Renormalization group approach to inhomogeneous cosmology

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Soliton solutions are recovered as scale-invariant asymptotic states of vacuum inhomogeneous cosmologies using renormalization group method. The stability analysis of these states is also given.

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I. INTRODUCTION

The study of influence of initial inhomogeneities upon the evolution of cosmological models is an important issue in cosmology, both in order to understand the formation of large-scale structures as well the smoothing away of these inhomogeneities. One way to deal with the inhomogeneities is to consider perturbations of known solutions, such as Friedmann-Robertson-Walker models or homogeneous Bianchi-type solutions. Another way is to assume inhomogeneous solutions from the beginning and study their dynamical evolution; it is this that we are concerned with here. The simplest inhomogeneous models are the so-called diagonal G_2 cosmologies for which the spacetime admits two commuting Killing vectors whose orbits are two-dimensional spacelike surfaces. They represent inhomogeneities of spatially homogeneous models and can be considered as gravitational waves of a single polarization propagating over a homogeneous background [1]. Using a few solution generating techniques, a large number of exact solutions with different sources have been found [2].

The asymptotic evolution of homogeneous Bianchi-type models and both their behavior near the initial singularity as well as their future state have been widely studied [3]. One of the facts relevant to our paper is that at late times Bianchi models can be described as self-similar solutions. However, the asymptotic behavior of inhomogeneous solutions, particularly G_2 metrics, revealed a task much more difficult than that of homogeneous models. Contrary to the homogeneous metrics for which the Einstein equation reduce to an autonomous system of differential equations that can be analyzed using techniques from the theory of dynamical systems, the field equations of inhomogeneous spacetimes are partial differential equations. In Ref. [3], the evolution of a particular class of G_2 metrics is studied showing that in that case all the solutions are asymptotically self-similar. The same result was obtained in [4] and in [5] where special families of G_2 solutions were considered with a scalar field. On the other side, it has been suggested that fluctuations might evolve from arbitrary initial conditions to a self-similar form [6]. From all these results, it is reasonable to regard the self-similar solutions as describing the long-time asymptotic of inhomogeneous metrics. Hence, it would be worthy to analyze the asymptotic behavior of a general class of G_2 metrics by using a method that emphasizes the scaling properties of the underlying field equations.

In this paper, we study the asymptotic evolution of vacuum G_2 metrics using a different approach than that used in the above referred papers: We make use of renormalization group (RG) tools to study the structurally stable characteristics of the inhomogeneous metrics. Recently, RG techniques have been exhibited as a powerful implement to study the asymptotic behavior of partial differential equations [7–10]. The RG method has been applied to study a homogeneous and isotropic universe, a spherically symmetric dust collapse [11], critical phenomena related with gravitational collapse [12], Newtonian cosmology [13], homogeneous flat causal bulk viscous cosmological models [14], and the theory of perturbations of an isotropic universe with dynamically evolving Newton constant and cosmological constant [15,16].

The plan of the paper is the following: In Sec. II we illustrate the application of the RG method to a homogeneous Bianchi-type metric with a scalar field. This case has been studied before and we recover the attractors of the system by means of the RG method. In Sec. III we apply the RG technique to the vacuum, diagonal, G_2 metric. We find the fixed points and analyze their stability as well. We conclude with Sec. IV.

THE RG METHOD: AN ILLUSTRATIVE EXAMPLE

It is well known that the asymptotics of partial differential equations can often be found from the consideration of scaling solutions (the equivalence of RG theory and the theory of intermediate asymptotics was shown by Goldenfeld *et al.* [17]). Though it is usual practice to find the similarity variable to analyze scaling solutions from a combination of variables using dimensional arguments, however, there is a large class of problems where this cannot be done [18]. RG provides a systematic approach for finding the scaling variables as well as the asymptotics of partial differential equations.

The general (RG) method adapted to partial differential equations is well known [7]. Therefore, instead of reproducing the whole procedure, we explain this method using an illustrative example which has been studied with other methods and has a nontrivial asymptotic structure. We consider a class of anisotropic cosmological models given by [19]

$$ds^2 = -dt^2 + a(t)^2 dx^2 + b(t)^2 e^{2mx} dy^2 + c(t)^2 e^{2x} dz^2. \tag{2.1}$$

This represents a one parameter (m) family of Bianchi models with Bianchi III, V, and VI_0 for, respectively, $m = 0, 1$, and -1, and Bianchi VI_h ($h = m + 1$) for all other values.

For a direct comparison with earlier works, we choose the same variables, namely, shear (σ) and expansion (θ) for our analysis. Note, however, that the feasibility of the earlier analysis depends crucially on finding the "right" set of variables to work with; otherwise the system is too complex to analyze completely, even in this homogeneous case. Moreover, the choice of these variables is model dependent and rather *ad hoc*. Therefore, one also needs to be lucky to be able to analyze. With RG, one can directly work with the metric functions, as will be done in the next section.

The shear and expansion for this spacetime are given by

$$\theta = \frac{\dot{a}}{a} + \frac{\dot{b}}{b} + \frac{\dot{c}}{c}, \tag{2.2}$$

$$\sigma^2 = \frac{1}{3}\left[\left(\frac{\dot{a}}{a}\right)^2 + \left(\frac{\dot{b}}{b}\right)^2 + \left(\frac{\dot{c}}{c}\right)^2\right]. \tag{2.3}$$

Here an overdot signifies the derivative with respect to t. We consider the energy momentum tensor describing a minimally coupled homogeneous scalar field $\phi(t)$, given by

$$T_{\mu\nu} = \phi_{,\mu}\phi_{,\nu} - g_{\mu\nu}[\tfrac{1}{2}\phi_{,\alpha}\phi^{,\alpha} + V(\phi)], \tag{2.4}$$

with an exponential potential $V(\phi) = \lambda \exp(k\phi)$. Here, $\lambda > 0$ and k are both constants. The evolution equations are then given by

$$\dot{\theta} = -2\sigma^2 - \frac{\theta^2}{3} - \dot{\phi}^2 + V(\phi), \tag{2.5}$$

$$\dot{\sigma} = -\sigma\theta + p(m)\left[\theta^2 - 3\sigma^2 - \frac{3}{2}\dot{\phi}^2 - 3V(\phi)\right], \tag{2.6}$$

$$\ddot{\phi} = -\theta\dot{\phi} - kV(\phi), \tag{2.7}$$

$$\dot{V} = k\dot{\phi}V, \tag{2.8}$$

where $p(m) = (1 - m)/[3\sqrt{3(1 + m + m^2)}]$. Since the above system is an autonomous system of differential equations, in the previous study [19] the theory of dynamical systems was used to analyze its asymptotic behavior using "expansion-normalized variables." We will use, as an alternative method, the RG technique and recover the same results. Again, our goal is to illustrate the method in this section and apply it to a new scenario in the next section.

We find it useful to work with a compact notation and, therefore, define a new indexed variable $u_i(t)$ that signifies the set $\{\theta, \sigma, \dot{\phi}, V\}$, for $i = 1, 2, 3, 4$, respectively. Our interest is in the asymptotics of the solution of the form

$$\lim_{t\to\infty} u_i(t) = t^{-\alpha_i} u_i^*(1), \tag{2.9}$$

where argument "1" signifies the initial value of the quantity. It is convenient to fix the initial time as $t = 1$. Now we will illustrate how the RG method gives a systematic procedure to fix α_i and determine the scaling function $u_i^*(1)$ as fixed points of RG equations (in a inhomogeneous case the argument of u_i^* will not be a constant but will depend on a combination of time and spatial variable, the scaling variable).

Let us consider transformations of the form

$$t \to Lt, \qquad u_i(t) \to U_i(t) = L^{\alpha_i} u_i(Lt), \tag{2.10}$$

which leave the equations invariant; i.e., if $u_i(t)$ is a solution so is $U_i(t)$. We use a number $L > 1$ as a parameter of scale transformation. Note that, unlike in the application of the RG method to quantum field theories or statistical mechanics, there is no natural way to choose a scale L here. Moreover, L never appears explicitly in the equations.

From equation set (2.5), (2.6), (2.7), (2.8), and (2.10), it is straightforward to recover

$$\alpha_1 = \alpha_2 = \alpha_3 = \frac{\alpha_4}{2} = 1. \tag{2.11}$$

Now, defining $L = \exp(\tau)$, and using Eq. (2.10) in (2.5), (2.6), (2.7), and (2.8), we get a new set of evolution equations:

$$\frac{dU_i}{d\tau} = \alpha_i U_i + \frac{\partial U_i}{\partial t}. \tag{2.12}$$

The procedure of defining τ is, in a sense, analogous to summing over all degrees of freedom corresponding to fluctuations of scale less than L and then rescaling everything by L^{-1}. This new set of evolution equations defines the RG transformations. Since scaled quantities satisfy the same evolution equations, the time derivative in the equation set above can be replaced with Eq. (2.5), (2.6), (2.7), and (2.8), and we have

$$\frac{dU_1}{d\tau} = U_1 - 2U_2^2 - \frac{U_1^2}{3} - U_3^2 + U_4, \tag{2.13}$$

$$\frac{dU_2}{d\tau} = U_2 - U_1U_2 + p(m)\left(U_1^2 - 3U_2^2 - \frac{3}{2}U_3^2 - 3U_4\right), \tag{2.14}$$

$$\frac{dU_3}{d\tau} = (1 - U_1)U_3 - kU_4, \tag{2.15}$$

$$\frac{dU_4}{d\tau} = (2 + kU_3)U_4. \tag{2.16}$$

All the quantities on the right-hand side are evaluated at $t = 1$. Scale-invariant solutions emerge now from the fixed-point structure of the RG map, which is defined by

$$\frac{dU_i^*}{d\tau} = 0, \tag{2.17}$$

with U_i^* being the fixed points. The complete set of fixed points for this system is given in Table I.

Therefore, we have four exact scale-invariant solutions of the form (2.9) with exponents α_i given by (2.11) and $u_i^*(1)$ being the fixed points given above. Since the stability properties of these solutions are well studied in literature [19], we move now to the more general inhomogeneous case. We analyze the fixed-point structure as well as their stability properties, which is not done before.

III. INHOMOGENEOUS CASE

We begin with a family of metrics which have been a very useful tool for studying inhomogeneous cosmological models, known as the generalized Einstein-Rosen spacetimes (see, for example, [20,21]). The line element is of the form

$$ds^2 = f^2(dz^2 - dt^2) + \gamma_{ab}dx^a dx^b, \qquad a, b = 1, 2. \tag{3.1}$$

Here $x^1 = x$, $x^2 = y$, and both f and γ_{ab} are functions only of z and t. Metric (3.1) is fairly general and includes Bianchi-type models I to VII. In this paper we are interested in vacuum solutions only. We can impose now

$$\mathrm{det}\gamma_{ab} = t^2, \tag{3.2}$$

since in the vacuum case one of the field equation takes the form

$$(\mathrm{det}\gamma_{ab})^{(1/2)}_{,tt} - (\mathrm{det}\gamma_{ab})^{(1/2)}_{,zz} = 0. \tag{3.3}$$

t and z are two independent solutions of this equation. Under this condition, the coordinates t and z are called the canonical coordinates, and there is no loss of generality in this choice [20]. To simplify the analysis, we now specialize to the diagonal metrics, i.e., metrics with a single polarization.

The spacetime can be written in the form

$$ds^2 = f^2(dz^2 - dt^2) + t(h^2 dx^2 + h^{-2}dy^2), \tag{3.4}$$

where f and h are functions of t and z only. These metrics admit an Abelian G_2 group of isometries with two spacelike commuting killing vectors ∂_x and ∂_y. The set of Einstein field equations in the vacuum case reduces to

$$h_{,t} = \frac{1}{2t}\frac{f_{,z}}{f}\frac{h^2}{h_{,z}}, \tag{3.5}$$

$$f_{,t} = -\frac{1}{4t}f + tf\left(\frac{h_{,z}}{h}\right)^2 + \frac{1}{4t}\frac{(f_{,z})^2}{f}\left(\frac{h}{h_{,z}}\right)^2. \tag{3.6}$$

Equations (3.5) and (3.6) are the evolution equations for h and f , respectively, and it is easy to check that the remaining field equations are identically satisfied. We now follow the prescription given in the previous section to recover the exact scaling solutions.

Let us consider the following scale transformation:

$$\begin{gathered} z \longrightarrow Lz, \qquad t \longrightarrow L^{\epsilon}t, \\ h(t, z) \longrightarrow \phi(t, z) = L^{\alpha}h(L^{\epsilon}t, Lz), \\ f(t, z) \longrightarrow \psi(t, z) = L^{\beta}f(L^{\epsilon}t, Lz). \end{gathered} \tag{3.7}$$

Here ϕ and ψ are the scaled quantities. Since scaled quantities also satisfy the original equations, Eq. (3.6) fixes

$$\epsilon = 1. \tag{3.8}$$

The scaling relations (3.7) along with (3.8) and successive transformations, first $t \longrightarrow 1$ and then $L \longrightarrow t$, give a scale-invariant solution of the form

$$h(t, z) = t^{-\alpha}\phi(1, z/t), \qquad f(t, z) = t^{-\beta}\psi(1, z/t). \tag{3.9}$$

The equations above express an arbitrary solution in terms of initial data (at $t = 1$). As stated earlier, we work with $t = 1$ as our initial time and evolution is in the sense of scaled time Lt with $L > 1$.

From simple dimensional analysis, we would have $\beta = 0$ and $\alpha = \pm 1/2$ (introducing a dimensional constant multiplying either dx or dy). However, as we will see later this would lead to a trivial solution: the homogeneous

TABLE I. Fixed points of Bianchi $\mathrm{VI_h}$ models.

	U_1	U_2	U_3	U_4
1.	1	$\pm\sqrt{\frac{2-3U_3}{6}}$	U_3	0
2.	$\frac{6}{k^2}$	0	$-\frac{2}{k}$	$2\frac{(6-k^2)}{k^4}$
3.	$\frac{2(1-m)^2}{27\mathrm{p(m)}^2(1+m^2)}$	$\frac{(1-m)^2}{9\mathrm{p(m)}(1+m^2)}$	0	0
4.	$(1+\frac{k^2}{27\mathrm{p(m)}^2})\frac{2(1-m)^2}{(k)^2(1+m^2)}$	$\frac{(k^2-2)(1-m)^2}{9k^2\mathrm{p(m)}(1+m^2)}$	$-\frac{2}{k}$	$\frac{2(1+m)^2}{k^2(1+m^2)} + \frac{4(1-m)^2}{k^4(1+m^2)}$

Kasner metric. The fact that we recover a Kasner solution as a fixed point of a general inhomogeneous G2 metric though is nontrivial, and it elucidates the appearance of this solution in earlier studies; it is used as a seed metric in different solution generating techniques, and more importantly, is known to describe "generic" cosmological singularity in the analysis of Belinskii *et al.* [22]. In order to get a nontrivial structure in fix-point analysis, we need to analyze the anomalous dimensions for both the functions f and h. It is well known [23] that the anomalous dimensions $\alpha \neq \pm 1/2$ and $\beta \neq 0$ are fixed by initial or boundary conditions. In our case, cosmological vacuum solutions, initial condition refers to geometry at $t = 0$. Since, metrics of the type (3.4) are singular at $t = 0$ there is no strict functional constraint on the system described by (3.5) and (3.6). Nevertheless, interpreting *a posteriori* the fixed points, we can give some hints to understand the role of anomalous dimensions α and β.

Denoting $L = \exp(\tau)$, the RG equations are

$$\frac{d\phi}{d\tau} = \alpha\phi + \phi' z + \frac{1}{2}\left(\frac{\psi'}{\psi}\right)\left(\frac{\phi^2}{\phi'}\right), \tag{3.10}$$

$$\frac{d\psi}{d\tau} = \left(\beta - \frac{1}{4}\right)\psi + \psi' z + \psi\left(\frac{\phi'}{\phi}\right)^2 + \frac{1}{4}\frac{(\psi')^2}{\psi}\left(\frac{\phi}{\phi'}\right)^2. \tag{3.11}$$

We now investigate the fix-point structure of the equation set above, i.e.,

$$\frac{d\phi^*}{d\tau} = 0 \Rightarrow \alpha = -\left(\frac{\phi^{*\prime}}{\phi^*}\right)z - \frac{1}{2}\left(\frac{\psi^{*\prime}}{\psi^*}\frac{\phi^*}{\phi^{*\prime}}\right), \tag{3.12}$$
$$\frac{d\psi^*}{d\tau} = 0 \Rightarrow \frac{1}{4} - \beta = \left(\frac{\psi^{*\prime}}{\psi^*}\right)z + \left(\frac{\phi^{*\prime}}{\phi^*}\right)^2 + \frac{1}{2}\left(\frac{\psi^{*\prime}}{\psi^*}\frac{\phi^*}{\phi^{*\prime}}\right)^2.$$

The system decouples to give

$$\left(\frac{\phi^{*\prime}}{\phi^*}\right) = \pm\sqrt{\frac{\triangle}{z^2 - 1}}, \tag{3.13}$$

$$\left(\frac{\psi^{*\prime}}{\psi^*}\right) = \frac{2\sqrt{\triangle}}{1 - z^2}(\sqrt{\triangle}z \pm \alpha\sqrt{z^2 - 1}), \tag{3.14}$$

where $\triangle = \alpha^2 + \beta - 1/4$. The real solutions correspond to $\triangle > 0$ for $z^2 > 1$ and $\triangle < 0$ for $z^2 < 1$. Equations (3.13) and (3.14) can be easily integrated to give the fixed points

$$\phi^* = (z + \sqrt{z^2 - 1})^{\pm\sqrt{\triangle}}, \tag{3.15}$$

$$\psi^* = c_f(z + \sqrt{z^2 - 1})^{\mp 2\alpha\sqrt{\triangle}}(z^2 - 1)^{-\triangle}, \tag{3.16}$$

where c_f is an integration constant. We have dropped the constant of integration from ψ^* since this can be absorbed simply by scaling of x and y. Also, we note here that the new parameter $\triangle$ relates our spacetime metric with the Kasner metric in the following way. The Kasner metric can be written in the form

$$ds^2 = t^{(d^2-1)/2}(dz^2 - dt^2) + t^{1+d}dx^2 + t^{1-d}dy^2, \tag{3.17}$$

where parameter d can be chosen positive or negative. A direct comparison with the metric (3.4) using (3.9) gives the Kasner relationship

$$4\alpha^2 + 4\beta - 1 = 0. \tag{3.18}$$

Therefore, the $\triangle = 0$ case corresponds to the Kasner models. When $\triangle > 0$ ($z^2 > t^2$), the critical solution is

$$h(t, z) = t^{-\alpha\mp\sqrt{\triangle}}(z + \sqrt{z^2 - t^2})^{\sqrt{\triangle}},$$
$$f(t, z) = c_f t^{(\alpha\pm\sqrt{\triangle}+1/2)(\alpha\pm\sqrt{\triangle}-1/2)}(z + \sqrt{z^2 - t^2})^{\mp 2\alpha\sqrt{\triangle}} \times (z^2 - t^2)^{-\triangle}. \tag{3.19}$$

and when $\triangle < 0$ ($z^2 < t^2$) the critical solution is

$$h(t, z) = t^{-\alpha}\exp\left(\mp\sqrt{-\triangle}\arccos\frac{z}{t}\right),$$
$$f(t, z) = c_f' t^{\alpha^2+\triangle-1/4}(t^2 - z^2)^{-\triangle} \times \exp\left(\pm 2\alpha\sqrt{-\triangle}\arccos\frac{z}{t}\right). \tag{3.20}$$

The spacetime is split into two regions separated by the light cone $z = t$. In each region, the solution takes one of the forms given by the expressions above. For $\triangle = 0$, we have the Kasner metric. In the general case ($\triangle \neq 0$), solution (3.19) is the soliton metric that had been obtained earlier by the inverse scattering transformation with real poles from the Kasner metric, and the solution (3.20) is the cosoliton solution generated also from the Kasner metric. Both solutions have been studied in [24] (and references therein). Depending of the parameters, the light cone $z = t$ is singular for the solution (3.19) but is always singular for (3.20). This means that, even though the metric and its first derivatives are continuous across the light cone, the solutions cannot be matched across the light cone (for a discussion on the matching of these metrics, see [24]). We would like to note here that the $\pm$ sign in the above equations is actually related to the symmetry under $x \leftrightarrow y$, and as we will see later in the stability analysis the same results hold for both signs. Moreover, we would like to stress here that the solutions which we have obtained are actually the future asymptotic states, i.e., to which spacetime "prefers" to settle down. This makes this analysis very powerful since not only do we recover in a very simple fashion a whole class of scale-invariant solutions but also, due to "universality," these solutions actually are the preferred asymptotic states.

We will consider now the linear stability analysis. Since these solutions emerge as fixed points of the RG map, the stability of these solutions is the stability of these fix points. Let us define

$$\phi = \phi^*(1 + \delta\phi), \tag{3.21}$$

$$\psi = \psi^*(1 + \delta\psi), \tag{3.22}$$

where $\delta\phi \ll 1$ and $\delta\psi \ll 1$. The above form of perturbations is chosen to facilitate the analysis since both unperturbed functions ϕ^* and ψ^* can diverge at $z = \pm 1$ and $z = \pm\infty$.

The perturbation equations take the form

$$\begin{aligned} \frac{d\delta\phi}{d\tau} &= \left(\frac{\phi^*}{\phi^{*\prime}}\right)\left[\left(\alpha + 2z\frac{\phi^{*\prime}}{\phi^*}\right)\frac{d\delta\phi}{dz} + \frac{1}{2}\frac{d\delta\psi}{dz}\right], \\ \frac{d\delta\psi}{d\tau} &= -\left(\frac{\phi^*}{\phi^{*\prime}}\right)\left[2\left(\alpha^2 + \triangle + 2\alpha z\frac{\phi^{*\prime}}{\phi^*}\right)\frac{d\delta\phi}{dz} + \alpha\frac{d\delta\psi}{dz}\right]. \end{aligned} \tag{3.23}$$

We shall compute normal modes assuming the following form for the perturbations:

$$\delta\phi = e^{\omega\tau}\rho(z), \qquad \delta\psi = e^{\omega\tau}\sigma(z), \tag{3.24}$$

where ω is a constant. The perturbation Eqs. (3.23) can be written as

$$\begin{aligned} \left(\alpha + 2z\frac{\phi^{*\prime}}{\phi^*}\right)\rho' + \frac{1}{2}\sigma' &= \frac{\phi^{*\prime}}{\phi^*}\omega\rho, \\ -2\left(\alpha^2 + \triangle + 2\alpha z\frac{\phi^{*\prime}}{\phi^*}\right)\rho' - \alpha\sigma' &= \frac{\phi^{*\prime}}{\phi^*}\omega\sigma. \end{aligned} \tag{3.25}$$

From these equations it is easy to see that

$$\omega\sigma = -2\sqrt{\triangle}\sqrt{z^2 - 1}\rho' - 2\alpha\omega\rho. \tag{3.26}$$

Taking the derivative of this equation and substituting in (3.25), we get

$$\rho'' + \frac{z}{z^2 - 1}(1 - 2\omega)\rho' + \frac{\omega^2}{z^2 - 1}\rho = 0. \tag{3.27}$$

Thus, the problem of solving linear perturbations around the fixed points has been reduced to finding solutions of the above equation, and (3.26) completes the solution. The general solution of (3.27) is given by

$$\rho(z) = (z^2 - 1)^{\omega/2+1/4}[c_1 P_{-1/2}^{\omega+1/2}(z) + c_2 Q_{-1/2}^{\omega+1/2}(z)], \tag{3.28}$$

where P_ν^μ and Q_ν^μ are Legendre functions of first and second kind, respectively, and c_1 and c_2 are arbitrary constants. From (3.26), the solution for σ can be easily obtained:

$$\begin{aligned} \sigma(z) = &-2\sqrt{\triangle}(z^2 - 1)^{\omega/2-1/4}\{c_1[zP_{-1/2}^{\omega+1/2}(z) \\ &- P_{1/2}^{\omega+1/2}(z)] + c_2[zQ_{-1/2}^{\omega+1/2}(z) - Q_{1/2}^{\omega+1/2}(z)]\} \\ &- 2\alpha(z^2 - 1)^{\omega/2+1/4}[c_1 P_{-1/2}^{\omega+1/2}(z) + c_2 Q_{-1/2}^{\omega+1/2}(z)]. \end{aligned} \tag{3.29}$$

We require that the perturbations be regular for all z. Since the differential Eq. (3.27) has four regular singular points at $z = \pm 1, \pm\infty$, we must consider three different regions: $1 < z < \infty$, $-\infty < z < -1$, and $-1 < z < 1$. The first and the second region are equivalent. Let us first analyze the behavior in the first region. It is easy to see that the leading terms of the two linearly independent solutions ρ_1 and ρ_2 of (3.27) in the neighborhood of $z = \infty$ are, respectively, $\rho_1 \approx z^\omega + O(z^{\omega-2})$ and $\rho_2 \approx \rho_1 \ln z + O(z^{\omega-2})$. Both solutions are regular if $\omega \leq 0$. Furthermore, from (3.26), σ is also regular provided that $\omega \leq 0$. At $z = 1$, the leading terms of the two linearly independent solutions for ρ are $(z-1)^{\omega+1/2} + O[(z-1)^{\omega+3/2}]$ and const $+ O[(z-1)]$. For σ, the leading terms are $(z-1)^\omega$ and const $+ O[(z-1)]$. Since ω is negative, the first independent solution must be neglected so that we have regular solutions. For instance, tacking $c_2 = 0$ in (3.28) the solution is regular.

Making the scale transformation given by (3.7), (3.8), and (3.9), we get the behavior of the solution near the fixed point :

$$\delta\phi = t^\omega\rho(z/t), \qquad \delta\psi = t^\omega\sigma(z/t), \tag{3.30}$$

with $\omega \leq 0$ and $z > t$. Since the solution above is valid in the region $z > t$, taking limit $t \to \infty$ means $z \to \infty$ as well. We can therefore distinguish two different situations. First, when we approach infinity in such a way that we encounter the light cone $z = t$ without crossing it. In this case asymptotically $z/t = 1$ and, bearing in mind the behavior of the regular solutions of (3.27) close to the point $z = 1$, the above perturbations behave as:

$$\delta\phi \approx \delta\psi \approx \text{const} \times t^\omega. \tag{3.31}$$

In the second situation, we approach infinity through a path that does not encounter the light cone. In this case z/t tends to a constant greater than 1, and the perturbations behave in a similar way than that given by the above expressions. So, in both cases the perturbations tend to zero when $t \to \infty$ which means that the fixed solution outside the light cone is stable.

Let us perform this analysis in the region $-1 < z < 1$. The leading terms of the independent solutions of (3.27) close to $z = 1$ are those described earlier for the same point changing $z - 1$ by $1 - z$. When $z = -1$ the behavior is the same changing $z - 1$ by $z + 1$. From this we cannot obtain a definite conclusion about the values of ω which make the solution regular. To do that is convenient to transform Eq. (3.27) to a hypergeometric differential equation with coefficients $a = b = 1/2$ and $c = 1/2 - \omega$. It is not difficult to see that one of the independent solutions, the solution of (3.27) and (3.26), is given by

$$\rho(z) = c_1(1+z)^{\omega+1/2}F\left(\frac{1}{2},\frac{1}{2};\frac{1}{2}-\omega;\frac{1-z}{2}\right),$$

$$\begin{aligned}\sigma(z) = &-2c_1\frac{\sqrt{|\triangle|}}{\omega}\sqrt{1-z^2}(1+z)^{\omega-1/2}\left[\left(\omega+\frac{1}{2}\right)\right.\\ &\times F\left(\frac{1}{2},\frac{1}{2};\frac{1}{2}-\omega;\frac{1-z}{2}\right)-\frac{1}{4-8\omega}(1+z)\\ &\left.\times F\left(\frac{3}{2},\frac{3}{2};\frac{3}{2}-\omega;\frac{1-z}{2}\right)\right]-2\alpha c_1(1+z)^{\omega+1/2}\\ &\times F\left(\frac{1}{2},\frac{1}{2};\frac{1}{2}-\omega;\frac{1-z}{2}\right).\end{aligned} \quad (3.32)$$

The form of the second independent solution depends on the values of ω. In any case, the above solution is regular when ω is positive, except when $\omega = n + 1/2$ (n a positive integer). Therefore, with $|z| < t$, the modes with ω positive dominate and the solution in the interior of the light cone is not stable.

IV. DISCUSSION

In this paper we have shown how the RG method can be used to obtain the asymptotic regime of cosmological solutions. We have first illustrated the method applying it to homogenous cosmological models with a scalar field. These models have been extensively analyzed using, mainly, the so-called "expansion-normalized variables," which are a set of dimensionless variables [3]. In those works the role of self-similar solutions in describing the asymptotic regime has been stressed. Since a fixed point of the RG transformation is a scale-invariant solution, it is reasonable to think that the RG method will naturally render the same results (and we have manifestly shown this here, Sec. II) as those obtained using the "expansion-normalized variables." It is worthy to stress also that those normalized variables are well adapted to the dimensional and similarity analysis [18]. These results bring the asymptotic behavior of the homogeneous cosmological models in a new perspective. Moreover, its simplicity as well as systematic approach show that it is a practical idea to implement RG for studying asymptotic behavior of spacetimes in general relativity.

We have also applied the RG method to a diagonal, vacuum inhomogeneous G_2 metric. In this case, the system reduces to a set of coupled partial differential equations whose analysis using the RG method is not difficult. The fixed point is an exact solution depending on two parameters. This solution belongs to the class of soliton solutions. Soliton solutions are intended as those solutions which can be obtained by the inverse scattering transformation from a known one [20]. In particular, the found fixed point is a soliton solution with real poles with origin at $z = 0$, whose "seed" metric is the Kasner metric (soliton origin marks the origin of the light cone $z^2 = t^2$). Since the metric (3.4) is invariant under a z translation, the fact that the origin is at $z = 0$ is not important. Moreover, a more general class of soliton solutions is that corresponding to a sum of solitons each with a different origin. Since the RG method gives the long-time behavior of the solution, the difference in the origins tends to zero as t tends to infinity leaving only one origin. It is interesting to note that, generically, the solution does not homogenize, let alone isotropize, for the final state is inhomogeneous. There is, however, a few particular cases for which the metric tends to a homogeneous solution: When $\triangle = 0$ the solution is the Bianchi-type I Kasner metric, and for a particular value of the parameters α and $\triangle$ the solution is Ellis and McCallum family of vacuum Bianchi models [25]. Finally, we would like to note that the fixed points recovered in the RG technique give all the scale-invariant exact solutions of vacuum G_2 cosmologies.

Now we can understand the relation of the anomalous dimensions α and β with initial conditions: Exponent α gives the Kasner parameter of the seed metric to generate the soliton solution by means of the inverse scattering technique. The parameter $\triangle$ (which fixes β) determines the number of solitons in our solution. So, the two "initial conditions" used in the inverse scattering technique, number of solitons and the Kasner seed metric, fixes the anomalous dimension of system comprising of Eqs. (3.5) and (3.6).

The RG method allows us to study the linear perturbations around the fixed points as well. We have shown that the solution outside the light cone, which corresponds to the soliton solution with real poles, is stable against bounded perturbations, contrary to the solution inside the light cone (cosoliton solution) which we find to be unstable. Let us note that, although the stability analysis does not have any apparent dependence on the $\triangle$ parameter, one should be very cautious to extend these results to the case $\triangle = 0$. This case corresponds to the Kasner solution and this is the only case for which the fixed-point results in $\phi^{*\prime} = \psi^{*\prime} = 0$, which makes the system of equations for the perturbations singular.

There remain a few open questions that we hope to deal with in future works. First is related to the role of soliton solutions with complex poles. These can be obtained as a complex translation along the z axis of the soliton solution with real poles and they are regular in all the spacetime, even on the light cones. Since this complexification does not alter the structure of the field equations, it would be worthy to investigate whether the complex pole soliton solutions can represent an asymptotic state of the inhomogeneous metrics. Second, it would be interesting to extend the analysis performed in this paper to general inhomogeneous nonvacuum metrics. Solutions with a scalar field are of particular interest in order to investigate issues such as isotropization, scaling solutions, etc. Finally, this technique can be used to analyze long-time behavior of spacetimes with more that four dimensions, for example, brane world scenarios.

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