNMs. Therefore there exists the lowest bound ( $m > (d-1)/2l$ ) of the mass of scalar field that permits QNMs to survive. We rewrite the corresponding QNM frequencies as follows:

$$\omega = \pm \frac{1}{l} \left[ m^2 l^2 - \left( \frac{d-1}{2} \right)^2 \right]^{1/2} - \frac{i}{l} \left( 2n + \ell + \frac{d-1}{2} \right) \quad (16)$$

or

$$\begin{aligned}
 \omega &= \pm \frac{1}{l} \left[ m^2 l^2 - \left( \frac{d-1}{2} \right)^2 \right]^{1/2} \\
 &\quad - \frac{i}{l} \left( 2n - \ell + 3 - \frac{d+1}{2} \right) \quad (17)
 \end{aligned}$$

This result confirms the argument that QNMs cannot exist for massless scalar field[14]. Here we further present the lowest mass bound for the scalar field to possess QNMs.

### III. SCALAR PERTURBATION IN TOPOLOGICAL DS SPACE

In AdS spacetime, the QNMs of different topological black holes have been studied recently. It was found that black hole topology influence a lot on the QNMs of scalar perturbations[8]. Here we would like to extend the investigation to the topological dS space.

The metric of the topological dS space is showed as (1) with[18]

$$f(r) = k + \frac{2Gm}{r^{d-3}} - \frac{r^2}{l^2} \quad (18)$$

where  $k = 0, -1$  correspond to two kinds of hypersurfaces  $\Omega_{d-2}$ : Ricci flat space  $R^{d-2}$  and negative constant curvature space  $H^{d-2}$ , respectively. Note that  $d \geq 4$  since  $d = 3$  is trivial.

The wave equation for the scalar perturbation still has the form as (3), but the effective potential

$$V(r) = \frac{K_\Omega}{r^2} + \frac{d-2}{2} \frac{ff'}{r} + \frac{d-2}{2} \left( \frac{d-2}{2} - 1 \right) \frac{f^2}{r^2} + m^2 f(r) \quad (19)$$

where  $K_\Omega$  is the eigenvalue of Laplace-Beltrami operator  $\nabla_{d-2}^2$ . It is easy to find that the effective potential tends to zero at cosmological horizon, while it goes to negative infinity when  $r \rightarrow 0$ . Because of the existence of negative effective potential for the scalar field equation of motion, the bound states can be formed leading to the growing modes instead of decaying modes[19].

The growing modes imply that these topological dS space are not stable. This result is not so surprising, since both of these topological dS spaces possess naked singularity.

In AdS case, though the wave amplification behavior exists in hyperbolic space, for flat space it still exhibits decaying wave for scalar perturbation[8]. The growing modes for the flat hypersurface in dS case can be considered as one more difference between dS and AdS spacetimes.

## IV. FERMION PERTURBATION

### A. Three dimensional case

The fermion perturbation in a BTZ background was studied in [7,20,21]. We are interested to generalize the investigation of the fermion perturbation in a pure dS space. There is a little difference between three and four dimensional case due to the different representations of gamma matrixes. We first consider the three dimensional case. After the coordinate transformation  $t = \tau$ ,  $r = l \cos \mu$  and  $\phi = \varphi/l$ , metric (1) becomes

$$ds^2 = -\sin^2 \mu d\tau^2 + l^2 d\mu^2 + \cos^2 \mu d\varphi^2. \quad (20)$$

We begin with the Dirac equation

$$\gamma^a e_a^\nu (\partial_\nu + \Gamma_\nu) \Psi + m \Psi = 0, \quad (21)$$

where  $\gamma^a$  are the conventional spin matrixes. In the three dimensional case we choose the representation of them in term of Pauli matrixes:  $\gamma^0 = i\sigma^2$ ,  $\gamma^1 = \sigma^1$ ,  $\gamma^2 = \sigma^3$ . The triads  $e_a^\nu$  are

$$e_\tau^0 = \sin \mu, e_\mu^1 = l, e_\varphi^2 = \cos \mu. \quad (22)$$

The spin connection, defined by  $\Gamma_\nu = \frac{1}{8} \gamma^{[a} \gamma^{b]} e_a^\lambda e_{b\lambda;\nu}$ , has only two nonvanishing components

$$\Gamma_\tau = -\frac{1}{4l} \cos \mu \gamma^{[0} \gamma^1], \quad \Gamma_\varphi = -\frac{1}{4l} \sin \mu \gamma^{[2} \gamma^1]. \quad (23)$$

We write out Eq. (21) explicitly

$$\left[ \frac{1}{l} \sigma^1 \left( \partial_\mu + \frac{\cos \mu}{2 \sin \mu} - \frac{\sin \mu}{2 \cos \mu} \right) + \frac{i\sigma^2}{\sin \mu} \partial_\tau + \frac{\sigma^3}{\cos \mu} \partial_\varphi + m \right] \Psi = 0. \quad (24)$$

Separating the equation by

$$\Psi = \frac{e^{-i\omega\tau} e^{-i\ell\varphi/l}}{\sqrt{\sin \mu \cos \mu}} \begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix}$$

we arrive at the radial part equation

$$\left( \partial_\mu + \frac{i\omega l}{\sin \mu} \right) \Psi_1 = - \left( ml + \frac{i\ell}{\cos \mu} \right) \Psi_2, \quad (25)$$

$$\left( \partial_\mu - \frac{i\omega l}{\sin \mu} \right) \Psi_2 = - \left( ml - \frac{i\ell}{\cos \mu} \right) \Psi_1, \quad (26)$$

Introducing a new set of wave functions  $\psi_1, \psi_2$ , which relate to  $\Psi_1, \Psi_2$  by

$$\Psi_1 + \Psi_2 = (1 + \tan^2 \mu)^{-1/4} \sqrt{1 + i \tan \mu} (\psi_1 + \psi_2) \quad (27)$$

$$\Psi_1 - \Psi_2 = (1 + \tan^2 \mu)^{-1/4} \sqrt{1 - i \tan \mu} (\psi_1 - \psi_2) \quad (28)$$

and using a new variable  $y = \tan \mu$  for the sake of convenience, we turn Dirac equations into

$$(1 + y^2) \partial_y \psi_1 + \left( -\ell y + \frac{i\ell\omega}{y} \right) \psi_1 = - \left[ \left( ml + \frac{i}{2} \right) + (i\ell - l\omega) \right] \psi_2, \quad (29)$$

$$(1 + y^2) \partial_y \psi_2 - \left( -\ell y + \frac{i\ell\omega}{y} \right) \psi_2 = - \left[ \left( ml + \frac{i}{2} \right) - (i\ell - l\omega) \right] \psi_1. \quad (30)$$

Further by setting  $z = -y^2$  and choosing the ansatz  $\psi_1 = z^\alpha (1-z)^\beta F(z)$ , from the coupled Eq. (33) and (34), we find the purely outgoing solutions in terms of hypergeometric function:

$$\psi_1 = B_1 z^\alpha (1-z)^\beta {}_2F_1(a, b, c, z), \quad (31)$$

$$\psi_2 = B_2 z^{1/2+\alpha} (1-z)^\beta \left[ \left( \frac{\ell}{2} - \alpha + \beta \right) {}_2F_1(a, b, c, z) + \frac{ab}{c} (1-z) {}_2F_1(a+1, b+1, c+1, z) \right], \quad (32)$$

where the constants  $\alpha, \beta$ , the hypergeometric parameters

$a, b, c$  and coefficients  $B_1, B_2$  are

$$\begin{aligned} \alpha &= -\frac{i\omega l}{2}, & \beta &= \pm\left(\frac{1}{4} - \frac{iml}{2}\right), \\ a &= -\frac{\ell}{2} + \alpha + \beta, & b &= \frac{1+\ell}{2} + \alpha + \beta, \\ c &= \frac{1}{2} + 2\alpha, & B_1 &= B_2\left(\frac{\ell}{2} - \alpha \pm \beta\right). \end{aligned} \quad (33)$$

We now consider the flux along the radial direction

$$\mathcal{F} = \sqrt{|g|} e_a^\mu \Psi^+ \gamma^a \Psi = (|\psi_1|^2 - |\psi_2|^2). \quad (34)$$

Using the property of hypergeometric function[17], we can obtain the asymptotic behavior of  $\psi_1$  when  $z \rightarrow -\infty$

$$\begin{aligned} \psi_1 &= B_1 \left(1 - \frac{1}{z}\right)^\beta (-1)^\alpha \left\{ (-z)^{\alpha+\beta-a} \frac{\Gamma(c)\Gamma(b-a)}{\Gamma(c-a)\Gamma(b)} \right. \\ &\quad \times {}_2F_1\left(a, a-c+1, a-b+1, \frac{1}{z}\right) + (-z)^{\alpha+\beta-b} \\ &\quad \times \left. \frac{\Gamma(c)\Gamma(a-b)}{\Gamma(c-b)\Gamma(a)} {}_2F_1\left(b, b-c+1, b-a+1, \frac{1}{z}\right) \right\} \\ &\simeq B_1 (-1)^\alpha \left[ (-z)^{\alpha+\beta-a} \frac{\Gamma(c)\Gamma(b-a)}{\Gamma(c-a)\Gamma(b)} \right. \\ &\quad \left. + (-z)^{\alpha+\beta-b} \frac{\Gamma(c)\Gamma(a-b)}{\Gamma(c-b)\Gamma(a)} \right]. \end{aligned} \quad (35)$$

Similarly for  $\psi_2$  we have

$$\begin{aligned} \psi_2 &\simeq B_2 (-1)^{\alpha+1/2} \left[ \left(\frac{\ell}{2} - \alpha + \beta\right) \left[ (-z)^{1/2+\alpha+\beta-a} \right. \right. \\ &\quad \times \left. \frac{\Gamma(c)\Gamma(b-a)}{\Gamma(c-a)\Gamma(b)} + (-z)^{\frac{1}{2}+\alpha+\beta-b} \frac{\Gamma(c)\Gamma(a-b)}{\Gamma(c-b)\Gamma(a)} \right] \\ &\quad + \frac{ab}{c} \left[ (-z)^{1/2+\alpha+\beta-a} \frac{\Gamma(c+1)\Gamma(b-a)}{\Gamma(c-a)\Gamma(b+1)} \right. \\ &\quad \left. + (-z)^{1/2+\alpha+\beta-b} \frac{\Gamma(c+1)\Gamma(a-b)}{\Gamma(c-b)\Gamma(a+1)} \right]. \end{aligned} \quad (36)$$

To make the flux  $\mathcal{F}$  vanish at the origin, we should have poles of gamma functions

$$c-a = -n, \quad \text{or} \quad b+1 = -n, \quad \text{for } \ell > 0 \quad (37)$$

$$c-b = -n, \quad \text{or} \quad a+1 = -n, \quad \text{for } \ell < 0 \quad (38)$$

Thus the quasinormal frequencies are

$$\begin{aligned} \omega &= -m - \frac{i}{l} \left(2n + \frac{3}{2} + \ell\right), \quad \text{or} \\ \omega &= m - \frac{i}{l} \left(2n + \frac{1}{2} + \ell\right), \quad \text{for } \ell > 0 \end{aligned} \quad (39)$$

$$\begin{aligned} \omega &= -m - \frac{i}{l} \left(2n + \frac{1}{2} - \ell\right), \quad \text{or} \\ \omega &= m - \frac{i}{l} \left(2n - \frac{1}{2} - \ell\right), \quad \text{for } \ell < 0 \end{aligned} \quad (40)$$

From (38)–(40), we know that when  $\ell = 0$  the flux would not vanish at the origin, in contradictory to the QNM's definition. So it is true for the fermion perturbation as well as the scalar perturbation that there is no QNMs for the s-wave in the three dimensional pure dS space.

## B. Four dimensional case

We now turn our discussion to the four dimensional case. The metric and the Dirac equation are showed as (1) and (21) but with  $d = 4$ . In this spherical coordinate, a general formalism was provided in [22](see also [23]). We will follow this setup. With the separation

$$\Psi = \frac{e^{-i\omega t}}{\sqrt{f}} \begin{pmatrix} \frac{iG^\pm(r)}{r} & \varphi_{jm}^\pm & (\theta, \phi) \\ \frac{F^\pm(r)}{r} & \varphi_{jm}^\mp & (\theta, \phi) \end{pmatrix},$$

where  $\varphi_{jm}^\pm$  are the two component spinors with  $j = l \pm \frac{1}{2}$  [24], we arrive at the radial part equation. If we use a new variable  $r = l \sin \mu$ , the equations are like

$$\left(\partial_\mu + \frac{\kappa_\pm}{\sin \mu}\right) G^\pm = -\left(ml - \frac{\omega l}{\cos \mu}\right) F^\pm, \quad (41)$$

$$\left(\partial_\mu - \frac{\kappa_\pm}{\sin \mu}\right) F^\pm = -\left(ml + \frac{\omega l}{\cos \mu}\right) G^\pm, \quad (42)$$

where  $\kappa_+$  and  $\kappa_-$  are positive and negative integers. These equations resemble Eqs. (25) and (26) in the form, if one replaces  $i\omega l, \ell, \Psi_1, \Psi_2$  there with  $\kappa_\pm, i\omega l, G^\pm, F^\pm$  here. So the general solutions are similar. But things are a bit complicate here. We cannot find the purely outgoing solutions as we did in the three dimensional case. Thus we first consider the flux along the radial direction at the origin  $r = 0$ (or equivalently  $z = 0$  here)

$$\begin{aligned} \mathcal{F} &= \sqrt{1-z} \sin \theta (|G^\pm|^2 |\varphi_{jm}^\pm|^2 - |F^\pm|^2 |\varphi_{jm}^\mp|^2) \\ &\simeq \sin \theta (|\psi_1^\pm|^2 |\varphi_{jm}^\pm|^2 - |\psi_2^\pm|^2 |\varphi_{jm}^\mp|^2). \end{aligned} \quad (43)$$

We can find the solutions that make the flux vanish at the origin. For the case  $(G^+, F^+, \kappa_+)$ , they are

$$\psi_1^+ = B_1^+ z^{(1+\kappa_+)/2} (1-z)^\beta {}_2F_1(a_1, b_1, c_1, z), \quad (44)$$

$$\begin{aligned} \psi_2^+ &= B_2^+ z^{\kappa_+/2} (1-z)^\beta \\ &\quad \times \left[ \left[ \frac{1}{2} + \kappa_+ - \left(\frac{1+\kappa_+}{2} - \frac{i\omega l}{2} + \beta\right) z \right] \right. \\ &\quad \times {}_2F_1(a_1, b_1, c_1, z) + \frac{a_1 b_1}{c_1} z(1-z) \\ &\quad \left. \times {}_2F_1(a_1+1, b_1+1, c_1+1, z) \right], \end{aligned} \quad (45)$$

where the hypergeometric parameters  $a, b$  and  $c$  and the

relationship between the coefficients  $B_1^+, B_2^+$  are

$$\begin{aligned} a_1 &= \frac{\kappa_+ + 1 - i\omega l}{2} + \beta, & b_1 &= \frac{2 + \kappa_+ + i\omega l}{2} + \beta, \\ c_1 &= \kappa_+ + \frac{3}{2}, & B_1^+ &= B_2^+ \left( \frac{\kappa_+}{2} + \frac{i\omega l}{2} \pm \beta \right). \end{aligned}$$

For the case  $(G^-, F^-, \kappa_-)$ , they are

$$\psi_1^- = B_1^- z^{-\frac{\kappa_-}{2}} (1-z)^\beta {}_2F_1(a_2, b_2, c_2, z), \quad (46)$$

$$\begin{aligned} \psi_2^- &= B_2^- z^{(1-\kappa_-)/2} (1-z)^\beta \\ &\times \left[ \left( -\frac{1+\kappa_-}{2} + \frac{i\omega l}{2} - \beta \right) {}_2F_1(a_2, b_2, c_2, z) \right. \\ &\left. + \frac{a_2 b_2}{c_2} (1-z) {}_2F_1(a_2 + 1, b_2 + 1, c_2 + 1, z) \right]. \end{aligned} \quad (47)$$

where the hypergeometric parameters and coefficients are

$$\begin{aligned} a_2 &= -\frac{\kappa_- + i\omega l}{2} + \beta, & b_2 &= \frac{1 - \kappa_- + i\omega l}{2} + \beta, \\ c_2 &= -\kappa_- + \frac{1}{2}, & B_1^- &= B_2^- \left( \frac{\kappa_-}{2} + \frac{i\omega l}{2} \pm \beta \right). \end{aligned}$$

Note that  $\beta$  is the solution of equation  $\beta^2 + \frac{1}{4}(ml + \frac{i}{2})^2 = 0$ .

To obtain the QNMs, we require that the waves are purely outgoing at the horizon. In other words, the wave function is of the form  $(-\frac{1}{z})^{-i\omega l/2}$  around the horizon ( $z \rightarrow -\infty$ ). To eliminate the ingoing part in the wave function  $\psi_{1,2}$ , again, we appeal to the property of hypergeometric function illustrated in (35). The incoming and outgoing parts are then detached which enable us to remove the incoming one thoroughly by setting poles, which are

$$\begin{aligned} c_1 - b_1 &= -n, \quad \text{or} \quad a_1 + 1 = -n, \\ &\text{for } (G^+, F^+, \kappa_+); \end{aligned} \quad (48)$$

$$\begin{aligned} c_2 - b_2 &= -n, \quad \text{or} \quad a_2 + 1 = -n, \\ &\text{for } (G^-, F^-, \kappa_-). \end{aligned} \quad (49)$$

The corresponding frequencies of QNMs are obtained as:

For  $(G^+, F^+, \kappa_+)$

$$\begin{aligned} \omega &= -m - \frac{i}{l} \left( 2n + \kappa_+ + \frac{3}{2} \right), \quad \text{or} \\ \omega &= m - \frac{i}{l} \left( 2n + \kappa_+ + \frac{1}{2} \right). \end{aligned} \quad (50)$$

For  $(G^-, F^-, \kappa_-)$

$$\begin{aligned} \omega &= -m - \frac{i}{l} \left( 2n - \kappa_- + \frac{1}{2} \right), \quad \text{or} \\ \omega &= m - \frac{i}{l} \left( 2n - \kappa_- - \frac{1}{2} \right). \end{aligned} \quad (51)$$

These results are very similar to that of the three dimensional case, showing consistent behavior of the fermion perturbation in pure dS space.

## V. CONCLUSIONS AND DISCUSSIONS

We have studied the QNMs of the scalar and fermion perturbations in the pure dS space. For the scalar perturbations, we have confirmed the argument (see [14]) that no QNM exists for the massless scalar perturbations in four dimensional space or, more generally, in arbitrary dimensional case. To allow the existence of QNMs of scalar perturbation, we have found that there is a constraint on the mass of scalar field, that is  $m > \frac{d-1}{2l}$ . Moreover, we have found that in the three dimensional pure dS space QNM does not exist for the s-wave scalar perturbation. This is a special result in the three dimensional case and does not exist in other dimensions.

We have also extended the discussion of scalar perturbation to the topological dS spaces. In two special cases, the flat hypersurface and the hyperbolic hypersurface, perturbations experience amplification behaviors. Here again we see that topology influence the behavior of the perturbation as that we observed in AdS situation [8]. In AdS cases, the wave amplification was only found in the hyperbolic spaces [8], however in dS spaces the growing modes are obtained in all topological spaces. This serves as an additional difference between the dS and AdS spacetimes.

We have also investigated the fermion perturbations in the three and four dimensional pure dS spaces. The well-defined QNM frequencies were obtained. Similar to the scalar perturbation, we again found that for the three dimensional case no QNMs exists for the s-wave perturbation.

Examining the QNM frequencies of both scalar and Fermion perturbations, we found that though the real part of scalar perturbation is proportional to the surface gravity of the cosmological horizon, this dependence disappears in Fermion perturbation. In addition to the difficulty in fully understanding the quantum connection between QNM and loop gravity [12,25], it is also too early to see the relation between the QNMs and thermodynamics of horizons.

## ACKNOWLEDGMENTS

This work was partially supported by NNSF of China, Ministry of Education of China and Shanghai Science

and Technology Commission. R. K. Su's work was also supported by National Basic Research Programme of

China under Grant No. 2003cb716300. B. Wang would like to thank R. G. Cai for helpful discussions.

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