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Three charge supertubes and black hole hair

Iosif Bena and Per Kraus
Department of Physics and Astronomy, UCLA, Los Angeles, California 90095-1547, USA
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We construct finite size, supersymmetric, tubular D-brane configurations with three charges, two angular momenta and several brane dipole moments. In type IIA string theory these are tubular configurations with D0, D4 and F1 charge, as well as D2, D6 and NS5 dipole moments. These multicharge generalizations of supertubes might have interesting consequences for the physics of the D1-D5-P black hole. We study the relation of the tubes to the spinning Breckenridge-Myers-Peet-Vafa black hole, and find that they have properties consistent with describing some of the hair of this black hole.

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I. INTRODUCTION

One of the more novel brane configurations considered in recent years is the so-called supertube [1]: this is a tubular D2 brane with world-volume electric and magnetic fields turned on such that it carries nonzero values of D0-brane charge, fundamental string charge, and angular momentum (see Refs. [2, 3] for a sampling of further work). The resulting configuration is supersymmetric, and remains so even when the cross section of the tube describes an arbitrary curve, and when several tubes are considered simultaneously. The crucial ingredient in the construction of the supertubes is the presence of a critical electric field, $2\pi\alpha' E=1$.[1] This leads to the disappearance of the D2 brane from the equations determining the preserved supersymmetry of the system as well as its tension; indeed the tension just becomes that of the D0 and F1 constituents.

The original supertube carries two independent conserved charges (D0 and F1), but it is only natural to consider the generalization to three charges.[2] This is one of the motivations for the present work. Up to U duality we can take the three charges to be those of D0 branes, D4 branes, and F1 strings, and this is the description we will find most convenient. The finite size of the resulting configuration leads to dipole moments for other branes. If we consider the three possible pairings of charges, and dualize the statement that D0 and F1 charges lead to a D2 dipole moment, we are lead to expect that our configuration will carry nonzero D2, D6, and NS5 dipole moments.

We will present two independent constructions of the three charge tubes. In the first we start with a tubular D6 brane, as described by the Born-Infeld action, and turn on fluxes so as to induce the correct lower brane charges. This is a straightforward generalization of the original supertube construction in Ref. [1]. The second construction is based on the non-Abelian theory of the D4 branes, and involves exciting the transverse scalars appropriately. This generalizes the construction in Ref. [5]. In both cases our considerations will be entirely classical, which implies that we cannot see the expected NS5 dipole moments. We furthermore expect that upon passing to the quantum theory our configurations will correspond to marginally bound states.

Besides their intrinsic interest, supertubes are beginning to play an important role in black hole physics, based on the work of Mathur, Lunin, and collaborators [6–11]; for a recent review see Ref. [12]. After a chain of dualities, the various configurations of the two charge supertubes are in one-to-one correspondence with the supersymmetric ground states of the D1-D5 system (with vanishing momentum). Furthermore, the corresponding supergravity solutions have been derived and turn out to be free of singularities, thus yielding a direct map between classical geometries and brane microstates. In this sense, the supertubes can be thought of as the hair of the D1-D5 system.

The story becomes even more interesting when we add the third charge, which corresponds to momentum in the D1-D5 description, since the system acquires a macroscopically large entropy for large charges. It has been conjectured that the supersymmetric states of the three charge system will continue to be in one-to-one correspondence with classical geometries, although this has so far only been checked for a single unit of momentum [11]. In analogy with the above discussion, we would then like to associate these states with the three charge supertubes which we study in this paper.

Since angular momentum plays an important role in the supertube construction, we will compare the properties of our supertubes with the properties of the rotating D1-D5-P black hole—the Breckenridge-Myers-Peet-Vafa (BMPV) black hole. According the story developed in Refs. [9–12] the BMPV black hole should represent, roughly speaking, the statistical average of the microstates of the D1-D5-P system with fixed angular momentum. By comparing the size and angular momentum bounds of our tubes with those of the BMPV black hole we will see that a consistent picture emerges. We will also use the tubes as a probe of the BMPV geometry in order to give support to the idea that the black hole can be thought of as being made up of tubes. A nice consistency check is to see how one is prevented from overspinning the black hole (which would result in closed timelike curves) by dropping high angular momentum tubes into the horizon. This provides a rather remarkable example of

[1]To be precise, this is the critical value in the absence of a magnetic field.

[2]Some three charge configurations have also been considered in Ref. [4].

chronology protection at work.

We should remark that these supertubes are unlike other configurations used in studying black hole entropy. Usually one computes the microscopic entropy at weak coupling, where the system is of string scale in extent, and its Schwarzschild radius even smaller. As the gravitational coupling is increased, the Schwarzschild radius grows, becoming comparable to the size of the brane configuration at the "correspondence point" [13], and larger thereafter. There are thus two descriptions of the system: as a microscopic string theory object for small g_s, and as a black hole for large g_s. One then compares the entropy in the two regimes and finds an agreement, which is precise if supersymmetry forbids corrections during the extrapolation. The supertubes are different. The size of a tube is determined by a balance between the angular momentum of the system and the tension of the tubular brane. As the string coupling is increased, the D-brane tension decreases, and thus the size of the tube grows, much like one would expect if these configurations directly represent the black hole microstates even at large g_s.

The remainder of this paper is organized as follows. In Sec. II we present the construction of the tubes from the D6-brane point of view; this is followed by the construction in terms of D4 branes in Sec. III. Connections with black hole physics are studied in Sec. IV. In Sec. V we add some concluding thoughts. For convenience, we have included an appendix on the BMPV black hole. Throughout this paper we will using the word "supertube" to denote any of the U duals of the 2 or 3 charge configurations we construct, even if in the D1-D5-P case these configurations do not look tubular (they are rotating helical branes).

II. CONSTRUCTION OF THE TUBES—THE D6 BRANE PICTURE

We begin with a single tubular D6 brane, and attempt to turn on world-volume fluxes such that we describe a Bogolmonyi-Prasad-Sommerfield (BPS) configuration carrying D4, D0 and F1 charges. Using a single D6 brane also leads to the presence of D2-brane charges, but we will subsequently introduce a second D6 brane to cancel this.

The D6 brane is described by the Born-Infeld action

$$S=-T_6\int d^7\xi\sqrt{-\det(g_{ab}+\mathcal{F}_{ab})}, \tag{2.1}$$

where g_{ab} is the induced world-volume metric, $\mathcal{F}_{ab}=2\pi F_{ab}$, and we have set $\alpha'=1$. The induced D4-brane and D0-brane charge densities are given by

$$Q_4=2\pi T_6\mathcal{F}$$

$$Q_0=2\pi T_6\mathcal{F}\wedge\mathcal{F}\wedge\mathcal{F}. \tag{2.2}$$

The F1 charge density is proportional to the canonical momentum conjugate to the vector potential:

$$Q_1=\vec{\pi}=\frac{\partial\mathcal{L}}{\partial\dot{\vec{A}}}. \tag{2.3}$$

Factors of 2π in Eqs. (2.2) and (2.3) are deserving of comment. One direction of our D6 brane will be an S^1, and we have defined the charges after integrating over the corresponding angular coordinate. So the D-brane charges are really charge densities per unit five dimensional area, and the fundamental string charge is a charge density per unit four dimensional area. Note also that the charges Q are the ones which appear in the Hamiltonian, and are related to the number of strings or branes by the corresponding tensions. These conventions will be convenient later on.

Our construction will essentially follow that of the original D2-brane supertube, except that we include four extra spatial dimensions and corresponding fluxes. We take our D6 brane to have geometry $R^{1,1}\times S^1\times T^4$. We take $R^{1,1}$. We take $R^{1,1}$ to span $x^{0,1}$; S^1 to be a circle in the $x^{2,3}$ plane of radius r and angular coordinate θ; and T^4 to span $x^{6,7,8,9}$. The configuration carries no net D6-brane charge due to its tubular shape. We should note that we are considering a circular D6 brane just for simplicity, but a general curve in R^4 can be considered as in Ref. [3], or as in the probe analysis we perform in the last section of this paper.

To induce D0 branes we turn on constant values of $\mathcal{F}_{1\theta}$, $\mathcal{F}_{67}$, and $\mathcal{F}_{89}$. $\mathcal{F}_{1\theta}$ then induces a density of D4 branes in the $x^{6,7,8,9}$ plane. To induce F1 charge in the x^1 direction we turn on a constant value of $\mathcal{F}_{01}$. As mentioned above, this single D6-brane configuration also carries D2-brane charges in the $x^{6,7}$ and $x^{8,9}$ directions, but these will eventually be cancelled by introducing a second D6 brane.

With these fluxes turned on we find

$$S=-T_6\int d^7\xi\sqrt{(1-\mathcal{F}_{01}^2)r^2+\mathcal{F}_{1\theta}^2}\sqrt{(1+,\mathcal{F}_{67}^2)(1+\mathcal{F}_{89}^2)}. \tag{2.4}$$

By differentiating with respect to $\mathcal{F}_{01}$ we find

$$Q_1=2\pi T_6\frac{\mathcal{F}_{01}r^2}{\sqrt{(1-\mathcal{F}_{01}^2)r^2+\mathcal{F}_{1\theta}^2}}\sqrt{(1+\mathcal{F}_{67}^2)(1+\mathcal{F}_{89}^2)}. \tag{2.5}$$

The key point to observe now is that if we choose

$$\mathcal{F}_{01}=1 \tag{2.6}$$

then r^2 drops out of the action (2.4). Let us further choose

$$\mathcal{F}_{67}=\mathcal{F}_{89}. \tag{2.7}$$

We can then work out the energy from the canonical Hamiltonian as

$$\begin{aligned}H &= \int Q_1 \mathcal{F}_{01} - L \\ &= \int [Q_1 + 2\pi T_6 |\mathcal{F}_{1\theta}| + 2\pi T_6 |\mathcal{F}_{1\theta}\mathcal{F}_{67}\mathcal{F}_{89}|] \\ &= \int [Q_1 + Q_4 + Q_0]. \end{aligned} \tag{2.8}$$

The final two integrals are over the five noncompact directions of the D6 brane. The radius of the system is determined by inverting Eq. (2.5):

$$r^2 = \frac{Q_1}{2\pi T_6} \frac{\mathcal{F}_{1\theta}}{1+\mathcal{F}_{67}\mathcal{F}_{89}} = \frac{1}{(2\pi T_6)^2} \frac{Q_1 Q_4^2}{Q_0+Q_4}. \tag{2.9}$$

From Eq. (2.8) we see that we have saturated the BPS bound, and so our configuration must solve the equations of motion, as can be verified directly. Supersymmetry can also be verified precisely as for the original D2-brane supertube. The presence of the electric field, $\mathcal{F}_{01}=1$, causes the D6 brane to drop out of the equations determining the tension and the unbroken supersymmetry. If we set $Q_0=0$ then Eq. (2.9) reduces (with the obvious relabelings) to the radius formula found for the original D2-brane supertube [1].

As we have noted, the above configuration also carries nonvanishing D2-brane charge associated with $\mathcal{F}_{1\theta}\mathcal{F}_{67}$ and $\mathcal{F}_{1\theta}\mathcal{F}_{89}$. To remedy this we can introduce a second D6 brane with flipped signs of $\mathcal{F}_{67}$ and $\mathcal{F}_{89}$ [14]. This simply doubles the D4, D0, and F1 charges, while cancelling the D2 charge. The S^1 of the second tube need not lie in the same $x^{2,3}$ plane as the first, and we instead generalize by taking it to lie in the $x^{4,5}$ plane. Even more generally, nothing requires the second S^1 to have the same radius as the first (the only constraint is the cancellation of the D2-brane charges), and so we will take it to have radius $\tilde{r}$.

More generally, let us introduce k D6 branes, with fluxes described by diagonal $k\times k$ matrices. $\mathcal{F}_{01}$ is equal to the unit matrix. We again set $\mathcal{F}_{67}=\mathcal{F}_{89}$, and take $F_{1\theta}$ to have nonnegative diagonal entries to preclude the appearance of $\overline{D4}$ branes. The condition of vanishing D2-brane charge is given by

$$\mathrm{Tr}\, \mathcal{F}_{1\theta}\mathcal{F}_{67}=0. \tag{2.10}$$

Finally, the F1 charge is described by taking Q_1 to be an arbitrary diagonal matrix with nonnegative entries.[3] This results in a BPS configuration of k D6 branes. In general, each D6 brane has a different radius; the radius formula is now given by Eq. (2.9) but with the entries replaced by their corresponding matrices. Since our matrices are all diagonal, the Born-Infeld action is unchanged except for the inclusion of an overall trace. Similarly, the energy is given by $H = \int \mathrm{Tr}[Q_1+Q_4+Q_0]$.

In analogy with the behavior of other branes, if we take the k D6 branes to sit on top of each other we expect that they can form a marginally bound state. In the classical description we should then demand that the radius matrix (2.9) be proportional to the unit matrix. Given a choice of magnetic fluxes, this determines the F1 charge matrix Q_1 up to an overall multiplicative constant which parametrizes the radius of the combined system.

As a special case, consider taking all k D6 branes to be identical modulo the sign of $\mathcal{F}_{67}$ and $\mathcal{F}_{89}$, so that both $\mathcal{F}_{1\theta}$ and $\mathcal{F}_{67}\mathcal{F}_{89}$ are proportional to the unit matrix.[4] Then in terms of the total charges, the radius formula is

$$r^2 = \frac{1}{k^2(2\pi T_6)^2} \frac{Q_1^{\mathrm{tot}}(Q_4^{\mathrm{tot}})^2}{Q_0^{\mathrm{tot}}+Q_4^{\mathrm{tot}}}. \tag{2.11}$$

We observe that after fixing the conserved charges and imposing equal radii for the component tubes, there is still freedom in the values of the fluxes. These can be partially parametrized in terms of various nonconserved ''charges,'' such as brane dipole moments. Due to the tubular configuration, our solution carries nonzero D6, D4, and D2 dipole moments, proportional to

$$\begin{aligned} Q_6^D &= T_6 r k \\ Q_4^D &= T_6 r \mathrm{Tr}\mathcal{F}_{67} \\ Q_2^D &= T_6 r \mathrm{Tr}\mathcal{F}_{67}\mathcal{F}_{89} \equiv T_6 r k_2. \end{aligned} \tag{2.12}$$

When the k D6 branes which form the tube are coincident, k_2 measures the local D2 brane charge of the tube. It is also possible to see that both for a single tube, and for k tubes identical up to the sign of $\mathcal{F}_{67}$ and $\mathcal{F}_{89}$, the dipole moments are related:

$$\frac{Q_2^D}{Q_6^D} = \frac{k_2}{k} = \frac{Q_0}{Q_4}. \tag{2.13}$$

Furthermore, if $\mathcal{F}_{67}$ and $\mathcal{F}_{89}$ are traceless, this tube has no D2 charge and no D4 dipole moment. More general tubes are not described by Eq. (2.13), and need not have zero D4 dipole moment when the D2 charge vanishes. We should also remark that the D2 dipole moment is an essential ingredient in constructing a supersymmetric three charge tube of finite size. When this dipole moment goes to zero, the radius of the tube also becomes zero.

Our tube also carries angular momentum in the $x^{2,3}$ and $x^{4,5}$ planes in which the S^1 factors lie. The angular momentum densities of a configuration with k identical D6 branes in the $x^{2,3}$ plane and k' identical D6 branes in the $x^{4,5}$ plane are

$$J_{23} = 2\pi r T_{0\theta} = 2\pi T_6 k \sqrt{(1+\mathcal{F}_{67}^2)(1+\mathcal{F}_{89}^2) r^2},$$

[3] Quantum mechanically, we should demand that $\mathrm{Tr}\, Q_1$ be an integer to ensure that the total number of F1 strings is integral.

[4] One could furthermore choose $\mathrm{Tr}\,\mathcal{F}_{67}=\mathrm{Tr}\,\mathcal{F}_{67}=0$ to cancel the D2 charge, but this does not affect the radius formula.

$$J_{45}=2\pi\tilde{r}T_{0\tilde{\theta}}=2\pi T_6 k'\sqrt{(1+\mathcal{F}_{67}^2)(1+\mathcal{F}_{89}^2)}\tilde{r}^2. \tag{2.14}$$

Thus, when one adds D0 brane charge to a F1-D4 supertube, the maximum angular momentum does not change, even if the radius becomes smaller. For completeness, we should also mention that the shape of the most generic three charge tube is an arbitrary curve inside R^4. The angular momenta can be obtained rather straightforwardly from this shape.

A. T duality to the D1-D5-P system

A T duality along x^1 transforms our D0-D4-F1 tubes into the more familiar D1-D5-P configurations. This T duality is implemented by the replacement $2\pi A^1\to X^1$. The nonzero value of $\mathcal{F}_{1\theta}$ before the T duality translates into a nonzero $\partial_\theta X^1$ after. This means that the resulting D5 brane is in the shape of a helix whose axis is parallel to x^1. This is the same as the observation that the D2-brane supertube T dualizes into a helical D1 brane. Since this helical shape is slightly less convenient to work with than a tube, we have chosen to emphasize the D0-D4-F1 description instead.

B. More general tubes

Having constructed three charge supertubes with D6 and D2 dipole moments,[5] it is interesting to explore whether one can say anything about configurations with more dipole moments. Before proceeding, it is an instructive exercise to understand the physics behind the radius formula (2.11) for the supertube with two dipole moments.

Let us consider two simple (two charge) tubes, one of which is made from Q_0 D0 branes, Q_1' fundamental strings, and k_2 D2 branes, and the other from Q_4 D4 branes, Q_1'' fundamental strings, and k_6 D6 branes. If the radii of the two tubes are the same, then

$$(2\pi T_6)^2r^2=\frac{Q_1''Q_4}{k_6^2}=\frac{Q_1'Q_0}{k_2^2}. \tag{2.15}$$

Let us furthermore require that $Q_0/Q_4=k_2/k_6$. Then, a short algebraic manipulation brings us to

$$(2\pi T_6)^2r^2=\frac{(Q_1''+Q_1')Q_4}{k_6^2\left(1+\frac{Q_0}{Q_4}\right)}=\frac{(Q_1''+Q_1')Q_0}{k_2^2\left(1+\frac{Q_4}{Q_0}\right)}=\frac{Q_1^{\rm tot}Q_0Q_4}{k_2k_6(Q_0+Q_4)}. \tag{2.16}$$

This formula reproduces Eq. (2.11), and is moreover duality invariant. Thus, the three charge supertube with the property $Q_0/Q_4=k_2/k_6$ (2.13) has the same radius and charges as the superposition of a D0-F1 and D4-F1 supertube. Note that the individual F1 charges of the component tubes (Q_1' and Q_1'') need not be quantized. Only their sum is.

The tubes with three charges and D2 and D6 dipole moments we constructed can be mapped by a chain of dualities to tubes with D2 and NS5 dipole moments, or to tubes with D6 and NS5 dipole moments. These tubes can again be thought of as a bound state of two simple two charge tubes. It is quite natural therefore to expect that the three charge tube with three dipole moments can be obtained by putting together three simple tubes. The resulting configuration is still 1/8 BPS because each supertube has the supersymmetries of its components.

Let us take a D2 tube with charges Q_1', Q_0' and D2 dipole moment k_2; a D6 tube with charges Q_1'', Q_4' and D6 dipole moment k_6; and an NS5 tube with charges Q_0'', Q_4'' and NS5 dipole moment k_5. The condition that the radii be equal is

$$(2\pi T_6)^2r^2=\frac{Q_1''Q_4'}{k_6^2}=\frac{Q_1'Q_0'}{k_2^2}=\frac{Q_4''Q_0''}{k_5^2}. \tag{2.17}$$

The total charges are

$$Q_1^{\rm tot}=Q_1'+Q_1'',\quad Q_0^{\rm tot}=Q_0'+Q_0'',\quad Q_4^{\rm tot}=Q_4'+Q_4'', \tag{2.18}$$

and the angular momentum of the system is

$$2\pi T_6J=\frac{Q_1''Q_4'}{k_6}+\frac{Q_1'Q_0'}{k_2}+\frac{Q_4''Q_0''}{k_5}. \tag{2.19}$$

Thus, given the total charges, dipole moments, and angular momentum, one has six equations with six unknowns (2.17), (2.18), (2.19) which determine the radius of this multicharge tube.

III. CONSTRUCTION OF THE TUBES—THE D4 BRANE PICTURE

When the radius of the system becomes comparable to the string scale, rather than describing our configurations by the Born-Infeld action for the D6 branes, it is more appropriate to seek a description in terms of lower dimensional branes. In this section we find a solution representing the three charge supertube in terms of its constituent D4 branes; this is parallel to the description of the D2-brane supertube in terms of D0 branes (the matrix theory description).

We start with a collection of N_4 D4 branes, and turn on fields such that it carries D0 and F1 charge. Just as in the previous section, we first present a simple solution which also carries D2-brane charge, and then modify our solution to cancel this. As we will comment on later, with a simple relabelling, our solution also yields the familiar D1-D5-P system.

Let the D4 branes be aligned along $x^{6,7,8,9}$. As described below, we will distribute the branes over a distance $N_4\ell$ in the x^1 direction, thus the D4-brane charge density is

[5] The D4 dipole moment of the configurations described above can be put to zero without loss of generality, and we will not consider it in this section. In contrast, the D2 and D6 dipole moments cannot be put to zero.

$$Q_4=\frac{T_4}{\ell}. \tag{3.1}$$

We turn on the world-volume field strengths

$$\mathcal{F}_{67}=\mathcal{F}_{89}=B1_{N_4}. \tag{3.2}$$

The D4 branes thus carry lower brane charge densities,

$$Q_2=\frac{T_4}{N_4\ell}\mathrm{Tr}\,\mathcal{F}=\frac{BT_4}{\ell},\quad Q_0=\frac{T_4}{N_4\ell}\mathrm{Tr}\,\mathcal{F}\wedge\mathcal{F}=\frac{B^2T_4}{\ell}. \tag{3.3}$$

To induce $F1$ charge along the x^1 direction we turn on the transverse scalars as [15,16,5]

$$\begin{aligned}
X^1&=\ell(j\delta_{ij})\\
X^2&=\frac{1}{2}r(a+a^{\dagger})\\
X^3&=\frac{i}{2}r(a^{\dagger}-a)\\
X^4&=0\\
X^5&=0
\end{aligned} \tag{3.4}$$

where

$$a_{ij}=\exp\left(-i\frac{\ell}{2\pi}t\right)\delta_{i-1,j}. \tag{3.5}$$

The form of X^1 implies that we are distributing the N_4 D4 branes separated by a distance ℓ in the x^1 direction. For large N_4, up to "boundary effects" which are subleading in N_4, we have the nonvanishing "transverse field strengths"

$$\begin{aligned}
\mathcal{F}_{02}&=\dot{X}^2=\frac{\ell}{2\pi}X^3\\
\mathcal{F}_{03}&=\dot{X}^3=-\frac{\ell}{2\pi}X^2\\
\mathcal{F}_{12}&=\frac{i}{2\pi}[X^1,X^2]=-\frac{\ell}{2\pi}X^3\\
\mathcal{F}_{13}&=\frac{i}{2\pi}[X^1,X^3]=\frac{\ell}{2\pi}X^2.
\end{aligned} \tag{3.6}$$

Other commutators vanish:

$$[X^2,X^3]=[X^2,X^4]=\cdots=[X^4,X^5]=[A_6,X^1]\cdots=[A_9,X^5]=0. \tag{3.7}$$

An important property, which we will use below, is that up to boundary effects,

$$(X^2)^2+(X^3)^2=r^2 1_{N_4}. \tag{3.8}$$

Thus r plays the role of the radius.

Commutators of field strengths are vanishing, $[\mathcal{F}_{\mu\nu},\mathcal{F}_{\mu'\nu'}]=0$, and we can therefore use the minimally non-Abelian form of the Born-Infeld action:

$$S=-T_4\mathrm{Tr}\int d^5\xi\sqrt{-\det(\eta_{MN}+\mathcal{F}_{MN})} \tag{3.9}$$

where M and N run over all spacetime directions. The F1 string charge density in the x^5 direction is found by substituting $\mathcal{F}_{01}\rightarrow\mathcal{F}_{01}-B_{01}$ and expanding the action to first order in B_{01} as $S\rightarrow S+\int Q_1B_{01}$. This yields

$$Q_1=\frac{T_4(1+B^2)\ell r^2}{(2\pi)^2}. \tag{3.10}$$

We can now work out the Hamiltonian as

$$H=\int \mathrm{Tr}P^i\dot{X}^i-\int\mathcal{L}=\int|Q_4+Q_1+Q_0|. \tag{3.11}$$

Thus we have again found a BPS saturating configuration, which implies that the equations of motion must be satisfied, as can be verified explicitly.

Our configuration has the independent radial parameter r which essentially parametrizes the angular momentum in the $x^{2,3}$ plane. By computing the energy momentum tensor we find the angular momentum to be

$$J_{23}=\frac{T_4}{2\pi}(1+B^2)r^2=\frac{2\pi Q_1Q_4}{T_4}. \tag{3.12}$$

Alternatively, using Eqs. (3.1), (3.3), and (3.10), we can express the radius in terms of the charges as

$$r^2=\frac{(2\pi)^2}{T_4^2}\frac{Q_1Q_4^2}{Q_4+Q_0}. \tag{3.13}$$

Equations (3.12) and (3.13) agree with Eqs. (2.14) and (2.9) after we recall that $T_4=(2\pi)^2T_6$. We can see again that when on adds the third charge to the two charge supertube the maximal angular momentum does not change.

To describe the more familiar D1-D5-P system, we merely need to start with a collection of D5 branes aligned along $x^{1,6,7,8,9}$, and make the change in notation $X^1\rightarrow 2\pi A_1$. This is just an implementation of T duality along x^1.

We now proceed to add in a second collection of D4 branes to cancel the D2-brane charge appearing in Eq. (3.3). This is accomplished by simply flipping a few signs. To be a bit more general, we can allow the second set of branes to expand into the $X^{4,5}$ plane with radius $\tilde{r}$. The solution is

$$\begin{aligned}
\mathcal{F}_{67}&=\mathcal{F}_{89}=B\sigma_3\otimes 1_{N_4}\\
X^1&=1_2\otimes\ell(j\delta_{ij})\\
X^2&=\begin{pmatrix}1&0\\0&0\end{pmatrix}\otimes\frac{1}{2}r(a+a^{\dagger})
\end{aligned}$$

$$X^3=\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}\otimes\frac{i}{2}r(a^\dagger-a)$$

$$X^4=\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}\otimes\frac{1}{2}\tilde{r}(a+a^\dagger)$$

$$X^5=\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}\otimes\frac{i}{2}\tilde{r}(a^\dagger-a). \tag{3.14}$$

The analysis proceeds much as before. The formulas for the charges are now

$$Q_4=\frac{2T_4}{\ell}$$

$$Q_0=\frac{2B^2T_4}{\ell} \tag{3.15}$$

$$Q_1=\frac{2T_4}{(2\pi)^2}(1+B^2)\ell(r^2+\tilde{r}^2)$$

and the angular momenta are

$$J_{23}=\frac{2T_4}{2\pi}(1+B^2)r^2=(2\pi)2T_6(1+B^2)r^2,$$

$$J_{45}=\frac{2T_4}{2\pi}(1+B^2)\tilde{r}^2=(2\pi)2T_6(1+B^2)\tilde{r}^2. \tag{3.16}$$

One can also generalize the above construction by deforming the tube cross sections to be ellipsoidal. However, the most general tube cross section—an arbitrary closed line in R^4—does not seem to be easily obtainable from the non-Abelian approach.

IV. IMPLICATIONS FOR BLACK HOLE PHYSICS

We now shift gears and discuss the relation of our supertubes to the black hole physics based on the D1-D5-P system. One of our main goals is to argue for a picture in which the spinning BMPV black hole can be thought of as being made of supertubes. This proposal will pass several consistency checks, especially relating to bounds on sizes and angular momenta. For example, a supertube implementation of chronology protection will prevent us from overspinning the black hole (which would result in closed timelike curves). In the following, we will be working in the context of the D1-D5-P system, and denote the quantized charges as N_1, N_5, and N_p. We correspondingly U dualize our previous formulas via the substitutions $N_0\to N_p$, $N_4\to N_5$, and $N_1\to N_1$, and the same for the Q's.

A. Lightning review of the D1-D5-P system

First, it is helpful to recall a few facts about the $1+1$ dimensional superconformal field theory (SCFT) describing the D1-D5 system. We have N_1N_5 hypermultiplets comprising $4N_1N_5$ bosons and fermions. The theory has an $SU(2)_L\times SU(2)_R$ R symmetry, which can be identified with the $SO(4)$ rotation group in the four dimensions transverse to the branes. The leftmoving fermions transform as $2N_1N_5$ doublets under $SU(2)_L$, while the leftmoving bosons are neutral. A single leftmoving fermion thus has equal eigenvalues for the $SO(4)$ generators J_{ij}:

$$J\equiv J_{23}=J_{45}=\pm\frac{1}{2}. \tag{4.1}$$

We will be interested in BPS states in the Ramond-Ramond (RR) sector, as these are the states relevant to the study of BPS black holes. States preserving 8 supercharges are the RR vacua with $N_p=0$. These states carry angular momenta due to the fermion zero modes; by aligning the zero modes in different ways we get states with

$$-N_1N_5\leqslant J\leqslant N_1N_5. \tag{4.2}$$

The entropy of such states is most easily computed by dualizing to the F1-P system, and yields

$$S=2\pi\sqrt{N_1N_5-|J|},\quad \text{for}\quad |J|\sim N_1N_5. \tag{4.3}$$

States preserving 4 supercharges are obtained by considering purely leftmoving (or rightmoving) excitations. The maximal J is obtained by distributing the N_p units of momentum among as many carriers as possible. This is achieved by filling up the Fermi sea.[6] The counting is simple for $N_p\gg N_1N_5$, in which case we fill up the Fermi sea to the highest harmonic $n_F=\sqrt{N_p/N_1N_5}$. The angular momentum then obeys

$$-N_1N_5N_p\leqslant J^2\leqslant N_1N_5N_p, \tag{4.4}$$

which is much greater than Eq. (4.2) given our assumption about the magnitude of N_p. For $N_1N_5N_p\gg J^2$, the entropy was argued in Ref. [17] to be

$$S=2\pi\sqrt{N_1N_5N_p-J^2}. \tag{4.5}$$

On the supergravity side, the entropy formula (4.3) was given a stretched horizon type interpretation in Ref. [10], while Eq. (4.5) is equal to the Bekenstein-Hawking entropy of the rotating BMPV black hole [17,18]. The BMPV solutions indeed obey the bound (4.4); overspinning results in closed timelike curves.

B. Comparison with supertubes

Turning now to our supertubes, we note that their angular momenta are not restricted to be equal. For example, we can choose a closed curve such that the supertube cross section lies in the 23 plane, in which case $J_{23}\neq 0$ and $J_{45}=0$. The bound on the angular momentum of the tubes coincides with the $N_p=0$ bound (4.2). A single circular tube with $Q_0=0$ saturates this bound:

[6] For our purposes we can think of the conformal field theory (CFT) as a free theory.

$$J_{23}=\frac{2\pi Q_1 Q_4}{T_4}=N_1 N_4 \rightarrow N_1 N_5 \tag{4.6}$$

while circular tubes with D6 dipole moment k have

$$J=\frac{2\pi Q_1 Q_4}{kT_4}=\frac{N_1 N_4}{k}\rightarrow \frac{N_1 N_5}{k}. \tag{4.7}$$

By appropriately changing the shape and orientation of the tube cross section, we can span the entire range in Eq. (4.2).

Next consider the three charge supertubes with $r=\tilde{r}$. The angular momentum is still given by Eq. (4.7), which we write as $k=N_1 N_5/J$. The radius is obtained from Eq. (2.11) as

$$r^2=\frac{2\pi}{8k^2}\frac{N_1 N_4^2}{T_0 N_0+T_4 N_4}\rightarrow \frac{2\pi}{8k^2}\frac{N_1 N_5^2}{N_p+T_5 N_5}$$
$$=\frac{\pi}{4}\frac{J^2}{N_1(N_p+T_5 N_5)}. \tag{4.8}$$

From Eq. (4.7) we note that for sufficiently large N we can easily exceed the black hole angular momentum bound in Eq. (4.4). We should also compare the size of the tubes with the size of the black hole. For simplicity, we consider the case of equal charges: $N_5=N_1=N_p$ and $g_s \ll 1$, yielding

$$r_{\rm tube}^2 \sim g_s \frac{J^2}{N^2}. \tag{4.9}$$

On the other hand, one can use Ref. [17] to compute the proper length of the circumference of the horizon (as measured at one of the equator circles) to be

$$r_{\rm hole}^2 \sim g_s \frac{N^3-J^2}{N^2}. \tag{4.10}$$

For small J we have $r_{\rm tube}<r_{\rm hole}$, and so we can consider the tube to fit inside the horizon. As J is increased the tube eventually becomes larger than the horizon, and for $J^2>N^3$ the black hole ceases to exist. In fact, since the crossover point is also $J^2 \sim N^3$, the black hole is essentially always larger than the tube in the region of parameter space where both can exist.

It has been proposed [9–12] that black hole entropy can be accounted for by the multiplicity of possible configurations inside the horizon, all of which appear essentially identical outside the horizon. In this spirit, we can imagine writing down supergravity solutions for each of our tube configurations. It is then an important consistency check that the tubes indeed lie inside the would-be horizon radius, since otherwise the individual geometries would be distinguishable even outside the horizon. Although our tubes can become very large, they are almost never larger than the horizon radius of the corresponding black hole, and so it is consistent to regard them as comprising part of the hair of the black hole.

C. AdS/CFT interpretation

It is interesting to see what happens if we try to give an AdS/CFT interpretation to the supertubes with $J^2>N^3$. Recall that in the CFT of the D1-D5 system, for $N_p \gg N_1 N_5$, we have the strict bound (4.4). On the other hand, it would seem that we could violate this bound in the bulk by placing one of our $J^2>N^3$ tubes in $\mathrm{AdS}_3\times S^3\times T^4$. However, this does not happen, for reasons analogous to the familiar giant graviton phenomenon. In the near horizon metric of the D1-D5 system, the S^3 has a size

$$r_{S^3}^2=g_s(N_1 N_5)^{1/2}\sim g_s N. \tag{4.11}$$

In order to fit the tube in the near horizon region we need $r_{\rm tube}<r_{S^3}$, which from Eq. (4.9) requires $J^2<N^3$. So we see that tubes with $J^2>N^3$ are too large to fit in the near horizon region.

For $N_p<N_1 N_5$ the picture that emerges is even more interesting. In that case, there are many field theory states whose angular momentum [bounded above by Eq. (4.2)] is larger than the black hole angular momentum [bounded above by Eq. (4.4)]. These states cannot be described by the black hole alone. Instead, as it has been argued in Ref. [7] for the zero momentum case, these states should be dual to supertube configurations. Thus, some of the states of the field theory can be mapped to one black hole, whose entropy gives the multiplicity of these states. Other states however are dual to spacetimes containing supertubes.

It is quite obvious that this distinction is arbitrary, as there is nothing special about the bound (4.4) in the regime $N_p<N_1 N_5$. It is therefore likely that all the states of the field theory are dual to supertube configurations. It is a distinct possibility that in some regime of parameters these tubes would be rather small and therefore not describable in supergravity. However, in other regimes the tubes are supergravity objects, and the possibility of using them to account for the black hole entropy is very interesting.

If we assume that for a certain angular momentum $J^2>N_1 N_5 N_p$ each field theory state can be mapped to a supertube geometry, then we can ask if there exists a "black object" which represents the statistical average of these states. There are two possibilities for what this object can be. The first is to have a black hole together with a supertube at a certain distance away from the horizon. The other possibility is to have a BPS black ring solution with $J^2>N_1 N_5 N_p$, whose hair is given by these supertubes. Several attempts at constructing nonextremal and extremal black rings which carry three charges and several dipole charges have appeared in the literature [19], but the solutions found have pathologies. We should also note that a BPS black ring solution would be the first of its kind, since other known black rings [20,21] are nonextremal. It would be interesting to find if such a BPS black ring exists, and if its entropy matches the multiplicity of the corresponding CFT states.

D. Constructing generalizations of the BMPV black hole

We noted above that the BMPV black hole obeys $J_{23}=J_{45}$, while our tubes obey no such restriction. On the other hand, we expect our tubes to coexist with the BMPV black hole in a BPS fashion. This suggests that we can form a more general BPS black hole by throwing in a tube through the horizon of the BMPV black hole.

We should note that since the D1-D5-P supertube is not homogeneous along the P direction, it does not directly descend to a five dimensional configuration. Hence, the resulting black hole is not properly a black hole of minimal five dimensional supergravity,[7] but a six dimensional black string. In fact, some solutions of this type have been proposed and discussed in Ref. [23] (based on Refs. [24,25]) as gyration waves along the six dimensional lift of the BMPV black hole. It might be possible to argue that some of the ten dimensional type IIA black brane configurations we will be discussing should be dual to smeared gyration waves along the string.

With this motivation in mind, we now turn to a discussion of the tube treated as a probe of the BMPV black hole to see if we can indeed perform this "experiment." We will also discuss the possibility of creating black holes by putting together three charge tubes. As before, for convenience we will perform our probe computations in the D0-D4-F1 picture, but we will sometimes rewrite our results after the duality which interchanges $N_0 \to N_p$, $N_4 \to N_5$, and $N_1 \to N_1$.

As a warmup we first consider the simpler setup of a D2-brane supertube probing a D4-brane background. In fact, we will see that this actually reproduces the results of the more complicated setup. If we can slowly contract the tube down to $r=0$ it will indicate that a bound state can form; up to T duality this bound state is a D1-D5-P black hole, and we can use this experiment to understand the bound (4.4).

We use the string frame metric for a collection of D4 branes aligned along $x^{0,6,7,8,9}$ and smeared along x^5:

$$ds^2 = Z^{-1/2} dx_\parallel^2 + Z^{1/2} dx_\perp^2$$

$$e^{2\phi} = Z^{-1/2}$$

$$Z = 1 + \frac{2\pi g_s N_4}{r^2} = 1 + \frac{N_4}{2\pi T_2}\frac{1}{r^2}. \tag{4.12}$$

In the last line $1/r^3 \to 2/r^2$ due to the smearing, and we wrote the result in terms of T_2 for later convenience. Note that N_4 is the number of D4-branes per unit length in x^5. Our probe is a D2-brane supertube, with axis parallel to x^5, and corresponding S^1 in the $x^{2,3}$ plane. The world-volume fields are $\mathcal{F}_{05}=1$ and $\mathcal{F}_{5\theta}$. The Born-Infeld action of the tube is

$$S = -T_2 \int d^3x e^{-\phi}\sqrt{-\det(g_{ab}+\mathcal{F}_{ab})}$$

$$= -T_2 \int d^3x \mathcal{F}_{5\theta} = -\int d^2x T_0 N_0. \tag{4.13}$$

[7]This is also expected from the fact that the only BPS black hole of this supergravity is the BMPV black hole [22].

As usual, the cancellation of the Z factors follows from the BPS property.

As in flat spacetime, the radius is determined from the formula for the fundamental string charge, which in this case is

$$N_1 = \frac{2\pi T_2}{N_0} Z r^2. \tag{4.14}$$

Combining Eqs. (4.12) and (4.14) we arrive at

$$2\pi T_2 r^2 = N_1 N_0 - N_4. \tag{4.15}$$

Bound states are described by solutions with $r^2 \leqslant 0$, which requires $N_1 N_0 \leqslant N_4$. On the other hand, the angular momentum of the supertube is

$$J_{23} = N_0 N_1 \tag{4.16}$$

and so we learn that bound states obey the restriction

$$J_{23} \leqslant \sqrt{N_0 N_1 N_4} \to \sqrt{N_1 N_5 N_p} \tag{4.17}$$

which agrees with Eq. (4.4). It is quite remarkable that the BPS black holes we construct this way, even if they have no angular momentum in the 45 plane, still obey the BMPV angular momentum bound (4.17). One can also imagine putting together equal amounts of probe tubes such that the total J_{23} and J_{45} are equal. Most likely the resulting black hole is the BMPV black hole. The fact that we can only create BMPV black holes which obey the bound (4.17) and hence have no closed timelike curves, is a remarkable example of chronology protection at work.

To see how to saturate the bound by bringing in tubes, consider the following process. Let the final bound state have charges N_0, N_1, and N_4 (all much larger than 1). We can always choose two charges such that their product is much larger than the third, and so up to duality we can take $N_0 N_1 \gg N_4$. Starting from N_4 D4 branes, we consider bringing in k tube probes each carrying

$$((N_0)_{\rm probe},(N_1)_{\rm probe}) = \left(\frac{N_0}{k},\frac{N_1}{k}\right) \Rightarrow J_{\rm probe} = \frac{N_0 N_1}{k^2}. \tag{4.18}$$

To get as much J as possible we would like k to be as small as possible, but at the same time we must obey $(N_0)_{\rm probe}(N_1)_{\rm probe} \leqslant N_4$ in order to be able to bring the probe to $r=0$. Therefore we should choose $k = \sqrt{N_0 N_1/N_4}$, leading to

$$J = k J_{\rm probe} = \sqrt{N_0 N_1 N_4} \to \sqrt{N_1 N_5 N_p}, \tag{4.19}$$

which indeed saturates the bound.

As we have discussed, the black hole we create from supertubes has $J_{45}=0$, while the BMPV black hole has $J_{45} = J_{23}$. We can of course introduce nonzero J_{45} by throwing in supertubes whose S^1 factors lie in this plane, and therefore it is clear that we can obtain BPS configurations with arbitrary and independent J_{23} and J_{45}, such that their sum is within the bound set by Eq. (4.4). It would be interesting to see if the explicit solution for these configurations can be obtained by smearing gyration waves on the six dimensional black string [23].

In the CFT description of the D1-D5-P system, angular momentum is carried by the fermions, the leftmoving species of which have $J_{23}=J_{45}$, and so the possible angular momentum is more restricted than our supertube thought experiment would suggest. Actually, even for BPS states we are not strictly required to have $J_{23}=J_{45}$, since angular momentum can also be carried by the rightmoving zero modes, which have $J_{23}=-J_{45}=1/2$. This means that we can have $|J_{23}-J_{45}|\leq 2N_1N_5$. On the other hand, we have seen that by throwing in tubes we can generate $|J_{23}-J_{45}|=\sqrt{N_1N_5N_p}$, which can be much larger than $2N_1N_5$. This mismatch can presumably be accounted for by including the vector multiplets in the D1-D5 CFT, since angular momentum is also carried by the Bosonic components of these multiplets.

E. Probing the BMPV black hole

Some of the conclusions we have drawn in the previous section actually rely on the probe analysis outside its domain of validity. Since the probe, by definition, should have a small effect on the ambient geometry, if we want to ask questions about the maximal angular momenta we should really be considering a supertube probing the BMPV black hole (or even better, the as yet unknown generalization of BMPV mentioned above). We therefore now carry out this analysis. As we will see, the probe potential will turn out to only depend on the harmonic function sourced by the D4 branes, and is independent on the D0 and F1 charges and the angular momenta. This proves that our previous inferences were in fact valid.

In the previous section we have also argued that by putting together tubes one cannot create an overspinning BMPV black hole. The probe analysis we now perform can also be used to show that one cannot overspin an already existing underspinning BMPV black hole by bringing in supertubes through the horizon.

As the first step, we need the lift of BMPV to 10 dimensions. This was written down in Ref. [26] and is reproduced in the Appendix. Next we T dualize along the D1 branes, since this is more convenient for the probe computation. Using the T-duality rules in Ref. [27] (see Refs. [28, 29] for earlier work) we obtain

$$\begin{aligned}
\tilde g_{tt}&=g_{tt}-g_{tz}^2/g_{zz}=-H_5^{-1/2}H_1^{-1/2}H_p^{-1}\\
\tilde g_{zz}&=1/g_{zz}=H_5^{1/2}H_1^{1/2}H_p^{-1}\\
\tilde B_{tz}&=-1+H_p^{-1}\\
\tilde B_{\phi_i z}&=J_iH_p^{-1}\\
\tilde g_{t\phi_i}&=g_{t\phi_i}-g_{tz}g_{z\phi_i}/g_{zz}=g_{t\phi_i}H_p^{-1}\\
&=-J_iH_5^{-1/2}H_1^{-1/2}H_p^{-1}\\
e^{2\tilde\phi}&=H_1^{3/2}H_5^{-1/2}H_p^{-1}
\end{aligned}\tag{4.20}$$

$$\begin{aligned}
\tilde g_{\phi_i\phi_j}&=g_{\phi_i\phi_j}-g_{z\phi_i}g_{z\phi_j}/g_{zz}\\
\tilde g_{\phi_1\phi_1}&=H_5^{1/2}H_1^{1/2}r^2\sin^2\vartheta-J_1^2H_5^{-1/2}H_1^{-1/2}H_p^{-1}\\
\tilde g_{\phi_2\phi_2}&=H_5^{1/2}H_1^{1/2}r^2\cos^2\vartheta-J_2^2H_5^{-1/2}H_1^{-1/2}H_p^{-1}
\end{aligned}\tag{4.21}$$

$$\begin{aligned}
\tilde g_{rr}&=H_5^{1/2}H_1^{1/2}\\
\tilde C_t^1&=H_1^{-1}-1\\
\tilde C_{\phi_i}^1&=J_iH_1^{-1}\\
\tilde C_{t\phi_i z}^3&=-(C_{tz}^2g_{\phi_i z}-C_{\phi_i z}^2g_{tz})/g_{zz}+C_{t\phi_i}^2=J_iH_p^{-1},
\end{aligned}$$

where H_p, H_1 and H_5 are the harmonic functions sourced by the F1 strings, D0 and D4 branes respectively,

$$J_1=\frac{j}{2r^2}\sin^2\vartheta,\quad J_2=\frac{-j}{2r^2}\cos^2\vartheta,\quad H_i=1+\frac{R_i^2}{r^2},\tag{4.22}$$

and we only give the components of the forms which we use in our calculations. This solution has a horizon at $r=0$. The horizon area is proportional to $\sqrt{N_1N_5N_p-J^2}$, and matches the entropy of the D1-D5-P system discussed in Sec. IV. If $J^2>N_1N_5N_p$ the solution has closed timelike curves. The solution can be also be continued behind the horizon by introducing the new coordinate $\rho^2=R^2+r^2$.

We probe this metric with a D2 supertube carrying D0 and F1 charge. On the worldvolume we turn on $\mathcal{F}_{tz}$ and $\mathcal{F}_{\theta z}$. The former will eventually be set to 1, but we keep it arbitrary for now since we want to differentiate with respect to it to get the F1 charge. Note that θ is the world-volume angular coordinate, distinct from the coordinate ϑ appearing in the supergravity solution. We allow r, ϑ, and ϕ to vary arbitrarily as functions of θ. On the other hand, we take t and z to coincide on the world volume and in spacetime.

The Wess-Zumino part of the brane action is

$$\begin{aligned}
S_{WZ}&=T_2\int \tilde C\wedge e^{\mathcal{F}+\tilde B}\\
&=T_2\int dt\,dz\,d\theta\Big[[\tilde C_{t\phi z}+\tilde C_t\tilde B_{\phi z}\\
&\quad-\tilde C_\phi(\tilde B_{tz}+\mathcal{F}_{tz})]\frac{\partial\phi}{\partial\theta}+\tilde C_t\mathcal{F}_{\theta z}\Big]\\
&=T_2\int dt\,dz\,d\theta\Big[jH_1^{-1}(1-\mathcal{F}_{tz})\frac{\partial\phi}{\partial\theta}+\mathcal{F}_{\theta z}H_1^{-1}-\mathcal{F}_{\theta z}\Big].
\end{aligned}\tag{4.23}$$

The Born-Infeld part of the action is

$$S_{BI}=-T_2\int dtdzd\theta e^{-\tilde{\phi}}\sqrt{-\det(\tilde{g}_{ab}+\tilde{B}_{ab}+\mathcal{F}_{ab})}$$
$$=-T_2\int dtdzd\theta e^{-\tilde{\phi}}\times\sqrt{-[\tilde{g}_{tt}\tilde{g}_{zz}\tilde{g}_{\theta\theta}-\tilde{g}_{zz}\tilde{g}_{t\theta}^2-2\tilde{g}_{t\theta}(\tilde{B}_{tz}+\mathcal{F}_{tz})(\tilde{B}_{\theta z}+\mathcal{F}_{\theta z})+\tilde{g}_{tt}(\tilde{B}_{\theta z}+\mathcal{F}_{\theta z})^2+\tilde{g}_{\theta\theta}(\tilde{B}_{tz}+\mathcal{F}_{tz})^2]}. \tag{4.24}$$

The induced world-volume metric is

$$\tilde{g}_{\theta\theta}=\tilde{g}_{\phi\phi}\frac{\partial\phi}{\partial\theta}\frac{\partial\phi}{\partial\theta}+\tilde{g}_{rr}\frac{\partial r}{\partial\theta}\frac{\partial r}{\partial\theta}. \tag{4.25}$$

The action then works out to be

$$S_{BI}=-T_2\int dtdzd\theta\Bigg[\bigg(\cos^2\vartheta r^2\frac{\partial\phi}{\partial\theta}\frac{\partial\phi}{\partial\theta}+\frac{\partial r}{\partial\theta}\frac{\partial r}{\partial\theta}\bigg)H_5H_1^{-1}[H_p^{-1}-H_p(H_p^{-1}-1+\mathcal{F}_{tz})^2]+H_1^{-2}\bigg(\mathcal{F}_{\theta z}+j(1-\mathcal{F}_{tz})\frac{\partial\phi}{\partial\theta}\bigg)^2\Bigg]^{1/2}. \tag{4.26}$$

As usual, the BPS configuration is realized for $\mathcal{F}_{tz}=1$. The total action then simplifies to

$$S=S_{WZ}+S_{BI}=-T_2\int dtdzd\theta\mathcal{F}_{\theta z}=-\int dtdzQ_0 \tag{4.27}$$

and the Hamiltonian is simply

$$H=\int dz(Q_1+Q_0) \tag{4.28}$$

where Q_1 is the canonical momentum conjugate to A_z as in Eq. (2.3). Thus, the configuration is BPS for any shape, just as expected. The shape of the tube is constrained by the explicit formula for Q_1, which is

$$Q_1=\frac{\partial L}{\partial\mathcal{F}_{tz}}\bigg|_{\mathcal{F}_{tz}=1}=T_2\int d\theta\frac{H_5}{\mathcal{F}_{\theta z}}\bigg(\cos^2\vartheta r^2\frac{\partial\phi}{\partial\theta}\frac{\partial\phi}{\partial\theta}+\frac{\partial r}{\partial\theta}\frac{\partial r}{\partial\theta}\bigg). \tag{4.29}$$

The first thing to notice is that Eq. (4.29) is independent of the black hole angular momentum. Indeed, it only depends on the induced metric on the tube in the absence of j. Furthermore, Eq. (4.29) only depends on the harmonic function H_5, and so we could just as well have been probing a pure D4 brane. Thus, the previous simplified probe computation captures the whole essence of the problem.

If we consider the simplest circular embedding $\phi=\theta$, with r and ϑ constant, we find

$$2\pi T_2r^2\cos^2\vartheta=\frac{N_1N_0}{H_5} \tag{4.30}$$

which implies for the D1-D5-P system:

$$(N_1N_p)_{\text{probe}}=2\pi T_2r^2\cos^2\vartheta+(N_5)_{\text{hole}}\cos^2\vartheta. \tag{4.31}$$

We can bring the tube into $r=0$ as long as $(N_1N_p)_{\text{probe}}\leqslant(N_5)_{\text{hole}}$. If we bring the tube in at constant r and ϑ to "crown" the black hole, the tube will cross the horizon at angle $\cos^2\vartheta=(N_1N_p)_{\text{probe}}/(N_5)_{\text{hole}}$.

Since the tube can be BPS for any radius one might think that there would exist configurations in which the tube straddles the horizon of the black hole, being partly inside and partly outside. However, Eq. (4.29) implies that a finite charge tube must have $\partial r/\partial\theta=0$ at the horizon, and so this cannot happen.

One can also use this probe computation to show that one cannot create closed timelike curves by overspinning a regular BMPV black hole. Let us take the charges of this black hole to be N_p, N_1, N_5 and its angular momenta to be J, satisfying $J\leqslant\sqrt{N_pN_1N_5}$. We can only bring a tube with $\Delta J=\Delta N_p\Delta N_1$ inside the horizon if $\Delta N_p\Delta N_1\leqslant N_5$. The resulting charges satisfy

$$\begin{aligned}(N_1+\Delta N_1)(N_p+\Delta N_p)N_5&=N_pN_1N_5+(N_p\Delta N_1+N_1\Delta N_p)N_5+\Delta N_p\Delta N_1N_5\\&\geqslant J^2+2N_5\sqrt{N_p\Delta N_1N_1\Delta N_p}+N_5\Delta J\\&\geqslant J^2+2J\Delta J+(\Delta J)^2=(J+\Delta J)^2.\end{aligned} \tag{4.32}$$

Thus, the resulting black hole is still underspinning, as expected from chronology protection.

V. CONCLUSIONS AND OUTLOOK

We have presented two ways to construct supersymmetric tubes with three charges and two or three dipole moments. We then analyzed the possibility that these configurations represent some of the hair of the spinning BMPV black hole. We found that this possibility passes some rather nontrivial tests. For example, the size of the tubes is always smaller than the horizon circumference in the regime where both exist. Also, we showed how to use tubes to construct a spinning three charge black hole; the maximal angular momentum this black hole can carry is exactly the BMPV maximal angular momentum.

Since the three charge supertubes can carry more angular momentum than a BMPV black hole with the same charges, we have also explored the possibility of using them to overspin the black hole. This would result in the creation of closed timelike curves. We have shown that this does not happen, providing a nice example of chronology protection.

As we have seen, the properties of the three charge supertubes mesh nicely with the properties of black holes. There are a number of interesting directions to pursue. By considering probes in the BMPV background we gave an argument for the existence of supersymmetric black solutions which would generalize those of BMPV to unequal angular momenta, and possibly to a tubular topology; it would be interesting to construct these.

It would of course be very interesting to find the supergravity solutions for arbitrary three charge supertubes, and to see if these can be put in one-to-one correspondence with the states of the D1-D5-P system, generalizing the work of Refs. [9–12]. From the brane point of view, it would be useful to be able to quantize the supertube configurations, and give an account of the entropy in this description. These are all problems to which we hope to return in the future.

Another BPS black hole in string theory is the four dimensional four charge black hole. Any three charges of this black hole can be also carried by a U dual of one of our three charge supertubes. Hence, it would be also interesting to construct four charge supertubes, and to see if they can account for the hair of this black hole.

APPENDIX: 10 DIMENSIONAL LIFT OF THE BMPV BLACK HOLE [26]

The BMPV black hole lifts to the following 10 dimensional type IIB supergravity solution:

$$g_{tt}=H_5^{-1/2}H_1^{-1/2}(H_p-2)$$

$$g_{zz}=H_5^{-1/2}H_1^{-1/2}H_p$$

$$g_{tz}=-H_5^{-1/2}H_1^{-1/2}(H_p-1)$$

$$g_{z\phi_i}=J_iH_5^{-1/2}H_1^{-1/2}$$

$$g_{t\phi_i}=-J_iH_5^{-1/2}H_1^{-1/2}$$

$$g_{\phi_1\phi_1}=H_5^{1/2}H_1^{1/2}r^2\sin^2\vartheta$$

$$g_{\phi_2\phi_2}=H_5^{1/2}H_1^{1/2}r^2\cos^2\vartheta$$

$$e^{2\phi}=H_1H_5^{-1}g_s^2$$

$$C_{tz}^2=H_1^{-1}-1$$

$$C_{\phi_iz}^2=J_iH_1^{-1}$$

$$C_{t\phi_i}^2=J_iH_1^{-1}$$

$$C_{\phi_1\phi_2}^2=(H_5-1)r^2\cos^2\vartheta \tag{A1}$$

where $i=1,2$,

$$J_1=\frac{j}{2r^2}\sin^2\vartheta,\quad J_2=\frac{-j}{2r^2}\cos^2\vartheta,\quad H_i=1+\frac{R_i^2}{r^2}, \tag{A2}$$

and C^6 is the same as C^2 with H_1 changed in H_5 and an extra 4-volume added. This solution has a horizon at $r=0$. The horizon area is proportional to $\sqrt{N_1N_5N_p-J^2}$, and matches the entropy of the D1-D5-P system discussed in Sec. IV. If $J^2>N_1N_5N_p$ the solution has timelike curves.

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