

Pion-nucleon scattering relations at next-to-leading order in $1/N_c$ Thomas D. Cohen,^{*} Daniel C. Dakin,[†] and Abhinav Nellore[‡]*Department of Physics, University of Maryland, College Park, Maryland 20742-4111, USA*Richard F. Lebed[§]*Department of Physics and Astronomy, Arizona State University, Tempe, Arizona 85287-1504, USA*

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We obtain relations between partial-wave amplitudes for $\pi N \rightarrow \pi N$ and $\pi N \rightarrow \pi \Delta$ directly from large N_c QCD. While linear relations among certain amplitudes holding at leading order (LO) in $1/N_c$ were derived in the context of chiral soliton models two decades ago, the present work employs a fully model-independent framework based on consistency with the large N_c expansion. At LO in $1/N_c$ we reproduce the soliton model results; however, this method allows for systematic corrections. At next-to-leading order (NLO), most relations require additional unknown functions beyond those appearing at LO and thus have little additional predictive power. However, three NLO relations for the $\pi N \rightarrow \pi \Delta$ reaction are independent of unknown functions and make predictions accurate at this order. The amplitudes relevant to two of these relations were previously extracted from experiment. These relations describe experiment dramatically better than their LO counterparts.

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I. INTRODUCTION

It is now 30 years since 't Hooft noted that treating the number N_c of QCD colors as an expansion parameter yields a limiting theory with substantial predictive power [1]; in 1979, Witten extended this general idea to the generic properties of baryons [2]. During the subsequent years, two main approaches to understanding the spin and flavor dependence of baryon properties arose. The first is the chiral soliton approach of Witten, Adkins, and Nappi [2–4] which reinterprets Skyrme's original soliton idea [5] in the context of large N_c QCD. It was noted early on that many relations among observables in such models depend only on the overall structure of the soliton models and are completely independent of the dynamical details [4]. This suggested that these relations directly reflect general results of large N_c QCD. An alternative fully model-independent approach based on consistent power counting of N_c factors in baryon-meson scattering processes was invented by Gervais and Sakita [6] and Dashen and Manohar [7], and then systematically developed by Dashen, Jenkins, and Manohar [8]. In this approach, an underlying contracted $SU(2N_f)$ spin-flavor symmetry (N_f being the number of light quark flavors) emerges as $N_c \rightarrow \infty$. The apparently model-independent relations of soliton models then automatically emerge at leading order in $1/N_c$ as results of the group structure of this emergent symmetry. The approach based on large N_c consistency conditions has two obvious advantages over the soliton

approach: It is manifestly model independent, and it allows for systematic $1/N_c$ corrections.

The systematic treatment of $1/N_c$ corrections comes at a cost. As in any effective theory, one must generally add new unknown coefficients at subleading orders. The power counting in the $1/N_c$ expansion implicitly constrains the typical size of these coefficients via naturalness criteria, but when an unknown coefficient enters a relation at next-to-leading order (NLO), one essentially has no predictive power beyond what is seen at leading order (LO). However, there exist certain relations that hold even after the inclusion of NLO coefficients; we denote such relations as “gold-plated.” The gold-plated relations hold at NLO and, hence, should have errors of $O(1/N_c^2)$ relative to the $O(N_c^0)$ amplitudes. Since $1/N_c^2 = 1/9$ for the physical world, these gold-plated relations may be taken to be semiquantitative predictions. In fact, these relations are often satisfied quite well. For example, one predicts $g_{\pi N \Delta}/g_{\pi NN} = \frac{3}{2}[1 + O(1/N_c^2)]$ [8], and the experimental value of the ratio deviates from $3/2$ by only a few percent. In contrast, ordinary “silver-plated” relations (those holding only at LO in $1/N_c$) are typically of a more qualitative nature.

In this paper we use the large N_c consistency condition approach to deduce relations among partial-wave amplitudes for the processes $\pi N \rightarrow \pi N$ and $\pi N \rightarrow \pi \Delta$, which involve only the two light quark flavors u and d . LO relations were derived long ago in the context of chiral soliton models [9,10]. Since these results were found to be independent of the dynamical details of any particular soliton model, it was generally assumed that they are fully model-independent consequences of large N_c QCD, holding at LO (N_c^0) for meson-baryon scattering amplitudes. It was recently noted that these LO relations can be obtained directly from the group structure arising

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from large N_c consistency conditions [11]. As observed above, one clear advantage of this approach is that it provides a straightforward formalism for working to higher order in $1/N_c$. If one insists upon model-independent constraints, one in fact gains no predictive power at higher order except through gold-plated relations, for which the LO correction terms cancel. In this paper, we show that such gold-plated relations do exist, but they necessarily involve the process $\pi N \rightarrow \pi \Delta$. We also show that these gold-plated relations hold moderately well experimentally, while the analogous silver-plated relations work quite poorly. Thus we find that one can at least semiquantitatively understand some aspects of the $\pi N \rightarrow \pi \Delta$ reaction from *ab initio* large N_c QCD considerations.

Before proceeding, it is useful to discuss the original derivation of the relations among partial-wave amplitudes from the soliton model and to explain why this result is considered to hold at LO in $1/N_c$ [9]. While we derive in this paper a general large N_c rather than merely a soliton model result, we believe that first seeing the result in a concrete realization is instructive, and makes a connection with the older literature. In chiral soliton models, baryons are supposed to arise from *hedgehog* configurations, which are the static, finite-energy solutions (*solitons*) of a (nearly) chirally symmetric tree-level pion Lagrangian. Such configurations can be assigned baryon number unity, but break both isospin and rotational symmetry while preserving “grand spin” $\vec{K} \equiv \vec{I} + \vec{J}$. Note that classical configurations are justified by large N_c considerations, since quantum fluctuations [associated with the excitation of only a few of the $O(N_c)$ constituents] contribute only at relative order N_c^{-1} . Static rotations of such classical soliton configurations lead to energetically degenerate solutions of the classical equations, implying the existence of a multiplet of degenerate states at large N_c . Slow rotational motion (with angular velocity $\sim N_c^{-1}$) among such states is orthogonal to intrinsic quantum excitations of the soliton and may be quantized separately. This quantization leads to nearly degenerate physical states with the usual physical quantum numbers and mass splittings of $O(1/N_c)$. The underlying hedgehog structure implies that each physical state has $I = J$, whose common value we label by R . The $R = \frac{1}{2}$ states are identified as nucleons and $R = \frac{3}{2}$ states are identified as Δ resonances for the $N_c = 3$ world, while states with higher values of R are generally assumed to be large N_c artifacts.

Physical pions are treated as fluctuations about the soliton; the action is expanded perturbatively in the number of pion fields, which is justified for large N_c since each additional pion field suppresses the amplitude by a factor $\sim 1/\sqrt{N_c}$. The scattering is then described in terms of the Green function of the pion-soliton system. The standard machinery of semiclassical projection then allows one to

obtain amplitudes for states with well-defined $I = J$ in both the initial and final states. The S matrix for such a channel in this formalism is given by

$$S_{LL'RR'I_s J_s} = \sum_K (-1)^{R'-R} \sqrt{[R][R']}[K] \begin{Bmatrix} K & I_s & J_s \\ R & L & 1 \end{Bmatrix} \times \begin{Bmatrix} K & I_s & J_s \\ R' & L' & 1 \end{Bmatrix} s_{KLL'} + O(N_c^{-1}), \quad (1.1)$$

where $[x] \equiv 2x + 1$, R (R') is the spin/isospin of the initial (final) baryon, L (L') is the relative orbital angular momentum of the initial (final) pion about the baryon, and J_s, I_s indicate the total spin and isospin, respectively, as measured in the pion-baryon s channel. Note that the S matrix is a reduced matrix element (in the sense of the Wigner-Eckart theorem) in terms of both angular momentum and isospin, in that dependence on the quantum numbers $(I_s)_3$ and $(J_s)_3$ has been factored out.

The explicit “1” in the $6j$ coefficients arises from the isospin of the pion. Although this formula holds for pion scattering, it has been generalized to mesons with spin one (e.g., ρ) and/or isospin zero (e.g., η) [12]. In such cases, Eq. (1.1) maintains the same basic form except the $6j$ coefficients are either replaced by $9j$ coefficients to account for the extra vector (spin 1) or collapse to Kronecker deltas (isospin 0).

The preceding derivation exploits the large N_c limit in multiple ways. As noted above, the use of the classical hedgehog itself is justified only for large N_c , so that quantum fluctuations are relatively unimportant. Moreover, baryon recoil is neglected in the scattering process since the baryon mass scales as N_c , while the characteristic scattering energy scale is $O(N_c^0)$. Similarly, the rotation of the soliton during the scattering event is neglected since the soliton moment of inertia, and, hence, the rotational period, also scales as N_c . These approximations are only valid to LO in $1/N_c$, and thus any predictions based on this formalism can be expected to hold only at LO in $1/N_c$.

The energy-dependent function $s_{KLL'}$ in Eq. (1.1) is called a *reduced amplitude* and contains all the dynamical information from the chiral soliton model. Note that these reduced amplitudes depend only on three variables, while the physical amplitudes depend on six: The same underlying soliton structure contributes to multiple physical states. It is precisely because there are fewer reduced amplitudes than physical amplitudes that one can obtain relations between the physical amplitudes. The physical interpretation of the label K is clear: It labels the grand spin of the given excitation.

Equation (1.1) has been used with considerable success to describe baryon spectroscopy. One approach uses a particular soliton model to evaluate explicitly the reduced matrix elements and then to predict fully the

physical scattering amplitudes. The detailed behavior of these amplitudes can then be used to predict values for baryon resonance observables [13,14]. Recently it was noted that Eq. (1.1) has model-independent applications in the study of baryon resonances [11]. There, it was noted that a resonance in a given channel corresponds to a pole in the S matrix, meaning that this pole must appear in one of the associated reduced amplitudes. Since the same reduced amplitudes occur in multiple scattering channels, degeneracies must exist in the excited baryon spectrum at leading order (N_c^0) in $1/N_c$.

While the prediction of degeneracies in the excited baryon spectrum at large N_c depends upon there being more S matrix elements than reduced amplitudes, the same fact implies the existence of linear relations among scattering amplitudes [9,10]. This result is made explicit by algebraically eliminating the reduced amplitudes, yielding linear relations among the physically measurable amplitudes. Such silver-plated relations were derived by Mattis and Peskin (MP) [9] for $\pi N \rightarrow \pi N$ and $\pi N \rightarrow \pi \Delta$ and are a focus of this paper. The $\pi N \rightarrow \pi N$ relations were first noted in the context of Skyrme models (but not large N_c *per se*) by Ref. [10]. We present them now for future reference, using the more compact notation (and noting that real initial target baryons are always nucleons, $R = 1/2$) $S_{LLRR'1_s J_s} \rightarrow S_{L,2I_s,2(J_s-L)}^{R'}$, or $S_{LL'RR'1_s J_s} \rightarrow S_{L,L',2I_s,2(J_s-L)}^{R'}$ if $L \neq L'$:

$$S_{L,3,-1}^{\pi N} = \frac{L-1}{4L+2} S_{L,1,-1}^{\pi N} + \frac{3(L+1)}{4L+2} S_{L,1,+1}^{\pi N} + O(N_c^{-1}), \quad (1.2)$$

$$S_{L,3,+1}^{\pi N} = \frac{3L}{4L+2} S_{L,1,-1}^{\pi N} + \frac{L+2}{4L+2} S_{L,1,+1}^{\pi N} + O(N_c^{-1}), \quad (1.3)$$

$$S_{L,3,-1}^{\pi \Delta} = \frac{4(L-1)}{\sqrt{10}(2L+1)} S_{L,1,-1}^{\pi \Delta} + \frac{3}{2L+1} \times \left[\frac{(L+1)(2L+3)(2L-1)}{10L} \right]^{1/2} S_{L,1,+1}^{\pi \Delta} + O(N_c^{-1}), \quad (1.4)$$

$$S_{L,3,+1}^{\pi \Delta} = \frac{3}{2L+1} \left[\frac{L(2L+3)(2L-1)}{10(L+1)} \right]^{1/2} S_{L,1,-1}^{\pi \Delta} + \frac{4(L+2)}{\sqrt{10}(2L+1)} S_{L,1,+1}^{\pi \Delta} + O(N_c^{-1}), \quad (1.5)$$

which are MP Eqs. (3.22a), (3.22b), (3.23a), and (3.23b), respectively,

$$\sqrt{L+1} S_{L,L+2,1,+1}^{\pi \Delta} = -\sqrt{L+2} S_{L+2,L,1,-1}^{\pi \Delta} + O(N_c^{-1}), \quad (1.6a)$$

$$\sqrt{L+1} S_{L,L+2,3,+1}^{\pi \Delta} = -\sqrt{L+2} S_{L+2,L,3,-1}^{\pi \Delta} + O(N_c^{-1}), \quad (1.6b)$$

$$\sqrt{10(L+1)} S_{L,L+2,3,+1}^{\pi \Delta} = +\sqrt{L+2} S_{L+2,L,1,-1}^{\pi \Delta} + O(N_c^{-1}), \quad (1.7a)$$

$$S_{L+2,L,1,-1}^{\pi \Delta} = -\sqrt{10} S_{L+2,L,3,-1}^{\pi \Delta} + O(N_c^{-1}), \quad (1.7b)$$

which are MP Eq. (3.24) (and only three of the four preceding relations are independent), and

$$S_{L,1,-1}^{\pi N} - S_{L,1,+1}^{\pi N} = \sqrt{\frac{2L-1}{L+1}} S_{L,1,-1}^{\pi \Delta} + \sqrt{\frac{2L+3}{L}} S_{L,1,+1}^{\pi \Delta} + O(N_c^{-1}), \quad (1.8)$$

which is MP Eq. (3.25).

Since these relations were derived from an underlying expression that is only justified at LO in $1/N_c$, the relations are *a priori* only known to be justified at LO in $1/N_c$. Our purpose in this work is to include the effects of NLO corrections in the $1/N_c$ expansion in order to obtain relations that hold at this higher order. As noted above, treatments based on the soliton model do not easily lend themselves to systematic higher-order corrections. Accordingly, we work with the formalism based on large N_c consistency rules. As a first step, in Sec. II, we rederive Eq. (1.1) [the LO expression from which Eqs. (1.2), (1.3), (1.4), (1.5), (1.6), (1.7), and (1.8) are obtained] directly from the model-independent formalism based on large N_c consistency conditions. Next, we use the large N_c framework to compute the NLO corrections. We present three new relations in Sec. III and compare them with the experimental data in Sec. IV.

II. DERIVATION

The relations among S matrix elements for different partial waves presented in Sec. I were derived in the context of a chiral soliton model in the large N_c limit. However, it is possible to show that these relations follow directly from large N_c QCD in a fully *model-independent* manner [11]. Of course, this model independence is not surprising since the relations in Eqs. (1.2), (1.3), (1.4), (1.5), (1.6), (1.7), and (1.8), although derived in the context of a soliton model, are completely insensitive to the dynamical details of the model. We explicitly demonstrate model independence using the methods of Ref. [8], which

is useful since the procedure illuminates the method by which we extend the linear relations to higher order in $1/N_c$.

The key point is the connection between the s -channel amplitudes of physical interest and the same reactions

$$\left\{ \begin{array}{ccc} K & I_s & J_s \\ R & L & 1 \end{array} \right\} \left\{ \begin{array}{ccc} K & I_s & J_s \\ R' & L' & 1 \end{array} \right\} = \sum_J (-1)^{I_s+J_s+L+L'+R+R'+K+J} [J] \left\{ \begin{array}{ccc} 1 & R' & I_s \\ R & 1 & J \end{array} \right\} \left\{ \begin{array}{ccc} L' & R' & J_s \\ R & L & J \end{array} \right\} \left\{ \begin{array}{ccc} 1 & L' & K \\ L & 1 & J \end{array} \right\}. \quad (2.1)$$

Inserting this into Eq. (1.1) yields

$$S_{LL'RR'I_s J_s} = \sum_J \left[\begin{array}{ccc} 1 & R' & I_s \\ R & 1 & J \end{array} \right] \left[\begin{array}{ccc} L' & R' & J_s \\ R & L & J \end{array} \right] s_{JLL'}^t + O(N_c^{-1}), \quad (2.2)$$

where

$$s_{JLL'}^t \equiv \frac{(-1)^{2J} [J]}{3([L][L'])^{1/2}} \sum_K [K] \left[\begin{array}{ccc} 1 & L' & K \\ L & 1 & J \end{array} \right] s_{KLL'}, \quad (2.3)$$

and for simplicity of presentation we have replaced the standard $6j$ symbol with a new symbol denoted by square brackets that folds in useful phase factors and overall constants, but retains all the usual triangle rules:

$$\left\{ \begin{array}{ccc} a & b & e \\ c & d & f \end{array} \right\} \equiv \frac{(-1)^{-(b+d+e+f)}}{([a][b][c][d])^{1/4}} \left[\begin{array}{ccc} a & b & e \\ c & d & f \end{array} \right]. \quad (2.4)$$

Use of the modified $6j$ symbols, henceforth called $[6j]$ symbols, leaves Eqs. (2.2) and (2.3) much more compact than the corresponding expressions using ordinary $6j$ symbols.

The energy-dependent function $s_{JLL'}^t$ is the t -channel reduced amplitude; it depends only on the pion orbital momentum and the SU(2) index J . Applying triangle rules to each of the $[6j]$ symbols in Eq. (2.2) reveals the physical significance of J . The first $[6j]$ symbol implies that J is the total isospin (I_t) exchanged between the meson and baryon in the t channel, while the second implies that J is the total angular momentum (J_t) exchanged in the t channel. Together, they demand the equality of isospin and angular momentum in the t -channel exchange, in accordance with the $I_t = J_t$ rule of Mattis and Mukerjee [12,17].

While this rule was originally derived in the Skyrme model, it was shown to be a result of large N_c QCD by Kaplan and Manohar [15] through the model-independent spin-flavor approach based on large N_c consistency conditions (which in turn follows from the pioneering work of Refs. [6–8]). They demonstrate that the matrix element of a general n -quark operator $\hat{O}_{I_0, J_0}^{(n)}$ with baryon number equal to zero, isospin I_0 , and spin J_0

expressed in terms of t -channel amplitudes. As discussed below, large N_c QCD severely limits the form of these t -channel amplitudes [15]. Thus, we rewrite Eq. (1.1) in terms of t -channel amplitudes rather than s -channel exchanges using the following $6j$ symbol identity [16]:

scales as

$$\langle B' | \hat{O}_{I_0, J_0}^{(n)} / N_c^n | B \rangle \lesssim 1/N_c^{|I_0 - J_0|}. \quad (2.5)$$

The significance of this result becomes manifest when one realizes that the “quarks” in this derivation need not be associated with dynamical quarks in any particular quark model. Rather, they merely reflect fields transforming according to the fundamental representation of the contracted SU($2N_f$) symmetry [8]. Thus, the rule applies to all baryon matrix elements. The operator that connects the pions and baryons in a t -channel exchange, from the point of view of the baryon, is simply a single current insertion that couples to external pions; its matrix element between baryon states then qualifies as the type described above. One sees from Eq. (2.5) that the largest contribution to the scattering comes from matrix elements with $I_t = J_t$; thus, the famed $I_t = J_t$ rule is a direct result of large N_c QCD without model input. Since Eq. (2.2) is the most general form for a scattering amplitude consistent with the $I_t = J_t$ rule, we have established that Eq. (1.1) is a model-independent, large N_c QCD result. This general argument was originally presented in Ref. [11].

The rederivation of Eq. (1.1) by nonsolitonic means is of only modest interest. However, the crucial point is that the general large N_c derivation can be extended to higher order in $1/N_c$. The method by which one extends the earlier LO results to NLO is clear: Since the LO t -channel constraint on the amplitudes, $|I_t - J_t| = 0$, implies Eq. (1.1), the first linearly independent $1/N_c$ correction arises from t -channel amplitudes with $|I_t - J_t| = 1$, since Eq. (2.5) implies that such amplitudes are the ones suppressed by a single factor $1/N_c$. All t -channel amplitudes with $|I_t - J_t| > 1$ can be excluded at NLO since the suppression is $1/N_c^2$ or more. As we now show, an expansion to this order remains predictive since only two t -channel amplitudes with $|I_t - J_t| = 1$ appear.

Writing the pion-baryon partial-wave amplitude in terms of reduced t -channel amplitudes and including the first subleading contributions from amplitudes with $|I_t - J_t| = 1$ generalizes Eq. (2.2):

$$\begin{aligned}
 S_{LL'RR'I_s J_s} = & \sum_J \begin{bmatrix} 1 & R' & I_s \\ R & 1 & J \end{bmatrix} \begin{bmatrix} L' & R' & J_s \\ R & L & J \end{bmatrix} s_{JLL'}^t - \frac{1}{N_c} \sum_x \begin{bmatrix} 1 & R' & I_s \\ R & 1 & x \end{bmatrix} \begin{bmatrix} L' & R' & J_s \\ R & L & x+1 \end{bmatrix} s_{xLL'}^{t(+)} \\
 & - \frac{1}{N_c} \sum_y \begin{bmatrix} 1 & R' & I_s \\ R & 1 & y \end{bmatrix} \begin{bmatrix} L' & R' & J_s \\ R & L & y-1 \end{bmatrix} s_{yLL'}^{t(-)} + O(N_c^{-2}),
 \end{aligned} \tag{2.6}$$

where the $s_{xLL'}^{t(\pm)}$ functions are the reduced t -channel amplitudes corresponding to $s_{JLL'}^t$ for the two possible ways of combining I_t and J_t such that $|I_t - J_t| = 1$. In Sec. III this formula is used to derive linear relations among partial-wave amplitudes for $\pi N \rightarrow \pi N$ and $\pi N \rightarrow \pi \Delta$ at NLO. As noted previously, any relations that depend explicitly on the higher-order amplitudes $s_{xLL'}^{t(\pm)}$ have essentially the same predictive power as the LO relations. However, if gold-plated relations can be found in which the effects of the $s_{xLL'}^{t(\pm)}$ cancel, then we have predictions that hold at NLO and thus are expected to describe nature far better than the generic LO relations of Eqs. (1.2), (1.3), (1.4), (1.5), (1.6), (1.7), and (1.8).

III. LINEAR RELATIONS

Before deriving gold-plated NLO linear relations, it is helpful to discuss restrictions on the reduced amplitudes and the pion angular momentum due to symmetry. Time-reversal invariance of the scattering process dictates that the S matrix is symmetric under the exchange of initial and final states (characterized by LR and $L'R'$, respectively). We see that the symmetry properties of the [6j]

symbols (inherited from the usual 6j symbols) imply that they are invariant under this exchange. Thus, all types of reduced amplitudes must also be symmetric (e.g., $s_{JLL'}^t = s_{J'L'L'}^t$) in order to maintain the symmetries of QCD. The [6j] symbols also encode important restrictions on $\Delta L \equiv |L' - L|$. For $\pi N \rightarrow \pi N$, the allowed change is $\Delta L = 0, 1$; while for $\pi N \rightarrow \pi \Delta$, the allowed change is $\Delta L = 0, 1, 2$. The $\Delta L = 1$ possibility is eliminated by parity conservation since $P = (-1)^{L+1} = (-1)^{L'+1}$. To summarize, the permitted cases are $\Delta L = 0$ for $\pi N \rightarrow \pi N$ and $\Delta L = 0, 2$ for $\pi N \rightarrow \pi \Delta$.

Let us first consider the reactions $\pi N \rightarrow \pi N$ and $\pi N \rightarrow \pi \Delta$ when the pion orbital angular momentum is unchanged: $L = L'$. There are eight physical amplitudes corresponding to the different ways to add the spin and isospin of the pion and the nucleon in the two reactions: $I_s = \frac{1}{2}, \frac{3}{2}$ and $J_s = L \pm \frac{1}{2}$. We can expand these in terms of seven reduced amplitudes: three LO and four first order in $1/N_c$. Therefore, there is only one relation among the physical amplitudes with all references to the NLO reduced amplitudes eliminated. This gold-plated relation is

$$\begin{aligned}
 S_{L,1,-1}^{\pi N} - S_{L,1,+1}^{\pi N} = & \sqrt{\frac{2L-1}{L+1}} S_{L,1,-1}^{\pi \Delta} + \sqrt{\frac{2L+3}{L}} S_{L,1,+1}^{\pi \Delta} + \left[\frac{2}{3} (S_{L,3,-1}^{\pi N} - S_{L,3,+1}^{\pi N}) + \frac{1}{3} (S_{L,1,-1}^{\pi N} - S_{L,1,+1}^{\pi N}) \right. \\
 & \left. + \sqrt{\frac{5}{18}} \left(\sqrt{\frac{2L-1}{L+1}} S_{L,3,-1}^{\pi \Delta} + \sqrt{\frac{2L+3}{L}} S_{L,3,+1}^{\pi \Delta} \right) - \frac{5}{6} \left(\sqrt{\frac{2L-1}{L+1}} S_{L,1,-1}^{\pi \Delta} + \sqrt{\frac{2L+3}{L}} S_{L,1,+1}^{\pi \Delta} \right) \right] + O(N_c^{-2}).
 \end{aligned} \tag{3.1}$$

The first four terms resemble one of the original MP relations, Eq. (1.8), but there is a correction term in the square brackets. Note that this correction term itself vanishes as $N_c \rightarrow \infty$ after substituting in Eqs. (1.2), (1.3), (1.4), and (1.5). However, the $1/N_c$ corrections to the terms in the square bracket from Eqs. (1.2), (1.3), (1.4), and (1.5) precisely cancel the corrections to Eq. (1.8), yielding a result that holds to $O(1/N_c^2)$. Equation (1.8) empirically works rather well, and we defer a discussion of the possible effects of the correction term to Sec. IV.

Now we consider the reactions for which the pion orbital angular momentum is changed by two units, $L = L' \pm 2$; the symmetry arguments given above restrict this case to the $\pi N \rightarrow \pi \Delta$ reaction. There are four physical amplitudes for this case. They can be expressed in terms

of two reduced amplitudes: one leading order and one first order in $1/N_c$. This implies the existence of two gold-plated linear relations:

$$\sqrt{L+1} S_{L,L+2,1,+1}^{\pi \Delta} = -\sqrt{L+2} S_{L+2,L,1,-1}^{\pi \Delta} + O(N_c^{-2}), \tag{3.2}$$

$$\sqrt{L+1} S_{L,L+2,3,+1}^{\pi \Delta} = -\sqrt{L+2} S_{L+2,L,3,-1}^{\pi \Delta} + O(N_c^{-2}). \tag{3.3}$$

These resemble two of the MP relations [cf. Eqs. (1.6a) and (1.6b)]. However, we have now shown that they hold at NLO, and thus are gold- rather than silver-plated. Thus, to the extent that the $1/N_c$ expansion applies to these observables, one expects that these relations hold far better than the generic silver-plated LO predictions. As

discussed in the following section, we show that this is, in fact, true.

IV. EXPERIMENTAL TESTS

In principle, all three linear relations derived in Sec. III (for each allowed value of L) can be tested by comparison with available experimental data. The numbers used result from partial-wave analysis applied to raw data from experiments in which pions are scattered off nucleon targets. An important feature complicates our task: All of the gold-plated relations involve the reaction $\pi N \rightarrow \pi\Delta$. While the extraction of partial-wave amplitudes for the $\pi N \rightarrow \pi N$ reaction from the large amount of reliable data is essentially straightforward, the extraction of partial-wave amplitudes for $\pi N \rightarrow \pi\Delta$ is complicated by the fact that the Δ decays strongly to πN . The $\pi N \rightarrow \pi\Delta$ partial waves must be extracted in the context of a model that distinguishes events in the observed reaction $\pi N \rightarrow \pi\pi N$ that pass through an intermediate Δ resonance and which do not. Therefore, the $\pi N \rightarrow \pi\Delta$ partial-wave amplitude data necessarily contains some model dependence, making it somewhat less reliable. Because of this uncertainty, much less attention has been paid to these reactions, and the set of analyzed data is far more sparse. Fortunately, the Δ is an extremely prominent resonance (understandable in the context of large N_c), and hence the model dependence should be rather modest.

For the comparison presented below, we use results from the analysis of Manley, Arndt, Goradia, and Teplitz [18], which is readily available through the SAID program at George Washington University [19]. The analysis is presented in terms of the T matrix [$\mathbf{T} \equiv (\mathbf{S} - \mathbf{1})/2i$] rather than the S matrix. This causes no complications, since any extra factors and terms cancel in our formulas. The results of Ref. [18] are presented in terms of the center-of-mass energy W of the πN system.

We first consider Eq. (3.1) and restrict attention to $1 \leq L \leq 3$. The lower bound is an elementary consequence of angular momentum conservation, while the upper bound reflects limitations of the available data. Even with this restriction we see that, for each L , Eq. (3.1) requires partial-wave amplitudes that are, unfortunately, not available in the data set. For example, the amplitudes PP_{31} , PP_{13} , DD_{33} , and FF_{17} (the notation is $LL'_{l_1 l_2 l}$) are not given. MP in their LO comparisons were able to circumvent this problem by rewriting the unknown amplitudes in terms of known ones using Eqs. (1.4) and (1.5). We have no such luxury; inserting Eqs. (1.4) and (1.5) into our gold-plated relations simply converts them to silver-plated relations. We make no assumptions about these unknown amplitudes and thus cannot test the validity of Eq. (3.1) at the present time.

We now consider Eqs. (3.2) and (3.3). Fortunately, there is sufficient analyzed data to study these relations, provided one restricts attention to the $L = 0$ case. It is

instructive to contrast the quality of the agreement of these gold-plated NLO relations with the $L = 0$ silver-plated LO Eqs. (1.7a) and (1.7b), since both sets involve only the $\pi N \rightarrow \pi\Delta$ amplitudes. We view the loss of predictive power due to the need to identify the Δ in the final state as a comparable systematic uncertainty for the two classes of relations. Our predictions are as follows:

$$SD_{11} = -\sqrt{2}DS_{13} + O(1/N_c^2), \quad (4.1)$$

$$SD_{31} = -\sqrt{2}DS_{33} + O(1/N_c^2), \quad (4.2)$$

$$SD_{11} = +\sqrt{20}DS_{33} + O(1/N_c), \quad (4.3)$$

$$SD_{31} = +\frac{1}{\sqrt{5}}DS_{13} + O(1/N_c), \quad (4.4)$$

where the first two relations are the gold-plated NLO relations (Fig. 1) and the second two are the silver-plated LO relations (Fig. 2).

It is immediately apparent that the gold-plated relations agree with experiment considerably better than their silver-plated analogs. For the gold-plated relations the gross structure of the amplitudes is clearly discerned on both the left- and right-hand sides of the relation. In contrast, the silver-plated relations are much less robust in describing the data.

V. CONCLUSION

We have demonstrated the utility and power of the large N_c expansion for describing pion-nucleon scattering. It has made a number of nontrivial predictions that can be tested with experimental data. The expansion in powers of $1/N_c$ allows one to compare predictions holding at different orders, and the quality of the agreement for the $O(1/N_c^2)$ relations is markedly better than the $O(1/N_c)$ relations. It is unfortunate that sufficient ana-

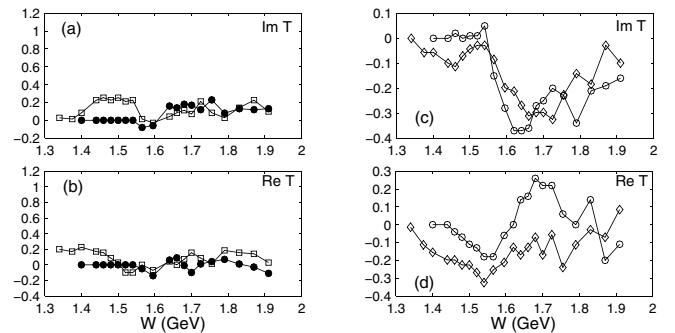


FIG. 1. Experimentally determined $\pi N \rightarrow \pi\Delta$ amplitudes SD_{11} and SD_{31} compared to the predictions of Eqs. (4.1) and (4.2). In plots (a) and (b), the closed circle (\bullet) is SD_{11} and the box (\square) is $-\sqrt{2}DS_{13}$. In plots (c) and (d), the open circle (\circ) is SD_{31} and the diamond (\diamond) is $-\sqrt{2}DS_{33}$. The data is provided by SAID [19].

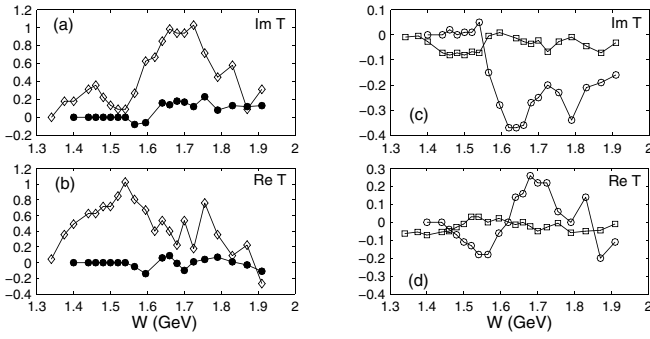


FIG. 2. Experimentally determined $\pi N \rightarrow \pi\Delta$ amplitudes SD_{11} and SD_{31} compared to the predictions of Eqs. (4.3) and (4.4). In plots (a) and (b), the closed circle (\bullet) is SD_{11} and the diamond (\diamond) is $+\sqrt{20}DS_{33}$. In plots (c) and (d), the open circle (\circ) is SD_{31} and the box (\square) is $+1/\sqrt{5}DS_{13}$. The data is provided by SAID [19].

lyzed data do not exist for our gold-plated relation Eq. (3.1). In principle, the relevant $\pi N \rightarrow \pi\Delta$ partial waves might be extracted from the raw data. However, this requires a formidable (and model-dependent) analysis. Previously there was, perhaps, little motivation to carry out this analysis, but in light of these large N_c predictions the incentive is now more compelling.

It is exciting to see that some rather complicated features of QCD, such as the $\pi N \rightarrow \pi\Delta$ reaction, can be understood semiquantitatively in terms of rather simple microscopic considerations based on large N_c .

In principle, our method can be applied again to derive the $1/N_c^2$ terms in the S matrix expansion. However, we note that such a procedure is of minimal utility for describing pion-nucleon scattering in the physical $N_c = 3$ world. The resulting triangle rules appearing in the $1/N_c^2$ corrections, applied to terms with a nucleon ($R = \frac{1}{2}$), cannot be satisfied for any baryon in the $R' = I = J$ multiplet of the large N_c world; this forces the $6j$ symbols to vanish, thus terminating the expansion. Therefore, it appears that we have exhausted the number of experimentally accessible gold-plated relations in pion-nucleon scattering, and we see that there are no “super”-gold-plated relations that hold at next-to-next-to-leading order.

This approach can clearly be extended to other processes. For example, one may relate partial waves in Compton scattering, electron scattering, and pion-electron production, or photoproduction. We defer such considerations to later work.

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