Polynomial interpretation of multipole vectors

Gabriel Katz*
Bennington College, Bennington, Vermont 05201, USA

Jeff Weeks†
15 Farmer Street, Canton, New York 13617, USA
(Received 1 June 2004; published 21 September 2004)

Copi, Huterer, Starkman, and Schwarz introduced multipole vectors in a tensor context and used them to demonstrate that the first-year Wilkinson microwave anisotropy probe (WMAP) quadrupole and octopole planes align at roughly the 99.9% confidence level. In the present article, the language of polynomials provides a new and independent derivation of the multipole vector concept. Bézout's theorem supports an elementary proof that the multipole vectors exist and are unique (up to rescaling). The constructive nature of the proof leads to a fast, practical algorithm for computing multipole vectors. We illustrate the algorithm by finding exact solutions for some simple toy examples and numerical solutions for the first-year WMAP quadrupole and octopole. We then apply our algorithm to Monte Carlo skies to independently reconfirm the estimate that the WMAP quadrupole and octopole planes align at the 99.9% level.

DOI: 10.1103/PhysRevD.70.063527 PACS numbers: 98.80.–k

I. INTRODUCTION

The first-year Wilkinson microwave anisotropy probe (WMAP) data [1] reveal a somewhat planar octopole, which approximately aligns with the quadrupole [2]. More recent studies confirm these conclusions at roughly the 99.9% level [3] while revealing mysterious alignments with the ecliptic plane [4], suggesting either a hitherto unknown solar system effect on the microwave background or an error in the collection and/or processing of the WMAP data. Other researchers find that the $\ell = 4$ multipole is generic, the $\ell = 5$ multipole is unusually nonplanar at the 99.8% level, and the $\ell = 6$ multipole is unusually planar at the 98.6% level [5]. No explanation is yet known for these strange results.

Multipole vectors provide a convenient means to quantify the planarity of a given multipole as well as to compare the alignment of two different multipoles [6]. The present authors, coming from a background in pure mathematics, were unable to decipher the formalism and terminology of Ref. [6] and chose instead to recreate the multipole vector concept from scratch. The real-valued spherical harmonics of order ℓ are precisely the homogeneous harmonic polynomials of degree ℓ in the variables x, y, and z (for example, Y_2^0 is the polynomial $x^2 + y^2 - 2z^2$, up to normalization), so the present authors sought to understand the multipole vectors of Copi, Huterer, and Starkman (CHS) from a polynomial point of view.

Translated to the language of polynomials, CHS's motivating goal [see Eq. (10) of Ref. [6]] was to factor every homogeneous harmonic polynomial P of degree ℓ into a product of linear factors

$$\begin{aligned} P(x, y, z) = \lambda &\cdot (a_1 x + b_1 y + c_1 z) \\ &\cdot (a_2 x + b_2 y + c_2 z) \\ &\cdots \\ &\cdot (a_\ell x + b_\ell y + c_\ell z). \end{aligned} \tag{1}$$

Such a factorization is, of course, impossible in general, as CHS implicitly acknowledge by their introduction of suitable error terms. In the language of polynomials the correct statement of the theorem is

Theorem 1.—Every homogeneous polynomial P of degree ℓ in x, y, and z may be written as

$$\begin{aligned} P(x, y, z) = \lambda &\cdot (a_1 x + b_1 y + c_1 z) \\ &\cdot (a_2 x + b_2 y + c_2 z) \\ &\cdots \\ &\cdot (a_\ell x + b_\ell y + c_\ell z) \\ &+(x^2 + y^2 + z^2) \cdot R, \end{aligned} \tag{2}$$

where the remainder term R is a homogeneous polynomial of degree $\ell - 2$. The decomposition is unique up to reordering and rescaling the linear factors.

Notes: (a) Theorem 1 lives entirely in the realm of real polynomials: the coefficients of P, R, and all the linear factors $a_i x + b_i y + c_i z$ are assumed to be real. (b) Theorem 1 does not require the polynomial P to be harmonic.

In cosmological applications, we are interested only in the value of the polynomial on the unit sphere S^2; we ignore its value on the rest of Euclidean 3-space. On the unit sphere, the factor $x^2 + y^2 + z^2$ is identically 1, so in this case Theorem 1 says that any homogeneous polynomial P may be written as a product of linear factors $\lambda(a_1 x + b_1 y + c_1 z) \cdots (a_\ell x + b_\ell y + c_\ell z)$ plus a remainder term R of lower degree. Applying this reasoning recursively gives the easy

*Electronic address: gkatz@bennington.edu
†Electronic address: www.geometrygames.org/contact.html

Corollary 2.—When restricted to the unit sphere, every polynomial P of degree ℓ in x, y, and z may be written as

$$\begin{aligned} P(x,y,z) = \; & \lambda_\ell \cdot (a_{\ell,1}x + b_{\ell,1}y + c_{\ell,1}z) \\ & \cdot (a_{\ell,2}x + b_{\ell,2}y + c_{\ell,2}z) \\ & \cdots (a_{\ell,\ell}x + b_{\ell,\ell}y + c_{\ell,\ell}z) \\ & + \ldots \\ & + \lambda_2 \cdot (a_{2,1}x + b_{2,1}y + c_{2,1}z) \\ & \cdot (a_{2,2}x + b_{2,2}y + c_{2,2}z) \\ & + \lambda_1 \cdot (a_{1,1}x + b_{1,1}y + c_{1,1}z) \\ & + \lambda_0. \end{aligned} \tag{3}$$

The decomposition is unique up to reordering and rescaling the linear factors within each term.

Note: Corollary 2 does not require the polynomial P to be either homogeneous or harmonic.

Proof of Corollary 2.—Write P as a sum of homogeneous terms $P = P_\ell + P_{\ell-1} + \ldots + P_1 + P_0$. First apply Theorem 1 to the highest order term P_ℓ, yielding a factorization $\lambda_\ell(a_{\ell,1}x + b_{\ell,1}y + c_{\ell,1}z) \cdots (a_{\ell,\ell}x + b_{\ell,\ell}y + c_{\ell,\ell}z)$ along with a remainder term $R_{\ell-2}$ of homogeneous degree $\ell - 2$. (The factor $x^2 + y^2 + z^2$ may be ignored on the unit sphere.) Fold $R_{\ell-2}$ in with $P_{\ell-2}$ and proceed recursively, applying Theorem 1 to $P_{\ell-1}$, then $P_{\ell-2}$, and so on.

To prove uniqueness, consider the even and the odd parts of P separately. That is, write $P = P_{\text{even}} + P_{\text{odd}}$, where P_{even} contains all the even-powered terms and P_{odd} contains all the odd-powered terms. Say we have two potentially different decompositions for the even part

$$\begin{aligned} P_{\text{even}} &= \Pi_\ell + \Pi_{\ell-2} + \Pi_{\ell-4} + \ldots + \Pi_0 \\ &= \Pi'_\ell + \Pi'_{\ell-2} + \Pi'_{\ell-4} + \ldots + \Pi'_0. \end{aligned} \tag{4}$$

where each Π_i is the ith term in a decomposition (3) and where the leading index will be ℓ or $\ell - 1$ according to whether ℓ is even or odd. To make these decompositions homogeneous, multiply through by appropriate powers of $Q = x^2 + y^2 + z^2$,

$$\begin{aligned} P_{\text{even}} &= \Pi_\ell + Q\Pi_{\ell-2} + Q^2\Pi_{\ell-4} + \ldots + Q^{\ell/2}\Pi_0 \\ &= \Pi'_\ell + Q\Pi'_{\ell-2} + Q^2\Pi'_{\ell-4} + \ldots + Q^{\ell/2}\Pi'_0. \end{aligned} \tag{5}$$

This does not affect the value of P on the unit sphere, because $Q = 1$ there. The uniqueness part of Theorem 1 implies that the leading order terms Π_ℓ and Π'_ℓ must be equal. So subtract off those leading terms, divide through by Q, and apply Theorem 1 again to conclude $\Pi_{\ell-2} = \Pi'_{\ell-2}$. Continue recursively to finally reach $\Pi_0 = \Pi'_0$. The same argument then proves that the odd part of P has a unique decomposition as well. Q.E.D.

II. PROOF OF THE MAIN THEOREM

Even though the statement of Theorem 1 lives wholly in the world of real polynomials, its proof will dive deeply into the world of complex polynomials. So let the variables x, y, and z range over the complex numbers, while insisting that the coefficients of the polynomial P remain real. Because P has homogeneous degree ℓ, whenever one point (x_0, y_0, z_0) satisfies $P(x, y, z) = 0$, every nonzero constant multiple $(\alpha x_0, \alpha y_0, \alpha z_0)$ satisfies it as well. Thus, the equation $P = 0$ is well defined on each equivalence class of points $\{\alpha(x_0, y_0, z_0) | \alpha \in \mathbb{C} - \{0\}\}$. In other words, the complex curve $P = 0$ is well defined on the complex projective plane $\mathbb{C}P^2$, which is the quotient of $\mathbb{C}^3 - \{(0,0,0)\}$ under the equivalence relation $(x_0, y_0, z_0) \sim \alpha(x_0, y_0, z_0)$. This leads us into the realm of algebraic geometry and puts its powerful tools at our disposal.

The most useful tool for our purposes is

Bézout's theorem.—If P and Q are homogeneous polynomials of degree m and n, respectively, then the curves $P = 0$ and $Q = 0$ intersect in $\mathbb{C}P^2$

(i) in exactly mn points, counted with multiplicity, if P and Q share no common factor, or
(ii) in infinitely many points, if P and Q do share a common factor.

For an elementary exposition of Bézout's theorem, see [7].

In the present case, the only way the polynomial P may share a factor with the irreducible polynomial $Q(x, y, z) \equiv x^2 + y^2 + z^2$ is for P to contain Q as a factor, in which case Theorem 1 is trivially satisfied (take $\lambda = 0$). So henceforth assume P does not contain Q as a factor. Bézout's theorem now tells us that the degree ℓ complex curve $P(x, y, z) = 0$ intersects the quadratic curve $Q(x, y, z) = 0$ in exactly 2ℓ points, counted with multiplicities. None of the intersection points may be purely real, because real values cannot possibly satisfy $x^2 + y^2 + z^2 = 0$—recall that the definition of $\mathbb{C}P^2$ explicitly excludes $(0, 0, 0)$. Furthermore, because P and Q both have real coefficients, whenever (x_0, y_0, z_0) lies in the intersection $P = Q = 0$, its complex conjugate $(\bar{x}_0, \bar{y}_0, \bar{z}_0)$ must lie there, too. So the 2ℓ points of intersection consist of ℓ pairs of nonreal complex conjugates $\{p_1, \bar{p}_1, \ldots, p_\ell, \bar{p}_\ell\}$.

We claim that each pair $\{p_i, \bar{p}_i\}$ determines a unique line $a_i x + b_i y + c_i z = 0$ with real coefficients. The proof is easy. The conjugate pair $\{p_i, \bar{p}_i\}$ lies on the line $a_i x + b_i y + c_i z = 0$ if and only if the real and imaginary parts satisfy the following two totally real equations:

$$\begin{aligned} a_i \text{Re} p_{i,x} + b_i \text{Re} p_{i,y} + c_i \text{Re} p_{i,z} = 0, \\ a_i \text{Im} p_{i,x} + b_i \text{Im} p_{i,y} + c_i \text{Im} p_{i,z} = 0. \end{aligned} \tag{6}$$

Geometrically, those two equations represent planes in R^3. If the coefficient vectors $(\text{Re} p_{i,x}, \text{Re} p_{i,y}, \text{Re} p_{i,z})$ and $(\text{Im} p_{i,x}, \text{Im} p_{i,y}, \text{Im} p_{i,z})$ are noncollinear, then the two planes are distinct and their intersection, which defines the solution set for (a_i, b_i, c_i), is a line through the origin in R^3. In other words, the line $a_i x + b_i y + c_i z = 0$ is unique. Normalize the coefficients to unit length, i.e.,

$a_i^2 + b_i^2 + c_i^2 = 1$, and the only remaining ambiguity is an overall factor of ± 1.

But what if the coefficient vectors $\text{Re}p_i = (\text{Re}p_{i,x}, \text{Re}p_{i,y}, \text{Re}p_{i,z})$ and $\text{Im}p_i = (\text{Im}p_{i,x}, \text{Im}p_{i,y}, \text{Im}p_{i,z})$ had been collinear? In this case the line $a_i x + b_i y + c_i z = 0$ would be ill defined. Fortunately, this case does not arise. For, if $\text{Im}p_i$ were proportional to $\text{Re}p_i$, say, $\text{Im}p_i = \beta \text{Re}p_i$, then the point p_i, as an element of $\mathbb{C}P^2$, could be rewritten as a scalar multiple $p_i \sim \frac{1}{1+i\beta} p_i = \text{Re}p_i$, showing that p_i is totally real. In other words, p_i would lie in $\mathbb{R}P^2 \subset \mathbb{C}P^2$. In particular, p_i would be its own complex conjugate, and we can hardly expect a single point $p_i = \overline{p}_i$ to determine a unique line. Fortunately, this case cannot occur, because $Q(x, y, z) = x^2 + y^2 + z^2 = 0$ admits no nontrivial real solutions.

So let L_i denote the unique line $a_i x + b_i y + c_i z = 0$ containing the conjugate pair $\{p_i, \overline{p}_i\}$. More precisely, let $L_i = a_i x + b_i y + c_i z$ be the unique (modulo rescaling) real linear polynomial whose roots include both p_i and $\overline{p}_i$. The desired decomposition (2) becomes

$$P = \lambda L_1 L_2 \cdots L_\ell + QR. \tag{7}$$

To prove that this equality holds, we again turn to Bézout's theorem. First, recall that the complex curve $P = 0$ intersects the complex curve $Q = 0$ in precisely the 2ℓ points $\{p_1, \overline{p}_1, \ldots, p_\ell, \overline{p}_\ell\}$. By construction, the product curve $L_1 L_2 \cdots L_\ell = 0$ also intersects $Q = 0$ in those same 2ℓ points, and by Bézout's theorem there are no other points of intersection. Now pick any other point $q \in \{Q = 0\}$ and define λ to be the ratio

$$\lambda = \frac{P(q)}{L_1(q) L_2(q) \cdots L_\ell(q)}. \tag{8}$$

Write a new polynomial

$$F \equiv P - \lambda L_1 L_2 \cdots L_\ell. \tag{9}$$

This new polynomial F has degree ℓ, yet has zeros at the $2\ell + 1$ distinct points $\{q, p_1, \overline{p}_1, \ldots, p_\ell, \overline{p}_\ell\} \subset Q$. In other words, the complex curve $F = 0$ intersects the complex curve $Q = 0$ at (at least) $2\ell + 1$ distinct points. By Bézout's theorem the polynomials F and Q must share a common factor; because Q is irreducible, the common factor must perforce be Q itself. Thus, we may factor F as

$$F = QR \tag{10}$$

for some remainder term R. Combining (9) and (10) yields the desired decomposition (7).

Let us now prove that λ is real. In light of the factorization (10), the polynomial F is clearly zero on the whole complex curve $Q = 0$. In particular, for the point q chosen earlier,

$$F(q) = F(\bar{q}) = 0. \tag{11}$$

On the one hand,

$$F(\bar{q}) = P(\bar{q}) - \lambda L_1(\bar{q}) L_2(\bar{q}) \cdots L_\ell(\bar{q}). \tag{12}$$

On the other hand, because P and L_i all have real coefficients,

$$\overline{F(q)} = P(\bar{q}) - \bar{\lambda} L_1(\bar{q}) L_2(\bar{q}) \cdots L_\ell(\bar{q}). \tag{13}$$

Comparing (11), (12), and (13), and recalling that q was chosen to ensure $L_i(q) \neq 0$, proves that $\lambda = \bar{\lambda}$; in other words, λ is real.

An elementary argument then shows that for all real values of x, y, and z, $R(x, y, z) = \overline{R(x, y, z)} = \overline{R}(x, y, z)$, implying that the coefficients of the polynomial R must all be real.

This completes the proof of the existence part of Theorem 1.

Let us now prove that the decomposition (2) is unique. Assume we have two decompositions

$$\begin{aligned} P(x, y, z) &= \lambda L_1 L_2 \cdots L_\ell + QR \\ &= \lambda' L'_1 L'_2 \cdots L'_\ell + QR'. \end{aligned} \tag{14}$$

Our goal is to show that each linear factor $L'_{i'}$ in the second decomposition occurs as a factor L_i in the first decomposition as well, modulo a possible rescaling. A given line $L'_{i'} = 0$ intersects the quadratic $Q = 0$ in a pair of conjugate points p and $\bar{p}$. Because p and $\bar{p}$ satisfy both $L'_{i'} = 0$ and $Q = 0$, they satisfy $P = 0$ as well. Turning our attention to the first decomposition, because p and $\bar{p}$ satisfy both $P = 0$ and $Q = 0$, they satisfy $L_1 L_2 \cdots L_\ell = 0$ as well. Hence, p must satisfy one of the lines $L_i = 0$, and because the line's coefficients are real, $\bar{p}$ must satisfy that same line. But we saw earlier that a pair of conjugate points p and $\bar{p}$ determines a unique line modulo normalization [recall the essentially unique solution to Eqs. (6)]. Therefore, L_i is a constant multiple of $L'_{i'}$, and if the coefficients of each have been normalized to length 1, then $L_i = \pm L'_{i'}$. This proves the uniqueness of the factorization.

If we evaluate the two decompositions (14) on the complex curve $Q = 0$, we get

$$\lambda L_1 L_2 \cdots L_\ell = \lambda' L'_1 L'_2 \cdots L'_\ell, \tag{15}$$

proving that, if the coefficients of L_i and $L'_{i'}$ are consistently normalized, then $\lambda = \lambda'$. It then follows easily that $R = R'$ as well.

This completes the proof that the decomposition (2) is unique, thus completing the proof of Theorem 1.

III. COMPUTATIONAL CONSIDERATIONS

The proof presented in Sec. II is almost constructive, but not quite. It relies on Bézout's theorem for the existence of the root pairs $\{p_1, \overline{p}_1, \ldots, p_\ell, \overline{p}_\ell\}$ but does not say how to find them. This section fills the gap.

The key observation is that the quadratic curve $Q = x^2 + y^2 + z^2 = 0$ is topologically a 2-sphere. More to the point, it is a copy of the complex projective line $\mathbb{C}P^1$, which happens to be homeomorphic to the 2-sphere. Let us parametrize the curve $Q = 0$ as

$$(u, v) \mapsto (x(u, v), y(u, v), z(u, v)) = (i(u^2 - v^2), -2iuv, u^2 + v^2), \quad (16)$$

where u and v are homogeneous coordinates in $\mathbb{C}P^1$. Clearly, the mapping (16) takes all points $(u, v) \in \mathbb{C}P^1$ to the curve $Q = 0$, by construction. The question is, which of those points happen to satisfy the given polynomial P as well? Write

$$P(x, y, z) = P(i(u^2 - v^2), -2iuv, u^2 + v^2) \quad (17)$$

to express P as a function on $\mathbb{C}P^1$.

If $v \neq 0$, then (u, v) and $(\frac{u}{v}, 1)$ represent the same point in $\mathbb{C}P^1$. If we define $\alpha \equiv \frac{u}{v}$, then expression (17) effectively reduces to a polynomial in a single variable,

$$P(x, y, z) = P(i(\alpha^2 - 1), -2i\alpha, \alpha^2 + 1). \quad (18)$$

The roots of this polynomial are the desired root pairs $\{p_1, \overline{p}_1, \ldots, p_\ell, \overline{p}_\ell\}$.

If, on the other hand, $v = 0$, then $(u, v) = (u, 0) \sim (1, 0)$. Thus, $(u, v) = (1, 0)$ may represent an additional root, which would not be found as a root of $P(\alpha)$ in (18).

Once we have found the parameters (u, v) for all 2ℓ roots of P, the easiest way to group them into conjugate pairs is to observe that the parametrization (16) maps "antipodal points" $(u, v), (-\bar{v}, \bar{u}) \in \mathbb{C}P^1$ to conjugate points $(x, y, z), (\bar{x}, \bar{y}, \bar{z}) \in \mathbb{C}P^2$. In other words, $(\alpha, 1)$ and $(-1, \bar{\alpha}) \sim (-1/\bar{\alpha}, 1)$ map to a pair of conjugate points in $\mathbb{C}P^2$.

IV. EXAMPLES

To illustrate how the algorithm works in practice, let us apply Theorem 1 to several concrete examples.

A. Toy quadrupole

Consider the quadratic polynomial

$$P(x, y, z) = xy + yz + zx - x^2 - z^2. \quad (19)$$

First, dismiss the special case $(u, v) = (1, 0)$ by noting that the parametrization (16) maps $(u, v) = (1, 0)$ to $(x, y, z) = (i, 0, 1)$ for which (19) gives $P(i, 0, 1) = i \neq 0$.

Now consider the general case, for which Eq. (18) becomes

$$i\alpha^4 + 2(1 - i)\alpha^3 - 4\alpha^2 - 2(1 + i)\alpha - i = 0 \quad (20)$$

with roots

$$\alpha_1 = 1 + \sqrt{2}, \qquad \alpha_2 = 1 - \sqrt{2}, \qquad \alpha_3 = i(1 + \sqrt{2}), \qquad \alpha_4 = i(1 - \sqrt{2}), \quad (21)$$

corresponding, respectively, to the four points of $\mathbb{C}P^2$,

$$p_1 = (1, -1, -i\sqrt{2}), \qquad p_2 = (-i\sqrt{2}, 1, -1), \qquad \overline{p}_1 = (1, -1, +i\sqrt{2}), \qquad \overline{p}_2 = (+i\sqrt{2}, 1, -1). \quad (22)$$

Solving the line equations (6) converts the preceding two pairs of conjugate points to the two lines

$$L_1 = \sqrt{\tfrac{1}{2}}x + \sqrt{\tfrac{1}{2}}y = 0, \qquad L_2 = \sqrt{\tfrac{1}{2}}y + \sqrt{\tfrac{1}{2}}z = 0, \quad (23)$$

which give us the two multipole vectors $(\sqrt{1/2}, \sqrt{1/2}, 0)$ and $(0, \sqrt{1/2}, \sqrt{1/2})$.

To find the correct λ, evaluate Eq. (8) for, say, $q = (1, i, 0)$, giving

$$\lambda = \frac{P(q)}{L_1(q)L_2(q)} = \frac{-1 + i}{-\frac{1}{2} + \frac{i}{2}} = 2. \quad (24)$$

Of course, any other choice for q would have given the same answer $\lambda = 2$, just so we make sure q lies on the curve $Q(q) = x^2 + y^2 + z^2 = 0$ and exclude $q \in \{p_1, \overline{p}_1, p_2, \overline{p}_2\}$.

We may now write down the polynomial F from Eq. (9), namely,

$$\begin{aligned} F &= P - \lambda L_1 L_2 \\ &= (xy + yz + zx - x^2 - z^2) \\ &\quad -2(\sqrt{\tfrac{1}{2}}x + \sqrt{\tfrac{1}{2}}y)(\sqrt{\tfrac{1}{2}}y + \sqrt{\tfrac{1}{2}}z) \\ &= -x^2 - y^2 - z^2, \end{aligned} \quad (25)$$

and divide by $Q = x^2 + y^2 + z^2$ to get the remainder term $R = F/Q = -1$. Thus, the final decomposition promised by Theorem 1 becomes

$$xy + yz + zx - x^2 - z^2 = 2(\sqrt{\tfrac{1}{2}}x + \sqrt{\tfrac{1}{2}}y)(\sqrt{\tfrac{1}{2}}y + \sqrt{\tfrac{1}{2}}z) + (x^2 + y^2 + z^2)(-1). \quad (26)$$

B. Toy octopole

The cubic polynomial

$$P(x, y, z) = x^2y + y^3 \quad (27)$$

illustrates some nongeneric behavior which may arise, namely, the possibilities of (a) a "missing root" and (b) multiple roots. We will follow the same algorithm as in Sec. IV A, pointing out only the differences.

The first difference is that the special case $(u, v) = (1, 0)$, corresponding to $(x, y, z) = (i, 0, 1)$, is indeed a root of P in (27). So we record that root and proceed onward in search of the other roots.

The next difference we encounter is that the polynomial

$$2i\alpha^5 + 4i\alpha^3 + 2i\alpha = 0 \tag{28}$$

has degree only 5, not degree $2\ell = 2 \times 3 = 6$ as one expects in the generic case. This polynomial's five roots supplement the one exceptional root $(i, 0, 1)$ we found in the previous paragraph, to give the required total of six roots. In other words, the existence of the exceptional root $(i, 0, 1)$ forces the degree of the polynomial from 2ℓ down to $2\ell - 1$.

The roots of (28) turn out to be $\{-i, i, -i, i, 0\}$. Unlike more generic polynomials, this one has multiple roots, implying a repeated factor in the product $L_1L_2L_3$. Specifically, those five roots correspond to

$$p_1 = (+i, 1, 0), \qquad \overline{p}_1 = (-i, 1, 0), \qquad p_2 = (+i, 1, 0),$$
$$\overline{p}_2 = (-i, 1, 0), \qquad p_3 = (-i, 0, 1), \tag{29}$$

and then the one exceptional root $(i, 0, 1)$ completes the pattern

$$\overline{p}_3 = (+i, 0, 1). \tag{30}$$

From here the algorithm is routine. The lines are

$$L_1 = z = 0, \qquad L_2 = z = 0, \qquad L_3 = y = 0, \tag{31}$$

the scalar multiple is $\lambda = -1$, and the final factorization is

$$x^2y + y^3 = -1(y)(z)(z) + (x^2 + y^2 + z^2)(y). \tag{32}$$

WMAP quadrupole and octopole

Our first task here is to convert a given set of coefficients $a_{\ell m}$ to a homogeneous harmonic polynomial. Converting the standard spherical harmonics Y_ℓ^m to polynomials in x, y, and z is easy. For example, for the quadrupole,

$$Y_2^{-2} = \sqrt{\frac{15}{32\pi}}\sin^2\theta e^{-2i\varphi} = \sqrt{\frac{15}{32\pi}}(x - iy)^2,$$
$$Y_2^{-1} = \sqrt{\frac{15}{8\pi}}\sin\theta\cos\theta e^{-i\varphi} = \sqrt{\frac{15}{8\pi}}(x - iy)z,$$
$$Y_2^0 = \sqrt{\frac{5}{16\pi}}(3\cos^2\theta - 1) = \sqrt{\frac{5}{16\pi}}(3z^2 - 1)$$
$$= \sqrt{\frac{5}{16\pi}}[3z^2 - (x^2 + y^2 + z^2)]$$
$$= \sqrt{\frac{5}{16\pi}}(-x^2 - y^2 + 2z^2),$$
$$Y_2^1 = -\sqrt{\frac{15}{8\pi}}\sin\theta\cos\theta e^{i\varphi} = -\sqrt{\frac{15}{8\pi}}(x + iy)z,$$
$$Y_2^2 = \sqrt{\frac{15}{32\pi}}\sin^2\theta e^{2i\varphi} = \sqrt{\frac{15}{32\pi}}(x + iy)^2. \tag{33}$$

Using the coefficients $a_{2,m}$ for the Doppler-quadrupole-corrected (DQ-corrected) Tegmark map of the first-year WMAP quadrupole gives

$$P(x, y, z) = -0.012\,627\,39x^2 + 0.023\,020\,19xy + 0.006\,776\,25y^2 + 0.009\,506\,98xz + 0.010\,640\,14yz + 0.005\,851\,13z^2. \tag{34}$$

Following the same algorithm, as illustrated in Secs. IV A and IV B, we get the polynomial

$$(0.018\,478\,52 - i0.009\,506\,98) - (0.046\,040\,38 + i0.021\,280\,27)\alpha - 0.040\,657\,52\alpha^2 + (0.046\,040\,38 - i0.021\,280\,27)\alpha^3 + (0.018\,478\,52 + i0.009\,506\,98)\alpha^4 \tag{35}$$

leading to the lines

$$L_1 = 0.939\,660x + 0.187\,066y + 0.286\,437z = 0,$$
$$L_2 = -0.437\,088x + 0.792\,820y + 0.424\,724z = 0. \tag{36}$$

Converting the coefficients of these lines to spherical coordinates gives multipole vectors

$$\hat{v}^{(2,1)} = (11.26°, 16.64°), \qquad \hat{v}^{(2,2)} = (118.87°, 25.13°), \tag{37}$$

in full agreement with those that CHS found using their tensor algorithm [Eq. (3) of Ref. [4]].

An analogous computation for the octopole yields multipole vectors $\hat{v}^{(3,1)}$, $\hat{v}^{(3,2)}$, and $\hat{v}^{(3,3)}$, again in full agreement with those reported in Eq. (3) of Ref. [4].

V. HOW WELL DO THE WMAP QUADRUPOLE AND OCTOPOLE ALIGN?

Following Ref. [4], we define the *quadrupole plane* normal vector

$$w^{2,1,2} \equiv \hat{v}^{(2,1)} \times \hat{v}^{(2,2)} \tag{38}$$

and the three *octopole plane* normal vectors

$$w^{3,1,2} \equiv \hat{v}^{(3,1)} \times \hat{v}^{(3,2)}, \qquad w^{3,2,3} \equiv \hat{v}^{(3,2)} \times \hat{v}^{(3,3)}, \qquad w^{3,3,1} \equiv \hat{v}^{(3,3)} \times \hat{v}^{(3,1)}. \tag{39}$$

Still following Ref. [4], we judge the alignment of the quadrupole plane with the three octopole planes via the dot products

$$A_1 = |w^{2,1,2} \cdot w^{3,2,3}|, \qquad A_2 = |w^{2,1,2} \cdot w^{3,3,1}|, \qquad A_3 = |w^{2,1,2} \cdot w^{3,1,2}|. \tag{40}$$

Finally, in contrast to Ref. [4] (whose statistics we question—see Ref. [3]), we let the sum

$$S = A_1 + A_2 + A_3 \tag{41}$$

provide a numerical measure of how well the quadrupole plane aligns with the octopole planes. For the DQ-corrected Tegmark map, the sum evaluates to $S_0 = 2.395$.

To judge how unusually large S_0 is, we evaluated S for 100 000 random quadrupoles and octopoles and found that only 118 trials produced $S > S_0$. This 99.9% confidence level, while weaker than the controversial confidence levels of Ref. [4], is completely consistent with Huterer and Starkman's revised statistical analysis [8].

Like Schwarz, Starkman, Huterer, and Copi, we find this result astonishing. In particular, we find it difficult to believe that the quadrupole and octopole align so well merely by chance. Whether the alignment is imposed by the global topology of a small finite universe, is due to some previously unknown solar system effect, or is merely the result of an error in the collection and/or processing of the WMAP data remains to be seen.

In the meantime, we emphasize that our simulations use an entirely different algorithm from that of Refs. [4,6] as well as completely independent computer code. This effectively rules out the possibility of error in computing the 1-in-1000 estimate, forcing us to take that estimate quite seriously.

We should point out that our Monte Carlo simulations chose random quadrupoles and octopoles independently of each other, relative to spherically symmetric distributions on the spaces of all spherical harmonics of degree 2 and 3, respectively. In other words, we used independent Gaussian coefficients $a_{\ell m}$.

VI. AN OPEN QUESTION

Corollary 2 motivates a broader question, first raised by Copi, Huterer, and Starkman [9]. One would like to decompose an arbitrary real-valued function $f: S^2 \to R$, for example, the temperature function on the microwave sky, as a sum $f = \sum_{\ell=0}^{\infty}(\lambda_\ell \prod_{i=1}^{\ell} L_{\ell,i})$. In other words, this approach would bypass the spherical harmonics entirely and, instead, write the function f directly as the sum of totally factored polynomials $\lambda_\ell \prod_{i=1}^{\ell} L_{\ell,i}$, one for each degree ℓ.

Corollary 2 almost makes such a factorization possible. For example, if we approximate the microwave sky temperature by the sum of its first 837 multipoles, $T = \sum_{\ell=0}^{837} \sum_{m=-\ell}^{\ell} a_{\ell,m} Y_\ell^m$, then Corollary 2 lets us reexpress it as $T = \sum_{\ell=0}^{837}(\lambda_\ell \prod_{i=1}^{\ell} L_{\ell,i})$. The question is, what happens when we add in the 838th spherical harmonic? For sure we will add an 838th term $\lambda_{838} \prod_{i=1}^{838} L_{838,i}$ to our factored-polynomial decomposition. Almost surely the 836th term will change significantly as it inherits the remainder R from the newly added 838th term. But what about the lower order terms? Will the second, fourth, and sixth terms remain stable? Or will they swing wildly every time we add a new high-order term to the series? In other words, is the decomposition stable?

[1] C. Bennett *et al.*, Astrophys. J. Suppl. Ser. **148**, 1 (2003).
[2] M. Tegmark, A. de Oliveira-Costa, and A. J. S. Hamilton, Phys. Rev. D **68**, 123523 (2003).
[3] The statistical test of Ref. [4] assigns a score of 99.97% to the first-year WMAP data. However, far more than 0.03% of all appropriate Gaussian random skies, when substituted for the WMAP data as the starting point of the algorithm, would achieve a score of 99.97% or better. Indeed, it is conceivable that 1% or more of the random skies would score 99.97% or better on that test. Therefore, even though Schwarz *et al.* compute the score of 99.97% correctly, this score cannot be interpreted as a confidence level. Fortunately, when Schwarz *et al.* apply different statistical tests to their data, they obtain a score of 99.9% that can be directly interpreted as a confidence level. Thus, the statistical significance of their claims remains excellent.
[4] D. J. Schwarz, G. D. Starkman, D. Huterer, and C. J. Copi, astro-ph/0403353.
[5] H. K. Eriksen, A. J. Banday, K. M. Górski, and P. B. Lilje, astro-ph/0403098.
[6] C. J. Copi, D. Huterer, and G. D. Starkman, Phys. Rev. D **70**, 043515 (2004).
[7] M. Reid, *Undergraduate Algebraic Geometry* (Cambridge University, Cambridge, England, 1988).
[8] D. Huterer and G. D. Starkman (private communication).
[9] G. D. Starkman (private communication).