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Mixed inflaton and curvaton perturbations

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A recent variant of the inflationary paradigm is that the primordial curvature perturbations come from quantum fluctuations of a scalar field, subdominant and effectively massless during inflation, called the "curvaton," instead of the fluctuations of the inflaton field. We consider the situation where the primordial curvature perturbations generated by the quantum fluctuations of an inflaton and of a curvaton field are of the same order of magnitude. We compute the total curvature perturbation and its spectrum in this case and we discuss the observational consequences.

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I. INTRODUCTION

The first observations of the WMAP satellite [1] have confirmed the basic predictions of the inflation scenarios [2] (see also [3]), namely, the existence of super-Hubble primordial fluctuations, mostly adiabatic and characterized by a quasi-scale-invariant spectrum. Because direct information on the primordial Universe is scarce, it is essential, when trying to interpret the cosmological data, to determine to which extent a precise measurement of the primordial spectrum gives us information on the inflationary scenario itself.

In this perspective, whereas in the standard inflationary paradigm, the primordial perturbations are generated by quantum fluctuations of the *inflaton*, it has been realized recently that the observed primordial fluctuations could be accounted for by quantum fluctuations of a scalar field other than the inflaton, dubbed the *curvaton* field [4–6] (and see also [7,8]). The curvaton is a scalar field σ whose effective mass must be much smaller than the Hubble parameter during inflation so that it acquires quantum fluctuations with a quasi-scale-invariant spectrum. After inflation, the primordial curvature perturbations are generated by the conversion of the isocurvature perturbations due to the fluctuations of the subdominant curvaton field, into adiabatic perturbations.

In the curvaton scenario, inflation is still necessary since this is the source of the curvaton quantum fluctuations but the separation of the rôles played by the inflaton that drives the evolution of the Universe and by the curvaton which generates the perturbations, liberates the inflaton from some observational constraints. This alternative possibility is mixed news for the inflationary scenarios. It is good news in the sense that it allows us to construct more easily explicit inflationary models with a firmer footing from a particle physicist's point of view [9]. But it is also bad news in the sense that it shows a new limitation to extract information on the nature of the inflaton field from the measurement of the primordial fluctuations.

In the curvaton scenario, the curvature fluctuations generated by the inflaton are usually assumed to be negligible, and the primordial fluctuations are thus essentially due to the curvaton. However, as was pointed out in [10], one can envisage cases where the fluctuations generated by both the inflaton and a curvatonlike field are relevant. This is the situation that we explore in the present work.

It can occur, for instance, when σ_* of the order of the Planck mass m_P during inflation, still assuming that the curvaton energy density is much smaller than that of the inflaton (otherwise we would be in the context of double inflation [11]), and implies that the curvaton starts to oscillate when its energy density already contributes significantly to the total energy density. This is in contrast with the usual curvaton scenario where the curvaton starts oscillating long before it dominates and can thus be treated as a pressureless fluid during the first radiation dominated phase.

Curvaton and inflaton-generated perturbations can also be of the same order of magnitude for small σ_*, i.e., $\sigma_* \ll m_P$, but with a slow-roll inflation parameter ϵ sufficiently small during inflation to produce big inflaton perturbations.

This paper is organized as follows. In the next section we write the background evolution equations and we derive their initial conditions explaining the context of our present analysis. In Sec. III we discuss the perturbation equations and their initial conditions. We can analytically derive the curvature perturbation after curvaton decay, i.e., at the onset of the standard big bang cosmology, in two limits. In the pure curvaton limit, for $\sigma_* \ll m_P$, we recover the curvature perturbation already derived in previous works. In the limit where the curvaton acts as a secondary inflaton, for $\sigma_* \gtrsim m_P$, we recover the perturbation of the double inflation model. In the intermediate case we resort to a numerical analysis to derive the mixed inflaton and curvaton perturbation, which is the main result of our work. In Sec. IV we derive the spectrum of the curvature perturbation, its spectral index, and discuss the observational consequences of the model. Finally we conclude in the last section.

II. HOMOGENEOUS EQUATIONS OF MOTION

In the curvaton scenario, the inflationary phase is followed by a *first* radiation dominated era, where the dominant species are the decay products of the inflaton. The curvaton is a scalar field, σ, that is assumed to be subdominant during the whole inflationary epoch and at the beginning of the radiation dominated post-inflationary era. Its effective mass is assumed to be much smaller than the Hubble parameter during inflation. Here, for simplicity, we take the quadratic potential

$$V(\sigma) = \tfrac{1}{2}m^2\sigma^2. \tag{1}$$

In the post-inflationary era, two components coexist: the radiation fluid and the curvaton. The evolution of the homogeneous background is thus governed by the Friedmann equation

$$H^2 = \frac{8\pi G}{3}\left(\rho_r + \frac{1}{2}\dot{\sigma}^2 + \frac{1}{2}m^2\sigma^2\right), \tag{2}$$

the continuity equation for the radiation energy density ρ_r,

$$\dot{\rho}_r + 4H\rho_r = 0, \tag{3}$$

and the equation of motion for the curvaton,

$$\ddot{\sigma} + 3H\dot{\sigma} + m^2\sigma = 0. \tag{4}$$

When $H \gg m$ the cosmological friction is so strong that the curvaton is essentially frozen. This is the case during inflation since the curvaton must be effectively massless to acquire the usual scale-invariant quantum fluctuations. After inflation, H keeps decreasing until it reaches the value $H \sim m$, at which point the curvaton starts oscillating.

In the standard curvaton scenario, σ is supposed to start oscillating while it is still strongly subdominant with respect to radiation. Using (2), this implies

$$H^2 \sim m^2 \gg \frac{m^2}{m_{\rm P}^2}\sigma_*^2, \tag{5}$$

where we have introduced the reduced Planck mass $m_{\rm P} \equiv (8\pi G)^{-1/2}$. This relation means that the initial amplitude of the curvaton must satisfy $\sigma_* \ll m_{\rm P}$. Conversely, assuming an initial value $\sigma_* \sim m_{\rm P}$ during inflation implies that the epochs of oscillation and domination roughly coincide for the curvaton.

During the radiation dominated era, when σ is subdominant, $H = (2t)^{-1}$, and the curvaton equation of motion (4) reads

$$\ddot{\sigma} + \frac{3}{2t}\dot{\sigma} + m^2\sigma = 0. \tag{6}$$

One recognizes a Bessel equation, whose nondecaying solution is given by

$$\sigma_{(i)} = \sigma_* A \frac{J_{1/4}(mt)}{(mt)^{1/4}}, \qquad A \equiv \frac{\pi}{2^{1/4}\Gamma(3/4)}, \tag{7}$$

where J_ν stands for a Bessel function of rank ν. In the limit $mt \ll 1$, one gets $\sigma_{(i)} \simeq \sigma_*$. The expression $\sigma_{(i)}$ given above embodies the initial conditions for the curvaton field.

Note that the solution (7) differs from the slow-roll solution that would apply during inflation when the Hubble parameter is slowly varying. Here, the acceleration is not small with respect to the other terms in the equation of motion (6). In particular, in the limit $mt \ll 1$, one finds

$$H\dot{\sigma}_{(i)} \simeq -\tfrac{1}{5}m^2\sigma_{(i)}. \tag{8}$$

III. ANALYSIS OF THE PERTURBATIONS

A. Equations of motion

The equations governing the evolution of the scalar-type perturbations are well-known and can be found in review papers such as [12] or [13]. We adopt here the *longitudinal gauge* in which the perturbed metric reads

$$ds^2 = -(1+2\Phi)dt^2 + a^2(t)(1-2\Psi)\delta_{ij}dx^i dx^j. \tag{9}$$

The perturbation of the curvaton field is denoted $\delta\sigma$ and the perturbations of the radiation fluid are described by the perturbation of the energy density, $\delta\rho_r$, or the density contrast $\delta_r = \delta\rho_r/\rho_r$, and the peculiar velocity v_r.

Taking into account the relation $\Psi = \Phi$, imposed by the Einstein equations in absence of anisotropic stress, the perturbed equations of motion are the energy and momentum constraints in Einstein's equations, respectively

$$-3H(H\Phi + \dot{\Phi}) - \frac{k^2}{a^2}\Phi = 4\pi G(\dot{\sigma}\,\dot{\delta\sigma} + m^2\sigma\delta\sigma - \dot{\sigma}^2\Phi + \rho_r\delta_r) \tag{10}$$

and

$$\dot{\Phi} + H\Phi = 4\pi G(\dot{\sigma}\delta\sigma - \tfrac{4}{3}\rho_r v_r), \tag{11}$$

the continuity and Euler equations for the radiation fluid, respectively

$$\dot{\delta}_r - \frac{4}{3}\frac{k^2}{a^2}v_r - 4\dot{\Phi} = 0 \tag{12}$$

and

$$\dot{v}_r - Hv_r + \Phi + \tfrac{1}{4}\delta_r = 0, \tag{13}$$

and finally the Klein-Gordon equation for the curvaton,

$$\delta\ddot{\sigma} + 3H\delta\dot{\sigma} + \left(\frac{k^2}{a^2} + m^2\right)\delta\sigma = 4\dot{\sigma}\,\dot{\Phi} - 2m^2\sigma\Phi. \quad (14)$$

Since the cosmological perturbations of observational interest today were on super-Hubble scales in the early Universe, we will be interested only in the long wavelength limit $k/(aH) \to 0$ of the above equations, which reduce to the following system:

$$-3H(H\Phi + \dot{\Phi}) = 4\pi G(\dot{\sigma}\,\delta\dot{\sigma} + m^2\sigma\delta\sigma - \dot{\sigma}^2\Phi + \rho_r\delta_r), \quad (15)$$

$$\dot{\delta}_r - 4\dot{\Phi} = 0, \quad (16)$$

and

$$\delta\ddot{\sigma} + 3H\delta\dot{\sigma} + m^2\delta\sigma = 4\dot{\sigma}\,\dot{\Phi} - 2m^2\sigma\Phi, \quad (17)$$

where we have gotten rid of the perturbed velocity v_r.

B. Initial conditions

Our initial conditions are defined early in the first radiation dominated era when the curvaton is at rest and subdominant. In this limit the Bardeen potential Φ is constant. In the standard curvaton case, it is further assumed to be zero, but here it cannot be neglected since it represents the contribution of the inflaton to the "primordial" curvature perturbation. Our initial conditions are thus given by

$$\Phi = \Phi_*, \quad \dot{\Phi} = 0, \quad \delta\sigma = \delta\sigma_*, \quad \delta\dot{\sigma} = 0, \quad (18)$$

where Φ_* stands for the curvature perturbation generated during inflation. Using the energy constraint (15), these initial conditions imply

$$\delta_r = -6\frac{m_P^2H^2}{\rho_r}\Phi_* - \frac{m^2\sigma}{\rho_r}\delta\sigma_*. \quad (19)$$

Moreover, since the curvaton is initially subdominant, the second term on the right hand side can be dropped and the initial condition for δ_r reduces to

$$\delta_r^{(i)} = -2\Phi_*. \quad (20)$$

The evolution of $\delta\sigma$ is governed by the equation of motion Eq. (17), which can be approximated in our limit by

$$\delta\ddot{\sigma} + \frac{3}{2t}\delta\dot{\sigma} + m^2\delta\sigma = -2m^2\Phi_*\sigma. \quad (21)$$

One recognizes the same Bessel equation as for the background but now with a source term on the right hand side, which is proportional to the background solution for σ given in Eq. (7). The solution for the perturbation is then found to be given by

$$\delta\sigma_{(i)} = \frac{A}{(mt)^{1/4}}\left\{\left(\delta\sigma_* - \frac{1}{2}\Phi_*\sigma_*\right)J_{1/4}(mt) + \Phi_*\sigma_* mtJ_{-3/4}(mt)\right\}. \quad (22)$$

Note also that using Eq. (7) we can rewrite Eq. (22) as

$$\delta\sigma_{(i)} = \frac{\delta\sigma_*}{\sigma_*}\sigma_{(i)} + t\dot{\sigma}_{(i)}\Phi_*. \quad (23)$$

This yields $\delta_{\sigma_{(i)}} = \delta\sigma_* + \mathcal{O}(m^2t^2)$, which justifies the initial conditions for the curvaton perturbation given in (18).

As the energy density perturbation for the curvaton is given explicitly by

$$\delta\rho_\sigma = m^2\sigma\delta\sigma + \dot{\sigma}\,\delta\dot{\sigma} - \dot{\sigma}^2\Phi, \quad (24)$$

we can compute, using (23), the initial density contrast

$$\delta_\sigma^{(i)} = 2\frac{\delta\sigma_*}{\sigma_*} - \frac{3}{2}\frac{\dot{\sigma}^2}{\rho_\sigma}\bigg|_{(i)}\Phi_*. \quad (25)$$

C. Evolution of the perturbations

The above analytical expressions are valid only during the early radiation dominated era when the curvaton contribution to the background energy density is negligible. This contribution however keeps increasing with time until it eventually becomes significant, and a more general analysis is required.

The purpose of this section is to compute the curvature perturbation after the curvaton decay, i.e., to establish the initial conditions at the start of the *second radiation dominated era*, which must be identified with the usual radiation era of the standard cosmological scenario. Since we assume the decay to be instantaneous (it has been shown recently that this is a very good approximation [14]), it is enough for our purpose to determine the curvature perturbation when the curvaton is both dominating and oscillating, which will be denoted Φ_f. The corresponding curvature perturbation in the subsequent radiation phase is then simply obtained by applying the usual transfer coefficient between a matter dominated phase and a radiation dominated phase.

In order to determine the *final* curvature perturbation in our scenario, i.e., the *primordial* perturbation for the standard cosmological model, we have solved numerically the system of equations that consists of (2)–(4) for the background and of (15)–(17) for the perturbations. For the homogeneous system, the only parameter that can be varied is the initial amplitude σ_* of the curvaton. As for the perturbations, the initial conditions depend on two parameters, the curvature perturbation Φ_* and the curvaton perturbation $\delta\sigma_*$. Since this is a linear problem, the final result can be written, quite generally as

$$\Phi_f = \alpha\Phi_* + \beta\delta\sigma_*, \quad (26)$$

where the two coefficients α and β, to be determined, depend only on the homogeneous quantities. This means that one can analyze independently the cases $\Phi_* = 0$ and $\delta\sigma_* = 0$, the general result being given by the sum of these particular cases.

Let us start with the case $\delta\sigma_* = 0$. We will now show that this case corresponds to a purely adiabatic initial condition. Let us first recall that the so-called isocurvature perturbation between two matter components (here radiation and curvaton) can be defined as (see, e.g., [14])

$$S \equiv 3(\zeta_r - \zeta_\sigma), \tag{27}$$

where for a given matter species X, ζ_X is the curvature perturbation on uniform energy density hypersurfaces, namely

$$\zeta_X \equiv -\Phi - H\frac{\delta\rho_X}{\dot{\rho}_X} = -\Phi + \frac{\delta_X}{3(1+w_X)}, \tag{28}$$

with $w_X = P_X/\rho_X$. For the curvaton, the pressure is $P_\sigma = (\dot{\sigma}^2 - m^2\sigma^2)/2$ and the energy density is $\rho_\sigma = (\dot{\sigma}^2 + m^2\sigma^2)/2$ so that the entropy perturbation is given by

$$S = \frac{\delta_r}{1+w_r} - \frac{\delta_\sigma}{1+w_\sigma} = \frac{3}{4}\delta_r - \frac{\rho_\sigma}{\dot{\sigma}^2}\delta_\sigma. \tag{29}$$

Substituting the initial conditions (20) and (25), one gets for the *initial* entropy perturbation

$$S^{(i)} = -2\frac{\rho_\sigma}{\dot{\sigma}^2}\frac{\delta\sigma_*}{\sigma_*}. \tag{30}$$

Therefore, $\delta\sigma_* = 0$ corresponds to $S^{(i)} = 0$, i.e., to a purely "adiabatic" initial perturbation.

The perturbation then remains adiabatic throughout the evolution from the radiation dominated era to the oscillating curvaton dominated era. Note that, in contrast with the case of two uncoupled adiabatic perfect fluids where the individual ζ_X are separately conserved, the entropy perturbation does not in general remain constant on super-Hubble scales because a nonzero intrinsic entropy of the scalar field implies $\dot{\zeta}_\sigma \neq 0$. However, if the initial entropy vanishes it will remain so subsequently. Since we have a pure adiabatic perturbation, the total ζ is conserved on super-Hubble scales. Consequently the curvature perturbation is simply governed by the global equation of state of the matter content, or in other words the total ζ is conserved in time. During a cosmological era with $w = \text{const}$, the Bardeen potential Φ is constant and is related to ζ via the relation (see, e.g., [13])

$$\zeta = -\frac{5+3w}{3(1+w)}\Phi. \tag{31}$$

Using this relation both in the initial radiation dominated era with $w = 1/3$ and in the oscillating curvaton dominated era with $w = 0$, the conservation of ζ implies that the coefficient α in Eq. (26) is given by

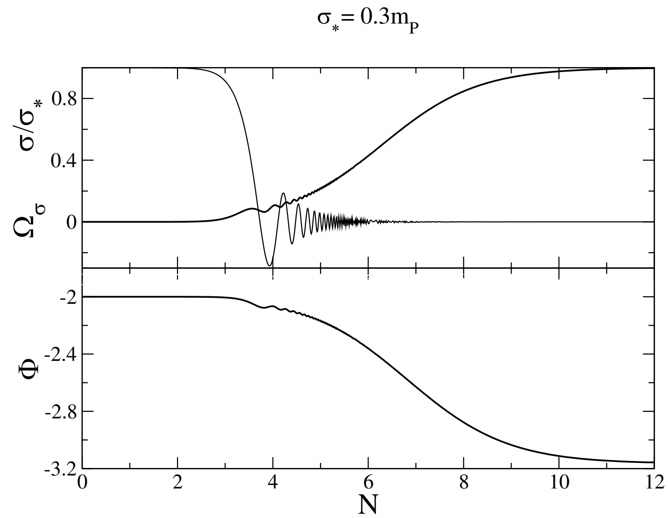

FIG. 1. The standard curvaton limit, for $\sigma_* = 0.3m_P$. Evolution of the curvaton, its energy density fraction $\Omega_\sigma \equiv \rho_\sigma/(\rho_\sigma + \rho_r)$ (upper part), and of the curvature perturbation Φ (lower part). The initial conditions for the perturbations are $\delta\sigma_* = m_P$ and $\Phi_* = -2$.

$$\alpha = \tfrac{9}{10}. \tag{32}$$

We have checked numerically that this result indeed holds for all types of initial conditions (not only $\delta\sigma_* = 0$).

To determine the other coefficient, β, it is now sufficient to consider the initial configurations with $\Phi_* = 0$. We distinguish below several cases: first the limiting case where σ behaves as a standard curvaton, i.e., oscillates long before domination (see upper part of Fig. 1); then the other limiting case where σ starts oscillating only after it completely dominates and thus generates a second inflationary phase (see upper part of Fig. 2); and finally the

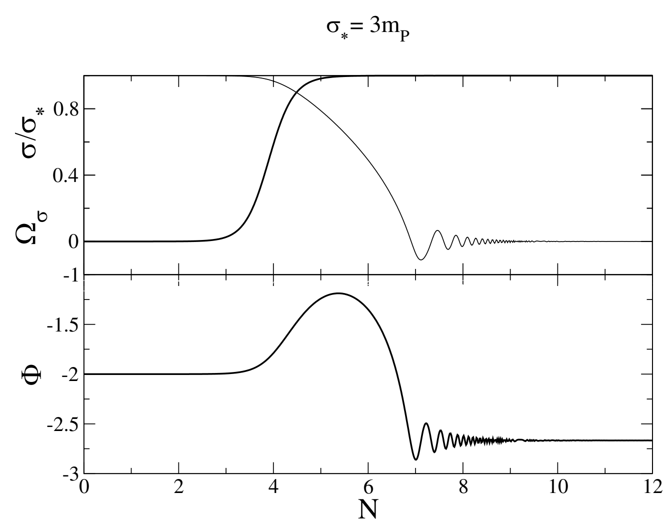

FIG. 2. The secondary inflaton limit, for $\sigma_* = 3m_P$. Evolution of σ, its energy density fraction (upper part), and of the curvature perturbation (lower part). The initial conditions for the perturbations are $\delta\sigma_* = m_P$ and $\Phi_* = -2$.

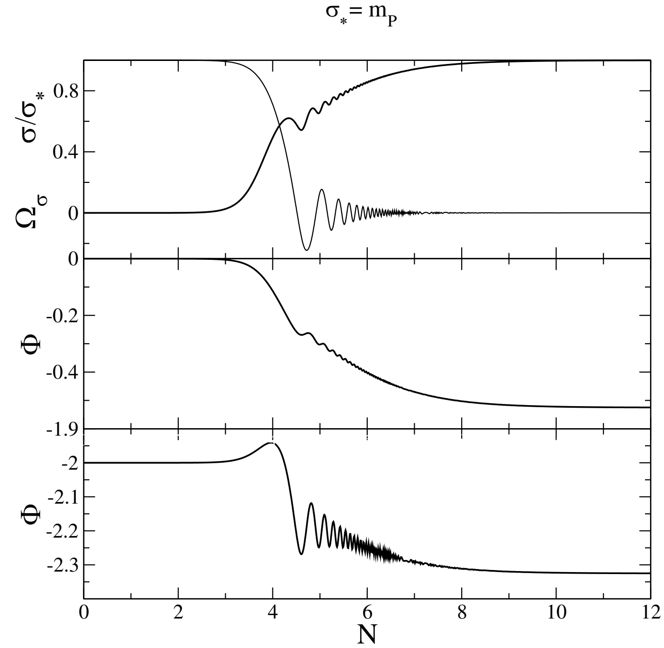

FIG. 3. The intermediate case, for $\sigma_* = m_P$. Evolution of σ, its energy density fraction (upper part), and of the curvature perturbation (lower part) for two different initial conditions: $\Phi_* = 0$ and for $\Phi_* = -2$ (both cases with $\delta\sigma_* = m_P$).

general intermediate case, as illustrated in the upper part of Fig. 3.

1. The standard curvaton limit

This is the limit corresponding to $\sigma_* \ll m_P$. As discussed before, this implies that the curvaton oscillates long before its domination. One thus recovers the standard curvaton scenario, in which case the curvaton can be identified with a pressureless fluid during the post-inflation phase.

Let us briefly recall the usual analysis in the standard curvaton scenario [4]. The curvaton is assimilated to a dustlike fluid with density contrast δ_m. Since the two fluids are decoupled, ζ_r and ζ_m are separately conserved, whereas the total ζ is given by

$$\zeta = \frac{4\rho_r\zeta_r + 3\rho_m\zeta_m}{4\rho_r + 3\rho_m}. \quad (33)$$

Assuming no initial curvature perturbation, i.e., $\Phi_* = 0$ and thus $\zeta_r = 0$, one finds that during the curvaton domination

$$\zeta = \zeta_m = \tfrac{1}{3}\delta_m^{(i)}, \quad (34)$$

where the second equality comes from the definition of ζ given in Eq. (28). Using the relation (31) between ζ and Φ during a $w = 0$ era, one gets

$$\Phi_f = -\tfrac{3}{5}\zeta = -\tfrac{1}{5}\delta_m^{(i)}. \quad (35)$$

The final step is to relate the density contrast to the initial curvaton fluctuation, by using Eq. (25). For $\Phi_* = 0$, this is simply $\delta_\sigma^{(i)} = 2\delta\sigma_*/\sigma_*$. Therefore, in the standard curvaton limit, one finds for the coefficient β of Eq. (26) the value

$$\beta = -\frac{2}{5\sigma_*}. \quad (36)$$

Although the expression for α has already been obtained by the general argument given earlier, it is nevertheless instructive to see how this value is recovered in the curvaton limit by considering a non vanishing Φ_*. When averaged over several oscillations, Eq. (25) yields

$$\delta_m^{(i)} \equiv \delta_\sigma^{(i)} = 2\frac{\delta\sigma_*}{\sigma_*} - \frac{3}{2}\Phi_*. \quad (37)$$

During the curvaton dominated era, one finds [note the difference with Eq. (34)]

$$\zeta = \zeta_m = -\Phi_* + \tfrac{1}{3}\delta_m^{(i)}. \quad (38)$$

Substituting in $\Phi_f = -(3/5)\zeta$, derived from (31), one finally gets the expected result

$$\Phi_f = \frac{9}{10}\Phi_* - \frac{2}{5}\frac{\delta\sigma_*}{\sigma_*}. \quad (39)$$

We have also checked this result by evolving numerically the perturbation equations. As an illustration, we show the evolution of Φ in Fig. 1 for initial perturbations $\delta\sigma_* = m_P$ and $\Phi_* = -2$ [1].

Note finally that the above equation can be interpreted as a particular case of the more general equation

$$\Phi_f = \tfrac{9}{10}\Phi_* + \tfrac{1}{5}S, \quad (40)$$

relating the final curvature perturbation, in a matter dominated era, to the initial curvature perturbation and entropy perturbation defined in a radiation dominated era. Equation (40) can be easily derived from $\Phi_f = -(3/5)\zeta_m$ by substituting $\zeta_m = \zeta_r - S/3$, which follows from (29), with $\zeta_r = -\Phi_* + \delta_r^{(i)}/4 = -(3/2)\Phi_*$. The expression (40) can be used only when the entropy perturbation is conserved on large scales. This is the case for the curvaton only when it is oscillating or when it dominates (and its intrinsic entropy thus vanishes), because otherwise ζ_σ is not conserved. In the early radiation dominated era, the expression for the entropy perturbation is given in Eq. (30). When σ oscillates, $\rho_\sigma/\dot{\sigma}^2 = 1$ and S is constant with the value $S = -2\delta\sigma_*/\sigma_*$. Substituting in (40) one indeed recovers (39).

[1] Within perturbation theory, the overall amplitude of the perturbations can be arbitrarily rescaled and we have chosen to set $\delta\sigma_* = m_P$. Moreover, the initial condition $\Phi_* = -2$ corresponds to the curvature perturbation produced by a chaotic quadratic model with $\phi_* = 6m_P$ and $\delta\phi_* = m_P$.

2. *The secondary inflaton limit*

Another limiting case corresponds to the situation where σ starts oscillating only after it completely dominates the Universe. This implies that the curvaton σ is responsible for an additional phase of inflation: this is why we call σ the *secondary inflaton* in this situation.

We now attempt to evaluate analytically the final curvature perturbation. Let us consider again the entropy perturbation S. As mentioned already, S is not constant because ζ_σ is not conserved. This can be checked explicitly with the expression

$$S = -2\frac{\rho_\sigma}{\dot\sigma^2}\frac{\delta\sigma_*}{\sigma_*}, \quad (41)$$

already given in Eq. (30). Indeed, in the early radiation dominated era, using the explicit solution given in (7), one finds the behavior $\rho_\sigma/\dot\sigma^2 \simeq (25/8)(mt)^{-2}$.

In the subsequent phases dominated by the curvaton, i.e., the secondary inflation and the oscillating phase, the entropy S remains constant. Although the expression (41) is rigorously valid only during the phase when the curvaton is negligible, we can evaluate it at the transition between the (first) radiation dominated era and the curvaton dominated era to get an estimate of the subsequently constant value of S.

During the curvaton dominated phase, σ follows a slow-roll motion, characterized by

$$3H^2 \simeq \frac{m^2\sigma^2}{2m_{\rm P}^2}, \quad \dot\sigma = -\frac{m^2\sigma}{3H}, \quad (42)$$

and therefore, the coefficient $\rho_\sigma/\dot\sigma^2$ takes the constant value

$$\frac{\rho_\sigma}{\dot\sigma^2} \simeq \frac{3}{4}\frac{\sigma_*^2}{m_{\rm P}^2}, \quad (43)$$

where we have used the fact that σ is essentially frozen at its inflationary value σ_* throughout the radiation dominated era. Inserting this ratio in Eq. (41), we get

$$S = -\frac{3\sigma_*}{2m_{\rm P}^2}\delta\sigma_*. \quad (44)$$

Substituting the above value in Eq. (40), we then find for the curvature perturbation in the strongly oscillating phase

$$\Phi_f = \frac{9}{10}\Phi_* - \frac{3\sigma_*}{10m_{\rm P}^2}\delta\sigma_*. \quad (45)$$

We have checked numerically that our analytical derivation gives a very good approximation. This is illustrated in Fig. 3 for two different values of Φ_* (and with $\delta\sigma_* = m_{\rm P}$).

3. *The general case*

We now consider the general case, which includes the intermediate situations where the curvaton starts oscillating at about the same time it becomes dominant. We have computed the dependence of the coefficient β on the initial value σ_*, by solving numerically the full system of equations for different values of σ_*: the intermediate case $\sigma_* = m_{\rm P}$ is plotted in Fig. 3 as an illustration.

The resulting function, up to a normalization constant defined just now, is plotted in Fig. 4. Instead of using β, we prefer to work with the coefficient that appears in the expression for the curvature perturbation *after the decay of the curvaton*, i.e., in the second radiation phase of our scenario, identified with the radiation dominated era of the standard big bang model. The relation between the curvature perturbation Φ_f at the end of the curvaton dominated phase and the curvaton perturbation Φ_{RD} in the post-curvaton radiation phase is given by

$$\Phi_{RD} = \tfrac{10}{9}\Phi_f, \quad (46)$$

the transfer coefficient being simply due to the transition from a matter dominated phase to a radiation dominated phase. It is then convenient to define the function f as

$$\Phi_{RD} = \Phi_* - \frac{f(\sigma_*)}{m_{\rm P}}\delta\sigma_*. \quad (47)$$

As we can see on Fig. 4, for small values of σ_*, we are in the standard curvaton regime with

$$f(\sigma_*) = \frac{4}{9}\frac{m_{\rm P}}{\sigma_*}, \quad \text{pure curvaton}, \quad (48)$$

whereas for large values of σ_*, we recover the secondary inflaton limit characterized by

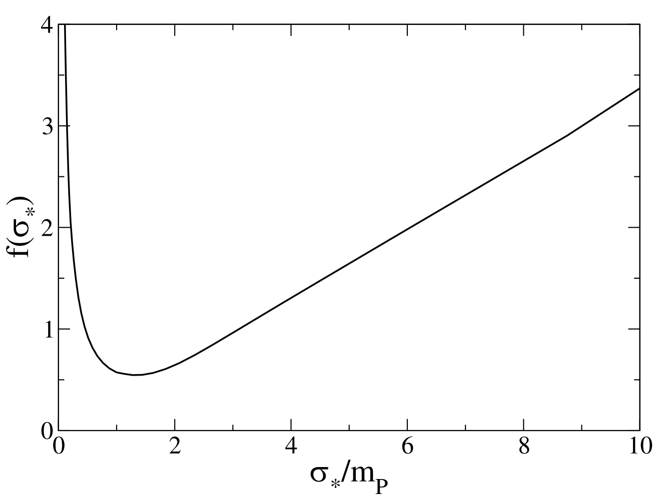

FIG. 4. The function $f(\sigma_*)$, which characterizes the amplitude of the contribution of σ to the curvature perturbation. For $\sigma_* \ll m_{\rm P}$ one recognizes the $\propto 1/\sigma_*$ contribution of the pure curvaton model. For $\sigma_* \gg m_{\rm P}$, one recognizes the contribution of a secondary inflaton, proportional to σ_*.

$$f(\sigma_*) = \frac{\sigma_*}{3m_{\rm P}}, \quad \text{secondary inflation.} \tag{49}$$

In the intermediate range, f reaches its minimum value, $f \simeq 0.55$ around $\sigma_* \simeq 1.2m_{\rm P}$.

IV. OBSERVATIONAL CONSEQUENCES

A. Power spectrum of the primordial fluctuations

So far, we have computed the primordial fluctuations, defined in the second radiation era after the decay of the curvaton, without specifying the initial curvature perturbation Φ_*. However, in the context of an inflationary era driven by a single slow-rolling scalar field, one can relate Φ_* to the fluctuation of the inflaton. The perturbations due to the inflaton can be computed following the standard technique for a slow-rolling single-field inflation model (see, e.g., [15]). One finds

$$\Phi_* = -[1 - (1 + 3C)\epsilon + C\eta]\frac{2}{3m_{\rm P}^2}\frac{V}{V'}\bigg|_* \delta\phi_*, \tag{50}$$

where $V = V(\phi)$ is the potential of the inflaton, $V' \equiv dV/d\phi$, ϵ and η are the first two slow-roll parameters,

$$\epsilon = \frac{m_{\rm P}^2}{2}\left(\frac{V'}{V}\right)^2, \tag{51}$$

$$\eta = m_{\rm P}^2\frac{V''}{V}, \tag{52}$$

and $C = -2 + \ln 2 + \gamma \simeq -0.73$, γ being the Euler constant. It is important to include the slow-roll corrections in the normalization of the curvature perturbation since these can be of the same order as the curvaton contribution. The subscript $*$ means that all quantities have to be evaluated at Hubble crossing during inflation. Note that we are implicitly assuming that large scales that are observable today, i.e., those of order H_0^{-1}, did not reenter the Hubble radius between the end of inflation and the curvaton domination.

Substituting in our result of the previous section, Eq. (47), the primordial curvature fluctuations in the second radiation era are thus given by the combination

$$\Phi_{RD} = -[1 - (1 + 3C)\epsilon + C\eta]\frac{2}{3m_{\rm P}^2}\frac{V}{V'}\bigg|_* \delta\phi_* - \frac{1}{m_{\rm P}}f(\sigma_*)\delta\sigma_*. \tag{53}$$

The fluctuations $\delta\phi_*$ and $\delta\sigma_*$ must be seen as *independent* random fields with the same amplitude for their power spectra and can be written as

$$\delta\phi_* = \frac{H_*}{2\pi}e_\phi, \qquad \delta\sigma_* = \frac{H_*}{2\pi}e_\sigma, \tag{54}$$

with $\langle e_\phi e_\sigma\rangle = 0$. It is also worth noticing that our result, in the secondary inflaton limit discussed before, with $f(\sigma_*) = \sigma_*/(3m_{\rm P})$ is in agreement with the expression obtained in [11] (see also [16]) for the curvature perturbation given in terms of the perturbations of the two scalar fields in a double inflation model.

Let us drop the subscript $*$. Using (53) and (54), the amplitude of the power spectrum for Φ is

$$\mathcal{P}_\Phi = [1 + \tilde{f}^2\epsilon - 2(1 + 3C)\epsilon + 2C\eta]\frac{H^2}{18\pi^2\epsilon m_{\rm P}^2}, \tag{55}$$

where we have introduced

$$\tilde{f} \equiv \frac{3}{\sqrt{2}}f. \tag{56}$$

The contribution of the curvaton to the total curvature perturbation appears here on the same footing as the slow-roll corrections in the amplitude of the inflaton power spectrum. However, when $\tilde{f} \gg 1$ the curvaton contribution dominates over the slow-roll corrections and the term $-2(1 + 3C)\epsilon + 2C\eta$ can be neglected.

The relative importance of the curvaton over the inflaton perturbation is parametrized by

$$R \equiv \frac{\tilde{f}^2\epsilon}{1 - 2(1 + 3C)\epsilon + 2C\eta} \simeq \tilde{f}^2\epsilon. \tag{57}$$

This is given by the product of two parameters that depend on two *a priori* independent physical processes: ϵ, which must be smaller than 1, is determined by the inflationary scenario, while $\tilde{f}^2 \gtrsim 1.36$ is determined by the curvaton expectation value. The slow-roll parameter ϵ is typically of order $\sim 1/60$ in chaotic models of inflation but it can be much smaller in other inflationary models. Therefore, there are cases where the curvaton value during inflation is small, $\sigma \ll m_{\rm P}$, and yet the perturbations generated by the inflaton are of the order of or larger than the curvaton perturbations, i.e., $R \lesssim 1$.

When the curvaton perturbation is absent, i.e., for $\tilde{f} = 0$, we recover the pure inflationary spectrum (see, e.g., [15]),

$$\mathcal{P}_\Phi = [1 - 2(1 + 3C)\epsilon + 2C\eta]\frac{H^2}{18\pi^2\epsilon m_{\rm P}^2}. \tag{58}$$

Conversely, when the curvaton perturbation dominates the power spectrum, i.e., for $R \gg 1$ and $\sigma \ll m_{\rm P}$, we recognize the spectrum of perturbations of the curvaton [4],

$$\mathcal{P}_\Phi = \frac{4H^2}{81\pi^2\sigma^2}, \tag{59}$$

where we have used $\tilde{f} = (2\sqrt{2}/3)(m_{\rm P}/\sigma)$ from Eqs. (48) and (56).

The presence of mixed curvaton and inflaton perturbations also affects the scalar spectral index, which is defined as

$$n_s - 1 \equiv \frac{d\ln\mathcal{P}_\Phi}{d\ln k}. \qquad (60)$$

Inserting in this expression the spectrum (55) and assuming that the curvaton mass is much smaller than the inflaton mass, $m \ll V''$, one finds

$$n_s - 1 = -2\epsilon + \frac{2\eta - 4\epsilon}{1 + \tilde{f}^2\epsilon} + [-2(7 + 12C)\epsilon^2 + 2(3 + 8C)\epsilon\eta - 2C\xi^2], \qquad (61)$$

where we have introduced the second-order slow-roll parameter

$$\xi^2 \equiv m_{\rm P}^4 \frac{V'V'''}{V^2}. \qquad (62)$$

In the square bracket we have isolated terms which are second order in the slow-roll parameters. These can be as important as the curvaton contribution if $\tilde{f} \sim 1$.

Neglecting second-order terms, in the case of pure inflation, $\tilde{f}^2 = 0$, one recovers the well-known result

$$n_s - 1 = 2\eta - 6\epsilon, \qquad (63)$$

whereas in the case of a pure curvaton, $\tilde{f}^2\epsilon \gg 1$, one gets [4]

$$n_s - 1 = -2\epsilon. \qquad (64)$$

The power spectrum for gravitational waves is given by the usual expression (see, e.g., [15])

$$\mathcal{P}_T = [1 - 2(1 + C)\epsilon]\frac{2H^2}{\pi^2 m_{\rm P}^2}. \qquad (65)$$

Using (55), we thus find for the tensor-scalar ratio

$$r \equiv \frac{4}{9}\frac{\mathcal{P}_T}{\mathcal{P}_\Phi} = \frac{16\epsilon}{1 + \tilde{f}^2\epsilon}[1 + 4C\epsilon - 2C\eta] \qquad (66)$$

(we have defined r as in [17]). Neglecting second-order terms in the slow-roll parameters, for $\tilde{f} = 0$ we recover $r = 16\epsilon$, the tensor-scalar ratio for a pure inflationary model. For a pure curvaton perturbation, $\tilde{f}^2\epsilon \gg 1$, one finds $r \simeq 16/\tilde{f}^2 \simeq 18(\sigma_*/m_{\rm P})^2$. In the usual curvaton context, where $\sigma_* \ll m_{\rm P}$, this implies that gravity waves are negligible [7].

Adding a curvaton perturbation to a pure inflationary perturbation always decreases the tensor-scalar ratio r. When $\tilde{f}$ is large, the effect on the scalar index depends on the sign of $2\eta - 4\epsilon$: if it is negative (this is the case for the so-called small field and chaotic models [18]), the scalar index is increased, which means that the spectrum is bluer; if it is positive, the effect is opposite.

Models of inflation are often classified according to the value of their first two slow-roll parameters ϵ and η [18]. Since for $\tilde{f} = 0$ there is a one-to-one correspondence between the (ϵ, η)-plane and the (n_s, r)-plane, it is customary to use the latter in order to easily confront models with observations. In the mixed case, $\tilde{f} \neq 0$, the presence of the curvaton introduces a degeneracy between these two planes, which depends on the parameter σ_*. In the next subsection we will use Eqs. (61) and (66) to show the effects of this degeneracy for the special case of $\lambda\phi^4$-inflation.

Let us now discuss the consistency relation. We restrict the discussion to the large $\tilde{f}$ case and we neglect terms which are second order in the slow-roll parameters. The tensor spectral index being given by the usual expression

$$n_T \equiv \frac{d\ln\mathcal{P}_T}{d\ln k} = -2\epsilon, \qquad (67)$$

the consistency relation between the tensor-scalar ratio and the tensor spectral index is modified into

$$r = \frac{-8n_T}{1 - \tilde{f}^2 n_T/2}. \qquad (68)$$

For large $\tilde{f}$ we find a modification of the standard consistency relation valid for a pure single-field inflation. This represents a distinctive feature of mixed curvaton and inflaton perturbations, which breaks the degeneracy of mixed models. Indeed, this modification depends solely on the parameter σ_*, which determines the value of $\tilde{f}$: since the relation (68) does not depend on a particular inflaton potential, it can be seen as a way of measuring σ_*.

For completeness, we briefly discuss the running of the scalar spectral index. We find

$$\frac{dn_s}{d\ln k} = 4\epsilon(\eta - 2\epsilon) - \frac{16\epsilon^2 - 12\epsilon\eta + 2\xi^2}{1 + \tilde{f}^2\epsilon} + \frac{4\tilde{f}^2\epsilon(\eta - 2\epsilon)^2}{(1 + \tilde{f}^2\epsilon)^2}. \qquad (69)$$

For $\tilde{f} = 0$ we recover the pure inflationary running [17],

$$\frac{dn_s}{d\ln k} = -24\epsilon^2 + 16\epsilon\eta - 2\xi^2, \qquad (70)$$

while for $\tilde{f}^2\epsilon \gg 1$ we find the running for a pure curvaton spectral index,

$$\frac{dn_s}{d\ln k} = 4\epsilon(\eta - 2\epsilon). \qquad (71)$$

In the next subsection, we will illustrate the general formulas derived here by considering the particular case of an inflaton with a quartic potential.

B. Mixed perturbations with $V(\phi) = \lambda\phi^4$

The quartic potential $V = \lambda\phi^4$ is a simple inflationary potential which has attracted some attention lately as it lies in a region excluded by the WMAP data analysis [17,19,20]. For this particular inflationary potential, we now illustrate how mixed curvaton and inflaton perturbations can be constrained by the data and how the observational predictions of a pure inflationary model change in the presence of a curvaton.

The slow-roll parameters for this potential are

$$\epsilon = 8\frac{m_P^2}{\phi^2}, \quad \eta = 12\frac{m_P^2}{\phi^2}, \quad \xi^2 = 96\frac{m_P^4}{\phi^4}. \tag{72}$$

The number of e-foldings before the end of inflation when the perturbations at our present Hubble scale exited the Hubble radius during inflation is given by

$$N_* = \ln\frac{a_{\rm end}}{a_*} = \frac{\phi_*^2}{8m_P^2}, \tag{73}$$

where we have assumed that $\phi_{\rm end} \ll \phi_*$. Using this expression we can write the slow-roll parameters in terms of the number of e-foldings N_*.

Determining the appropriate number of e-foldings N_* is very important, since it tells us which part of the potential corresponds to the present observable fluctuations. This depends on the history of the Universe from the end of inflation until today. In particular, it is very sensitive on the duration of the reheating phase: this is usually the largest uncertainty in its estimation. However, in the case of pure inflation with a quartic potential, N_* can be evaluated very accurately because the expansion during reheating is the same as in a radiation dominated era. Indeed, the energy density of an oscillating scalar field in a quartic potential behaves as radiation, and the duration of the reheating phase is thus no longer important. In this case, the number of e-foldings can be evaluated precisely [21] and is

$$N_*^{\rm quartic} \simeq 64. \tag{74}$$

This number assumes radiation domination from the end of inflation until equality. It does not take into account a possible intermediate phase of curvaton domination between reheating and nucleosynthesis, which lasts

$$\Delta N = \frac{1}{3}\ln\frac{\rho_{\rm CD}}{\rho_{\rm dec}} \tag{75}$$

e-foldings, where $\rho_{\rm CD}$ is the total energy density at curvaton and radiation equality and $\rho_{\rm dec} \simeq 3\Gamma^2 m_P^2$ is the energy density of the Universe when the curvaton decays with decay rate Γ. Curvaton domination thus *decreases* the number of e-foldings which becomes

$$N_* \simeq 64 - \frac{1}{12}\ln\frac{\rho_{\rm CD}}{\rho_{\rm dec}}. \tag{76}$$

Let us compute this correction in more details. If $\sigma_* \sim m_P$ the domination phase starts with a small period of inflation. When the curvaton and radiation energy densities are the same, we have for the total energy density

$$\rho_{\rm CD} = 2\rho_\sigma \simeq m^2\sigma_*^2. \tag{77}$$

In this case we obtain

$$N_* \simeq 64.1 - \frac{1}{6}\ln\frac{m}{\Gamma} - \frac{1}{6}\ln\frac{\sigma_*}{m_P}, \quad \text{for } \sigma_* \sim m_P. \tag{78}$$

It is also interesting to consider the pure curvaton case corresponding to $\sigma_* \ll m_P$. In this case Eq. (77) does not apply, but we can still compute the ratio $\rho_{\rm dec}/\rho_{\rm CD}$. Once the curvaton starts oscillating at $H \sim m$, we have

$$\rho_\sigma \simeq m^2\sigma_*^2\left(\frac{a_{\rm osc}}{a}\right)^3, \quad \rho_r \simeq 3m^2 m_P^2\left(\frac{a_{\rm osc}}{a}\right)^4, \tag{79}$$

where $a_{\rm osc}$ is the scale factor at the onset of the curvaton oscillating period. This implies

$$a_{\rm CD}/a_{\rm osc} \simeq 3m_P^2/\sigma_*^2. \tag{80}$$

On the other hand, when the curvaton decays ($H \sim \Gamma$), it dominates the Universe and

$$m^2\sigma_*^2\left(\frac{a_{\rm osc}}{a_{\rm dec}}\right)^3 \simeq 3\Gamma^2 m_P^2. \tag{81}$$

On combining Eqs. (80) and (81) we can compute the correction factor in Eq. (76), and we obtain finally

$$N_* \simeq 64.4 - \frac{1}{6}\ln\frac{m}{\Gamma} - \frac{2}{3}\ln\frac{\sigma_*}{m_P}, \quad \text{for } \sigma_* \ll m_P. \tag{82}$$

The ratio m/Γ depends on the specific model for the curvaton. It parameterizes the duration of the curvaton oscillating phase and represents here the main uncertainty, although limited by the prefactor $1/6$. Also the last term can be important for very small σ_*: the smaller σ_*/m_P is, the later the curvaton starts dominating the Universe, and thus the shorter the duration of the curvaton domination phase is. This increases the number of e-foldings and partially compensates the $\ln(m/\Gamma)$ correction.

Now we have all the ingredients to study the predictions of mixed models as compared to a pure quartic inflationary model. We use Eqs. (72) and (73) to express the slow-roll parameters ϵ and η in terms of the number of e-foldings N_*. Furthermore, on using Eqs. (61) and (66), we can express the observables n_s and r in terms of N_*, and we can compare the predictions on the (n_s, r)-plane of a pure quartic model and mixed models with cosmological data.

In Fig. 5 the two-dimensional likelihood contours at 68% and 95% confidence level of the WMAP data (combined with other data as from the analysis of [20]) are shown on the (n_s, r)-plane. For a pure inflationary model, the $N_* = 64$ realization is marginally excluded by the data. When adding the perturbation of the curvaton, the predictions on the (n_s, r)-plane are changed and the $N_* = 64$ realization of the inflationary model can be safely included into the 95% confidence level contour for a mixed model with $\sigma_* \lesssim 0.5m_P$, and into the 68% confidence level contour for a mixed model with $\sigma_* \lesssim 0.1m_P$.

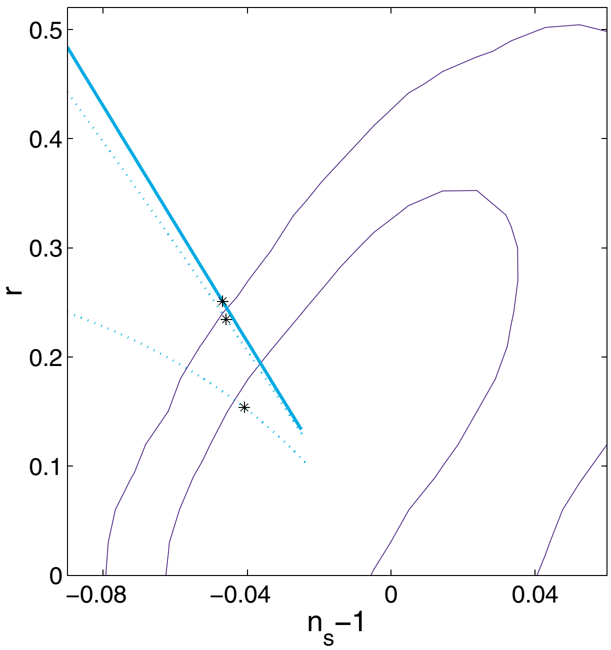

FIG. 5 (color online). Two-dimensional likelihood contours at 68% and 95% confidence level of the WMAP data on the (n_s, r)-plane, as compared to the predictions of a pure inflationary model (solid N-trajectory), a mixed model with $f = 1$ (upper dotted N-trajectory), corresponding to $\sigma_* \sim 0.5 m_{\mathrm{P}}$, and one with $f = 3$ (lower dotted N-trajectory), corresponding to $\sigma_* \sim 0.1 m_{\mathrm{P}}$, respectively. The asterisks denote $N = 64$ on the N-trajectories. The interval $60 \leq N \leq 64$ is represented by bold dots on the N-trajectories of the mixed models. The likelihood contours are from the analysis of S. Leach and A. Liddle [20].

However, one must also take into account the fact that the number of e-foldings N_* decreases because of the phase of curvaton domination, according to (76). As an illustration of this effect, we consider the interval $60 \leq N_* \leq 64$ on the curvaton trajectory and we plot it in Fig. 5. Decreasing the number of e-foldings tends to exclude the model.

As anticipated in the introduction, the extraction of information from the data is limited now by the introduction of a new field, the curvaton, and a corresponding new parameter, the curvaton expectation value during inflation. The degeneracy of the data can be broken by making use of the consistency relation, Eq. (68), although we are still far from measuring n_T [22].

V. CONCLUSION

In the present work, we have explored one particular facet of the curvaton mechanism: we have considered the case where primordial curvature perturbations generated by the curvaton are of the same order of magnitude as the traditional inflaton-generated curvature perturbations. This can happen when the vacuum expectation value for the curvaton is of the order of the Planck mass during inflation. This situation can occur in some scenarios, such as in the string axion scenario discussed in [10].

Independently of the specific model of inflation, we have obtained the expression for the primordial curvature perturbation, given as a linear combination of the post-reheating curvature perturbation (i.e. generated during inflation) and of the curvaton perturbation. In terms of the curvature perturbation on the uniform energy density hypersurface this is given as

$$\zeta = \left[\frac{1}{\sqrt{\epsilon}} e_\phi + \tilde{f}(\sigma) e_\sigma\right] \frac{H}{2\sqrt{2}\pi m_{\mathrm{P}}}, \quad \langle e_\phi e_\sigma \rangle = 0, \quad (83)$$

where H, σ, and ϵ are the Hubble parameter, the vacuum expectation value of the curvaton, and the first slow-roll parameter, all evaluated at Hubble crossing during inflation, respectively [see Eq. (53)]. This expression is only valid for large $\tilde{f}$, otherwise corrections which are second order in the slow-roll parameters must be introduced. The function $\tilde{f}(\sigma)$ depends on the initial background value for the curvaton and interpolates between two regimes, which have been investigated previously in the literature: the standard curvaton regime, where the curvaton starts oscillating long before it dominates, and for which $\tilde{f} = (2\sqrt{2}/3)(m_{\mathrm{P}}/\sigma)$, and the secondary inflaton regime, where the "curvaton" is still frozen when it dominates and thus produces an additional period of inflation, and $\tilde{f} = (1/\sqrt{2})(\sigma/m_{\mathrm{P}})$.

In the case of slow-roll inflation, we have computed the spectral amplitude and index associated with curvature perturbations of mixed origins and shown how the resulting expressions nicely interpolate between the pure inflaton and pure curvaton cases. As an illustration, we have considered the case of a quartic potential for the inflaton and shown how allowing for a mixing with a curvaton changes the usual predictions.

The results derived here are quite generic and can be applied to any inflationary model. Although we have considered, for simplicity, only a quadratic potential for the curvaton, which is a good approximation near a local minimum of most potentials, it is in principle straightforward to extend our results to other potentials. As we have shown, a generic consequence of mixed perturbations is a modification of the usual consistency relation, which gives some hope to be able to distinguish observationally between the standard inflationary scenario and its mixed version.

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