Nonlinear N-parameter spacetime perturbations: Gauge transformations

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We introduce N-parameter perturbation theory as a new tool for the study of nonlinear relativistic phenomena. The main ingredient in this formulation is the use of the Baker-Campbell-Hausdorff formula. The associated machinery allows us to prove the main results concerning the consistency of the scheme to any perturbative order. Gauge transformations and conditions for gauge invariance at any required order can then be derived from a generating exponential formula via a simple Taylor expansion. We outline the relation between our novel formulation and previous developments.

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In the theory of spacetime perturbations [1–5] (see [6–8] for an introduction), one usually deals with a family of spacetime models which, in most cases, depends on a single parameter λ: $M_\lambda = (\mathcal{M}, \{\mathcal{T}_\lambda\})$, where $\mathcal{M}$ is a manifold that accounts for the topological and differential properties of spacetime, and $\{\mathcal{T}_\lambda\}$ is a set of fields on $\mathcal{M}$, representing its geometrical and physical content (this formulation does not depend on the gravitational field equations). The parameter λ labels the elements of the family and gives an indication of the "size" of the perturbations, regarded as deviations of M_λ from a background model M_0. It can either be a formal parameter, as in cosmology [3,9,10], in backreaction problems (see, e.g., [11,12]), or in the study of quasinormal modes of stars and black holes [13,14], or it can have a specific physical meaning, as in the study of black hole mergers via the close limit approximation, in the analysis of quasinormal mode excitation by a physical source, or in the modeling of perturbations generated by the collapse of a rotating star (see [14–17], and references therein).

There are, however, physical applications in which it may be convenient to use a perturbation formalism based on two [5] or more parameters. For instance, in order to study general time-dependent perturbations of stationary axisymmetric rotating stars [18] using a spherical background. In this case one can separately consider the stationary axisymmetric rotational perturbations, for example, up to second order in a parameter Ω, then considering the coupling of these with the first-order time dependent ones (see [5] for further discussion). As it should be clear from this example, the advantage of an N-parameter non-linear perturbation theory (NLPT) is that it allows us to make distinctions between different types of perturbations corresponding to different parameters, so that we can study their coupling and some nonlinear effects without having to compute the whole set of higher-order perturbations. Such a framework may provide flexibility by allowing us to look at a given problem from different points of view. It may also allow us to choose a simpler (typically more symmetrical) background to model a given physical scenario. Given that, even in NLPT, the differential operators are those defined on the chosen background, this can drastically reduce the computations and even change the way to perform them.

The aim of the present article is to introduce a novel approach to the study of the gauge dependence of perturbations in NLPT which (i) deals with an arbitrary number of parameters, (ii) provides a closed formula for the action of a general gauge transformation, valid to any order, (iii) the construction and derivation of formulas of practical interest is simpler than in previous frameworks [3–5]. This new approach is mainly based on the application of the Baker-Campbell-Hausdorff (BCH) formula [19]. This has been used previously [20], in the context of the backreaction problem in cosmology, to derive second order one-parameter gauge transformations. Here we show how to make use of the full power of the BCH formula deriving the transformation between two given N-parameter gauges, each represented by an N-parameter group of diffeomorphisms, at an arbitrary order. Our formalism therefore also contains the conditions for gauge invariance for every perturbation order in N-parameters.

We start by summarizing some relevant results regarding the mathematical structure of the single parameter NLPT (see [3,4]). The Taylor expansion of a general nonlinear gauge transformation can be expressed in terms of Lie derivatives with respect to a set of vector fields which, order by order, constitute the generators of the trans formation. A closed formula for this expansion, valid at all orders, was found in [4], using a new object dubbed *Knight diffeomorphism* (KD), first defined in [3] (cf. also [11,21]). The analysis in [3,4,22] gives also the conditions for gauge invariance at any given order, and provides the framework for the construction of gauge-invariant

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formalisms [23]. A similar construction has been attempted for the two-parameter case in [5], where the action of a general gauge transformation on arbitrary tensor quantities was expressed in terms of the Lie derivatives with respect to a set of vector fields. However, since no natural generalizations of the KD approach were found, these expressions were derived up to fourth order in the parameters by imposing, order by order, consistency conditions (see [24] for a related analysis and [25] for an application to gravitating strings). It must be pointed out that, although that derivation is not as elegant and compact as in [3,4] or the one based on the BCH formula presented here, it still leads to the right formulas of practical interest, as we shall discuss.

The basic assumption for the construction of a multiparameter relativistic NLPT is the existence of a multiparameter family of spacetime models $M_{\vec{\lambda}} = (\mathcal{M}, \{\mathcal{T}_{\vec{\lambda}}\})$, where $\mathcal{M}$ denotes the spacetime manifold and $\{\mathcal{T}_{\vec{\lambda}}\}$ is a set of fields on $\mathcal{M}$, describing their geometrical and physical content, which we assume to be analytic. These spacetime models are labeled by a set of N parameters $\vec{\lambda} = (\lambda_1, \ldots, \lambda_N)$ that control the strength of the perturbations with respect to the *background* spacetime model $M_{\vec{0}}$, which describes an idealized situation. In order to construct the physical spacetime model $M_{\vec{\lambda}}$ as a deviation from the background model $M_{\vec{0}}$, we need to establish a correspondence between them; what, in the context of relativistic perturbation theory, is called a *gauge choice*. This correspondence is established, for all $\vec{\lambda}$, through the action of a diffeomorphism of $\mathcal{M}$: $\varphi_{\vec{\lambda}}: \mathcal{M} \to \mathcal{M}$. The set of diffeomorphisms $\mathcal{G}[\varphi] = \{\varphi_{\vec{\lambda}} | \vec{\lambda} \in \mathbb{R}^N\}$ is chosen in such a way they constitute an N-parameter group of diffeomorphisms of $\mathcal{M}$:

$$\begin{aligned} \varphi: \mathcal{M} \times \mathbb{R}^N &\longrightarrow \mathcal{M} \\ (p, \vec{\lambda}) &\longmapsto \varphi(p, \vec{\lambda}) := \varphi_{\vec{\lambda}}(p). \end{aligned} \tag{1}$$

The identity element corresponds to $\vec{\lambda} = \vec{0}$, i.e. $\varphi_{\vec{0}}(p) = p$. Moreover, a consistent perturbation scheme should have the property that perturbing first with respect to a given parameter, say λ_P, and afterwards with respect to another parameter, say λ_Q, must be equivalent to the converse operation. We can implement this idea by imposing the following composition rule for the group $\mathcal{G}[\varphi]$:

$$\forall\ \vec{\lambda}, \vec{\lambda}', \quad \varphi_{\vec{\lambda}} \circ \varphi_{\vec{\lambda}'} = \varphi_{\vec{\lambda}+\vec{\lambda}'}. \tag{2}$$

This property implies that the group is Abelian. It also implies that we can decompose $\varphi_{\vec{\lambda}}$ into N one-parameter groups of diffeomorphisms (flows) that remain implicitly defined by the equalities (we have $N!$ equalities)

$$\begin{aligned} \varphi_{\vec{\lambda}} &= \varphi_{(\lambda_1,0,\ldots,0)} \circ \varphi_{(0,\lambda_2,\ldots,0)} \circ \cdots \circ \varphi_{(0,0,\ldots,\lambda_N)} \\ &= \varphi_{(0,\lambda_2,\ldots,0)} \circ \varphi_{(\lambda_1,0,\ldots,0)} \circ \cdots \circ \varphi_{(0,0,\ldots,\lambda_N)} \\ &= \cdots. \end{aligned} \tag{3}$$

The action of the flow $\varphi_{(0,\ldots,\lambda_M,\ldots,0)}$ is generated by a vector field, ζ_M ($M = 1, \ldots, N$), acting on the tangent space of $\mathcal{M}$, and the Lie derivative of an arbitrary tensor field T with respect to ζ_M is given by

$$\mathcal{L}_{\zeta_M} T = \left[\frac{\partial \varphi^*_{(0,\ldots,\lambda_M,\ldots,0)} T}{\partial \lambda_M}\right]_{\lambda_M=0}, \tag{4}$$

where the superscript $*$ denotes the pullback. Since the group is Abelian, the vector fields ζ_M commute

$$[\zeta_P, \zeta_Q] = 0 \quad (\mathrm{P,Q} = 1, \ldots, \mathrm{N}). \tag{5}$$

The Taylor expansion around $\vec{\lambda} = \vec{0}$ of the pullback associated with the flow $\varphi_{(0,\ldots,\lambda_M,\ldots,0)}$ is given by

$$\varphi^*_{(0,\ldots,\lambda_M,\ldots,0)} T = \sum_{k=0}^{\infty} \frac{\lambda_M^k}{k!} \mathcal{L}^k_{\zeta_M} T. \tag{6}$$

This expression can be written in a more compact way by using the formal exponential notation

$$\varphi^*_{(0,\ldots,\lambda_M,\ldots,0)} T = \exp(\lambda_M \mathcal{L}_{\zeta_M}) T = e^{\lambda_M \mathcal{L}_{\zeta_M}} T, \tag{7}$$

which provides a clear operational way for working with groups of diffeomorphisms. From expressions (6) and (3) we can derive the Taylor expansion of the pullback $\varphi^*_{\vec{\lambda}} T$

$$\varphi^*_{\vec{\lambda}} T = \sum_{k_1,\ldots,k_N=0}^{\infty} \left(\prod_{P=1}^{N} \frac{\lambda_P^{k_P}}{k_P!} \mathcal{L}^{k_P}_{\zeta_P}\right) T. \tag{8}$$

Using the exponential notation and the commutation relations (5) we can write it as follows:

$$\varphi^*_{\vec{\lambda}} T = \left[\prod_{P=1}^{N} \exp(\lambda_P \mathcal{L}_{\zeta_P})\right] T = \exp\left(\sum_{P=1}^{N} \lambda_P \mathcal{L}_{\zeta_P}\right) T. \tag{9}$$

In a given gauge φ, the perturbation of an arbitrary tensorial quantity T is defined as

$$\Delta T^{\varphi}_{\vec{\lambda}} := \varphi^*_{\vec{\lambda}} T_{\vec{\lambda}} - T_{\vec{0}}. \tag{10}$$

The first term on the right-hand side of (10) can be Taylor-expanded around $\vec{\lambda} = \vec{0}$ using (8) to get

$$\Delta T^{\varphi}_{\vec{\lambda}} = \sum_{k_1,\ldots,k_N=0}^{\infty} \left(\prod_{P=1}^{N} \frac{\lambda_P^{k_P}}{k_P!}\right) \delta^{\vec{k}}_{\varphi} T - T_{\vec{0}}, \tag{11}$$

where $\vec{k} = (k_1, \ldots, k_N)$ and

$$\delta^{\vec{k}}_{\varphi} T := \left[\frac{\partial^{k_1+\cdots+k_N}}{\partial \lambda_1^{k_1} \cdots \partial \lambda_N^{k_N}} \varphi^*_{\vec{\lambda}} T\right]_{\vec{\lambda}=\vec{0}} = \prod_{P=1}^{N} \mathcal{L}^{k_P}_{\zeta_P} T, \tag{12}$$

which defines the perturbation of order $(k_1, \ldots, k_N)$ of T ($\delta^{\vec{0}}_{\varphi} T := T_{\vec{0}}$). The total order of a perturbation can be defined as $n_T := k_1 + \cdots + k_N$. As a consequence of these definitions, we have that $\Delta T^{\varphi}_{\vec{\lambda}}$ and $\delta^{\vec{k}}_{\varphi} T$ are fields that belong to the background spacetime model $M_{\vec{0}}$.

Let us consider two different gauges φ and ψ, with generators $({}^{\varphi}\zeta_1, \ldots, {}^{\varphi}\zeta_N)$ and $({}^{\psi}\zeta_1, \ldots, {}^{\psi}\zeta_N)$, respectively. For all $\vec{\lambda}$, the objects defined in these two gauges can be related by a diffeomorphism $\Phi_{\vec{\lambda}}:\mathcal{M}\rightarrow\mathcal{M}$ given by

$$\Phi_{\vec{\lambda}} := \varphi_{\vec{\lambda}}^{-1}\circ\psi_{\vec{\lambda}} = \varphi_{-\vec{\lambda}}\circ\psi_{\vec{\lambda}}. \quad (13)$$

This is what is called a *gauge transformation* in perturbation theory. The family of all the possible gauge transformations for two given gauges φ and ψ

$$\begin{aligned}\Phi:\mathcal{M}\times\mathbb{R}^{\rm N} &\longrightarrow \mathcal{M}\\ (p,\vec{\lambda}) &\longmapsto \Phi(p,\vec{\lambda}) = \Phi_{\vec{\lambda}}(p),\end{aligned} \quad (14)$$

is not in general a group of diffeomorphisms. The action of the gauge transformation $\Phi_{\vec{\lambda}}$ can be written as

$$\begin{aligned}\Phi^{*}_{\vec{\lambda}}T &= (\varphi_{-\vec{\lambda}}\circ\psi_{\vec{\lambda}})^{*}T = \psi^{*}_{\vec{\lambda}}\circ\varphi^{*}_{-\vec{\lambda}}T\\ &= \exp\left(\sum_{\rm P=1}^{\rm N}\lambda_{\rm P}\mathcal{L}_{\psi_{\zeta_{\rm P}}}\right)\exp\left(-\sum_{\rm P=1}^{\rm N}\lambda_{\rm P}\mathcal{L}_{\varphi_{\zeta_{\rm P}}}\right)T.\end{aligned} \quad (15)$$

Using group theory techniques we can write (15) as the action of a single exponential operator. This can be explicitly done by using the BCH formula [19] (cf. [20]). This formula, which can be applied to any two *linear operators* A and B, reads

$$e^{A}e^{B} = e^{f(A,B)}, \quad (16)$$

$$f(A,B) = \sum_{m\geq1}^{\infty}\frac{(-1)^{m-1}}{m}\sum_{\substack{p_i,\,q_i\\ p_i+q_i\geq1}}\frac{[\overbrace{A\cdots A}^{p_1}\overbrace{B\cdots B}^{q_1}\cdots\overbrace{A\cdots A}^{p_m}\overbrace{B\cdots B}^{q_m}]}{\left[\sum_{\alpha=1}^{m}(p_j+q_j)\right]\prod_{\alpha=1}^{m}p_\alpha!q_\alpha!}, \quad (17)$$

where the following notation has been used:

$$[X_1X_2X_3\cdots X_n] := [\cdots[[X_1,X_2],X_3],\cdots,X_n]. \quad (18)$$

Then, the BCH formula can be seen as an expansion in commutators of the initial operators A and B. Up to two commutators, this expansion is given by

$$\begin{aligned}f(A,B) = A + B + \frac{1}{2}[A,B] + \frac{1}{12}[[A,B],B]\\ + \frac{1}{12}[[B,A],A] + \cdots.\end{aligned} \quad (19)$$

This infinite expansion can be truncated and becomes finite when some commutators vanish (for a solvable Lie algebra). For instance, if $[[A,B],A] = [[A,B],B] = 0$, then the BCH formula only contains the three first terms shown in (19).

The application of the BCH formula, (16) and (17), to the construction of the gauge transformation $\Phi_{\vec{\lambda}}$ (15) constitutes the main point in our approach. As we are going to see, it provides an operational apparatus to compute all the perturbation orders as well as expressions for the vector fields that generate the transformation between different gauges. This supposes an important advantage with respect to the approach to two-parameter NLPT considered in [5], which is based on a construction order by order and no close expressions to every order can be obtained. In what follows, we show how to use the BCH formula to obtain closed expressions at every order, in particular, for the gauge transformation generators.

To apply the BCH formula to Eq. (15) we have to choose A and B in (16) and (17) as follows:

$$A = \sum_{\rm P=1}^{\rm N}\lambda_{\rm P}\mathcal{L}_{\psi_{\zeta_{\rm P}}} \text{ and } B = -\sum_{\rm P=1}^{\rm N}\lambda_{\rm P}\mathcal{L}_{\varphi_{\zeta_{\rm P}}}. \quad (20)$$

Since A and B are linear in the parameters, the number of commutators in a given term in the expansion of $f(A,B)$ coincides with the total perturbation order n_T.

Using the properties of Lie derivatives we can then express the gauge transformation $\Phi_{\vec{\lambda}}$ in the following way:

$$\Phi^{*}_{\vec{\lambda}}T = \exp\left\{\mathcal{L}_{f(\sum_{\rm P=1}^{N}\lambda_{\rm P}\psi_{\zeta_{\rm P}},-\sum_{\rm Q=1}^{\rm N}\lambda_{\rm Q}\varphi_{\zeta_{\rm Q}})}\right\}T. \quad (21)$$

This can be rewritten as:

$$\Phi^{*}_{\vec{\lambda}}T = \exp\left\{\sum_{k_1,\ldots,k_{\rm N}=0}^{\infty}\left(\prod_{\rm P=1}^{\rm N}\frac{\lambda_{\rm P}^{k_{\rm P}}}{k_{\rm P}!}\right)\mathcal{L}_{\xi_{\vec{k}}} - I\right\}T, \quad (22)$$

where $\mathcal{L}_{\xi_{\vec{0}}}$ denotes the identity operator I and the rest of terms are Lie derivatives. This is a consequence of the fact that A and B are linear combinations of Lie derivatives (20) and that the functional $f(A,B)$ is a linear combination of A, B, and commutators formed out of A and B (19). Then, this introduces an infinite set of vector fields $\{\xi_{\vec{k}}|\vec{k}\in\mathbb{N}^{\rm N}-\{\vec{0}\}\}$. By direct comparison of (19), (20), and (22) we can find the explicit expressions of these vector fields $\xi_{\vec{k}}$ directly in terms of the generators of the gauges φ and ψ.

We can then derive an expression for the gauge transformation up to a given order in the perturbation parameters by simply expanding the exponential (22). Up to third total order ($n_T = 3$) we obtain

$$\Phi^*_{\vec\lambda}T = T + \sum_{\rm P=1}^{\rm N}\lambda_{\rm P}\mathcal{L}_{\xi_{\vec k_{\rm P}}}T + \frac{1}{2}\sum_{\rm P,Q=1}^{\rm N}\lambda_{\rm P}\lambda_{\rm Q}\times\left(\mathcal{L}_{\xi_{\vec k_{\rm P}+\vec k_{\rm Q}}} + \mathcal{L}_{\xi_{\vec k_{\rm P}}}\mathcal{L}_{\xi_{\vec k_{\rm Q}}}\right)T + \frac{1}{6}\sum_{\rm P,Q,R=1}^{\rm N}\lambda_{\rm P}\lambda_{\rm Q}\lambda_{\rm R}\times\left(\mathcal{L}_{\xi_{\vec k_{\rm P}+\vec k_{\rm Q}+\vec k_{\rm R}}} + \frac{3}{2}\left\{\mathcal{L}_{\xi_{\vec k_{\rm P}+\vec k_{\rm Q}}}, \mathcal{L}_{\xi_{\vec k_{\rm R}}}\right\} + \mathcal{L}_{\xi_{\vec k_{\rm P}}}\mathcal{L}_{\xi_{\vec k_{\rm Q}}}\mathcal{L}_{\xi_{\vec k_{\rm R}}}\right)T + O^4(\vec\lambda), \quad (23)$$

where $\vec k_{\rm P} = (\overbrace{0,\ldots,0}^{P-1}, 1, \overbrace{0,\ldots,0}^{N-P})$ and $\{A, B\}$ denotes the anticommutator of A and B. From (23) we can easily derive the transformation of a given perturbation from the gauge φ to the gauge ψ, $\delta^{\vec k}_{\psi}T - \delta^{\vec k}_{\varphi}T$. First, from (13), the pullbacks $\varphi^*_{\vec\lambda}T$ and $\psi^*_{\vec\lambda}T$ are related by

$$\psi^*_{\vec\lambda}T_{\vec\lambda} = \Phi^*_{\vec\lambda}\varphi^*_{\vec\lambda}T_{\vec\lambda}. \quad (24)$$

Then, using (10) and (11) we have that

$$\varphi^*_{\vec\lambda}T_{\vec\lambda} = \sum_{k_1,\ldots,k_{\rm N}=0}^{\infty}\left(\prod_{\rm P=1}^{\rm N}\frac{\lambda_p^{k_{\rm P}}}{k_{\rm P}!}\right)\delta^{\vec k}_{\varphi}T, \quad (25)$$

and for $\psi^*_{\vec\lambda}T_{\vec\lambda}$ we only have to replace φ by ψ. From (23)–(25) we can extract the expressions for the $\delta^{\vec k}_{\psi}T$'s in terms of Lie derivatives of the $\delta^{\vec k}_{\varphi}T$'s. In the particular case $N = 2$ and $\vec k = (1, 1)$ we find

$$\delta^{(1,1)}_{\psi}T = \delta^{(1,1)}_{\varphi}T + \mathcal{L}_{\xi_{(1,0)}}\delta^{(0,1)}_{\varphi}T + \mathcal{L}_{\xi_{(0,1)}}\delta^{(1,0)}_{\varphi}T + \left[\mathcal{L}_{\xi_{(1,1)}} + \frac{1}{2}\{\mathcal{L}_{\xi_{(1,0)}}, \mathcal{L}_{\xi_{(0,1)}}\}\right]T_0. \quad (26)$$

With the aim of comparing the formulation here introduced with previous approaches to NLPT, we will show now how to recover standard one-parameter NLPT. Let us consider two gauge choices: φ and ψ. For a given λ, the action of their associated pullbacks can be written in the exponential notation as: $\varphi^*_\lambda T = e^{\lambda\mathcal{L}_{\varphi\zeta}}T$, and $\psi^*_\lambda T = e^{\lambda\mathcal{L}_{\psi\zeta}}T$. A gauge transformation between these two gauges is then described by $\Phi_\lambda = \varphi^{-1}_\lambda \circ \psi_\lambda$. Using the exponential notation, its action can expressed as follows

$$\Phi^*_\lambda T = e^{\lambda\mathcal{L}_{\psi\zeta}}e^{-\lambda\mathcal{L}_{\varphi\zeta}}T. \quad (27)$$

The result of using the BCH formula can be written as

$$\Phi^*_\lambda T = \exp\left(\sum_{n=1}^{\infty}\frac{\lambda^n}{n!}\mathcal{L}_{\xi_n}\right)T, \quad (28)$$

where $\{\xi_n | n \in \mathrm{IN} - \{0\}\}$ is a set of generators of Φ. These can be expressed in terms of the gauge generators ${}^{\varphi}\zeta$ and ${}^{\psi}\zeta$. From (19) the three first generators are:

$$\xi_1 = {}^{\psi}\zeta - {}^{\varphi}\zeta, \qquad \xi_2 = [{}^{\varphi}\zeta, {}^{\psi}\zeta], \qquad \xi_3 = \frac{1}{2}[{}^{\varphi}\zeta + {}^{\psi}\zeta, [{}^{\varphi}\zeta, {}^{\psi}\zeta]]. \quad (29)$$

Up to third order, (28) gives [the N=1 subcase of (23)] :

$$\Phi^*_\lambda T = T + \lambda\mathcal{L}_{\xi_1}T + \frac{\lambda^2}{2}(\mathcal{L}_{\xi_2} + \mathcal{L}^2_{\xi_1})T + \frac{\lambda^3}{3!}\times\left(\mathcal{L}_{\xi_3} + \frac{3}{2}\{\mathcal{L}_{\xi_1}, \mathcal{L}_{\xi_2}\} + \mathcal{L}^3_{\xi_1}\right)T + O^4(\lambda). \quad (30)$$

This form of $\Phi^*_\lambda T$, derived through the BCH approach, is not the same as the one obtained using KDs in [3,4]. However, as we are now going to show, the resulting gauge transformations are—order by order—equivalent, as expected, since both cases are expansions of the same exact expression (27). The KD is defined as [3,4]:

$$\Phi^{(k)}_\lambda = \phi^{(k)}_{\lambda^k/k!} \circ \cdots \circ \phi^{(2)}_{\lambda^2/2} \circ \phi^{(1)}_\lambda, \quad (31)$$

where the $\phi^{(n)}$ are one-parameter groups of diffeomorphisms (flows). The main idea behind this concept is that a family of diffeomorphisms $\{\Phi_\lambda | \lambda \in \mathrm{IR}\}$ can be approximated at a given order in λ, say k, by a KD of order k. Therefore one can approximate Φ_λ by $\Phi^{(k)}_\lambda$ in the following sense [4]:

$$\Phi^*_\lambda T - \Phi^{(k)*}_\lambda T = O^{k+1}(\lambda). \quad (32)$$

The action of the pullback of $\Phi^{(k)}_\lambda$ can be expressed, using the exponential notation, as

$$\Phi^{(k)*}_\lambda T = e^{\lambda\mathcal{L}_{\chi_1}} \cdots e^{\lambda^k/_{k!}\mathcal{L}_{\chi_k}}T, \quad (33)$$

where $\{\chi_n\}_{n=1,\ldots,k}$ is the set of generators of the family $\Phi^{(k)}$, and each χ_n is the generator of the flow $\phi^{(n)}$. Like the ξ_n's, they can be expressed in terms of the gauge generators ${}^{\varphi}\zeta$ and ${}^{\psi}\zeta$. Hence, we can find the relations between the ξ_n's and χ_n's. Up to third order we have

$$\xi_1 = \chi_1, \qquad \xi_2 = \chi_2, \qquad \xi_3 = \chi_3 + \frac{3}{2}[\chi_1, \chi_2]. \quad (34)$$

Therefore, the expansion for $\Phi^*_\lambda T$ that we obtain from the expansion of $\Phi^{(k)*}_\lambda T$ is (up to third order):

$$\Phi^{(k)*}_\lambda T = T + \lambda\mathcal{L}_{\chi_1}T + \frac{\lambda^2}{2}(\mathcal{L}_{\chi_2} + \mathcal{L}^2_{\chi_1})T + \frac{\lambda^3}{3!}\times(\mathcal{L}_{\chi_3} + 3\mathcal{L}_{\chi_1}\mathcal{L}_{\chi_2} + \mathcal{L}^3_{\chi_1})T + O^4(\lambda), \quad (35)$$

i.e. the result in [3,4]. Comparing the expansions (30) and (35) and we see that they have different structures. However, substituting χ_1, χ_2 and χ_3 from (34) into (35) we obtain (30); thus, these two expansions of the gauge transformation (27) are equivalent. Our formulation, Eq. (30), leads to an expansion with terms of the form $\cdots\mathcal{L}_{\xi_k}\cdots\mathcal{L}_{\xi_l}\cdots T$ with $l < k$, which do not appear in the formulations of [3] [due to the ordering introduced by the KDs, see (31)] and [5]. Comparing further our formulation with the order by order approach in [5] we see how the use of the BCH formula naturally selects a unique expansion for the gauge transformation, in contrast with [5], which contains freely specifiable parameters.

To sum up, we have presented a formulation of N-parameter NLPT in which we have a (unique) closed formula for the expansion of general gauge transformations and whose consistency is given by construction, shedding light onto the underlying mathematical structure. The importance of this result is even more clear if considered in the context of practical applications of relativistic perturbation theory, where the issue of comparing results obtained in different gauges and the related problem of constructing gauge-invariant formulations have always been crucial to obtain physically transparent results [1,22]. These issues become even more important when dealing with nonlinear perturbations [3,4,22] and more than one parameter. Our formalism provides the gauge transformations and the conditions for gauge invariance for every perturbation order in N-parameters (explicit conditions for the 2-parameter case where given in [5]).

In retrospect, one may wonder why our general results have not been previously derived, given that the BCH formula has long been known in differential geometry [19]. The likely answer is that, although the problem of gauge dependence is as old as relativistic perturbation itself [1], until recently, spacetime perturbations have mostly been considered at first order only and for a single parameter, and consequently gauge transformations have always been dealt with at the most elementary linear level, where the BCH formalism and the exponentiation (7) on which it is based are superfluous. When the problem of gauge transformations has been considered in NLPT for the case of one parameter, two routes have been followed. In [3,4] KDs have been introduced and used (see also [11] for an equivalent second order result and [21] for some basic fomulas), and, in particular, in [4] a closed formula was derived to generate gauge transformations at arbitrary order. In [20] on the other hand the BCH formula was used, for one parameter at second order. As we have illustrated above in the one parameter case and up to third order, the two routes are equivalent in that they provide equivalent gauge transformations at the required order. On the other hand, the gauge transformations derived in [5] contain freely specifiable constants (linked by sets of constraints) that are not present in the BCH derivation presented here. Again, order by order the gauge transformations are equivalent, with one specific choice of the constants corresponding directly to the BCH derivation, and other choices connected by appropriate relations between the two sets of generators of the gauge transformations. From the point of view of generality elegance and compactness of the derivation the BCH approach presented here is by far superior to that followed in [5]. However, for practical purposes one is interested in the gauge transformations at a given order, e.g., (26), and in this perspective we believe that the formulas with freely specifiable constants in [5] may still be useful. Indeed, the typical problem (see, e.g., [3,9]) is that one wants to know how to transform between two preassigned gauges. In this case the unknowns of the problem are the generators of the transformation. Then, one faces integration calculations, and given that two different choices of the freely specifiable constants correspond to integration in a different order, it may turn out that one specific choice of constants is better in solving the problem.

We want to finish by discussing the potential applications of N-parameter NLPT. First of all, it is important to remark that perturbation theory in general relativity, and in other spacetime theories (some of them of great relevance nowadays), remains the main alternative to fully numerical methods in a context in which one has to deal with sets of nonlinear field equations. Depending on the physical problem at hand, it is sometimes necessary to go beyond simple linear perturbations, considering higher-order contributions. In this sense, a multiparameter perturbative scheme as the one presented here allows us to select only the higher-order perturbative sectors relevant for the physical problem under consideration, simplifying in this way the calculations involved.

There are already quite few applications of the one-parameter NLPT at second order in the literature. In cosmology, the evolution of perturbations in two different gauges is explicitly worked out [3,9] and applications to the cosmic microwave background have been considered [26]. Further applications in the cosmological context can be found in [10,20]. In an astrophysical context there are a number of studies of sources of gravitational radiation: in [17,27] oscillations during gravitational collapse have been analyzed; in [15] the so called close limit approximation is used to study the outcome of black hole mergers; in [23] a second-order gauge-invariant perturbative scheme for the Kerr metric has been developed. The N-parameter NLPT opens the door for new applications in spacetime theories (see [25] for an example). In the general relativistic case, it can provide a new way of studying slowly rotating relativistic stars, and it can be the main tool to study other issues as for instance the nonlinear coupling of oscillation modes of relativistic stars, or in cosmology to study the combined effect of magnetic fields and linear perturbations in the matter distribution.

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