Twist of a stationary black hole or ring in five dimensions

Shinya Tomizawa,[1] Yuki Uchida,[1] and Tetsuya Shiromizu[1,2,3]

[1]*Department of Physics, Tokyo Institute of Technology, Tokyo 152-8551, Japan*

[2]*Department of Physics, The University of Tokyo, Tokyo 113-0033, Japan*

[3]*Advanced Research Institute for Science and Engineering, Waseda University, Tokyo 169-8555, Japan*

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It is unlikely that uniqueness theorem holds for stationary black holes in higher dimensional spacetimes. However, we will examine the possibility that the higher multipole moments classify vacuum solutions uniquely. Especially, we compute the potentials associated with rotational Killing vectors and look at the dependence on the total mass M and angular momentum J. Consequently, there is a potential σ which we cannot write down in terms of integer power of M and J explicitly. This may be regarded as an evidence for the uniqueness using multipole moments generated by σ.

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INTRODUCTION

TeV gravity/brane world [1] opened up the possibility of the production of higher dimensional black holes in accelerators [2]. Therefore the fundamental study on higher dimensional black holes came to be important. Important issues are uniqueness theorem for final equilibrium state of gravitational collapse [3] and no-hair theorem [4] like four dimensional cases. If these theorems hold, we can use exact solutions to have definite predictions for events in accelerators. Recently the uniqueness for nonrotating higher dimensional black holes in asymptotically flat spacetimes has been proven to be unique [5–7]. See Ref. [8] for a related issue of supersymmetric black holes. Perturbative uniqueness was also addressed in Ref. [9]. But, if we think of stationary cases, the situation is drastically changed. The uniqueness theorem for rotating cases does not hold due to the presence of the counter-example, black ring solution with $S^1 \times S^2$ event horizon, discovered by Emparan and Reall [10] in five dimensions (see also Ref. [11] for extended solutions). As a result, there are several solutions with the same total mass M and angular momentum J. In stationary vacuum black hole spacetimes with two commuting rotational Killing vectors with S^3 event horizon, the uniqueness of Myers-Perry solution [12] has been proven [13].

In this paper we want to discuss the conditions for uniqueness theorem. In the theorems the asymptotic boundary conditions are imposed. The apparent failure in the uniqueness theorem in higher dimensions seems to tell us some missing ingredients to prove it. Here we would propose that the missing one is higher multipole moments. In the four dimensional case, M and J are accidentally enough parameter set for the uniqueness theorem. If the higher order multipole moments are specified in higher dimensional spacetimes, we might be able to prove the uniqueness theorem. Indeed, four dimensional stationary spacetimes are unique under fixed multipole moments in the neighborhood of spatial infinity [14]. In higher dimensional spacetimes, on the other hand, the asymptotic structure is not so simple. In general the spacetimes will be not fixed by multipole moments in asymptotically flat spacetimes defined by conformal completion [15]. But, if we focus on asymptotically flat spacetimes where regular Cartesian coordinates can be spanned, we can expect that spacetimes near spatial infinity can be fixed by multipole moments in the same way with four dimensional cases. Following this expectation, it is realized that the multipole moments may be able to distinguish black ring solutions and Myers-Perry solution where the regular Cartesian coordinates are spanned in asymptotic region.

Multipole moments were firstly defined by Geroch for four dimensional static spacetimes in covariant way [16]. Then it was extended to stationary spacetimes by Hansen [17]. Furthermore it turned out that multipole moments uniquely determine the asymptotically flat and source-free solutions of the Einstein equation in four dimensions as mentioned above [14]. Geroch's multipole moments are defined in terms of the norm of timelike Killing vector and its derivatives at spatial infinity in the framework of conformal completion [18]. In the stationary case Hansen defined two new potential (Hansen potentials) which are composed of some combinations of the norm and twist potentials of the timelike Killing vector. The moments are defined so that they satisfy a certain transformation under the change of conformal factor, which corresponds to the change of an origin in Newtonian limit. Recently Geroch's definitions were extended to higher dimensional static spacetimes [15]. The extension to stationary cases has not been done yet. The problem is how to find Hansen potentials in higher dimensions. As a first step for the extension to stationary cases, therefore, we will compute a set of potentials associated with Killing vectors. It is also the starting point in four dimensions. When the exact solutions are given, it will be enough for the argument of uniqueness. This can be seen as follows. Let us write all parameters of potentials in terms of M and J. If potentials can be expanded by some integer powers of M and J near spatial infinity, the solutions are degenerated. On the other hand, we cannot distinguish solutions from each other if potentials cannot be expanded by only integer powers of M and J.

The rest of this paper is organized as follows. In Sec. II, we describe the black string/black hole solutions and introduce the polar coordinate systems. In Sec. III, we define some potentials in five dimensional stationary vacuum spacetimes with three commuting Killing vectors. Then we will compute the scalar functions for five dimensional black string/black hole solutions and discuss the uniqueness properties. Finally we will give summary and discussion in Sec. IV.

BLACK RING AND POLAR COORDINATE

We first describe the black ring/black hole solutions [10,12]. The metric of black ring/black hole solutions is written in the form [10]

$$\begin{aligned} ds^2 = &-\frac{F(x)}{F(y)}\left(dt + \sqrt{\frac{\nu}{\xi_1}}\frac{\xi_2 - y}{A}d\Psi\right)^2 + \frac{1}{A^2(x-y)^2} \\ &\times\left\{-F(x)\left[G(y)d\Psi^2 + \frac{F(y)}{G(y)}dy^2\right] + F(y)^2\right. \\ &\left.\times\left(\frac{dx^2}{G(x)} + \frac{G(x)}{F(x)}d\phi^2\right)\right\}, \end{aligned} \tag{1}$$

where

$$F(\xi) = 1 - \frac{\xi}{\xi_1}, \tag{2}$$

$$G(\xi) = \nu\xi^3 - \xi^2 + 1 = \nu(\xi - \xi_2)(\xi - \xi_3)(\xi - \xi_4). \tag{3}$$

The roots of $G(\xi) = 0$ satisfy $\xi_2 < \xi_3 < \xi_4$ and $\nu \le \nu_* = 2/3\sqrt{3}$. The coordinates Ψ and ϕ are identified with period

$$\Delta\phi = \Delta\Psi = \frac{4\pi\sqrt{\xi_1 - \xi_2}}{\nu\xi_1^{1/2}(\xi_3 - \xi_2)(\xi_4 - \xi_2)}. \tag{4}$$

For the black ring case, we must require

$$\xi_1 = \frac{\xi_4^2 - \xi_2\xi_3}{2\xi_4 - \xi_2 - \xi_3} \tag{5}$$

to make spacetime regular at $x = \xi_2$ and ξ_4. For black hole (Myers-Perry) case, on the other hand,

$$\xi_1 = \xi_3 \tag{6}$$

is imposed.

Mass and angular moment are given by

$$M = \frac{3\pi}{2GA^2}\frac{\xi_1 - \xi_2}{\nu\xi_1^2(\xi_3 - \xi_2)(\xi_4 - \xi_2)}, \tag{7}$$

$$J = \frac{2\pi}{GA^3}\frac{(\xi_1 - \xi_2)^{5/2}}{\nu^{3/2}\xi_1^3(\xi_3 - \xi_2)^2(\xi_4 - \xi_2)^2}. \tag{8}$$

We should note that M and J are uniquely determined by two independent parameters A and ν together with condition Eq. (5) or (6). Since ξ_1 depends on the solutions, M and J are multivalued functions of ν.

When one wants to address multipole moments, the polar coordinate is useful. The transformation into polar coordinate $(\rho, \chi, \theta, \mu)$ is expressed in

$$x = \frac{\sin^2\chi\sin^2\theta}{\tilde{A}^2\rho^2} + \xi_2, \tag{9}$$

$$y = -\frac{\sin^2\chi\cos^2\theta + \cos^2\chi}{\tilde{A}^2\rho^2} + \xi_2, \tag{10}$$

$$\tilde{\phi} = \frac{2\pi\phi}{\Delta\phi} = \mu, \tag{11}$$

$$\tan\tilde{\Psi} = \frac{\cos\chi}{\sin\chi\cos\theta}, \tag{12}$$

where $\tilde{\Psi}: = (2\pi/\Delta\Psi)\Psi$ and

$$\tilde{A}: = A\frac{\xi_1\sqrt{\nu(\xi_3 - \xi_2)(\xi_4 - \xi_2)}}{2(\xi_1 - \xi_2)}. \tag{13}$$

The period of $\tilde{\phi}$ and $\tilde{\Psi}$ are 2π. We can check that the metric approaches the five dimensional Minkowski spacetimes in polar coordinates, that is,

$$ds^2 \simeq -dt^2 + d\rho^2 + \rho^2 d\Omega_3^2, \tag{14}$$

where $d\Omega_3^2 = d\chi^2 + \sin^2\chi(d\theta^2 + \sin^2\theta d\mu^2)$.

TWIST POTENTIAL

In this section we will define scalar functions associated with Killing vectors. The computation of such functions for exact solutions can be regarded as a first step for defining multipole moments. Rather say the computation of them is enough for current purpose when the exact solutions are given.

The metric of Eq. (1) admits three Killing vectors;

$$\xi_3^a = \left(\frac{\partial}{\partial\Psi}\right)^a, \qquad \xi_4^a = \left(\frac{\partial}{\partial t}\right)^a, \qquad \xi_5^a = \left(\frac{\partial}{\partial\phi}\right)^a. \tag{15}$$

There are scalar potentials associated with the Killing vectors. One of them is

$$\lambda: = -g_{ab}\xi_4^a\xi_4^a = 1 - \frac{8GM}{3\pi\rho^2}\frac{1}{1 + \frac{8GM}{3\pi}\frac{\sin^2\chi\cos^2\theta + \cos^2\chi}{\rho^2}} \tag{16}$$

Note that λ depends on only M.

To see the feature related to angular momentum, we consider twist one-form. There are three kinds of twist one-forms

$$\omega_{ia}: = \epsilon_{abcde}\xi_4^b\xi_5^c\nabla^d\xi_i^e \quad (i = 4, 5), \tag{17}$$

$$\sigma_{Ia}: = \epsilon_{abcde}\xi_3^b\xi_5^c\nabla^d\xi_I^e \quad (I = 3, 5), \tag{18}$$

$$\tau_{\mu a}: = \epsilon_{abcde}\xi_4^b\xi_3^c\nabla^d\xi_\mu^e \quad (\mu = 3, 4). \tag{19}$$

These twist one-forms are evaluated in the coordinate basis (t, x, y, ϕ, Ψ) as follows

$$\sigma_{3a} = \frac{1}{A^3}\sqrt{\frac{\nu}{\xi_1}}\Bigg[\Bigg\{\frac{\nu}{\xi_1}(\xi_2 - y)^2 - (\xi_2 - y) \times \left(\frac{F(y)G(y)}{(x-y)^2}\right)_{,y} - \frac{F(y)G(y)}{(x-y)^2}\Bigg\}(dx)_a + 2F(x)G(x) \times \frac{\xi_2 - y}{(x-y)^3}(dy)_a\Bigg], \tag{20}$$

$$\omega_{4a} = -\frac{1}{A}\sqrt{\frac{\nu}{\xi_1}}(dx)_a, \tag{21}$$

and

$$\omega_{5a} = \sigma_{5a} = \tau_{\mu a} = 0. \tag{22}$$

By virtue of vacuum spacetime, then, there exists potentials σ and ω for each twist σ_{3a} and ω_{4a} as

$$\sigma_{3a} = \nabla_a\sigma \quad \text{and} \quad \omega_{4a} = \nabla_a\omega. \tag{23}$$

We obtain these twist potentials from the above twist one-forms by integrating them. The results are

$$\sigma = \frac{1}{\xi_1 A^3}\sqrt{\frac{\nu}{\xi_1}}\Bigg[\frac{(\xi_1 - x)(x - 2y + \xi_2)(\nu x^3 - x^2 + 1)}{(x-y)^2} + \nu x^3 - (\nu\xi_1 - \nu\xi_2 + 1)x^2 - (-\xi_1 + \xi_2 + \nu\xi_1\xi_2 - \nu\xi_2^2)x\Bigg] \tag{24}$$

and

$$\omega = -\frac{1}{A}\sqrt{\frac{\nu}{\xi_1}}(x - \xi_2). \tag{25}$$

Here it is reminded that the period of coordinates ϕ and Ψ are $\Delta\phi = \Delta\Psi \neq 2\pi$. Thus it is better to rewrite down σ and ω in new coordinates $\tilde{\phi}$ and $\tilde{\Psi}$ and the rescaled potentials are used. In polar coordinates they become

$$\tilde{\sigma} = \frac{\partial\phi}{\partial\tilde{\phi}}\frac{\partial\Psi}{\partial\tilde{\Psi}}\sigma = \frac{\Delta\phi}{2\pi}\frac{\Delta\Psi}{2\pi}\sigma = \frac{16G^2MJ}{3\pi^2\rho^2}\sqrt{\frac{\xi_1}{\xi_1 - \xi_2}}\sin^2\chi\sin^2\theta\{\sin^2\chi\sin^2\theta(1 + \sin^2\chi\cos^2\theta + \cos^2\chi) - 1\}\Bigg\{-\xi_1 + 3\xi_2 - 6\nu\xi_2^2 - 3\nu\xi_1\xi_2 + \frac{8GM}{3\pi\rho^2} \times (\xi_1 - \xi_2)(1 + \nu\xi_1 - 4\nu\xi_2) \times \sin^2\chi\sin^2\theta - \left[\frac{8GM}{3\pi\rho^2}(\xi_1 - \xi_2)\right]^2\nu \times \sin^4\chi\sin^4\theta\Bigg\} \tag{26}$$

and

$$\tilde{\omega} = \frac{\partial\phi}{\partial\tilde{\phi}}\omega = \frac{\Delta\phi}{2\pi}\omega = -\frac{4GJ}{\pi}\frac{\sin^2\chi\sin^2\theta}{\rho^2}. \tag{27}$$

Let us consider the situation with fixed M and J. The functions λ and $\tilde{\omega}$ defined above are written in terms of M and J. Moreover M and J are contained as the form of integer powers of them in asymptotic region. Therefore we cannot use λ and $\tilde{\omega}$ to classify the solutions with same M and J. On the other hand, it is easy to see from Eq. (26) that the dependence of σ on M and J are quite nontrivial. If we consider a certain exact solution, σ can be written in terms of M and J. However, M and J are not included as the form of integer powers in asymptotic region. Since the dependence depends on solutions, σ are not fixed even if we fixed M and J. Therefore we can split the degeneracy between black holes and black rings by twist potential σ. To confirm this argument, we should perform numerical evaluation of the value of coefficients, which cannot be explicitly written in terms of M and J in Eq. (26), for each solution. For example, pick up the coefficient of the term of ρ^{-6}, $\sqrt{\frac{\xi_1}{\xi_1 - \xi_2}}(\xi_1 - \xi_2)^2\nu$. The result is in Fig. 1. From this result we conclude the twist potential σ has different profiles depending on each solution for fixed mass M and angular momentum J.

CONCLUSION

In this paper we evaluated a twist potential σ for stationary five dimensional black ring/black hole. As a result we saw that its shape depends on solutions. Therefore this result means that using twist potentials we can distinguish the black ring solution from the black hole one with same mass and angular momentum. Yet, it indicates that a sort of uniqueness theorem may hold under the strong asymptotic conditions specified by the multipole moments defined via this σ.

As future work, the definition of the multipole moments in higher dimension will be important. It is expected that they uniquely determine stationary solutions.

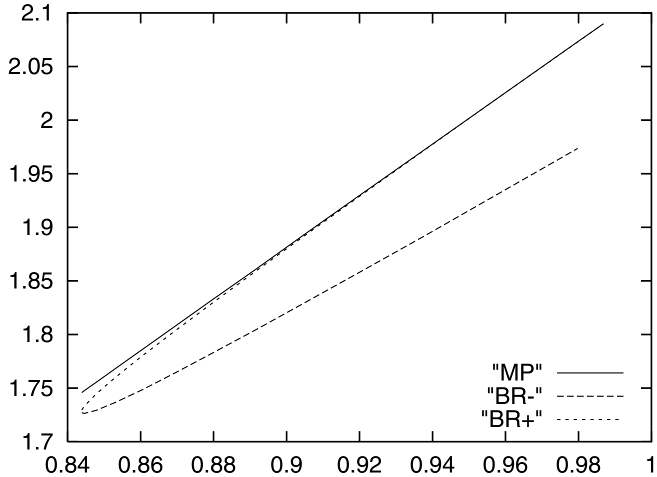

FIG. 1. The above numerical result shows the dependence on solutions of the twist potential σ. Here, we computed the coefficient of the term of ρ^{-6} in the twist potential σ. We denote it as $\alpha = \xi_1^{1/2}(\xi_1 - \xi_2)^{3/2}\nu$. The vertical axis is α and the horizontal axis is $(27\pi/32G)J^2/M^3$. The dotted line and the dashed line correspond to the two black ring solutions (these solutions are different in the values of ν $[\nu(\mathrm{BR}+) > \nu(\mathrm{BR}-)]$). The solid line expresses Myers-Perry solution. This result also shows that the twist potential of Myers-Perry solution is asymptotically close to the one of the black ring solution as $(27\pi/32G)J^2/M^3 \to 1$, where the black ring and Myers-Perry black hole degenerate [10].

It is an open question how Hansen's multipole moments are constructed from the combinations of twist potentials and gravitational potentials. We also should consider the multipole moments in Einstein-Maxwell (or higher form fields) systems in four or higher dimensional spacetimes because black holes produced in accelerator have charges in general.

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APPENDIX: CALCULATION

In this Appendix we write down useful formulae and sketch the derivation of twist potential σ of Eq. (26).

1. Inverse and determinant of metric

The inverse of metric of Eq. (1) is

$$\begin{aligned} g^{ab} = &-\left[\frac{\nu(x-y)^2(\xi_2-y)^2}{\xi_1 F(x)G(y)} + \frac{F(y)}{F(x)}\right]\left(\frac{\partial}{\partial t}\right)^a\left(\frac{\partial}{\partial t}\right)^b \\ &+ 2A\sqrt{\frac{\nu}{\xi_1}}\frac{(x-y)^2(\xi_2-y)}{F(x)G(y)}\left(\frac{\partial}{\partial t}\right)^{(a}\left(\frac{\partial}{\partial \Psi}\right)^{b)} \\ &- \frac{A^2(x-y)^2}{F(x)G(y)}\left(\frac{\partial}{\partial \Psi}\right)^a\left(\frac{\partial}{\partial \Psi}\right)^b + \frac{A^2(x-y)^2G(x)}{F(y)^2} \\ &\times\left(\frac{\partial}{\partial x}\right)^a\left(\frac{\partial}{\partial x}\right)^b - \frac{A^2(x-y)^2G(y)}{F(x)F(y)}\left(\frac{\partial}{\partial y}\right)^a\left(\frac{\partial}{\partial y}\right)^b \\ &\times\frac{A^2(x-y)^2F(x)}{F(y)^2G(x)}\left(\frac{\partial}{\partial \phi}\right)^a\left(\frac{\partial}{\partial \phi}\right)^b. \end{aligned} \tag{A1}$$

The determinant $g = \det g_{\mu\nu}$ of the metric $g_{\mu\nu}$ is

$$g = -\frac{F(x)^2F(y)^4}{A^8(x-y)^8}. \tag{A2}$$

2. Twist potential

In this subsection, we sketch the calculation of twist potential σ. From Eq. (20), the partial differential equations for σ are

$$\begin{aligned} \sigma_{,x} = &\frac{1}{A^3}\sqrt{\frac{\nu}{\xi_1}}\left\{\frac{\nu}{\xi_1}(\xi_2-y)^2 - (\xi_2-y)\right. \\ &\left.\times\left(\frac{F(y)G(y)}{(x-y)^2}\right)_{,y} - \frac{F(y)G(y)}{(x-y)^2}\right\} \end{aligned} \tag{A3}$$

and

$$\sigma_{,y} = \frac{2}{A^3}\sqrt{\frac{\nu}{\xi_1}}F(x)G(x)\frac{\xi_2-y}{(x-y)^3}. \tag{A4}$$

Integrating Eq. (A4), we obtain

$$\begin{aligned} \sigma = &\frac{1}{A^3\xi_1}\sqrt{\frac{\nu}{\xi_1}}\left[(\xi_1-x)(\nu x^3 - x^2 + 1)\right. \\ &\left.\times\left\{\frac{1}{(x-y)} + \frac{\xi_2-y}{(x-y)^2}\right\} + f(x)\right], \end{aligned} \tag{A5}$$

where $f(x)$ is an arbitrary function of x. Substituting Eq. (A5) into Eq. (A3), we obtain the ordinal differential equation for $f(x)$ as

$$\begin{aligned} f'(x) = &3\nu x^2 + (2\nu\xi_2 - 2\nu\xi_1 - 2)x + \xi_1 - \xi_2 \\ &- \nu\xi_1\xi_2 + \nu\xi_2^2. \end{aligned} \tag{A6}$$

This can be easily integrated and then

$$\begin{aligned} f(x) = &\nu x^3 + (\nu\xi_2 - \nu\xi_1 - 1)x^2 + (\xi_1 - \xi_2 \\ &- \nu\xi_1\xi_2 + \nu\xi_2^2)x + \mathrm{const.} \end{aligned} \tag{A7}$$

Finally we obtain Eq. (26) substituting Eq. (A7) into (A5).

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