Semiclassical and quantum field theoretic bounds
for traversable Lorentzian stringy wormholes

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A lower bound on the size of a Lorentzian wormhole can be obtained by semiclassically introducing the Planck cutoff on the magnitude of tidal forces (Horowitz-Ross constraint). Also, an upper bound is provided by the quantum field theoretic constraint in the form of the Ford-Roman Quantum Inequality for massless minimally coupled scalar fields. To date, however, exact static solutions belonging to this scalar field theory have not been worked out to verify these bounds. To fill this gap, we examine the wormhole features of two examples from the Einstein frame description of the vacuum low energy string theory in four dimensions which is the same as the minimally coupled scalar field theory. Analyses in this paper support the conclusion of Ford and Roman that wormholes in this theory can have sizes that are indeed only a few order of magnitudes larger than the Planck scale. It is shown that the two types of bounds are also compatible. In the process, we point out a ''wormhole'' analog of naked black holes.

I. INTRODUCTION

Recent years have seen an intense activity in the field of wormhole physics especially in the wake of the seminal works of Morris, Thorne and Yurtsever [1]. Wormholes are created by embedding into space topological handles that connect two distant otherwise disconnected regions of space. Theoretical importance of such geometrical objects is exemplified in several ways. For instance, they are invoked to interpret/solve many outstanding issues in the local as well as in cosmological scenarios or even for probing the interior of black holes [2–5]. Lorentzian wormholes could be threaded both by quantum and classical matter fields that violate certain energy conditions (''exotic matter'') at least at the throat. In the quantum regime, several negative energy density fields are already known to exist. For instance, they occur in the Casimir effect, and in the context of Hawking evaporation of black holes, and also in the squeezed vacuum states [1]. Classical fields playing the role of exotic matter also exist. They are known to occur in the $R + R^2$ theory [6], scalar tensor theories [7–11], Visser's cut and paste thin shell geometries [12]. On general grounds, it has recently been shown that the amount of exotic matter needed at the wormhole throat can be made arbitrarily small thereby facilitating an easier construction of wormholes [13].

A key issue in wormhole physics is the question of traversability. A wormhole could be traversable in princi-ple but not in practice, due to the occurrence of large tidal forces at and around the throat. Hence, to ensure the possibility of travel to be realistic and safe from the human point of view, several classical constraints are required to be imposed on the parameters of a Lorentzian wormhole as well as on the kinematics of the traveler. For instance, the conditions that the time of actual travel be reasonable and that the tidal accelerations remain less than one Earth gravity $g_\oplus$ constrain the speed of the traveler in a definite way. The most severe constraint occurs at the throat of the wormhole in the form of a radial tension which is inversely proportional to the square of the throat radius. If the size of the throat is small, the tension is large. Morris and Thorne [1] constructed a few wormhole solutions in Einstein's theory and showed that the velocity of the traveler v is also constrained linearly by the size b_0 of the throat, viz., $v_{th} \leq b_0$ with suitable dimensional adjustments.

In addition to the classical constraints, some of which are mentioned above, there are constraints that come from the quantum field theory. For instance, one has the Ford-Roman Quantum Inequality (FRQI) [14] that provides a constraint of intermediate nature between point-wise and integral (average) energy conditions. It has the form of an ''energy density-proper time'' quantum uncertainty type relation that constrains the magnitude and duration of the negative energy density of a massless minimally coupled scalar field seen by a timelike geodesic observer. The validity of these constraints can be illustrated only at the level of specific, but appropriate, solutions. To this end, Ford and Roman applied their bound to the stress energy of static, traversable wormhole spacetimes that were discussed as examples in Ref. [1].

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The calculations demonstrate that the wormholes can only be microscopic with sizes being a few orders of magnitudes larger than the Planck scale. Alternatively, if the wormhole is macroscopic, its geometry must be characterized by large discrepancy in length scales. Kuhfittig [15] has developed latter kind of model traversable wormholes by suitably adjusting different parameters that allow large discrepancies in the Ford-Roman length scales. However, the solutions considered in the examples in Ref. [14] were originally designed in Ref. [1] in an artificial fashion with the primary aim to demonstrate easy traversability. Not unexpectedly, the resulting stress tensors for those solutions do not coincide with any *a priori* known form of the source stress tensor provided by some well-defined physical principles. Known forms of stress tensor could come from physically reasonable theories of gravity such as the minimally coupled scalar field theory or other field theories mentioned in the beginning. Apart from this, a desirable feature of any gravity theory should be that it explains all known tests of gravity to date. In view of these plausible requirements, we propose to tread here the reverse path, namely, we start from a premise where the form of stress energy is known and investigate the semiclassical and quantum field theoretic constraints on the corresponding wormhole solutions. We choose to work in the Einstein massless minimally coupled scalar field (EMS) theory since it is this theory for which the FRQI was originally intended. To our knowledge, the literature still seems to lack an investigation of this kind and the essential motivation of the present paper is to fill this gap.

In this paper, we shall consider two classes of static, spherically symmetric exact solutions of the EMS theory which is just the Einstein frame (EF) version of the low energy limit of vacuum string theory in four dimensions. That is the reason why we called such solutions "stringy" in the title. To go along, the immediate question to be addressed is whether the considered solutions truly represent traversable wormholes. This is necessary in order for any constraint including FRQI to be meaningful. A detailed analysis shows that, under suitable choices of parameters, the two classes of solutions do indeed represent Lorentzian wormholes that are traversable in principle. Practical traversability, on the other hand, requires that the magnitude of tidal forces at the throat be less than the Planck scale. This condition sets a lower bound (Horowitz-Ross constraint [16]) to the size of the wormhole throat, which we designate here as a semiclassical bound in order to distinguish it from the quantum field theoretic bound. The latter we consider next, namely, the FRQI and we find, in accord with the conclusions of Ford and Roman, that the size of the wormholes in the EMS are also bounded above by values only slightly larger than the Planck scale. Since both the lower and upper bounds turn out to be of the Planck order, it is necessary to check

that the two bounds are compatible. It follows that this is also the case. The two examples that are considered here differ substantially in character yet, interestingly, they show similar wormhole behaviors. A couple of limiting cases together with the interesting wormhole analog of naked black holes are briefly touched upon. The developments in this paper could be useful also from the pedagogical point of view.

The paper is organized as follows: In Sec. II, we investigate the wormhole characteristics of the first example while Sec. III briefly touches upon some classical constraints on traversability and the discussion continues through Sec. IV until we arrive at the Horowitz-Ross semiclassical constraint. In Sec. V, FRQI is calculated. In Sec VI, relevant details of the second example are presented. Sec. VII summarizes the contents. An appendix at the end contains some useful expressions for the second example.

II. EMS THEORY: EXAMPLE 1

The EMS field equations are given by [9]

$$R_{\mu\nu} = \alpha \Phi_{,\mu} \Phi_{,\nu} \tag{1}$$

$$\Phi_{;\mu}^{;\mu} = 0. \tag{2}$$

where α is an arbitrary constant, Φ is the scalar field, $R_{\mu\nu}$ is the Ricci tensor and the semicolon denotes covariant derivatives with respect to the metric $g_{\mu\nu}$. If α is negative, or Φ is imaginary, then the Ricci tensor is negative and the stress tensor of Φ is said to represent exotic matter. We shall concentrate here on the solution set given by (We use units $G = c = \hbar = 1$ and the signature convention $-, +, +, +$):

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu$$
$$= -e^{2\phi(r)} dt^2 + e^{-2\psi(r)}[dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\varphi^2], \tag{3}$$

where

$$\phi(r) = \psi(r) = -\frac{M}{r}, \tag{4}$$

and the scalar field is given by

$$\Phi(r) = -\frac{M}{r}. \tag{5}$$

This solution was proposed by Yilmaz [17] decades ago. Integrating the Einstein complex for the stress energy, we find that the total conserved mass for the solution is given by M, and it is also the tensor mass that exhibits all the desirable properties of a mass [18]. Most importantly, the metric (3) exactly coincides up to second order with the Robertson expansion [19] of a centrally symmetric field. Hence, it describes all the well known tests of general

relativity just as exactly as does the Schwarzschild metric for $r > 2M$.

To examine if the solution (3) represents a traversable wormhole spacetime, it is convenient to employ the five geometric conditions put forward by Visser [20] which state that:

  (i) The functions $\phi(r)$ and $\psi(r)$ are everywhere finite. (We call $\phi(r)$ the redshift function).
  (ii) The function $C(r) \equiv 2\pi r e^{-\psi(r)}$ has a minimum at $r_0 \neq 0$. This provides the location of the throat at $r = r_0$.
  (iii) The two asymptotically flat regions are at $r = +\infty$ and at $r = 0$.
  (iv) $\phi(0)$ and $\phi(\infty)$ must both be finite.
  (v) $\psi(\infty)$ must be finite while $e^{-2\psi(r)} \to r^{-4}$ as $r \to 0$.

The condition (i) is obviously satisfied everywhere except at the origin. The application of the condition (ii) allows us to locate the wormhole throat at the isotropic coordinate radius $r_0 = M$. As for (iii), note that the solution is asymptotically flat at $r = +\infty$. However, to discover another flat region at $r = 0$, let us calculate the curvature scalars. The Ricci, Kretschmann and Weyl scalars, respectively, turn out to be

$$R = g^{\mu\nu}R_{\mu\nu} = \frac{2M^2}{r^4}e^{-\frac{2M}{r}}, \qquad (6)$$

$$R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta} = \left[\frac{28M^4}{r^8} - \frac{64M^3}{r^7} + \frac{48M^2}{r^6}\right]e^{-\frac{4M}{r}}, \qquad (7)$$

$$C_{\alpha\beta\gamma\delta}C^{\alpha\beta\gamma\delta} = \frac{16M^2}{3r^8}(3r - 2M)^2 e^{-\frac{4M}{r}}. \qquad (8)$$

All these curvature scalars vanish in the limit $r \to 0$ and so the spacetime is really flat there. To find the kind of metric form that exhibits manifest flatness at the origin $r = 0$, that is, a form that satisfies especially the conditions (iv) and (v) above, we transform the metric (3) under inversion $r \to 1/r$ to get

$$ds^2 = -e^{-2Mr}dt^2 + r^{-4}e^{2Mr}[dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\varphi^2]. \qquad (9)$$

Now, with regard to condition (iv), note from (9) that $\phi(r) = e^{-Mr} = 1$ at $r = 0$ and from (4), $\phi(r) = e^{-\frac{M}{r}} = 1$ at $r = \infty$. Similarly, from (4) again, we see that $\psi(r) = e^{\frac{M}{r}} = 1$ at $r = \infty$, while from (9), it is evident that $e^{-2\psi(r)} \to r^{-4}$ as $r \to 0$ accounting for the condition (v). Thus, finally, we can conclude that the solution (3) represents a Lorentzian wormhole that is traversable at least in principle.

The proper radial distance l away from the throat (where $l = 0$) is given by

$$l(r) = \pm \int_M^r e^{\frac{M}{r}}dr = \pm\left[re^{\frac{M}{r}} - M \times Ei\left(\frac{M}{r}\right)\right]_M^r, \qquad (10)$$

where $Ei(x)$ is the exponential-integral function given by

$$Ei(x) = C + \ln x + \sum_{k=1}^{\infty}\frac{x^k}{k!k}, \quad x > 0 \qquad (11)$$

$$= C + \ln(-x) + \sum_{k=1}^{\infty}\frac{x^k}{k!k}, \quad x < 0, \qquad (12)$$

where C is an arbitrary constant. Clearly, it is not possible to invert Eq. (10) and obtain $r = r(l)$ in a closed form and cast the metric (3) in the proper distance language. However, the function l is well behaved everywhere since $l \to \pm\infty$ as $r \to \pm\infty$. The equation for the embedded surface $z = z(r)$ is given by

$$z(r) = \pm\int\sqrt{\frac{2M}{r} - \frac{M^2}{r^2}} \times e^{\frac{M}{r}}dr. \qquad (13)$$

Again, the right hand side can not be integrated into a closed form. Nevertheless, as required, the embedding surface becomes flat very far from the throat: $dz/dr = 0$ as $r \to \pm\infty$ which corresponds to $l \to \pm\infty$.

As to the question of violation of energy conditions, note that the solutions (3)–(5) satisfy the field equations for the value $\alpha = -2$ [17]. This implies that there is a negative sign before the kinetic term in the Einstein-Hilbert action. Consequently, almost all energy conditions are violated providing a situation that is very conducive for the creation of wormholes. Indeed, the energy density ρ, the radial pressure p_r, the lateral pressures p_θ and p_φ in the static orthonormal frame turn out to be

$$\rho = -\left(\frac{1}{8\pi}\right) \times \frac{M^2}{r^4 e^{\frac{2M}{r}}}, \qquad p_r = -\left(\frac{1}{8\pi}\right) \times \frac{M^2}{r^4 e^{\frac{2M}{r}}},$$

$$p_\theta = p_\varphi = \left(\frac{1}{8\pi}\right) \times \frac{M^2}{r^4 e^{\frac{2M}{r}}}. \qquad (14)$$

Clearly, $\rho < 0$ for all values of r and the Weak Energy Condition (WEC) is violated. However, the Strong Energy Condition (SEC) is marginally satisfied since $\rho + p_r + p_\theta + p_\varphi = 0$. The massless limit $M = 0$ leads only to a trivially flat spacetime and is not physically interesting.

III. TRAVERSABILITY: CLASSICAL CONSTRAINTS

It is of some interest to discuss the classical constraints on practical traversability across the wormhole by humanoids. To begin with, note that our wormhole is attractive. The radially moving traveler that starts off from rest from an asymptotic location has the equation of motion:

$$\frac{d^2 r}{d\tau^2} \equiv a^r = \frac{M}{r^2} \times e^{-\frac{M}{r}} \times \left(1 - \frac{M}{r}\right), \qquad (15)$$

where τ is the proper time. Clearly, $a^r > 0$ for $r > M$ and $a^r = 0$ for $r = r_0 = M$. Therefore, the traveler will be

pulled in until he/she attains zero acceleration at the throat and in order to emerge at the other mouth, he/she has to maintain an outward directed radial acceleration from being pulled in again. At the throat, the *static* observers are also geodesic observers as $a^r = 0$ there, which is satisfied for a constant velocity including its zero value [14]. This is a basic feature of the wormhole example under present investigation.

Suppose that a human being travels radially with velocity v such that $v = 0$ at $l = -l_1$ and at $l = +l_2$ and $v > 0$ at $l_1 < l < l_2$, where l_1 and l_2 are the locations of two widely separated space stations. Then, in order that the journey is completed in a reasonable length of time, say, 1 yr, the velocity $v(r)$ has to satisfy the following constraints [1]:

$$\Delta \tau = \int_{-l_1}^{+l_2} \frac{dl}{v\gamma} \le 1 \text{ year}, \qquad \Delta t = \int_{-l_1}^{+l_2} \frac{dl}{v e^\phi} \le 1 \text{ year},$$
(16)

where $\gamma = [1 - v^2]^{-1/2}$ and $\Delta\tau$ is the proper time interval of the journey recorded by the traveler's clock, Δt is the coordinate time interval recorded by observers situated at the stations. These are also several other kinematic constraints. For instance, at the stations, the geometry must be nearly flat. This constraint can be easily satisfied by locating the stations at large r. Another constraint comes from the demand that the traveler not feel an acceleration greater than one $g_\oplus$. This leads to

$$\left| e^{-\phi} \frac{d(\gamma e^\phi)}{dl} \right| \le g_\oplus.$$
(17)

For our solution, the conserved total energy E per unit mass of the radially freely falling traveler is given by $E = \gamma(r)e^{\phi(r)} =$ constant, and therefore the constraint (17) is satisfied easily.

IV. TRAVERSABILITY: HOROWITZ-ROSS CONSTRAINT

There are also constraints coming from the dynamical considerations. For instance, traveler's velocity is constrained by the magnitudes of tidal forces that involve the curvature tensor. For our form of the solution, the only nonvanishing curvature components in the static observer's orthonormal basis are R_{0101}, R_{0202}, R_{0303}, R_{1212}, R_{1313} and R_{2323}. Radially freely falling travelers with conserved energy E per unit mass are connected to the static orthonormal frame by a local Lorentz boost with an instantaneous velocity given by

$$v = \frac{dr}{d\tau} = [1 - e^{2\phi} E^{-2}]^{1/2}.$$
(18)

Then the nonvanishing curvature components in the Lorentz-boosted frame ($\wedge$) are [16,21]:

$$R_{\hat{0}\hat{1}\hat{0}\hat{1}} = R_{0101},$$
(19)

$$R_{\hat{0}k\hat{1}k} = \cosh\alpha \sinh\alpha (R_{0k0k} + R_{1k1k}),$$
(20)

$$R_{\hat{0}k\hat{0}k} = R_{0k0k} + \sinh^2\alpha (R_{0k0k} + R_{1k1k}),$$
(21)

$$R_{\hat{1}k\hat{1}k} = R_{1k1k} + \sinh^2\alpha (R_{0k0k} + R_{1k1k}),$$
(22)

and R_{klkl}, where $k, l = 2, 3$ and $\sinh\alpha = \frac{v}{\sqrt{1-v^2}}$. The terms in the parentheses represent an enhancement of curvature in the traveler's frame. Incidentally, note that, in the Schwarzschild or Reissner-Nordström spacetime, the sums in the parentheses are exactly zero due to special cancellations. This might appear surprising at first sight, but actually this cancellation occurs only in the "standard" coordinates which hide the nontrivial enhancement that *actually* takes place. This is only to be expected as the two pieces in $(R_{0k0k} + R_{1k1k})$ transform differently at any spacetime point under transformations to different coordinate systems.

The differential tidal accelerations felt by the traveler are

$$\Delta a_j = -R_{\hat{0}j\hat{0}p}\xi^p,$$
(23)

where $j, p = 1, 2, 3$ and $\vec{\xi}$ is the vector separation between two parts of the body. Taking $|\vec{\xi}| \approx 2$ meters (the size of the body), the radial tidal constraint should be such as to satisfy $|R_{\hat{0}\hat{1}\hat{0}\hat{1}}| \le \frac{g_\oplus}{2m} \cong 10^{-20}$ cm^{-2}. For the solution (4), we have:

$$|R_{\hat{0}\hat{1}\hat{0}\hat{1}}| = |R_{0101}| = \frac{2Me^{-\frac{2M}{r}}}{r^3}\left(1 - \frac{M}{r}\right),$$
(24)

which vanishes at the throat $r = r_0 = M$. Evidently, the constraint is well satisfied throughout the journey. On the other hand, the requirement

$$|R_{\hat{0}2\hat{0}2}| \le \frac{g_\oplus}{2m} \cong 10^{-20} \text{ cm}^{-2}$$
(25)

constrains the velocity v of the traveler to values that are comfortably attainable [1]. The exact form of $|R_{\hat{0}2\hat{0}2}|$ will be shown below. However, from now on, we shall focus on the constraints engendered by physical requirements rather than by the requirement of human comfort. Using Eq. (21), we have

$$|R_{\hat{0}2\hat{0}2}| = \frac{Me^{-\frac{2M}{r}}}{r^3}\left(1 - \frac{M}{r}\right) + \frac{M^2 e^{-\frac{2M}{r}} v^2 \gamma^2}{r^4}.$$
(26)

Clearly, the first term on the right is the curvature measured in the static frame while the second represents excess in curvature measured by the geodesically falling observer with $v \ne 0$. Other curvature components follow from Eqs. (20) and (22) and they are:

$$|R_{\hat{0}\hat{3}\hat{0}\hat{3}}| = \frac{Me^{-\frac{2M}{r}}}{r^3}\left(1 - \frac{M}{r}\right) + \frac{M^2 e^{-\frac{2M}{r}} v^2 \gamma^2}{r^4},$$

$$|R_{\hat{0}\hat{2}\hat{1}\hat{2}}| = \frac{M^2 e^{-\frac{2M}{r}} v \gamma^2}{r^4}, \tag{27}$$

$$|R_{\hat{1}\hat{2}\hat{1}\hat{2}}| \equiv |R_{\hat{1}\hat{3}\hat{1}\hat{3}}| = \frac{Me^{-\frac{2M}{r}}}{r^3}\left(1 + \frac{Mv^2\gamma^2}{r}\right),$$

$$|R_{\hat{2}\hat{3}\hat{2}\hat{3}}| \equiv |R_{2323}| = \frac{Me^{-\frac{2M}{r}}}{r^2}\left(\frac{M}{r^2} - \frac{2}{r}\right), \tag{28}$$

and they can also be separated likewise into static and excess parts. It may be noted here that, at the throat, $r = r_0 = M$, all the values of the curvature remarkably coincide, up to an unimportant factor $(e^2/2)$, with those obtained for the case of "$\phi = 0, b = r_0 = $ const." zero density wormholes discussed in Refs. [1,14].

For a particle that is static at the throat, the radial and lateral tidal forces, given, respectively, by $|R_{0101}|$, $|R_{0202}|$ and $|R_{0303}|$, are exactly zero. But a radially falling particle could experience much larger tidal forces in the vicinity of the throat, either for its velocity $v \approx 1$ or for the wormhole geometry $r_0 \approx 0$ or for both reasons, than the one static at the throat that actually feels no tidal forces at all. Thus, we have here a wormhole analog of the idea of naked black holes proposed by Horowitz and Ross [16] for which the curvatures just above the horizon are much larger than those at the horizon. In the vicinity of the throat, the maximum value of the curvature felt by the falling particle [Eqs. (26)–(28)] is given by γ_0^2/r_0^2, where γ_0 is the Lorentz factor at the throat. In order to avoid the occurrence of infinite tidal forces, the physical requirement is that the magnitude of curvature be less than the Planck scale. This implies that the local radius of curvature (r_0/γ_0) be greater than the Planck length. This condition gives us a semiclassical lower bound or a Planck cutoff, on r_0, and we call it the Horowitz-Ross constraint [16], viz.,

$$r_0 > \gamma_0 \ell_P, \tag{29}$$

where ℓ_P is the Planck length, $\gamma_0 = 1/\sqrt{1 - v_0^2}$ and v_0 is the velocity of the particle at the throat. Because of the introduction of the Planck length, the right hand side of (29) remains microscopic even for values of v_0 very close to unity. Inequalities similar to (29), but without involving the Planck scale, have also been worked out by Morris and Thorne [1] in case of their examples of traversable wormholes. We shall now turn to FRQI to see what upper bound it offers on the throat size.

V. FORD-ROMAN QUANTUM INEQUALITY (FRQI)

This is a constraint coming essentially from the full quantum field theoretic considerations. The bound has the form of an uncertainty-principle-type constraint on the magnitude and duration of the negative energy density as seen by an observer fixed to a timelike geodesic particle. The quantum inequality is given by [14]:

$$\frac{\tau_0}{\pi} \int_{-\infty}^{+\infty} \frac{\langle T_{\mu\nu} u^\mu u^\nu \rangle d\tau}{\tau^2 + \tau_0^2} \geq -\frac{3}{32\pi^2 \tau_0^4}, \tag{30}$$

for all τ_0 where τ is the freely falling observer's proper time, $\langle T_{\mu\nu} u^\mu u^\nu \rangle$ is the expectation value of the stress energy of the minimally coupled scalar field in the observer's frame of reference. Although the inequality was basically derived in the Minkowski space quantum field theory, it can be applied also in the curved spacetime provided that τ_0 is taken sufficiently small, that is, much less than the size of the proper local radius of curvature.

To apply the FRQI to our solution, let us find the energy density in the geodesic frame of the radially falling observer. This can be obtained by applying a local Lorentz boost given by

$$\rho' = \gamma^2(\rho + v^2 p_r). \tag{31}$$

Using the relevant expressions from Eqs. (14), we have

$$\rho' = -\left(\frac{1}{8\pi}\right) \times \frac{M^2 \gamma^2}{r^4 e^{\frac{2M}{r}}} \times (1 + v^2) < 0. \tag{32}$$

Next, from the expressions of the components of Riemann tensor [Eqs. (26)–(28)], it follows that, at the throat, the maximum magnitude of curvature in the Lorentz-boosted frame is $R'_{\max} \leq \frac{\gamma_0^2}{r_0^2}$ and therefore the smallest local proper radius of curvature $r'_c \geq r_0/\gamma_0$. Thus the sampling time is taken as $\tau_0 = f r_0/\gamma_0 << r'_c$, for $f << 1$. The energy density does not significantly change over this time scale and FRQI says:

$$\frac{\tau_0}{\pi} \int_{-\infty}^{+\infty} \frac{\langle T_{\mu\nu} u^\mu u^\nu \rangle d\tau}{\tau^2 + \tau_0^2} \approx \rho'_0 \geq -\frac{3}{32\pi^2 \tau_0^4}, \tag{33}$$

where $\rho'_0 = -(\frac{1}{8\pi}) \times \frac{\gamma_0^2}{r_0^2 e^2} \times (1 + v_0^2)$ is the value of the energy density at the throat $r = r_0 = M$. Putting this value in FRQI (33), we have the upper bound on r_0:

$$r_0 \leq \left(\frac{e}{2f^2 \sqrt{1 - v_0^4}}\right) \ell_P. \tag{34}$$

For $v_0 = 0$ (recall that it is still geodesic motion), and $f \approx 10^{-4}$, we have $r_0 \approx 10^{-25}$ cm. Even if v_0 is extremely close to unity, say $1 - v_0^4 \approx 10^{-40}$, one has $r_0 \approx 10^{-5}$ cm. These results show that the FRQI bound is really robust. The solution (3) of the EMS theory does indeed represent a wormhole of microscopic size, even at the two near extreme values of observer's velocity. Considering a realistic motion (with energy E normalized to unity) that begins from rest at the asymptotic region and passes through the wormhole throat, we see that the

particle attains maximum velocity right at the throat and it is $v = v_0 = \sqrt{1 - e^{-2}}$. Then $r_0 \leq (1/2f^2) \times (e^3/\sqrt{2e^2 - 1})\ell_P$. Obviously, again the FRQI constrains the wormholes to have sizes that are just a few orders of magnitude larger than the Planck scale. Looking at the Horowitz-Ross constraint (29), we expect it to be compatible with the FRQI (34). That is, we expect the following inequality to hold good:

$$\gamma_0\ell_P < r_0 \leq \left(\frac{e}{2f^2\sqrt{1 - v_0^4}}\right)\ell_P. \tag{35}$$

This is true if $\sqrt{1 + v_0^2} < \frac{e^2}{2f^2}$, which is easily satisfied for $f < < 1$.

It was mentioned earlier that the class of solutions (4) is distinguished from other classes of solutions in the EMS theory in some important respects. In the next section, we consider one such class of solutions in the form of a second example pointing out how it differs in nature from that of Example 1.

VI. EMS THEORY: EXAMPLE 2

Consider the class of solutions, which, in isotropic coordinates, is given by [9,22]:

$$\phi(r) = \beta \ln\left[\frac{1 - \frac{m}{2r}}{1 + \frac{m}{2r}}\right],$$

$$\psi(r) = (\beta - 1)\ln\left(1 - \frac{m}{2r}\right) - (\beta + 1)\ln\left(1 + \frac{m}{2r}\right), \tag{36}$$

$$\Phi(r) = \left[\frac{2(1 - \beta^2)}{\alpha}\right]^{\frac{1}{2}}\ln\left[\frac{1 - \frac{m}{2r}}{1 + \frac{m}{2r}}\right], \tag{37}$$

where α is an arbitrary constant parameter. The two undetermined constants m and β are related to the source strengths of the gravitational and scalar parts of the configuration. To highlight the differences in nature between this solution and that in (4), we point out the following: Once the scalar component is set to zero ($\Phi = 0 \Rightarrow \beta = 1$), the solutions (36) and (37), reduce to the Schwarzschild black hole in accordance with Wheeler's "no scalar hair" conjecture. Physically, this indicates the possibility that the scalar field is radiated away during collapse and the end result is a Schwarzschild black hole [18]. On the other hand, in the case of our previous example, solution (4), there is no separate scalar parameter. The condition $\Phi = 0 \Rightarrow M = 0$, that is, one obtains only a flat space from the metric (3) and not a black hole. In this sense, (4) was a pure wormhole solution having no counterpart in the black hole regime. Another important difference is that, for $\beta \neq 1$, the solutions (36) and (37), represent a spacetime with naked singularity at $r = m/2$ in the sense that all curvature invariants diverge there. In contrast, such divergences do not occur in the solutions

(4) and (5). In spite of these basic differences, the calculations below show that the presence of a separate scalar parameter β does not alter the Horowitz-Ross or FRQI constraints.

The solution (36) can be interpreted as a traversable wormhole as it satisfies all of the Visser's conditions (i)–(v). We only mention that the metric is not only flat at $r = 0$ but is also form invariant under inversion. (Just choose $r = (m/2)\rho$, $\rho \rightarrow 1/\rho$.) The throat appears at the co-ordinate radii

$$r_0^{\pm} = \frac{m}{2}[\beta \pm (\beta^2 - 1)^{1/2}]. \tag{38}$$

Here we take only the positive sign (r_0^+). The requirement that the throat radii be real implies that $\beta^2 > 1$ and the reality of Φ in turn demands that $\alpha < 0$. Alternatively, one could have $\alpha > 0$ allowing for an imaginary Φ. The latter choice presents no pathology or inconsistency in the wormhole physics, as recently shown in Ref. [23]. In both cases, however, we have a negative sign before the stress tensor on the right hand side of Eq. (1) and consequently almost all energy conditions are violated. For instance, the energy density is given by

$$\rho = -\left(\frac{1}{8\pi}\right)$$
$$\times \left[\frac{256m^2r^4(\beta^2 - 1)(1 - m/2r)^{2\beta}(1 + m/2r)^{-2\beta}}{(m^2 - 4r^2)^4}\right]. \tag{39}$$

Thus, $\rho < 0$ at the throat and elsewhere satisfying the necessary wormhole condition that the Weak Energy Condition (WEC) be violated. The expressions for the pressure components are given in the Appendix. Note that the tensor mass of the solution is given by $M = m\beta$ and the expansion of the metric (36) indicates that it is also the Keplerian mass.

All curvature components in the Lorentz-boosted or-thonormal frame are given in the Appendix. We consider here only a representative one, viz., $R_{\hat{0}2\hat{0}2}$. From (A4)–(A6), it is evident that the static frame measure of the curvature at the throat is zero. The geodesic excess, at the throat $r = r_0^+ (> m/2)$ is given by the last term in (A5) which works out to a remarkably simple expression:

$$|R_{\hat{0}2\hat{0}2}| = \left(\frac{v_0\gamma_0}{r_0^+}\right)^2, \tag{40}$$

for $\beta^2 > 1$. So, once again, we get the same Horowitz-Ross constraint, that is, the inequality (29). Note that, for $\beta = 1$ (Schwarzschild), we have, $r_0^+ \equiv r_H = \frac{m}{2}$ (r_H is the horizon radius) and $v_0 = 1$, as expected. Only for these exact values, $|R_{\hat{0}2\hat{0}2}| \rightarrow \infty$, that is, an arbitrarily large tidal force is experienced by the test (light!) particle. But for slightly massive test particles ($v_0 \approx 1$), one can introduce a Planck cutoff as embodied in (29) and avoid infinities in the measurement of curvature.

As to the FRQI, we get, at the throat, using a little manipulation with Eqs. (31), (39), and (A1),

$$\rho' = -\frac{1}{32\pi} \times \frac{\gamma_0^2(1+v_0^2)}{r_0^{+2}} \times \left(\frac{\beta+\sqrt{\beta^2-1}}{\sqrt{\beta^2-1}}\right)^2$$

$$\times \left[\frac{(\beta-1)-\sqrt{\beta^2-1}}{(\beta-1)+\sqrt{\beta^2-1}}\right]^{2\beta}$$

$$\equiv -\frac{1}{32\pi} \times \frac{\gamma_0^2(1+v_0^2)}{r_0^{+2}} \times g(\beta). \qquad (41)$$

For $\beta \to 1$ it is clear from (39) and (A1) that both ρ and p_r vanish implying that $\rho' = 0$. For $\beta \to \infty$, the function $g(\beta)$ tends to 0. Therefore, only the coefficient of $g(\beta)$ is important for FRQI. Noting from Eq. (40), that $R'_{\max} \leq \gamma_0^2/r_0^{+2}$ and taking $\tau_0 = f r_0^+/\gamma_0$ with $f \ll < 1$, and using Eq. (41) in (33), we get the same upper bound as in (34). This concludes the discussion of bounds on wormholes.

Let us consider a couple of limiting cases. If $m \approx 0$, one can choose β sufficiently large and arrange to have any finite nonzero value for M so that

$$r_0^+ \approx M, \qquad \rho|_{r_0^+} \approx -\frac{1}{8\pi M^2}, \qquad |R_{\hat{0}\hat{2}\hat{0}\hat{2}}| = \frac{v_0^2 \gamma_0^2}{M^2}, \qquad (42)$$

with all other curvature components behaving similarly. This is the closest approximation to a Schwarzschild-like, but traversable wormhole that one can obtain in the EMS theory. If, on the other hand, we set $\beta = 0$ but $m \neq 0$ in the equation $M = m\beta$, we have a zero mass ($M = 0$) wormhole [23]. These solutions are not flat. In fact, in this case, we have

$$r_0^+ = \frac{m'}{2}, \qquad \rho|_{r_0^+} = -\frac{1}{8\pi m'^2}, \qquad \mathbf{R} = -\frac{2}{m'^2},$$

$$m = -im', \qquad (43)$$

where $\mathbf{R}$ is the Ricci scalar. This is an extreme case, since $M = 0$. Also, at the throat the velocity of the test particle for unit energy E, viz.,

$$v_0^2 = 1 - \left[\frac{(\beta-1)-\sqrt{\beta^2-1}}{(\beta-1)+\sqrt{\beta^2-1}}\right]^{2\beta} \qquad (44)$$

becomes zero for $\beta = 0$. (This also implies that $|R_{\hat{0}\hat{2}\hat{0}\hat{2}}| = 0$.) Therefore, the test particle is captured and kept at rest forever at the throat [24,25]. This is an interesting aspect of zero mass wormholes.

VII. SUMMARY

Quantum field theory calculations involving massless minimally coupled scalar field (EMS theory) imply that there are two possible alternatives: Either a wormhole threaded by this matter must only be of microscopic (Planck) size or that there should be large discrepancies in the length scales associated with macroscopic wormholes [14]. Ford and Roman applied their bound only to some artificial examples for which the stress tensors do not comply with those in the EMS theory. There is therefore the important logical need that the bound be applied in the proper setting. To this end, it is necessary to consider exact wormhole solutions in the EMS theory, investigate their traversability and see which of the two alternatives is allowed. The present paper is motivated essentially by these considerations.

We considered two wormhole examples from the EMS theory. The first example has been worked out in some detail while analogous calculations can be carried out for the second example, of which an outline is given above. In both the examples, we calculated the physical condition for traversability which provides the Horowitz-Ross lower bound [16] on the throat size of the wormhole. This bound is obtained by introducing the Planck cutoff on a classical quantity, viz., curvature and that is why we called this bound semiclassical. In the process, we arrived at the wormhole analog of naked black holes proposed by Horowitz and Ross [16]. The similarity is interesting given that the energy conditions are violated only in the former case, but not in the latter. The FRQI provides a quantum field theoretic upper bound on the throat size. It is shown that the two bounds are compatible. The main lesson that the two examples teach us is that traversable Lorentzian wormholes in the EMS theory could indeed be microscopic, which supports the conclusions of Ford and Roman [14] in a direct way. An analogous result has been advanced by Visser [26] in the context of minisuperspace models. He has shown that the expectation value of the throat radius is also of the order of Planck length. It is tempting to speculate that the EMS wormholes, in virtue of their sizes being microscopic, could be the *natural* candidates for the constituents of the spacetime "foam" of Wheeler [27,28].

Finally, although microscopic wormholes are of considerable theoretical interest, one question still remains. Recall that traversability is a basic criterion in order for FRQI to be defined since the negative energy density is measured in the proper frame of the traveling or static observer. If the wormhole throats are doomed to be of only Planck dimensions in the EMS theory, can one meaningfully define a *nonhypothetical* static and/or a traveling test particle through the wormhole? It seems, in general, one cannot since the Bohr radius of an elementary particle is several orders of magnitude higher than the Planck length. However, if the velocity is exceedingly close to that of light, an elementary particle can just pass through (see the discussion after Eq. (34)). For zero mass wormholes, the test particle is captured at the throat and kept at rest forever there. The possibility of interstellar travel by using these microscopic objects seems out of question [26].

ACKNOWLEDGMENTS

APPENDIX

For the metric (36), the pressure components are given by

$$p_r = -\left(\frac{1}{8\pi}\right)$$

$$\times \left[\frac{256m^2 r^4(\beta^2 - 1)(1 - m/2r)^{2\beta}(1 + m/2r)^{-2\beta}}{(m^2 - 4r^2)^4}\right], \tag{A1}$$

$$p_\theta = p_\varphi = -p_r. \tag{A2}$$

Using the expression for ρ from Eq. (39), we have

$$\rho + p_r + p_\theta + p_\phi = 0. \tag{A3}$$

The curvature components in the Lorentz-boosted orthonormal frame for the metric (36) read, using Eqs. (19)–(22):

$$R_{\hat{0}\hat{1}\hat{0}\hat{1}} = R_{0101} = \frac{128m\beta r^3(m^2 + 4r^2 - 4m\beta r)(1 - m/2r)^{2\beta}(1 + m/2r)^{-2\beta}}{(m^2 - 4r^2)^4}, \tag{A4}$$

$$R_{\hat{0}\hat{2}\hat{0}\hat{2}} = R_{\hat{0}\hat{3}\hat{0}\hat{3}} = R_{0202} + v^2\gamma^2(R_{0202} + R_{1212}), \tag{A5}$$

$$R_{\hat{1}\hat{2}\hat{1}\hat{2}} = R_{1212} + v^2\gamma^2(R_{0202} + R_{1212}), \tag{A6}$$

$$R_{\hat{2}\hat{3}\hat{2}\hat{3}} = R_{2323} = \frac{128mr^3[m^2\beta + 4r^2\beta - 2mr(1 + \beta^2)](1 - m/2r)^{2\beta}(1 + m/2r)^{-2\beta}}{(m^2 - 4r^2)^4}, \tag{A7}$$

$$R_{\hat{0}\hat{2}\hat{1}\hat{2}} = v\gamma^2(R_{0202} + R_{1212}), \tag{A8}$$

$$R_{0202} = R_{0303} = \frac{64m\beta r^3(m^2 + 4r^2 - 4m\beta r)(1 - m/2r)^{2\beta}(1 + m/2r)^{-2\beta}}{(m^2 - 4r^2)^4}, \tag{A9}$$

$$R_{1212} = R_{1313} = \frac{128mr^3(m^2\beta - 4mr + 4r^2\beta)(1 - m/2r)^{2\beta}(1 + m/2r)^{-2\beta}}{(m^2 - 4r^2)^4}.$$

The wormhole throat satisfies $r_0^{\pm 2} + \frac{m^2}{4} - m\beta r_0^\pm = 0$, and so, from (A4) and (A5), we see that in the static frame, the tidal accelerations (Eq. (23)) vanish at the throat.

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