Regularization techniques for the radiative corrections of Wilson lines and Kaluza-Klein states

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Within an effective field theory framework we compute the most general structure of the one-loop corrections to the 4D gauge couplings in one- and two-dimensional orbifold compactifications with nonvanishing constant gauge background (Wilson lines). Although such models are nonrenormalizable, we keep the analysis general by considering the one-loop corrections in three regularization schemes: dimensional regularization (DR), zeta-function regularization (ZR), and proper-time cutoff regularization (PT). The relations among the results obtained in these schemes are carefully addressed. With minimal redefinitions of the parameters involved, the results obtained for the radiative corrections can be applied to most orbifold compactifications with one or two compact dimensions. The link with string theory is discussed. We mention a possible implication for the gauge coupling unification in such models.

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I. INTRODUCTION

There currently exists great interest in the physics of compact dimensions in the context of experimental and theoretical efforts to understand the physics beyond the standard model (SM). Model building beyond the standard model is in general based on additional assumptions such as a higher amount of symmetry (supersymmetry, gauge symmetry), additional compact dimensions, and string theory, etc., which attempt to explain the physics at high energy scales and which must ''recover'' in the low energy limit the standard model physics. One way to ''relate'' these two very different energy scales and thus provide an insight into physics beyond the SM is to study the behavior of the gauge couplings of the model by considering their one-loop radiative corrections.

In this paper we use an effective field theory (EFT) approach to compute radiative corrections to the 4D gauge couplings induced by orbifold compactifications with Wilson line background. Such corrections are related to the ''threshold effects'' of Kaluza-Klein (KK) states associated with the compact dimensions. In general higher dimensional models also have a larger gauge symmetry than that in supersymmetric versions of SM-like models. Examples of breaking the higher-dimensional gauge symmetries are the Hosotani [1] or Wilson line [2,3] mechanism, which is natural for manifolds not simply connected. This symmetry breaking mechanism affects the 4D Kaluza-Klein masses and thus the one-loop corrections to the gauge couplings. We discuss the corrections to the couplings due to Kaluza-Klein modes in the presence of such symmetry breaking mechanisms.

Radiative corrections from compact dimensions were studied in the past in effective field theory approaches (see, for example, Refs. [4–7]) or in string theory (see, for example, Refs. [8–13]). On the field theory level the effect of Wilson lines on the 4D gauge couplings has been little explored even for the simplest field theory orbifolds, due to technical difficulties, and this motivated the present work. Further, field theory calculations are usually performed for a particular choice of the regularization scheme, and the link with other schemes is not always clear. Such a link is important because models with compact dimensions are nonrenormalizable, and comparing the results for radiative corrections in various regularizations provides additional valuable information on the UV behavior of the models.

Previous studies of the link between field and string theory results [14–16] for Kaluza-Klein radiative corrections suggest that in some cases the string ''prefers'' on the field theory side a proper-time cutoff regularization for the UV region. However, such a regularization is not gauge invariant in field theory. In this context our purpose is to provide for one- and two-dimensional field theory orbifolds, the most general structure of the one-loop corrections to gauge couplings in the presence of a Wilson lines background, in dimensional regularization (DR), and zeta-function regularization (ZR). Their link with results in the proper-time cutoff regularization (PT) and with string theory is also provided. Our results for the radiative corrections are very general and can be easily applied to specific models.

The analysis starts from the observation that while the field content, which contributes to the one-loop corrections, is strongly model dependent, the general structure of the mass spectrum of Kaluza-Klein modes is determined by the (eigenvalues of the Laplacian Δ in a constant gauge background for the) manifold/orbifold of compactification. For the particular but often considered cases of an orbicircle or two-dimensional orbifold T^2/Z_N, the integrals over compact dimensions and sums over associated nonzero Kaluza-Klein modes can be performed in a model-independent way. Once this is done, this leaves the much simpler task of determining the exact values of the beta functions to a model-by-model analysis.

More explicitly, note that the general structure of one-loop corrections to the inverse of the tree level (''bare'') gauge couplings α_i, induced by Kaluza-Klein modes, may be written *formally* as

$$\Omega_i^* = \mathrm{tr}\,\frac{\beta(\sigma)}{4\pi}\ln\det\Delta(\sigma), \tag{1}$$

where $\Delta(\sigma)$ is the (spectrum of the) Laplacian on the manifold/orbifold considered. $\beta(\sigma)$ is the one-loop beta function of a ''component'' state of charge σ under some symmetries of compactification (boundary conditions) or a constant gauge background, belonging to a particular multiplet/representation. The trace tr acts over all states/representations of the theory that have associated Kaluza-Klein modes. In the string context Ω_i^* can be related to the free energy of compactification [10] (see also Ref. [17]) and torsion [18,19].

In general, the dependence of the spectrum of the Laplacian Δ on the charge (σ) prevents one from factorizing the σ dependence (full beta function) in front of the logarithm (1). However, we regard σ as a *fixed parameter* and compute $\ln\det\Delta(\sigma)$ in general, for one- and two-dimensional orbifolds. Effectively this means to replace Δ by its eigenvalues expressed in some mass units. In an effective field theory the natural mass unit is that associated with its ultraviolet cutoff Λ. With this argument Eq. (1) gives the usual sum of logarithms $\Sigma_n \ln \Lambda/M_n(\sigma)$ known in field theory [21], with $M_n(\sigma)$ the mass of a Kaluza-Klein state of level n (for two dimensions n is replaced by a set of two integers $\{n_1, n_2\}$ associated each with one compact dimension). One then multiplies this sum by $\beta(\sigma)$ and performs the remaining model-dependent sum (tr) over σ.

In the presence of a constant gauge background/twist (Wilson lines) the eigenvalues of the Laplacian are changed by an amount function of σ, related to the Wilson line vacuum expectation value (VEV's). The correction of the Wilson lines to the gauge couplings may be regarded in some cases as an additional effect (''perturbation'') to that due to Kaluza-Klein modes alone, for vanishing Wilson VEV's. This idea may in principle be used for much more complex manifolds (for example, Calabi Yau, G_2 manifolds) with Wilson line background, to relate their associated one-loop corrections to those for vanishing background and the corresponding topological quantities (torsion) [19].

There remains the question of the regularization of (1). This equation only makes sense in the presence of a regularization both in the UV and IR regions. Indeed, $\det\Delta$ vanishes for massless modes and an IR regulator (mass shift) χ is in general required to ensure $\ln\det\Delta$ is well defined *before* proceeding further. Thus one should in fact compute $\ln\det(\Delta-\chi^2)$. This is ''avoided'' in the sense that one usually evaluates only the (IR finite) contribution of the *massive* (Fourier) modes alone, denoted $\ln\det(\Delta')$. This means that one implicitly takes the limit $\chi\rightarrow 0$ in the massive mode sector. This leaves only the IR regulator to be present, and which acts only in the sector of the massless modes. Further, the correction $\ln\det(\Delta')$ itself requires a regularization, this time in the UV region [14,15] since the contribution of the KK tower is in general UV divergent and a regulator denoted ξ ($\xi\rightarrow 0$) is introduced. The important point is that the limits $\chi\rightarrow 0$ and $\xi\rightarrow 0$ of the above UV and IR regularization of $\ln\det(\Delta'-\chi^2)$ do not necessarily commute in the *massive* mode sector. The two regularizations and the UV and IR regions may not be ''decoupled'' from each other and a UV-IR ''mixing'' (UV divergent, IR finite) is present. See Refs. [15,20] for an example with two compact dimensions and subsequent string theory interpretation. Such a situation can arise in nonrenormalizable theories due to summing over two *infinite*-level Kaluza-Klein towers, and is not present if the two sums are truncated to a finite number of modes. We will encounter this issue in Sec. III B.

In the following we compute the one-loop corrections due to *massive* modes to the 4D gauge couplings for one- and two-dimensional orbifolds in the DR, ZR, and PT regularization schemes of the UV region. As we shall see in our analysis, the former two are very closely related. In the last scheme (PT), the UV scale dependence appears naturally, in a form that—for the case of the two compact dimensions—agrees with the (heterotic) string. This is supported by findings in Refs. [14,16] where such a regularization recovered in a field theory approach the (limit of ''large'' radii of the) one-loop string thresholds to the gauge couplings in 4D $N=1$ toroidal orbifolds with $N=2$ subsectors in the absence [14] or presence [16] of Wilson lines.

The plan of the paper is the following. In the next section we review for one- and two-dimensional orbifolds the structure of the 4D KK mass spectrum in the presence of nonzero Wilson line VEV's that ''commute'' with the orbifold projection of the model. The structure of the 4D KK mass spectrum is the starting point for the main analysis of this work (Sec. III) where we compute the radiative corrections and their dependence on the UV regulator/scale. The Appendix provides extensive and self-contained technical details for the general series of Kaluza-Klein integrals that we encountered in one-loop calculations in dimensional, zeta-function, and proper-time cutoff regularizations. The exact mathematical relation among these schemes is also provided. Such results can be useful for other applications involving one-loop radiative corrections from compact dimensions.

II. ORBIFOLDS, WILSON LINES, AND THE 4D KALUZA-KLEIN MASS SPECTRUM

As an introduction we review the effect of Wilson lines on the general form of the 4D Kaluza-Klein masses for one- and two-dimensional field theory orbifolds. Although some details of the analysis may be different in specific models, the *structure* of the 4D Kaluza-Klein masses that we find in Eqs. (9) and (13) is general [22,23] and this is employed in Sec. III.

Consider a one- and a two-dimensional orbifold of discrete group Z_N. For the one-dimensional case, its action is $z\rightarrow z'=\theta_l z$ and z denotes the extra dimension. For two compact dimensions $z,\overline{z}$ one has $z\rightarrow z'=\theta_l z$, $\overline{z}\rightarrow\overline{z'}=\overline{\theta_l z}$, with $\theta_l=\exp(2i\pi l/N)$, $l=0,1,\ldots,N-1$. We denote $\tilde{\mu}=\{\mu,z\}$ and $\tilde{\mu}=\{\mu,z,\overline{z}\}$ for one and two compact dimensions, respectively, with $\mu=0,\ldots,3$. Then the gauge field $A_{\tilde{\mu}}$ and a scalar multiplet Φ in the fundamental representation transform as[1]

$$A_{\tilde{\mu}}(x,\theta_l z)=\gamma_\theta P_\theta A_{\tilde{\mu}}(x,z)P_\theta^\dagger \quad (x\in M^4),$$

$$\Phi(x,\theta_l z)=P_\theta \Phi(x,z), \tag{2}$$

where $\gamma_\theta=1$ for $\tilde{\mu}=\mu$, and $\gamma_\theta=\theta_l^{-1}$ for the compact dimension(s) index. Conditions (2) ensure that terms in the action as $|D_{\tilde{\mu}}\Phi D^{\tilde{\mu}}\Phi|^2$ are invariant under the orbifold action. Suppose the action has a symmetry G^* before the orbifold action (2) and is invariant under a gauge transformation $U(x,z)$:

$$A'_{\tilde{\mu}}(x,z)=U(x,z)A_{\tilde{\mu}}(x,z)U^\dagger(x,z)-iU(x,z)\partial_{\tilde{\mu}}U^\dagger(x,z),$$

$$\Phi'(x,z)=U(x,z)\Phi(x,z). \tag{3}$$

Equation (2) is invariant under a gauge transformation $U(x,z)$ provided that

$$U(x,\theta_l z)P_\theta=P_\theta U(x,z). \tag{4}$$

Equation (4) gives the remaining gauge symmetry after imposing the orbifold condition (2). At fixed points $z_f=\theta_l z_f$, this is generated by $G=\{T_a$, with $T_a=P_\theta T_a P_\theta^\dagger\}$. For broken generators (T_a^*) with $P_\theta T_a^* P_\theta^\dagger=\omega^{k_a}T_a^*$ $(\omega\equiv e^{i2\pi/N})$ and with $\omega^{k_a}=\theta_l$, the corresponding components A_z^a of the field A_z respect the relation $A_z^a(x,\theta z)=A_z^a(x,z)$, and their nonzero VEV's will break the group G further.

A. One compact dimension: General structure of 4D Kaluza-Klein masses

The initial fields satisfy periodicity conditions with respect to the compact dimension z,

$$A_{\tilde{\mu}}(x,z+2\pi R)=QA_{\tilde{\mu}}(x,z)Q^{-1},$$

$$\Phi(x,z+2\pi R)=Q\Phi(x,z), \tag{5}$$

where Q is a global transformation. Equations (5) are invariant under a gauge transformation $U(x,z)$ if

$$U(x,z+2\pi R)Q=QU(x,z). \tag{6}$$

We now assume that A_z of (2) has some nonzero components in the Cartan-Weyl basis of G^* [see discussion after Eq. (4)]. It is then easier to do calculations in a new gauge, with no background field, i.e., $A'_z=0$, which is achieved by a z-dependent, nonperiodic gauge transformation. Then Eq. (6) is not respected and Eq. (5) will change for the gauge-transformed (''primed'') fields. We consider A_z constant and, for simplicity, that it lies in the Cartan subalgebra of G^*, $A_z=A_z^I T_I^*$. The generators of the group G satisfy $[T_I,T_J]=0$; $[T_I,E_\alpha]=\alpha_I E_\alpha$; $I,J=1,\ldots,rkG$, with $\alpha=1,\ldots,\dim G-rkG$. The nonperiodic gauge transformation is

$$V(z)=e^{-izA_z}Q^{-1} \quad (A_z=A_z^I T_I^*). \tag{7}$$

We use $A_\mu=A_\mu^I T_I+A_\mu^\alpha E_\alpha$, $T_I\Phi_\lambda=\lambda_I\Phi_\lambda$ with Φ_λ the component λ of the multiplet Φ. With (3) for $U=V$, conditions (5) for the fields transformed under V become

$$A'^I_\mu(x,z+2\pi R)=A'^I_\mu(x,z), \quad A'_z=0,$$

$$A'^\alpha_\mu(x,z+2\pi R)=e^{i2\pi\rho_\alpha}A'^\alpha_\mu(x,z),$$

$$\rho_\alpha\equiv -RA_z^I\alpha_I,$$

$$\Phi'_\lambda(x,z+2\pi R)=e^{i2\pi\rho_\lambda}\Phi'_\lambda(x,z),$$

$$\rho_\lambda\equiv -RA_z^I\lambda_I, \tag{8}$$

where A_z originates from Eq. (2) and $\sigma=\alpha$ (λ) for the adjoint (fundamental) representation. In the following we refer to ρ_σ as the Wilson line or ''twist'' of higher-dimensional fields with respect to the compact dimension. From the Klein-Gordon equation with no gauge background (since $A'_z=0$) but with constraint (8), we find that component fields with twist ρ_σ $(\sigma=\alpha,\lambda)$ have 4D modes with mass

$$M_n^2(\sigma)=\chi^2+(n+\rho_\sigma)^2\frac{1}{R^2}. \tag{9}$$

This provides the structure of the 4D Kaluza-Klein mass spectrum, which takes account of nonzero background fields A_z^I or more generally of ρ_σ twists in the ''new'' boundary conditions (8). The contribution χ^2 is only present if higher-dimensional fields such as Φ are massive.[2] For the gauge fields $\chi=0$ and $M_0(\alpha)\neq 0$ if there is a nonzero ρ_α. As a result the corresponding generator E_α is ''broken'' and the symmetry G is reduced. See Refs. [24,26] for specific examples and related discussions. Equation (9) will be used in Sec. III A.

Although our derivation of the mass formula (9) is not necessarily general, the important point is that its structure is generic and appears in many orbifold compactifications S_1/Z_2, $S_1/Z_2\times Z_2$ [22,24] even in the *absence* of Wilson line VEV's ρ_σ. In many cases ρ_σ is just replaced by a constant (twist), while its value given in (8) is specific to the case of Wilson line symmetry breaking only.

For generality the one-loop corrections from the KK modes are computed in Sec. III A with ρ_σ an *arbitrary* parameter. Any model dependence will only involve minimal redefinitions of the parameters ρ_σ, R, and χ of the model.

[1]There is an inconsistency in the notation in Eqs. (2)–(4) in that for two compact dimensions the fields $A_{\tilde{\mu}}$ and Φ and operator U are actually functions of $(x,z,\overline{z})$ or $(x,\theta_l z,\overline{\theta_l z})$ rather than (x,z) or $(x,\theta_l z)$.

[2]In such a case χ will play the role of infrared regulator in the radiative corrections to gauge couplings.

B. Two compact dimensions: General structure of 4D Kaluza-Klein masses

We repeat the above analysis for two compact dimensions. For compactifications on a two-torus T^2, the higher-dimensional fields now satisfy periodicity conditions with respect to shifts along both dimensions. Under the following shifts of $(z,\overline{z})$ on the torus lattice, $(z',\overline{z'})\equiv(z+2\pi R_2 e^{i\theta},\overline{z}+2\pi R_2 e^{-i\theta})$, and $(z'',\overline{z''})\equiv(z+2\pi R_1,\overline{z}+2\pi R_1)$, one has

$$A_{\tilde{\mu}}(x;z',\overline{z'})=QA_{\tilde{\mu}}(x;z,\overline{z})Q^{\dagger},$$

$$\Phi(x;z',\overline{z'})=Q\Phi(x;z,\overline{z}),$$

$$A_{\tilde{\mu}}(x;z'',\overline{z''})=QA_{\tilde{\mu}}(x;z,\overline{z})Q^{\dagger},$$

$$\Phi(x;z'',\overline{z''})=Q\Phi(x;z,\overline{z}). \tag{10}$$

We assume that A_z and $A_{\overline{z}}$ of (2) have nonzero components in the Cartan-Weyl basis. For simplicity we take $A_z=A_z^I T_I^*$, $A_{\overline{z}}=A_{\overline{z}}^I T_I^*$ and A_z, $A_{\overline{z}}$ constant. A $(z,\overline{z})$-dependent gauge transformation $V(z,\overline{z})=\exp(-izA_z-i\overline{z}A_{\overline{z}})Q^{-1}$ removes the constant gauge "background," so $A'_z=0$ and $A'_{\overline{z}}=0$. After the transformation V the components in the Weyl-Cartan basis of the gauge-transformed fields satisfy

$$A'^{\alpha}_{\mu}(x;z',\overline{z'})=e^{i2\pi\rho_{2,\alpha}}A'^{\alpha}_{\mu}(x;z,\overline{z}),$$

$$\Phi'_{\lambda}(x;z',\overline{z'})=e^{i2\pi\rho_{2,\lambda}}\Phi'_{\lambda}(x;z,\overline{z}),$$

$$A'^{\alpha}_{\mu}(x;z'',\overline{z''})=e^{i2\pi\rho_{1,\alpha}}A'^{\alpha}_{\mu}(x;z,\overline{z}),$$

$$\Phi'_{\lambda}(x;z'',\overline{z''})=e^{i2\pi\rho_{1,\lambda}}\Phi'_{\lambda}(x;z,\overline{z}), \tag{11}$$

$$\rho_{1,\sigma}\equiv-R_1(A_z^I+A_{\overline{z}}^I)\sigma_I,$$

$$\rho_{2,\sigma}\equiv-R_2(A_z^I e^{i\theta}+A_{\overline{z}}^I e^{-i\theta})\sigma_I,\qquad \sigma=\alpha,\lambda \tag{12}$$

while A'^{I}_{μ} do not acquire any twist. Here $\sigma=\alpha$ ($\sigma=\lambda$) for adjoint (fundamental) representations, Φ_λ denotes a component λ of the multiplet Φ, and we used $T_I\Phi_\lambda=\lambda_I\Phi_\lambda$.

From the Klein-Gordon equation with no gauge background[3] but with "twisted" boundary conditions (11) it can be shown that the 4D modes of component fields A'^{α}_{μ}, Φ'_λ acquire a mass [16]

$$M^2_{n_1,n_2}(\sigma)=\frac{1}{\sin\theta^2}\left|\frac{1}{R_2}(n_2+\rho_{2,\sigma})-\frac{e^{i\theta}}{R_1}(n_1+\rho_{1,\sigma})\right|^2,$$

$$\sigma=\alpha\ \text{ or }\ \sigma=\lambda \tag{13}$$

or, in a different notation,

$$M^2_{n_1,n_2}(\sigma)=\frac{\mu^2}{T_2U_2}|n_2+\rho_{2,\sigma}-U(n_1+\rho_{1,\sigma})|^2,$$

$$\sigma=\alpha,\lambda, \tag{14}$$

with

$$U\equiv U_1+iU_2=R_2/R_1e^{i\theta}\quad(U_2>0),$$

$$T_2(\mu)\equiv\mu^2R_1R_2\sin\theta. \tag{15}$$

We introduced a finite nonzero mass scale μ to ensure a dimensionless definition for T_2; the dependence on μ cancels out in M_{n_1,n_2}. Equations (11)–(14) show that the symmetry G present after orbifolding is further broken by the Wilson lines (12) or "twist" $\rho_{i,\alpha}\neq0$ since then $M_{0,0}(\alpha)=0$, and the corresponding A'^{α}_{μ} becomes massive and the generator E_α is "broken."

Equation (13) gives the general structure of 4D KK masses for T^2 with Wilson lines or for T^2/Z_N. For example for T^2/Z_2 one has $\rho_{1,2}=0$, $\frac{1}{2}$ from orbifold parity conditions. Additional constraints may apply to A_z, $A_{\overline{z}}$ and thus to $\rho_{i,\sigma}$, $i=1,2$, which may take continuous/discrete values. However, for our analysis below we simply regard $\rho_{i,\sigma}$ as *arbitrary, fixed* parameters. This allows our results to be applied to specific models (see examples in Ref. [23]) with twisted boundary conditions, even in the absence of Wilson lines ($\rho_{i,\alpha}=0$). Model-dependent constraints can be implemented in the final results by using appropriate redefinitions of the parameters ρ_i, U, T.

III. GENERAL FORM OF ONE-LOOP CORRECTIONS

A. Case 1: One compact dimension

Using the general structure of the KK mass spectrum in one- and two-dimensional orbifolds with Wilson lines, Eqs. (9) and (14), we can address the implications for the radiative corrections to the 4D gauge couplings. The one-loop correction to the gauge couplings induced by the Kaluza-Klein states is given by the Coleman-Weinberg formula [see, for example, Ref. [27] for a general derivation of $\Omega_i(\sigma)$]

$$\left.\frac{1}{\alpha_i}\right|_{1\ loop}=\left.\frac{1}{\alpha_i}\right|_{tree}+\Omega_i^*,\quad \Omega_i^*=\sum_r\sum_{\sigma=\lambda,\alpha}\Omega_i(\sigma),$$

$$\Omega_i(\sigma)\equiv\frac{\beta_i(\sigma)}{4\pi}\sum_{m\in\mathbf{Z}}{}'\int_0^\infty\frac{dt}{t}e^{-\pi tM_m^2(\sigma)/\mu^2}{}_{\text{reg}}, \tag{16}$$

where μ is a finite, nonzero mass parameter that enforces a dimensionless equation for Ω_i. We would like to mention that the right-hand-side (rhs) formula for Ω_i is obtained by evaluating one-loop diagrams for vanishing momentum ($q^2=0$), such as that of $\Pi(q^2)$ shown in Fig. 1, with a tower of KK states, each of mass $M_m(\sigma)$ (m integer) present in the loop. For more technical details on how to obtain this expression for Ω_i, see, for example, Appendix A in Ref. [25], or Ref. [21]. Note the distinction between the dependence

[3]This was removed by V gauge transformation.

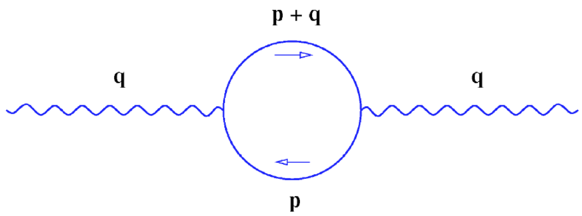

FIG. 1. Generic one-loop diagram contributing to Ω_i, with KK modes in the loop. Its value for $q^2=0$ can be read from Eqs. (16) and (26) for one and two compact dimensions. See also Appendix A in Ref. [25].

(''running'') of the couplings on the momentum scale q (see Fig. 1) for large q, given by $1/\alpha_i(q^2)-1/\alpha_i(0)=[\Pi_i(q^2)-\Pi_i(0)]/\alpha_i(0)$, and their dependence on the UV cutoff (regulator) of the theory that we compute in this work for $q^2=0$, given by Ω_i. We will only briefly discuss the dependence on $q^2\neq 0$ of the couplings; for a detailed analysis see Refs. [7,20].

$\Omega_i(\sigma)$ is thus the contribution of an infinite tower of Kaluza-Klein modes associated with a state of charge σ in the Weyl-Cartan basis and of mass ''shifted'' by real $\rho(\sigma)$, with $\sigma=\lambda,\alpha$ the weights or roots belonging to the representation r. The ''primed'' sum over m runs over all nonzero positive *and* negative integers (levels). The case when this sum is restricted to positive (negative) levels only will also be addressed. The effect of zero modes is not included in Ω_i since their presence is in general model dependent. Thus their contribution should be added separately to $1/\alpha_i$. The important point to note is that while the sums over r and $\sigma=\alpha,\lambda$ in Eq. (16) depend on the field content and are thus model dependent, the integral and the sum in Ω_i over Kaluza-Klein modes of nonzero level depend only on the geometry of compactification. It is this integral and sum over KK levels that are difficult to perform exactly, and they are evaluated below.

Supersymmetry is not a necessary ingredient in formula (16). Supersymmetry is however present in many models with compact dimensions that consider minimal supersymmetric standard model (MSSM)–like models as the viable ''low-energy'' limit. Regarding the beta functions β_i we have (we suppress the subscript i) $\beta(\sigma)=k_r(\sigma_I\sigma^I)/rkG$ for σ belonging to representation r; $k_r=\{-11/3,2/3,1/3\}$ for adjoint representations, Weyl fermion, and scalar. k_r essentially counts the degrees of freedom in the corresponding representations. The Dynkin index $T(r)=(\Sigma_\sigma\sigma_I\sigma^I)_r/(rkG)$, where the sum is over all weights or roots σ belonging to representation r, each occurring the number of times equal to its multiplicity [31]. With the definition $b_i(r)\equiv\Sigma_\sigma\beta_i(\sigma)$ for the weights σ belonging to r, one has $b_i=-11/3T_i(A)+2/3T_i(R)+1/3T_i(S)$, to account for the adjoint Weyl fermion in representation R and scalar in representation S. In the supersymmetric case massive $N=1$ Kaluza-Klein states are organized as $N=2$ hypermultiplets [vector supermultiplets] with $b_i=2T_i(R)[b_i=-2T_i(A)]$.

The subscript ''reg'' shows that formula (16) is not well defined in the UV region $t\to 0$, and a UV regularization is required. We assume $M_m(\sigma)\neq 0$ so no IR regularization (i.e., for $t\to\infty$) is needed.[4] The use of a particular regularization is in general dictated by the symmetries of the initial, higher-dimensional theory. If a string embedding exists for this theory, a PT regularization is in some cases the appropriate choice [14,16]. In the absence of such a fully specified theory and to keep the analysis general, we analyze Ω_i in three regularization schemes: DR, ZR, and PT regularization.

1. Dimensional regularization

In this scheme Ω_i of Eq. (16) has under the integral $1/t$ replaced by $1/t^{1+\epsilon}$ with $\epsilon\to 0$ the UV regulator. In such case μ is the arbitrary (finite, nonzero) mass scale introduced in the DR scheme in $d=4-\epsilon$ dimensions. The evaluation of Ω_i is rather long and is presented in detail in the Appendix, Eqs. (A1)–(A16). The calculation of Ω_i uses expansions in Hurwitz or Riemann zeta functions that do not necessarily involve a Poisson resummation of the ''original'' KK levels. This has the advantage that one may be able to identify which of the *original* KK levels brings the leading contribution to Ω_i. Using Eqs. (9), (A1), (A2), and (A16) one finds for Ω_i in the DR scheme

$$\Omega_i\big|_{DR}=\frac{\beta_i(\sigma)}{4\pi}\sum_{m\in\mathbf{Z}}{}' \int_0^\infty \frac{dt}{t^{1+\epsilon}}e^{-\pi t M_m^2(\sigma)/\mu^2}$$

$$=\frac{\beta_i(\sigma)}{4\pi}\left\{\frac{1}{\epsilon}-\ln\frac{(R\mu)^2}{\pi e^{\gamma}}-\ln\left|\frac{2\sin\pi(\rho_\sigma+i\chi R)}{\rho_\sigma+i\chi R}\right|^2\right\}. \tag{17}$$

The presence of the pole in ϵ accounts for an UV divergence. To find the *scale* dependence of this divergence in DR one may in general introduce a small/infrared mass shift χ of the momentum of the KK state. One would then expect the emergence in the final result of a term χ/ϵ to account for a linear divergence (in scale), given the extra dimension present. However, this procedure does not apply to the case with one compact dimension only.[5] Therefore, unlike the case of two compact dimensions to be discussed later, the presence of the pole alone does not tell us the nature of the scale dependence of the UV divergence. Note also that a single state (such as the zero mode, for example) gives a leading one-loop contribution proportional to $-1/\epsilon$, which is

[4]See, however, the discussion in Ref. [15] for the case of two compact dimensions.

[5]This is somewhat similar to computing $\int d^4p/p^2$, which in cutoff regularization is quadratically divergent while in DR is vanishing. However, a small mass shift χ of the momentum leads to $\int d^4p/(p^2+\chi^2)$, which has a pole in DR, which signals the usual quadratic divergence. In our case, even adding a small $(\text{mass})^2$ shift (accounted for by χ^2) does not introduce a scale dependence of the divergence in (17), such as χ/ϵ. For two (even number of) compact dimensions, this procedure in DR does lead to the scale dependence of the leading divergence, as opposed to the case of one (odd number of) extra dimension(s). See also Eq. (A21), which shows the emergence of a linear divergence in DR when summing over *positive (negative) modes only*.

of the same form but of *opposite sign* to that found in Eq. (17) for the whole Kaluza-Klein tower, excluding the zero mode [compare $\mathcal{R}_\epsilon$ vs *finite* $\mathcal{R}_\epsilon^T$ in Eq. (A16)]. Further, this $-1/\epsilon$ pole due to a single state is also known to correspond in four dimensions to a UV divergence only logarithmic in scale (rather than linear). Note, however, that the change of the couplings with momentum q in Fig. 1 is indeed linear in the momentum scale q and dominates if $(qR)^2 \geqslant \mathcal{O}(1)$ [7,20].

If ρ_σ is a nonzero integer, there exists a level n_0 such as $M_{n_0}=\chi$ and then χ plays the role of an IR regulator in Eqs. (16) and (17), and ensures that the term $\ln[\sin \pi(\rho_\sigma+i\chi R)]$ remains finite. If ρ_σ, χ vanish, the last term in (17) vanishes and one is left with the correction in the absence of the twist or Wilson line background ρ_σ.

In deriving Ω_i we summed over both positive and negative Kaluza-Klein levels, as shown in Eq. (16). However, it is interesting to analyze the effect of summing separately the contributions of the positive (negative) levels. In such a case the corresponding value of Ω_i, denoted Ω_i^+ (Ω_i^-), is computed in a similar way. The result, derived in the Appendix, Eqs. (A18) and (A21) is

$$\Omega_i^\pm|_{DR}=\frac{\beta_i(\sigma)}{4\pi}\left\{\frac{1}{2\epsilon}\pm\frac{\rho_\sigma}{\epsilon}+\ln|\Gamma(1\pm\rho_\sigma+i\chi R)|^2-\ln(2\pi) -\left[\frac{1}{2}\pm\rho_\sigma\right]\ln\frac{(R\mu)^2}{\pi e^\gamma}\right\}. \tag{18}$$

The divergent terms of $\Omega_i^\pm$ are then

$$\Omega_i^\pm|_{DR}\sim 1/(2\epsilon)\pm\rho_\sigma/\epsilon. \tag{19}$$

The presence of the additional divergence ρ_σ/ϵ is triggered by a nonzero background/twist ρ_σ, and is cancelled in the sum $\Omega_i^+ + \Omega_i^-$ of both positive and negative Kaluza-Klein levels, giving the overall result Ω_i in (17). If ρ_σ has the value given in (8) and is thus proportional to the VEV of A_z and to R, then ρ_σ/ϵ may be regarded as a divergence linear in scale. It is also possible that in some models one may actually have ρ_σ a *constant*, for example $\rho_\sigma=+1/2$ (or $-1/2$), then Ω_i^- (Ω_i^+) are *finite*, respectively, and the *overall* divergence in $\Omega_i=\Omega_i^+ + \Omega_i^-$ comes entirely from $\Omega_i^+(\Omega_i^-)$ respectively. To conclude, the positive and negative Kaluza-Klein levels propagating in opposite directions in the compact dimension, with a nonzero background/twist ρ_σ, contribute by different amounts to the overall divergence of Ω_i; in special cases the positive or negative levels alone give (one-loop) *finite* contributions.

2. ζ-function regularization

Alternatively, one can employ a ζ-function regularization of Ω_i. In this case the correction is given [up to a factor $\beta_i(\sigma)/(4\pi)$] by the derivative of the ζ function associated with the Laplacian, evaluated at the origin. As detailed in the Appendix this means that Ω_i in this scheme is just the derivative with respect to ϵ of the value obtained in the DR scheme (divided by $\Gamma[-\epsilon]$), and evaluated for $\epsilon=0$. From Eqs. (9), (16), (A27), and (A28) one obtains the value of Ω_i in the ZR scheme,

$$\Omega_i\bigg|_{ZR}=\frac{\beta_i(\sigma)}{4\pi}\frac{d}{d\epsilon}\times\left\{\frac{-\pi^{-\epsilon}}{\Gamma[-\epsilon]}\sum_{m\in\mathbf{Z}}{}' \int_0^\infty \frac{dt}{t^{1+\epsilon}}e^{-\pi t M_m^2(\sigma)/\mu^2}\right\}\Bigg|_{\epsilon\to 0} =\frac{\beta_i(\sigma)}{4\pi}\left\{-\ln(R\mu)^2-\ln\left|\frac{2\sin\pi(\rho_\sigma+i\chi R)}{\rho_\sigma+i\chi R}\right|^2\right\}. \tag{20}$$

This result is similar to that found in the DR scheme, with the notable difference that there is no pole structure present. The above result is only logarithmically dependent on the mass scale μ. As discussed in Appendix A2, μ plays, in the case of ζ-function regularization, the role of the UV cutoff of the model. Finally, note that the contribution of a zero mode—if included—would bring a similar dependence on μ but of *opposite* sign to cancel this dependence in the total sum (see also $\mathcal{R}_\zeta$ and $\mathcal{R}_\zeta^T$ in Appendix A2).

One can show that the separate contributions to Ω_i of positive and negative Kaluza-Klein modes are different due to the asymmetry introduced by the Wilson lines or twist ρ_σ. The results denoted $\Omega_i^+(\Omega_i^-)$, respectively, are given by Eq. (A31),

$$\Omega_i^\pm|_{ZR}=\frac{\beta_i(\sigma)}{4\pi}\left\{\ln|\Gamma[1\pm\rho_\sigma+i\chi R]|^2-\ln(2\pi) -\left[\frac{1}{2}\pm\rho_\sigma\right]\ln(R\mu)^2\right\}, \tag{21}$$

so the positive (negative) modes again bring a different UV behavior (μ dependence)

$$\Omega_i^\pm|_{ZR}\sim -(1/2\pm\rho_\sigma)\ln(R\mu)^2. \tag{22}$$

For ρ_σ just a *constant*, the ρ_σ-dependent term is just an additional logarithmic correction (in μ or R) to the couplings. However, in the case ρ_σ is indeed due to a nonzero Wilson line VEV (from initial A_z gauge fields), a *linear* dependence of the couplings on this VEV/scale emerges. This term can then have significant implications for the value of the gauge couplings. As it was the case in the DR scheme, such terms cancel in the sum of positive and negative mode contributions. A special case is $\rho_\sigma=\mp 1/2$ when the coefficient of the logarithmic UV divergence (in μ) of $\Omega_i^\pm$ is vanishing, and $\Omega_i^+(\Omega_i^-)$ has no μ dependence, with similarities to the DR case.

3. Proper-time regularization

The above results for Ω_i can be compared to that obtained in the proper-time regularization. In this regularization Ω_i of Eq. (16) has the lower limit of its integral set equal to $\xi>0$, where $\xi\to 0$ is a dimensionless UV regulator. For details of the calculation of Ω_i in this scheme see Appendix A3 and Ref. [16] (Appendix A1). From Eqs. (9), (A32), (A33), and (A36) and with the notation $\Lambda^2 \equiv \mu^2/\xi$, one obtains for Ω_i in the PT scheme

$$\Omega_i|_{PT} = \frac{\beta_i(\sigma)}{4\pi} \sum_{m\in\mathbf{Z}}{}' \int_\xi^\infty \frac{dt}{t} e^{-\pi t M_m^2(\sigma)/\mu^2}$$

$$= \frac{\beta_i(\sigma)}{4\pi} \left\{ 2R\Lambda - \ln \frac{(R\Lambda)^2}{\pi e^\gamma} - \ln \left| \frac{2\sin \pi(\rho_\sigma + i\chi R)}{\rho_\sigma + i\chi R} \right|^2 \right\}. \tag{23}$$

The ξ-dependent terms combine naturally with the scale μ to define the UV cutoff Λ of the model and one obtains a dependence on ΛR only. Unlike the DR and ZR cases, a zero-mode contribution to the above result—if included—would *not* cancel the leading linear divergence [(in $\Lambda \sim 1/\sqrt{\xi}$), but only the logarithmic one (for more details compare $\mathcal{R}_\xi$ and $\mathcal{R}_\xi^T$ in Appendix A3, Eq. (A36)].

What is the meaning of the individual contributions to Ω_i? Technical details show that the term $\ln|\rho_\sigma + i\chi R|$ is similar to a contribution corresponding to a massive Kaluza-Klein state of level zero. It may be interpreted as a one-loop effect of this state between the compactification scale $1/R$ and the scale set by the Wilson lines VEV's, $\sigma_I \langle A_z^I \rangle$ with σ accounting for a root/weight. The term $\ln[\sin(\cdots)]$ in (23) is an effect due to "Poisson re-summed" (PR) Kaluza-Klein states [see Eq. (A65)], with the dominant contribution from the lower PR levels. Further, the logarithm $\ln(\Lambda R)$ can be thought of as a one-loop effect from the compactification scale to the UV cutoff scale Λ. Finally, the term ΛR is due to the presence of a *large enough* number of Kaluza-Klein modes that enable the Poisson resummation. This term is due to the Poisson resummed mode of zero level. Thus one should expect $\Lambda R \gg 1$ because ΛR approximates the number of Kaluza-Klein modes. In fact the PT result (23) is valid provided that

$$\max\{1/R^2, \chi^2, (\langle A_z^I \rangle \sigma_I)^2 + \chi^2\} \ll \Lambda^2 \tag{24}$$

derived from Eq. (A35) of Appendix A3. Here we replaced ρ_σ in terms of the VEV's of A_z^I as in Eq. (8). More generally, for arbitrary ρ_σ this condition is

$$\max\{1/R^2, \chi^2, \rho_\sigma^2/R^2 + \chi^2\} \ll \Lambda^2. \tag{25}$$

Therefore the result in the PT scheme is valid if R is large (in UV cutoff units) and if the gauge symmetry breaking VEV's or $(\rho_\sigma/R)^2$ and the mass scale χ^2 have a sum much smaller than the UV cutoff. Note that these constraints are not shared by the DR or ZR counterparts computed above. This is important for in general to avoid a large UV sensitivity of the couplings one would like to have $\Lambda R \approx 1$ which is a region for which the PT result does not hold accurately. From comparing it with its DR counterpart, the presence of the pole $1/\epsilon$ of the latter may indicate that even if ΛR is made smaller, of order unity, a UV divergence is still manifest. Finally, if one considered a string embedding of these models, the string counterpart of $\Lambda R \approx 1$ would be $M_s R \approx 1$ with M_s the string scale. In this case string effects due to additional (winding) states not present in field theory may become important.

Comparing the three results for Ω_i obtained in these different regularization schemes one observes that the finite (regulator independent) part is the same in all regularizations. This is a strong consistency check of the calculation. Regarding the (i.e., regulator dependent) part, note that the $1/\epsilon$ term of DR is replaced in the PT cutoff regularization by the ξ (Λ) dependent divergent term, accounting for a linear divergence. Note that the ZR counterpart has only (rather "mild") a logarithmic UV divergence. Equations (17) to (23) generalize the results [25] for one compact dimension, in the presence of Wilson lines/twists ρ_σ.

B. Case 2: Two compact dimensions

We now consider the case of a two-dimensional compactification. With the structure of the mass spectrum of Eq. (14) we again compute the general form of the correction to the 4D gauge couplings due to nonzero level Kaluza-Klein modes in the presence of Wilson lines. This correction can be applied to a large class of models [23]. Formally, the correction is

$$\Omega_i^* = \sum_r \sum_{\sigma=\lambda,\alpha} \Omega_i(\sigma),$$

$$\Omega_i(\sigma) \equiv \frac{\beta_i(\sigma)}{4\pi} \sum_{n_1,n_2\in\mathbf{Z}}{}' \int_0^\infty \frac{dt}{t} e^{-\pi t M_{n_1,n_2}^2(\sigma)/\mu^2} \Bigg|_{\text{reg}}. \tag{26}$$

Similarly to the case of one extra dimension, Ω_i is obtained by computing one loop diagrams evaluated for $q^2=0$ (Fig. 1) with Kaluza-Klein states of mass $M_{n_1,n_2}(\sigma)$ in the loop.

In the following we perform—for σ fixed—the integral and the sums over $(n_1,n_2)\neq(0,0)$ in Eq. (26). Any model dependence [beta functions $\beta_i(\sigma)$, sums over weights σ, representations r] can then easily be implemented on the final result for Ω_i^*. The presence of the state $(n_1,n_2)=(0,0)$ is model dependent and its contribution should be considered separately. We again discuss the value of Ω_i in DR, ZR and PT regularization schemes for the UV divergence ($t\to 0$) of Eq. (26). We assume $M_{n_1,n_2}\neq 0$ for all integers, so no IR divergence (at $t\to\infty$) exists. However, if there exists a pair (n_1,n_2) for which $M_{n_1,n_2}=0$, see the results in the PT scheme of Ref. [15] and the discussion in the DR scheme to follow.

1. Dimensional regularization

In the DR scheme Ω_i is defined with $1/t$ under its integral replaced by $1/t^{1+\epsilon}$ where $\epsilon\to 0$ is the UV regulator. The calculation is rather technical and is presented in Appendix A 4, Eqs. (A37) to (A41), where the sums over $n_{1,2}$ and integral in (26) are evaluated. Using Eqs. (14), (26), (A38), (A41), and (A64), one obtains Ω_i in the DR scheme

$$\Omega_i\Big|_{DR} = \frac{\beta_i(\sigma)}{4\pi} \sum_{n_1,n_2\in\mathbf{Z}}{}' \int_0^\infty \frac{dt}{t^{1+\epsilon}} e^{-\pi t M^2_{n_1,n_2}(\sigma)/\mu^2}$$

$$= \frac{\beta_i(\sigma)}{4\pi}\left\{\frac{1}{\epsilon} - \ln\frac{T_2U_2}{\pi e^\gamma} - \ln\left|\frac{\vartheta_1(\rho_{2,\sigma}-U\rho_{1,\sigma}|U)}{(\rho_{2,\sigma}-U\rho_{1,\sigma})\,\eta(U)} e^{i\pi U\rho^2_{1,\sigma}}\right|^2\right\}. \qquad (27)$$

The special functions η, ϑ_1 are defined in Appendix A7. The pole $1/\epsilon$ accounts for divergences up to quadratic level. How can we see this? By introducing a small (mass)^2 shift $\mu^2\delta$ to $M^2_{n_1,n_2}$ (δ dimensionless, $\delta \ll 1$), i.e., $M^2_{n_1,n_2} \to M^2_{n_1,n_2} + \mu^2\delta$ under the integral in (26) and computing the integral in this more general case one obtains for Ω_i, in addition to the divergence $1/\epsilon$, a contribution $\pi\delta T_2/\epsilon$. This is a quadratic divergence in scale (T_2 "contains" a μ^2) that $1/\epsilon$ term effectively signals in Eqs. (27) and (A41). For additional technical details see Appendix A4, Eqs. (A43) and (A44).[6] The emergence of the additional scale-dependent contribution $\pi\delta T_2/\epsilon$ is to be contrasted with what happened in DR in the one extra dimension case already discussed, where a small mass shift did not introduce a scale dependence of the UV divergence. This is due to the different UV behavior of models with one (or odd number of) and two (or even number of) compact dimensions, respectively. Note that in the special case when there exists a pair (n_1,n_2) such as $M_{n_1,n_2}=0$, an IR regulator—in addition to the UV one—is required in Eqs. (26) and (27) to ensure the convergence of the integral at $t\to\infty$. The aforementioned shift $\mu^2\delta$ of the KK masses would in such special case act as an IR regulator in (26) and one would obtain in (27) a term $\pi\delta T_2/\epsilon$, which represents an IR-UV "mixing" term between the IR sector (δ) and UV sector (ϵ) of the theory. For a discussion on this UV-IR mixing see Refs. [15,20] where its string theory interpretation is also presented. Finally, considerations similar to those for one extra dimension apply for the separate role of negative or positive Kaluza-Klein levels, respectively.

2. ζ-function regularization

In this scheme Ω_i is related to the derivative of the zeta-function associated with the Laplacian, as discussed in Appendix A 5. In fact Ω_i in ZR is the derivative with respect to ϵ of Ω_i in DR divided by $\Gamma[-\epsilon]$, and evaluated for $\epsilon=0$. Using Eqs. (14), (A53), (A54), (A64) one finds Ω_i in the ZR scheme

$$\Omega_i\Big|_{ZR} = \frac{\beta_i(\sigma)}{4\pi}\frac{d}{d\epsilon} \times\left\{\frac{-\pi^{-\epsilon}}{\Gamma[-\epsilon]}\sum_{n_1,n_2\in\mathbf{Z}}{}' \int_0^\infty \frac{dt}{t^{1+\epsilon}} e^{-\pi t M^2_{n_1,n_2}(\sigma)/\mu^2}\right\}\Bigg|_{\epsilon\to 0}$$

$$= \frac{\beta_i(\sigma)}{4\pi}\left\{-\ln[T_2U_2] - \ln\left|\frac{\vartheta_1(\rho_{2,\sigma}-U\rho_{1,\sigma}|U)}{(\rho_{2,\sigma}-U\rho_{1,\sigma})\,\eta(U)} e^{i\pi U\rho^2_{1,\sigma}}\right|^2\right\}. \qquad (28)$$

This result has a form similar to that in the DR scheme from which the pole structure has been subtracted. The μ scale dependence "hidden" in T_2 should in this case be regarded as the UV cutoff as discussed in Appendix A 5, Eq. (A50). In this scheme there is thus only a logarithmic dependence on the UV cutoff. Finally, the finite part is similar to that obtained in the DR scheme. It would be of phenomenological interest to know which higher dimensional theories would require such a regularization, since in this case the UV cutoff dependence of the couplings is milder and the models would then have less amount of sensitivity to this cutoff scale, possibly similar to that of MSSM-like models.

3. Proper-time regularization

Finally, for a comparison we include here the value of Ω_i in the proper-time cutoff regularization scheme [16]. In this scheme Ω_i of (26) is defined with a (dimensionless) cutoff $\xi\to 0$ in the lower limit of its integral, which acts as an UV regulator. After a long calculation one obtains the result [for details see Eqs. (26), (A56), (A59), (A64), and also Eq. (52) in Ref. [16]]

$$\Omega_i\Big|_{PT} = \frac{\beta_i(\sigma)}{4\pi}\sum_{n_1,n_2\in\mathbf{Z}}{}' \int_\xi^\infty \frac{dt}{t} e^{-\pi t M^2_{n_1,n_2}(\sigma)/\mu^2}$$

$$= \frac{\beta_i(\sigma)}{4\pi}\left\{\frac{T_2}{\xi} - \ln\frac{[(T_2/\xi)U_2]}{\pi e^\gamma} - \ln\left|\frac{\vartheta_1(\rho_{2,\sigma}-U\rho_{1,\sigma}|U)}{(\rho_{2,\sigma}-U\rho_{1,\sigma})\,\eta(U)} e^{i\pi U\rho^2_{1,\sigma}}\right|^2\right\}. \qquad (29)$$

Equation (29) is valid if [see Eq. (A60) and definition (12)]

$$\max\left\{\frac{1}{R_1}, \frac{1}{R_2\sin\theta}, \langle A_z^I\rangle\sigma_I, \langle A_{\bar z}^I\rangle\sigma_I\right\} \ll \Lambda, \qquad \Lambda^2 \equiv \frac{\mu^2}{\xi}. \qquad (30)$$

This condition requires "large" compactification radii (in UV cutoff units) and symmetry breaking VEV's much smaller than Λ. Here we replaced $\rho_{i,\sigma}$ in terms of the VEV's of A_z^I, Eq. (12) but for arbitrary $\rho_{i,\sigma}$ this condition is $\max\{1/R_1, 1/R_2\sin\theta, |\rho_{1,\sigma}|/R_1, |\rho_{2,\sigma}-U\rho_{1,\sigma}|/(R_2\sin\theta)\} \ll \Lambda$.

Equation (29) shows the presence of a UV quadratic divergent term also known as "powerlike" threshold, given by $T_2/\xi = \Lambda^2 R_1R_2\sin\theta$ where $\Lambda^2 \sim 1/\xi$ is the UV cutoff scale.

[6] This also has consequences for the change of the gauge couplings with momentum q in Fig. 1 as discussed in Ref. [20].

A logarithmic correction is also present, $\ln(T_2/\xi) = \ln(\Lambda R_1 R_2 \sin\theta)$, as well as a $\ln U_2 = \ln(R_2 \sin\theta/R_1)$ part. The remaining term in Ω_i includes effects due to nonzero ρ_σ which bring in a finite, regulator independent correction.

The field theory result (29) has a great advantage over its DR and ZR counterparts in that it allows a straightforward comparison with results of 4D $N=1$ heterotic string orbifolds with $N=2$ sectors and Wilson lines [12], when this string result is considered in the limit of large compactification radii/area (in string units) [16], as required by Eq. (30). The UV regulator $\xi \sim 1/\Lambda^2$ has a natural counterpart in the (heterotic) string in $\alpha' \sim 1/M_s^2$ (M_s is the string scale). Therefore, T_2/ξ of (29) has a counterpart at the string level in T_2/α', where T_2/α' is the (imaginary part of the) Kähler structure moduli. With the correspondence of the fundamental lengths in field and string theory respectively, $\xi \leftrightarrow \alpha'$, the result (29) is indeed similar [16] to the limit of large radii of the heterotic string result [12]. Such agreement provides support for this regularization scheme in the field theory approach, although it is not gauge invariant. String theory also brings additional corrections, nonperturbative on the field theory side (world-sheet instantons) but their effect is exponentially suppressed, $\mathcal{O}(e^{-T_2/\alpha'})$ [12]. For more details on the exact link with the corrections to the gauge couplings due to the heterotic string with Wilson lines present, see Ref. [16].

The effective field theory result (29) has an interesting limit, that of vanishing Wilson lines VEV's or twists $\rho_{i,\sigma}$. For $\rho_{i,\sigma} \to 0$ (σ fixed) after using the relations in Eqs. (A61) and (A62) one finds

$$\Omega_i|_{PT}(\rho_{i,\sigma}\to 0) = -\frac{\beta_i(\sigma)}{4\pi}\ln[4\pi e^{-\gamma} e^{-T_2/\xi} \times (T_2/\xi) U_2 |\eta(U)|^4],$$

$$T_2/\xi = \Lambda^2 R_1 R_2 \sin\theta. \tag{31}$$

For two compact dimensions this result generalizes the "power-law" corrections (in the UV cutoff) of Ref. [25], by including the dependence on $U = R_2/R_1 e^{i\theta}$.

The field theory result (31) is itself the exact limit [14,15] of "large $R_{1,2}$" (in string units) of the result in 4D $N=1$ heterotic string orbifolds with $N=2$ sectors and without Wilson lines [9]. The only difference[7] between Ω_i of (31) and the above limit of the string result [9] is that the leading term T_2/ξ in Ω_i has a coefficient that depends on the regulator choice (ξ) while in string case at "large $R_{1,2}$" the leading term is[8] $(\pi/3) T_2/\alpha'$. With the correspondence $\xi \leftrightarrow \alpha'$ mentioned before, the exact matching of these two terms thus requires a redefinition of the PT regulator $\xi \to (3/\pi)\xi$ or equivalently $\Lambda^2 \to \pi/3 \Lambda^2$. Such specific normalization of ξ (or Λ) cannot be motivated on field theory grounds only.

It is interesting to mention that imposing on the field theory result (31) one of the string symmetries $T \leftrightarrow U$ or $T \leftrightarrow 1/T$ enables one to recover the *full* heterotic string result [9] from that derived using only field theory methods. Thus one may obtain full string results by using only field theory methods supplemented by some of the symmetries of the string, not respected by the field theory approach, but imposed on the final field theory result. For more details on the exact link with the heterotic string without Wilson lines, see Refs. [14,15]. This ends our discussion on the corrections in the PT regularization scheme and their relation to string theory.

Comparing the results for Ω_i in the three regularization schemes, Eqs. (27) to (29), one notices that the finite (regulator-independent) part of Ω_i is the same in all cases which is a consistency check of the calculation. An important point to mention is that the result in the PT scheme has the constraint that the compactification radii be large (in UV cutoff units). The results in the DR and ZR schemes show that the finite part of the one-loop correction has the value found without such restrictions.

Regarding the divergent part of the one-loop corrections, this is effectively dictated by the regularization choice one has to make, in agreement with the symmetries of the model. Our discussion above shows that for two compact dimensions the PT regularization is indeed appropriate in calculations seeking the link with their string counterparts. Further, the ζ-function regularization leads to an UV divergence that is milder (logarithmic) than in the PT scheme with possible phenomenological implications. This is important because models with "powerlike" regime require in general a significant amount of fine-tuning [32]. It is difficult to justify, without the knowledge of the full higher-dimensional theory, in which case the ζ-function regularization is the right choice. The results of Eqs. (27) to (31) generalize, in the presence of Wilson lines, early results [25] for the radiative corrections from two compact dimensions.

The one-loop corrections obtained in the DR, ZR, or PT schemes have strong similarities with their one-dimensional counterparts, Eqs. (17) to (23), with $T_2 U_2$ and $\rho_{2,\sigma} - U\rho_{1,\sigma}$ of Eqs. (27) to (29) replaced in the one-dimensional case by $R\mu$ and ρ_σ respectively, while $\ln(\vartheta_1/\eta)$ has as counterpart in the one-dimensional case the term $\ln[\sin\pi(\rho + i\chi)]$. A similar term appears in compactification on G_2 manifolds [19] suggesting that this latter correction is rather generic.

We end with a remark on possible phenomenological implications. The result for Ω_i has a divergence which depends—as expected—on the regularization choice. Since this is a nonrenormalizable theory, a natural question is whether one can make a prediction without the knowledge of the fundamental, underlying theory that would otherwise dictate the regularisation to use. If the gauge group G after orbifolding is a grand unified group which is further broken by Wilson lines to a SM-like group, the coefficient of the (regularization-dependent) divergent terms found in Ω_i^* is the same for all group factors into which G is broken (G-invariant). If so, such UV divergent terms of Ω_i^* can then be absorbed into the redefinition of the initial 4D tree

[7]See however Ref. [15] and the discussion in the DR scheme.

[8]The presence of $\pi/3$ is a "remnant" of the modular invariance symmetry of the string.

level coupling of the group[9] G. The newly defined coupling can be regarded as the 4D ''MSSM-like'' unified coupling. Further, the remaining, *finite* part of Ω_i^* brings a splitting term to this coupling, due to Wilson line VEV ρ_σ, but independent on the UV cutoff (regularization). Finally, the ''MSSM-like'' massless states not included so far would bring the usual logarithmic correction (UV scale dependent). This raises the possibility of allowing MSSM-like logarithmic unification even for ''large'' compact dimensions, and the aforementioned splitting of couplings would ''mimic'' (at a scale of the order of the compactification scale) what could be seen from a 4D point of view as further running[10] up to a high unification scale, such as that of the MSSM ($\approx 2 \times 10^{16}$ GeV) or higher.

IV. CONCLUSIONS

The general structure of radiative corrections to gauge couplings was investigated in generic 4D models with one- and two-dimensional compactifications in the presence of Wilson lines. The analysis was based on the following observation. Although one-loop corrections are dependent on the exact field content of the model, for the compactifications considered one can still perform in a general case, the one-loop integral and the infinite sums over (nonzero) Kaluza-Klein levels associated with a given state, component of a multiplet. This leaves the much simpler analysis of determining which states have associated Kaluza-Klein towers, to a model-by-model analysis.

The evaluation of the one-loop radiative corrections from compact dimensions summed up the individual effects of nonzero-level Kaluza-Klein modes. Although the models are nonrenormalizable, the calculation was kept general by considering the radiative effects in three regularization schemes: dimensional, zeta-function, and proper-time cutoff regularizations for the UV divergences and the exact link among these results was investigated. The results in DR and ζ-function regularization schemes are very similar with the notable difference that the (UV) pole structure of the DR scheme ($1/\epsilon$) is not present in the ζ-function regularization. This applies to both one and two extra dimension cases. In the ZR scheme for Ω_i only a logarithmic divergence in the UV cutoff scale is present. This is important, since it provides an amount of sensitivity of the radiative corrections to this scale smaller than that of other regularizations, which may be relevant for phenomenology. In the DR and ZR schemes the finite part of the results is valid for either large or small compactification radii, for both one and two compact dimension cases.

In proper-time regularization the leading divergences of the radiative corrections are for one and two compact dimensions linear and quadratic in scale, respectively. The finite (regulator-independent) part is the same as in the DR and zeta-function regularization, which is a strong consistency check of the calculation. The result in the proper-time regularization is only valid for ''large'' compactification radii (in UV cutoff units), a constraint not shared by the results in the DR and ZR schemes. The effect of zero modes (whose existence is model dependent) can easily be added to the results we obtained. In specific cases they may even cancel the divergence from the entire KK tower of *nonzero* modes. Finally, we also discussed the cases when for special values of the background/twists ρ_σ, one obtains one-loop *finite* results for the corrections due to positive or negative modes alone.

There remains the question of which regularization scheme to use in (nonrenormalizable) models with compact dimensions. Explicit calculations and comparison with the (heterotic) string show that proper-time cutoff regularization is in exact quantitative agreement with the limit of large compactification radii of the string results. This applies to the case of two compact dimensions that contribute to the radiative corrections to the gauge couplings. Therefore this regularization is an appropriate choice for computing radiative corrections *for the purpose of establishing the link* with results from string theory. However, this regularization may be of limited use in field theory since is not gauge invariant. For the case of one compact dimension the lack of string results prevents one from making a similar statement, and the choice of regularization should follow the usual guidelines such as its compatibility with the symmetries of the model.

We addressed the possibility of making phenomenological predictions that are independent of the UV divergence of the radiative corrections, which, in the case of a grand unified group G broken by Wilson line VEVs/twist ρ_σ, can be absorbed in the redefinition of the tree level coupling. This leaves a splitting of the couplings at the compactification scale possibly compatible with what can be regarded in a 4D (renormalizable) theory as further ''running'' up to a high, MSSM-like unification scale.

The paper provides all the technical details necessary in models with one and two compact dimensions that examine the one-loop corrections to the gauge couplings from Kaluza-Klein thresholds in the presence of Wilson lines. Although we discussed only the dependence of the corrections on the UV cutoff/regulator, the paper provides the technical results for investigating the change of the gauge couplings with respect to the (momentum) scale q as well. Extensive mathematical details of regularizations of integrals and series present in one-loop corrections due to compact dimensions were provided in the Appendix. Our results can be applied with minimal changes to many one- and two-dimensional orbifolds with Wilson lines, by making appropriate redefinitions of the parameters of the models, such as the compactification radii (R), the twist of the initial fields with respect to the compact dimensions or the Wilson line VEV's (ρ).

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[9]The method of ''absorbing'' the divergences in the initial tree level coupling also exists in heterotic string models [11] where gauge universal, gravitational effects are included in the tree-level coupling, in addition to the dilaton, with the remark that this is actually dictated by the symmetries of the (tree level coupling of) the string.

[10]In a 4D renormalizable theory.

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APPENDIX

We provide general results for series of integrals present in one-loop corrections to the gauge couplings, evaluated in DR, ζ-function, and proper-time cutoff regularizations, for one and two compact dimensions (see also Appendix A4 in Ref. [16]). The notation used is as follows: a "primed" sum $\Sigma'_m f(m)$ is a sum over $m\in\mathbf{Z}-\{0\}$; $\Sigma'_{m,n} f(m,n)$ is a sum over all pairs of integers (m,n) *excluding* $(m,n)=(0,0)$.

1. One compact dimension in dimensional regularization

(1) We compute the following integral:

$$\mathcal{R}_\epsilon \equiv \int_0^\infty \frac{dt}{t^{1+\epsilon}} \sum_{m\in\mathbf{Z}}{}' e^{-\pi t[(m+\rho)^2\beta+\delta]}, \quad \delta\geq 0,\ \beta>0. \tag{A1}$$

Ω_i of Eq. (17) is then given by

$$\Omega_i|_{DR} = \beta_i(\sigma)/(4\pi)\mathcal{R}_\epsilon(\beta\to 1/(R\mu)^2; \quad \delta\to(\chi/\mu)^2; \quad \rho\to\rho_\sigma) \tag{A2}$$

Proof. Consider first $0<\delta/\beta\leq 1$. With the notation $\rho=[\rho]+\Delta_\rho$, $[\rho]\in\mathbf{Z}$, $0\leq\Delta_\rho<1$ one has

$$\begin{aligned}
\mathcal{R}_\epsilon &= \int_0^\infty \frac{dt}{t^{1+\epsilon}}\left[-e^{-\pi t\beta\rho^2}+e^{-\pi t\beta\Delta_\rho^2}+\sum_{n\in\mathbf{Z}}{}' e^{-\pi t(n+\Delta_\rho)^2\beta}\right]e^{-\pi t\delta} \\
&= \Gamma[-\epsilon]\pi^\epsilon\left\{(\delta+\beta\Delta_\rho^2)^\epsilon-(\delta+\beta\rho^2)^\epsilon+\left[\sum_{n>0}[\beta(n+\Delta_\rho)^2+\delta]^\epsilon+(\Delta_\rho\to-\Delta_\rho)\right]\right\} \\
&= \Gamma[-\epsilon]\pi^\epsilon[(\delta+\beta\Delta_\rho^2)^\epsilon-(\delta+\beta\rho^2)^\epsilon]+\Gamma[-\epsilon](\pi\beta)^\epsilon\{\zeta[-2\epsilon,1+\Delta_\rho]+\zeta[-2\epsilon,1-\Delta_\rho]\} \\
&\quad +(\pi\beta)^\epsilon\sum_{k\geq 1}\frac{\Gamma[k-\epsilon]}{k!}\left[\frac{-\delta}{\beta}\right]^k\{\zeta[2k-2\epsilon,1+\Delta_\rho]+(\Delta_\rho\to-\Delta_\rho)\}, \quad 0<\delta/\beta\leq 1.
\end{aligned} \tag{A3}$$

which is convergent under conditions shown. In the last step we used the *binomial* expansion [29]

$$\sum_{n\geq 0}[a(n+c)^2+q]^{-s}=a^{-s}\sum_{k\geq 0}\frac{\Gamma[k+s]}{k!\Gamma[s]}\left[\frac{-q}{a}\right]^k\zeta[2k+2s,c], \quad 0<q/a\leq 1, \tag{A4}$$

Here $\zeta[q,a]$ with $a\neq 0,-1,-2,\ldots$ is the Hurwitz zeta function, with $\zeta[q,a]=\Sigma_{n\geq 0}(a+n)^{-q}$ for $\mathrm{Re}(q)>1$. The Hurwitz zeta-function has one singularity (simple pole) at $q=1$ and $\zeta[q,1]=\zeta[q]$ with $\zeta[q]$ the Riemann zeta function. Further, using Eq. (A3) and the identity

$$\zeta[q,a]=a^{-q}+\zeta[q,a+1],$$

we obtain that

$$\begin{aligned}
\mathcal{R}_\epsilon &= \pi^\epsilon\Gamma[-\epsilon][(\delta+\beta\Delta_\rho^2)^\epsilon-(\delta+\beta\rho^2)^\epsilon]+(\pi\beta)^\epsilon\Gamma[-\epsilon][\zeta[-2\epsilon,1+\Delta_\rho]+\zeta[-2\epsilon,1-\Delta_\rho]] \\
&\quad +\sum_{k\geq 1}\left[\frac{-\delta}{\beta}\right]^k\frac{1}{k}[\zeta[2k,1+\Delta_\rho]+\zeta[2k,2-\Delta_\rho]+(1-\Delta_\rho)^{-2k}] \\
&= \pi^\epsilon\Gamma[-\epsilon][(\delta+\beta\Delta_\rho^2)^\epsilon-(\delta+\beta\rho^2)^\epsilon]+(\pi\beta)^\epsilon\Gamma[-\epsilon][\zeta[-2\epsilon,1+\Delta_\rho]+\zeta[-2\epsilon,1-\Delta_\rho]] \\
&\quad -\ln\frac{|\sin\pi[\Delta_\rho+i(\delta/\beta)^{(1/2)}]|^2}{\pi^2(\Delta_\rho^2+\delta/\beta)}-2\ln[\Gamma[1-\Delta_\rho]\Gamma[1+\Delta_\rho]],
\end{aligned} \tag{A5}$$

provided that

$$0<\frac{\delta}{\beta}\leqslant 1, \quad \frac{\delta}{\beta}<(1+\Delta_\rho)^2, \quad \frac{\delta}{\beta}<(2-\Delta_\rho)^2. \qquad \text{(A6)}$$

Since $\Delta_\rho<1$ we conclude that (A5) is valid if the first condition (the strongest) is respected:

$$0<\frac{\delta}{\beta}\leqslant 1. \qquad \text{(A7)}$$

In the last step of deriving Eq. (A5) for each of the series in zeta functions we used [30]

$$\sum_{k\geqslant 1}\frac{t^{2k}}{k}\zeta[2k,a]=\ln\frac{\Gamma[a+t]\Gamma[a-t]}{\Gamma[a]^2}, \quad |t|<|a|, \qquad \text{(A8)}$$

with $t=i(\delta/\beta)^{1/2}$, $a=1+\Delta_\rho$, $2-\Delta_\rho$ and from which the last two conditions in (A6) emerged. Finally, in the last step in (A5) we also used

$$\prod_{\pm}\Gamma[1\pm x\pm iy]=\frac{\pi^2(x^2+y^2)}{|\sin\pi(x+iy)|^2}, \quad x,y \text{ real}, \qquad \text{(A9)}$$

where the product runs over all 4 combinations of plus/minus signs in the argument of Γ functions. Equation (A9) can be easily proved using that $\Gamma[1-z]\Gamma[1+z]=\pi z/\sin\pi z$.

In Eq. (A5) we now evaluate the ϵ dependent part for $\epsilon\rightarrow 0$ by using (see, for example, Ref. [28])

$$\zeta[-2\epsilon,q]=\frac{1}{2}-q-2\epsilon\frac{d}{dz}\zeta[z,q]|_{z=0}+\mathcal{O}(\epsilon^2)$$

$$\frac{d}{dz}\zeta[z,q]|_{z=0}=\ln\Gamma[q]-\frac{1}{2}\ln(2\pi),$$

$$\Gamma[-\epsilon]=-\frac{1}{\epsilon}-\gamma+\mathcal{O}(\epsilon),$$

$$x^\epsilon=1+\epsilon\ln x+\mathcal{O}(x). \qquad \text{(A10)}$$

We finally find from Eqs. (A5), (A7), and (A10) that (if $\rho\in\mathbf{Z}^*$, $\delta=0$ is excluded)

$$\mathcal{R}_\epsilon=\int_0^\infty\frac{dt}{t^{1+\epsilon}}\sum_{m\in\mathbf{Z}}{}' e^{-\pi t[(m+\rho)^2\beta+\delta]}$$
$$=\frac{1}{\epsilon}-\ln\frac{|2\sin\pi[\Delta_\rho+i(\delta/\beta)^{1/2}]|^2}{\pi e^\gamma\beta(\rho^2+\delta/\beta)}, \quad 0\leqslant\delta/\beta\leqslant 1 \qquad \text{(A11)}$$

(2) We now evaluate $\mathcal{R}_\epsilon$ for the case $\delta/\beta>1$ (with notation $\rho\equiv[\rho]+\Delta_\rho$, $[\rho]\in\mathbf{Z}$, $0\leqslant\Delta_\rho<1$)

$$\mathcal{R}_\epsilon=\int_0^\infty\frac{dt}{t^{1+\epsilon}}\sum_{m\in\mathbf{Z}}{}' e^{-\pi t[(m+\rho)^2\beta+\delta]}$$
$$=\int_0^\infty\frac{dt}{t^{1+\epsilon}}\left[\sum_{n\in\mathbf{Z}}e^{-\pi t(n+\Delta_\rho)^2\beta}-e^{-\pi t\beta\rho^2}\right]e^{-\pi t\delta}$$
$$=\Gamma[-\epsilon]\pi^\epsilon\left[\sum_{n\in\mathbf{Z}}[\beta(n+\Delta_\rho)^2+\delta]^\epsilon-(\delta+\beta\rho^2)^\epsilon\right]. \qquad \text{(A12)}$$

We further use the well-known expansion given below [for details see, for example, (4.13) in Ref. [29]]

$$\sum_{n\in\mathbf{Z}}[a(n+c)^2+q]^{-s}=\sqrt{\pi}\left[\frac{q}{a}\right]^{1/2}\frac{\Gamma[s-1/2]}{\Gamma[s]}q^{-s}+\frac{4\pi^s}{\Gamma[s]}\left[\frac{q}{a}\right]^{1/4}q^{-s/2}a^{-s/2}\times\sum_{n=1}^{\infty}n^{s-1/2}\cos(2\pi nc)\times K_{s-1/2}(2\pi n\sqrt{q/a}) \qquad \text{(A13)}$$

with $a>0$, $c\neq 0,-1,-2,\ldots$ and which is rapidly convergent for $q/a>1$. K_w is the modified Bessel function of index w. The first term proportional to q/a gives the leading contribution; the remaining ones give "instantonlike" corrections. This result is then used to evaluate (A12). Compare (A13) rapidly convergent for $q/a>1$ with (A4) valid for $q/a<1$. Alternatively, instead of (A13) one can simply use a Poisson resummation in (A12) and the definition of the modified Bessel functions to reach the same result. With $s=-\epsilon$ in (A13) and with

$$\Gamma[-\epsilon]=-\frac{1}{\epsilon}-\gamma+\mathcal{O}(\epsilon), \quad K_{-\frac{1}{2}}(z)=\sqrt{\frac{\pi}{2z}}e^{-z}, \qquad \text{(A14)}$$

one finds from (A12) and (A13)

$$\mathcal{R}_\epsilon=\int_0^\infty\frac{dt}{t^{1+\epsilon}}\sum_{m\in\mathbf{Z}}{}' e^{-\pi t[(m+\rho)^2\beta+\delta]}$$
$$=\frac{1}{\epsilon}-\ln\frac{|2\sin\pi[\Delta_\rho+i(\delta/\beta)^{1/2}]|^2}{\pi e^\gamma(\delta+\beta\rho^2)}, \quad \text{if } \delta/\beta>1 \qquad \text{(A15)}$$

To see the complementarity of (A11) and (A15) note that the latter is not valid for $\delta=0$ since (A13) is not valid in that case.

In conclusion from Eqs. (A11) and (A15) we have

$$\mathcal{R}_\epsilon \equiv \int_0^\infty \frac{dt}{t^{1+\epsilon}} \sum_{m\in\mathbf{Z}}{}' e^{-\pi t[(m+\rho)^2\beta+\delta]}$$

$$= \frac{1}{\epsilon} - \ln\frac{|2\sin\pi[\rho+i(\delta/\beta)^{1/2}]|^2}{\pi e^\gamma(\beta\rho^2+\delta)}, \quad \delta\geqslant 0, \beta>0.$$

$$\mathcal{R}_\epsilon^T \equiv \int_0^\infty \frac{dt}{t^{1+\epsilon}} \sum_{m\in\mathbf{Z}} e^{-\pi t[(m+\rho)^2\beta+\delta]}$$

$$= -\ln|2\sin\pi[\rho+i(\delta/\beta)^{1/2}]|^2, \quad \delta\geqslant 0, \beta>0. \tag{A16}$$

In Eq. (A16) we used the properties of the sine function to replace Δ_ρ by ρ. The pole $1/\epsilon$ cancels between zero-mode and nonzero mode contributions. Equation (A16) was used in Eq. (17).

(3) We compute the integral

$$\mathcal{R}_\epsilon^+ \equiv \int_0^\infty \frac{dt}{t^{1+\epsilon}} \sum_{m>0} e^{-\pi t[(m+\rho)^2\beta+\delta]}, \quad \delta\geqslant 0, \beta>0 \tag{A17}$$

which sums positive modes only. $\mathcal{R}_\epsilon^-$, which sums negative modes only, is then $\mathcal{R}_\epsilon^- = \mathcal{R}_\epsilon^+(\rho\to-\rho)$. $\Omega_i^\pm$ mentioned in the text, Eq. (19), and corresponding to summing only positive (negative) Kaluza-Klein modes is then given by

$$\Omega_i^\pm|_{DR} \equiv \beta_i(\sigma)/(4\pi)\mathcal{R}_\epsilon^\pm(\beta\to 1/(R\mu)^2; \delta\to(\chi/\mu)^2; \rho\to\rho_\sigma) \tag{A18}$$

The calculation proceeds almost identically to Appendix Sec. 1 (1). The result is

$$\mathcal{R}_\epsilon^+ = \frac{1}{2\epsilon} + \frac{\rho}{\epsilon} + \ln|\Gamma[1+\rho+i(\delta/\beta)^{1/2}]|^2 - \ln(2\pi) + \left[\frac{1}{2}+\rho\right]\ln(\pi\beta e^\gamma), \tag{A19}$$

which shows that a new divergence ρ/ϵ is present. One can easily verify that

$$\mathcal{R}_\epsilon^+ + \mathcal{R}_\epsilon^- = \mathcal{R}_\epsilon \tag{A20}$$

with $\mathcal{R}_\epsilon$ given in (A16). This shows that the divergence ρ/ϵ of separate contributions from the positive and negative modes is *cancelled in their sum*, which equals $\mathcal{R}_\epsilon$. While $\mathcal{R}_\epsilon$ corresponds to states propagating in both directions in the compact dimension in the "background" ρ, $\mathcal{R}_\epsilon^\pm$ account for effects propagating in one direction only.

Similar properties exist for the full one-loop radiative corrections $\Omega_i^\pm$ given below, corresponding to positive and negative modes, respectively. The radiative correction in DR due to positive (negative) modes only is

$$\Omega_i^\pm|_{DR} \equiv \frac{\beta_i(\sigma)}{4\pi}\mathcal{R}_\epsilon^\pm(\beta\to 1/(R\mu)^2; \delta\to(\chi/\mu)^2; \rho\to\rho_\sigma)$$

$$= \frac{\beta_i(\sigma)}{4\pi}\left\{\frac{1}{2\epsilon} \pm \frac{\rho_\sigma}{\epsilon} + \ln|\Gamma(1\pm\rho_\sigma+i\chi R)|^2 - \ln(2\pi) + \left[\frac{1}{2}\pm\rho_\sigma\right]\ln\frac{\pi e^\gamma}{(R\mu)^2}\right\}. \tag{A21}$$

One finds $\Omega_i^+ + \Omega_i^- = \Omega_i$ with Ω_i as in (17). The "linear" divergence ρ_σ/ϵ cancels between positive and negative modes' contributions.

2. One compact dimension in ζ-function regularization

(a) Here we define/evaluate Ω_i of Eq. (20) in the ZR scheme. The one-loop correction to the gauge couplings in zeta-function regularization is defined by (proportional to) the derivative of the zeta function associated with the Laplacian on the compact manifold and evaluated in 0. To see this note that ζ function of the Laplacian (eigenvalues $\lambda_m > 0$) is defined as

$$\zeta_\Delta[s] \equiv \sum_m{}' \frac{1}{\lambda_m^s} = \frac{1}{\Gamma[s]}\sum_m{}' \int_0^\infty \frac{dt}{t^{1-s}} e^{-\lambda_m t}, \tag{A22}$$

where we use

$$Q^{-s} = \frac{1}{\Gamma[s]}\int_0^\infty \frac{dt}{t^{1-s}} e^{-Qt}, \quad Q>0. \tag{A23}$$

From (A22) the *formal* derivative of the zeta function $\zeta'_\Delta[0]$ is an infinite sum of individual logarithms of λ_m. With λ_m expressed in some mass units μ, $(\lambda_m = M_m^2/\mu^2)$ one has the *formal* result

$$\left.\frac{d\zeta_\Delta[s]}{ds}\right|_{s=0} = -\sum_m{}' \ln\lambda_m = \sum_m{}' \ln(\mu/M_m)^2 \tag{A24}$$

and the link of Ω_i with the one-loop corrections is obvious; μ acts as the effective field theory UV cutoff.

From Eq. (A22) we have

$$\mathcal{R}_\zeta \equiv \left.\frac{d\zeta_\Delta[s]}{ds}\right|_{s=0} = \frac{d}{ds}\left[\frac{1}{\Gamma[s]}\sum_m{}' \int_0^\infty \frac{dt}{t^{1-s}} e^{-\lambda_m t}\right]_{s=0}, \tag{A25}$$

which relates the ζ-function regularization of an operator to its value in the DR scheme. One can also include the contribution of the zero mode $m=0$ (if $\lambda_0\neq 0$) in the definition of $\zeta_\Delta[s]$. Accordingly $\mathcal{R}_\zeta$ changes and is relabelled $\mathcal{R}_\zeta^T$.

With $\lambda_m = (m+\rho)^2\beta+\delta$ as general eigenvalues of Laplacian for one-dimensional case [see Eq. (9)] with boundary conditions given in the text, and using the results of Eq. (A16),

$$\mathcal{R}_\epsilon \equiv \int_0^\infty \frac{dt}{t^{1+\epsilon}} \sum_{m\in\mathbf{Z}}{}' \; e^{-\pi t[(m+\rho)^2\beta+\delta]} = \frac{1}{\epsilon} - \ln\frac{|2\sin\pi[\rho+i(\delta/\beta)^{1/2}]|^2}{\pi e^\gamma(\delta+\beta\rho^2)},$$

$$\mathcal{R}^T_\epsilon \equiv \int_0^\infty \frac{dt}{t^{1+\epsilon}} \sum_{m\in\mathbf{Z}} e^{-\pi t[(m+\rho)^2\beta+\delta]} = -\ln|2\sin\pi[\rho+i(\delta/\beta)^{1/2}]|^2, \tag{A26}$$

we finally find

$$\mathcal{R}_\zeta = -\frac{d}{d\epsilon}\left\{\frac{\pi^{-\epsilon}}{\Gamma[-\epsilon]}R_\epsilon\right\}_{\epsilon=0} = -\ln\frac{|2\sin\pi(\rho+i(\delta/\beta)^{(1/2)})|^2}{(\delta+\beta\rho^2)}$$

$$\mathcal{R}^T_\zeta = -\frac{d}{d\epsilon}\left\{\frac{\pi^{-\epsilon}}{\Gamma[-\epsilon]}R^T_\epsilon\right\}_{\epsilon=0} = -\ln|2\sin\pi(\rho+i(\delta/\beta)^{(1/2)})|^2. \tag{A27}$$

Comparing the results of the last two sets of equations, one notices that (up to a constant) the result in ζ-function regularization is equal to that in DR from which the pole contribution was subtracted.

Equations (A22), (A25), and (A27) allow us to evaluate Ω_i of Eq. (20). This is given by

$$\Omega_i|_{ZR} \equiv \beta_i(\sigma)/(4\pi)\mathcal{R}_\zeta(\delta\to\chi^2/\mu^2, \beta\to 1/(R\mu)^2, \rho\to\rho_\sigma). \tag{A28}$$

According to Eq. (A24) μ should be regarded as the effective field theory UV cutoff.

(b) Using the DR results (A17) of summing over positive (negative) modes only,

$$\mathcal{R}^\pm_\epsilon \equiv \int_0^\infty \frac{dt}{t^{1+\epsilon}} \sum_{m>0} e^{-\pi t[(m\pm\rho)^2\beta+\delta]} = \frac{1}{2\epsilon} \pm \frac{\rho}{\epsilon} + \ln|\Gamma[1\pm\rho+i(\delta/\beta)^{1/2}]|^2 - \ln(2\pi) + \left[\frac{1}{2}\pm\rho\right]\ln(\pi\beta e^\gamma), \tag{A29}$$

one finds the associated ζ-regularized result for positive (negative) mode contribution

$$\mathcal{R}^\pm_\zeta \equiv -\frac{d}{d\epsilon}\left\{\frac{\pi^{-\epsilon}}{\Gamma[-\epsilon]}\mathcal{R}^\pm_\epsilon\right\} = \ln|\Gamma[1\pm\rho+i(\delta/\beta)^{1/2}]|^2 - \ln(2\pi) + \left[\frac{1}{2}\pm\rho\right]\ln\beta. \tag{A30}$$

The effect of positive (negative) modes on the gauge couplings in ζ-function regularization is then

$$\Omega^\pm_i|_{ZR} \equiv \frac{\beta_i(\sigma)}{4\pi}\mathcal{R}^\pm_\zeta(\delta\to\chi^2/\mu^2; \beta\to 1/(R\mu)^2; \rho\to\rho_\sigma) = \frac{\beta_i(\sigma)}{4\pi}\left\{\ln|\Gamma[1\pm\rho_\sigma+i\chi R]|^2 - \ln(2\pi) + \left[\frac{1}{2}\pm\rho_\sigma\right]\ln\frac{1}{(R\mu)^2}\right\}. \tag{A31}$$

This result was used in Eq. (22).

3. One compact dimension in proper-time regularization

Here we provide technical details used to derive the result of Eq. (23). In the proper-time cutoff regularization, the generic structure of the one-loop corrections is

$$\mathcal{R}_\xi \equiv \int_\xi^\infty \frac{dt}{t} \sum_{n\in\mathbf{Z}}{}' \; e^{-\pi t[(n+\rho)^2\beta+\delta]}, \quad \xi\to 0 \ \ (\xi>0), \qquad \delta\geq 0, \ \ \beta>0. \tag{A32}$$

Ω_i of Eq. (23) is then given by

$$\Omega_i|_{PT} \equiv \beta_i(\sigma)/(4\pi)\mathcal{R}_\xi(\beta\to 1/(R\mu)^2; \rho\to\rho_\sigma; \delta\to\chi^2/\mu^2; \xi\to\xi). \tag{A33}$$

To obtain $\mathcal{R}_\xi$ we use Eq. (A9) of Appendix A1 of Ref. [16]. One has

$$\mathcal{R}_\xi = \int_{\beta\xi}^\infty \frac{dt}{t} \sum_{n\in\mathbf{Z}}{}' \; e^{-\pi t[(n+\rho)^2+\delta/\beta]} = \ln[(\rho^2+\delta/\beta)\pi e^\gamma] - \ln\frac{e^{-2/\sqrt{\xi\beta}}}{\xi\beta} - \ln|2\sin\pi[\Delta_\rho+i(\delta/\beta)^{1/2}]|^2 \tag{A34}$$

with Δ_ρ defined after Eq. (A2) and which is valid if

$$\xi\beta \ll \left\{1; \frac{1}{\pi\delta/\beta}; \frac{1}{\pi(\rho^2+\delta/\beta)}\right\}. \tag{A35}$$

One concludes that

$$\mathcal{R}_\xi \equiv \sum_{n\in\mathbf{Z}}{}' \int_\xi^\infty \frac{dt}{t} e^{-\pi t[(n+\rho)^2\beta+\delta]} = -\ln\frac{e^{-2/\sqrt{\xi\beta}}}{\xi} - \ln\frac{|2\sin\pi[\rho+i(\delta/\beta)^{1/2}]|^2}{[\pi e^\gamma(\delta+\beta\rho^2)]},$$

$$\mathcal{R}_\xi^T \equiv \sum_{n\in\mathbf{Z}} \int_\xi^\infty \frac{dt}{t} e^{-\pi t[(n+\rho)^2\beta+\delta]}$$

$$= \frac{2}{\sqrt{\xi\beta}} - \ln|2 \sin \pi[\rho + i(\delta/\beta)^{1/2}]|^2, \tag{A36}$$

with condition (A35). In the above equations we replaced Δ_ρ by ρ.

Note that adding the zero mode to $\mathcal{R}_\xi$ does not cancel the leading linear divergence unlike the cases of DR or ZR schemes. To understand the differences among the various regularization schemes it is useful to compare the above result of the PT regularization equations (A36) and (A35) with that of DR regularization equation (A16), and that of ζ-functions regularization equation (A27).

Equation (A36) was used in the text, Eq. (23).

4. Two compact dimensions in dimensional regularization

(a) For two compact dimensions we evaluate the integral

$$L_\epsilon \equiv \int_0^\infty \frac{dt}{t^{1+\epsilon}} \sum_{m_1,m_2\in\mathbf{Z}}{}' \; e^{-\pi t \tau |m_2+\rho_2-U(m_1+\rho_1)|^2},$$

$$\tau>0; \quad U=U_1+iU_2. \tag{A37}$$

Ω_i of Eq. (27) is then given by

$$\Omega_i|_{DR} = \beta_i(\sigma)/(4\pi) L_\epsilon(\tau\to 1/(T_2U_2), \rho_i \to \rho_{i,\sigma}). \tag{A38}$$

Proof: To compute L_ϵ we use the Poisson resummation Eq. (A65), so the integrand of L_ϵ becomes

$$\sum_{m_1,m_2}{}' \; e^{-\pi t\tau|m_2+\rho_2-U(m_1+\rho_1)|^2} = \sum_{m_2}{}' \; e^{-\pi t\tau|m_2+\rho_2-U\rho_1|^2} + \sum_{m_1}{}' \sum_{m_2\in\mathbf{Z}} e^{-\pi t\tau|m_2+\rho_2-U(m_1+\rho_1)|^2}$$

$$= \sum_{m_2}{}' \; e^{-\pi t\tau|m_2+\rho_2-U\rho_1|^2} + \frac{1}{\sqrt{t\tau}} \sum_{m_1}{}' \; e^{-\pi t\tau U_2^2(m_1+\rho_1)^2}$$

$$+ \frac{1}{\sqrt{t\tau}} \sum_{m_1}{}' \sum_{\tilde{m}_2}{}' \; e^{-\frac{\pi\tilde{m}_2^2}{t\tau} - \pi t\tau U_2^2(m_1+\rho_1)^2 + 2\pi i\tilde{m}_2[\rho_2 - U_1(\rho_1+m_1)]}. \tag{A39}$$

A prime on the double sum in the lhs indicates that the mode $(m_1,m_2)\neq(0,0)$ is excluded. If ρ_1 is non-integer the three series in the rhs of (A39) can be integrated separately over $(0,\infty)$ to find

$$L_\epsilon = L_1 + L_2 + L_3,$$

where

$$L_1 \equiv \int_0^\infty \frac{dt}{t^{1+\epsilon}} \sum_{m_2}{}' \; e^{-\pi t\tau|m_2+\rho_2-U\rho_1|^2} = \frac{1}{\epsilon} - \ln|2 \sin \pi(\rho_2 - U\rho_1)|^2 + \ln[\pi\tau e^\gamma|\rho_2-U\rho_1|^2],$$

$$L_2 \equiv \frac{1}{\sqrt{\tau}} \int_0^\infty \frac{dt}{t^{3/2+\epsilon}} \sum_{m_1}{}' \; e^{-\pi t\tau U_2^2(m_1+\rho_1)^2} = 2\pi U_2\left[|\rho_1| + \frac{1}{6} - \Delta_{\rho_1} + \Delta_{\rho_1}^2\right],$$

$$L_3 \equiv \frac{1}{\sqrt{\tau}} \int_0^\infty \frac{dt}{t^{3/2+\epsilon}} \sum_{m_1}{}' \sum_{\tilde{m}_2}{}' \; e^{-\pi\tilde{m}_2^2 \frac{1}{t\tau} - \pi t\tau U_2^2(m_1+\rho_1)^2 + 2\pi i\tilde{m}_2[\rho_2 - U_1(\rho_1+m_1)]}$$

$$= \ln|2 \sin \pi(\rho_2 - U\rho_1)|^2 - 2\pi U_2\left[|\rho_1| + \frac{1}{6} - \Delta_{\rho_1}\right] - \ln\left|\frac{\vartheta_1(\Delta_{\rho_2} - U\Delta_{\rho_1}|U)}{\eta(U)}\right|^2, \tag{A40}$$

where Δ_y denotes the positive definite fractional part of y defined as $y=[y]+\Delta_y$, $0<\Delta_y<1$, with $[y]$ an integer number. $\vartheta_1(z|\tau)$ and $\eta(U)$ are special functions defined in the Appendix, Eqs. (A61), (A62).

To evaluate L_1 we used Eq. (A15) with the following replacements for the arguments of this equation: $\beta\to\tau$, $\rho \to \rho_2 - U_1\rho_1$ and $\delta \to \tau U_2^2 \rho_1^2$. To compute L_2 we used the results of Appendix A of Ref. [16], Eq. (A22) or more generally Eqs. (A43) and (A45). Regarding L_3, taking the limit $\epsilon\to 0$ is allowed under the integral before performing the integral itself or the two sums. This is justified by technical calculations (not shown) which prove that L_3 is bound by an expression which has no poles in $\epsilon\to 0$. This is actually expected because the integrand is well defined for $t\to 0$ or $t\to\infty$ when $\epsilon=0$. After setting $\epsilon=0$ the integral equals that evaluated in Eqs. (A28)–(A31) in Appendix A 3 of Ref. [16].

Adding together L_1, L_2, L_3 we find for $U=U_1+iU_2$, $\tau>0$

$$L_\epsilon \equiv \int_0^\infty \frac{dt}{t^{1+\epsilon}} \sum_{m_1,m_2\in\mathbf{Z}}{}' e^{-\pi t\tau|m_2+\rho_2-U(m_1+\rho_1)|^2}$$
$$= \frac{1}{\epsilon} + \ln[\pi\tau e^{\gamma}] - \ln\left|\frac{\vartheta_1(\Delta_{\rho_2} - U\Delta_{\rho_1}|U)}{[\rho_2 - U\rho_1]\eta(U)}\right|^2 + 2\pi U_2 \Delta_{\rho_1}^2, \quad \tau>0. \tag{A41}$$

Further, one can make the replacement $\Delta_{\rho_i}\to\rho_i$, due to the identity given by Eq. (A64). Equation (A41) was used in the text, Eq. (27).

Using the properties of ϑ_1 one also finds an interesting limit of L_ϵ for $\rho_1=\rho_2=0$:

$$L_\epsilon(\rho_{1,2}=0) \equiv \int_0^\infty \frac{dt}{t^{1+\epsilon}} \sum_{m_1,m_2\in\mathbf{Z}}{}' e^{-\pi t\tau|m_2-Um_1|^2}$$
$$= \frac{1}{\epsilon} + \ln[\tau e^{\gamma}/(4\pi)] - \ln|\eta(U)|^4, \quad \tau>0 \tag{A42}$$

in agreement with Eq. (B12) of Ref. [14]. Note that the contribution of the (0,0) mode-if added to L_ϵ-would cancel the pole $1/\epsilon$ and $\ln\tau|\rho_2-U\rho_1|$ term above.

(b) One important observation is in place here. To find the scale dependence of the divergence ($1/\epsilon$) of L_ϵ in the DR scheme one can introduce a small/infrared $(\text{mass})^2$ parameter $\mu^2\delta$ (δ dimensionless, $\delta>0$) in addition to the $(\text{mass})^2$ of the Kaluza-Klein states in the exponent in Eqs. (26) and (A37). This amounts to multiplying the integrand in Eq. (26) by $e^{-\pi t\delta\mu^2}$ or that in (A37) by $e^{-\pi t\delta}$. After lengthy algebra, one obtains the following change for L_1, L_2, L_3:

$$L_1 \to L_1' = L_1, \quad \text{if } \delta\to 0$$
$$L_2 \to L_2' = L_2 + \frac{\pi\delta}{\epsilon}\frac{1}{\tau U_2}, \quad \text{if } \delta\to 0$$
$$L_3 \to L_3' = L_3, \quad \text{if } \delta\to 0. \tag{A43}$$

As a result

$$L_\epsilon' \equiv \int_0^\infty \frac{dt}{t^{1+\epsilon}} \sum_{m_{1,2}\in\mathbf{Z}}{}' e^{-\pi t\tau|m_2+\rho_2-U(m_1+\rho_1)|^2-\pi\delta t}$$
$$= L_1' + L_2' + L_3' = L_\epsilon + \frac{\pi\delta}{\epsilon}\frac{1}{\tau U_2}, \quad \text{with } \delta\to 0;$$
$$\delta,\tau>0; \quad U=U_1+iU_2 \tag{A44}$$

with L_ϵ given in (A41). Therefore a divergence is emerging $\delta/(\tau U_2\epsilon)$, induced by the change of L_2. With $\tau=1/(T_2U_2)$ the divergence is proportional to T_2/ϵ, and is quadratic in mass, given the definition of T_2. It is similar to that of proper-time regularization (T_2/ξ); see Appendix A (6). Note that L_2', which brings in this term, is a contribution from both compact dimensions, as Kaluza-Klein mode effects from one dimension and Poisson resummed Kaluza-Klein zero modes of the second compact dimension. Also note a particular and useful limit of Eq. (A44), that with $\rho_1=\rho_2=0$.

(c) For future reference we also give the result of computing the integral:

$$L_\epsilon^* \equiv \int_0^\infty \frac{dt}{t^{1+\epsilon}} \sum_{m_1,m_2\in\mathbf{Z}}{}' e^{-\pi t\tau|Um_1-m_2|^2-\pi t\delta} \quad \tau>0;$$
$$\delta\geq 0,\ U\equiv U_1+iU_2. \tag{A45}$$

Proof. Following the steps in Eq. (A39) one has $L_\epsilon^*=L_1^*+L_2^*+L_3^*$ with

$$L_1^* \equiv \int_0^\infty \frac{dt}{t^{1+\epsilon}} \sum_{m_2}{}' e^{-\pi t\tau m_2^2-\pi\delta t}$$
$$= \frac{1}{\epsilon} - \ln\left[\frac{|2\sinh\pi(\delta/\tau)^{1/2}|^2}{\pi e^{\gamma}\delta}\right],$$

$$L_2^* \equiv \frac{1}{\sqrt{\tau}} \int_0^\infty \frac{dt}{t^{3/2+\epsilon}} \sum_{m_1}{}' e^{-\pi t\tau U_2^2 m_1^2-\pi\delta t}$$
$$= \frac{\pi U_2}{3} + \frac{\pi\delta}{\epsilon\tau U_2} + \frac{\pi\delta}{\tau U_2}\ln[4\pi e^{-\gamma}\tau U_2^2] + 2\sqrt{\pi}U_2 \sum_{k\geq 1} \frac{\Gamma[k+1/2]}{(k+1)!}\left[\frac{-\delta}{\tau U_2^2}\right]^{k+1}\zeta[2k+1], \tag{A46}$$

$$L_3^* \equiv \frac{1}{\sqrt{\tau}} \int_0^\infty \frac{dt}{t^{3/2+\epsilon}} \sum_{m_1}{}' \sum_{\tilde{m}_2}{}' \times e^{-\pi\tilde{m}_2^2/(t\tau)-\pi t\tau U_2^2 m_1^2-2i\pi\tilde{m}_2 m_1 U_1-\pi\delta t}$$
$$= -\ln\prod_{m_1\geq 1} |1-e^{-2\pi(\delta/\tau+U_2^2m_1^2)^{1/2}} e^{2i\pi U_1 m_1}|^4.$$

For L_1^* we used Eq. (A16), for L_2^* see Eqs. (B11) to (B15) in Appendix B of Ref. [15]. For L_3^* one may set $\epsilon=0$ (no poles at $t\to 0$ or $t\to\infty$) and use the integral representation of Bessel function $K_{1/2}$ with $K_{1/2}(z)$ given in (A14). Adding together the above contributions one has

$$L_\epsilon^* = \frac{1}{\epsilon} + \frac{\pi\delta}{\epsilon}\frac{1}{\tau U_2} - \ln\left[4\pi e^{-\gamma}\frac{1}{\tau}|\eta(U)|^4\right] + \frac{\pi\delta}{\tau U_2}\ln(4\pi e^{-\gamma}\tau U_2^2) - 2\ln\frac{\sinh\pi(\delta/\tau)^{1/2}}{\pi(\delta/\tau)^{\frac{1}{2}}} + \mathcal{W}\left(\frac{\delta}{\tau}\right), \tag{A47}$$

with the constraint $0<\delta|U|^2/(U_2^2\tau)\leqslant 1$, $0<\delta/(\tau U_2^2)\leqslant 1$. Also $\mathcal{W}(y\to 0)\to 0$ and is defined as

$$\mathcal{W}(y) \equiv 2\sqrt{\pi}U_2\sum_{k\geqslant 1}\frac{\Gamma[k+1/2]}{(k+1)!}\left[\frac{-y}{U_2^2}\right]^{k+1}\zeta[2k+1] - \ln\prod_{m_1\geqslant 1}\frac{|1-e^{-2\pi(y+U_2^2m_1^2)^{1/2}+2i\pi U_1 m_1}|^4}{|1-e^{2i\pi U m_1}|^4}. \tag{A48}$$

5. Two compact dimensions in ζ-function regularization

(a) Here we derive the result for Ω_i of Eq. (28). The one-loop correction to gauge couplings in ζ-function regularization is proportional to the derivative of ζ function associated with the Laplacian on the compact manifold, evaluated in 0.

The ζ function of the Laplacian (eigenvalues $\lambda_{m,n}>0$) is defined as

$$\zeta_\Delta[s] \equiv \sum_{m,n\in\mathbf{Z}}{}' \frac{1}{\lambda_{m,n}^s} = \frac{1}{\Gamma[s]}\sum_{m,n\in\mathbf{Z}}{}' \int_0^\infty \frac{dt}{t^{1-s}}e^{-\lambda_{m,n}t}, \tag{A49}$$

where we used Eq. (A23) and the "primed" sum excluded the (0,0) mode. Note that as in the one-extra-dimension case, one can express $\lambda_{m,n}$ in some mass units μ, $\lambda_{m,n} = M_{m,n}^2/\mu^2$ and one has that, *formally*

$$\left.\frac{d\zeta_\Delta[s]}{ds}\right|_{s=0} = -\sum_{m,n}{}' \ln\lambda_{m,n} = \sum_{m,n}{}' \ln(\mu/M_{m,n})^2 \tag{A50}$$

and one can see the link of this derivative with the one-loop radiative corrections, given by a sum over individual logarithmic corrections, with μ acting as the UV cutoff of the model. Up to a beta function coefficient, Eq. (A50) is also in agreement with the formal expression in Eq. (1).

From Eq. (A49) one has

$$L_\zeta \equiv \left.\frac{d\zeta_\Delta[s]}{ds}\right|_{s=0} = \frac{d}{ds}\left[\frac{1}{\Gamma[s]}\sum_{m,n\in\mathbf{Z}}{}' \int_0^\infty \frac{dt}{t^{1-s}}e^{-\lambda_m t}\right]_{s=0}, \tag{A51}$$

which relates ζ-function regularization of an operator to (the derivative of) its DR result.

With general eigenvalues of Laplacian for the two-dimensional case [see Eq. (14)]

$$\lambda_{m,n} = \tau|(m_2+\rho_2) - U(m_1+\rho_1)|^2, \quad \tau>0 \tag{A52}$$

and using L_ϵ of Eq. (A41), one has

$$L_\zeta = -\frac{d}{d\epsilon}\left\{\frac{\pi^{-\epsilon}}{\Gamma[-\epsilon]}L_\epsilon\right\} = \ln[\tau] - \ln\left|\frac{\vartheta_1(\Delta_{\rho_2} - U\Delta_{\rho_1}|U)}{[\rho_2 - U\rho_1]\eta(U)}\right|^2 + 2\pi U_2\Delta_{\rho_1}^2. \tag{A53}$$

One can further replace $\Delta_{\rho_i}\to\rho_i$, due to the identity in Eq. (A64). The result in ζ-function regularization is equal to that in DR from which the contribution of the pole was subtracted.

(b) Equation (A53) was used to evaluate Ω_i in Eq. (28) with

$$\Omega_i|_{ZR} = \beta_i(\sigma)/(4\pi)L_\zeta(\tau\to 1/(T_2U_2), \rho_i\to\rho_{i,\sigma}). \tag{A54}$$

6. Two compact dimensions in proper-time regularization

In the PT regularization one evaluates (see the Appendix in Ref. [16])

$$L_\xi \equiv \int_\xi^\infty \frac{dt}{t}\sum_{m_{1,2}\in\mathbf{Z}}{}' e^{-\pi t\tau|m_2+\rho_2-U(m_1+\rho_1)|^2}, \qquad \xi\to 0, \quad \xi>0, \quad \tau>0 \tag{A55}$$

with $U\equiv U_1 + iU_2$. Therefore Ω_i of Eq. (29) is

$$\Omega_i|_{PT} = \beta_i(\sigma)/(4\pi)L_\xi(\tau\to 1/(T_2U_2)) \tag{A56}$$

Using the results of the Appendix in Ref. [16] one has

$$L_\xi = \left\{\frac{1}{\xi\tau U_2} + \ln\xi\right\} + \ln[\pi e^\gamma\tau] - \ln\left|\frac{\vartheta_1(\Delta_{\rho_2} - U\Delta_{\rho_1}|U)}{|\rho_2 - U\rho_1|\eta(U)}\right|^2 + 2\pi U_2\Delta_{\rho_1}^2, \tag{A57}$$

with the condition

$$\frac{1}{\tau\xi} \gg \{U_2^2, 1/U_2^2\}. \tag{A58}$$

The (divergent) expression in the curly braces is corresponding to $1/\epsilon$ in the DR result, Eq. (A41). Finally

$$L_\xi(\tau\to 1/(T_2U_2)) = \frac{T_2}{\xi} + \ln\frac{\pi e^\gamma}{(T_2/\xi)U_2} - \ln\left|\frac{\vartheta_1(\Delta_{\rho_2} - U\Delta_{\rho_1}|U)}{(\rho_2 - U\rho_1)\eta(U)}\right|^2 + 2\pi U_2\Delta_{\rho_1}^2, \tag{A59}$$

with

$$\max\left\{\frac{1}{R_1}, \frac{1}{R_2\sin\theta}, \frac{|\rho_1|}{R_1}, \frac{|\rho_2 - U\rho_1|}{R_2\sin\theta}\right\} \ll \Lambda, \tag{A60}$$

which was derived in Eq. (52) of [16]. Here $T_2 = \mu^2 R_1 R_2 \sin\theta$, $U_2 = R_2/R_1 \exp(i\theta)$ and $\Lambda^2 \equiv \mu^2/\xi$. One can make the replacement $\Delta_{\rho_i} \to \rho_i$, due to the identity given in Eq. (A64).

7. Mathematical appendix, definitions, and conventions

In the text we used the special function η

$$\eta(\tau) \equiv e^{\pi i \tau/12} \prod_{n\geqslant 1} (1 - e^{2i\pi\tau n}),$$

$$\eta(-1/\tau) = \sqrt{-i\tau}\,\eta(\tau), \qquad \eta(\tau+1) = e^{i\pi/12}\eta(\tau). \tag{A61}$$

We also used the Jacobi function ϑ_1,

$$\vartheta_1(z|\tau) \equiv 2q^{1/8}\sin(\pi z) \prod_{n\geqslant 1} (1-q^n)(1-q^n e^{2i\pi z}) \times (1-q^n e^{-2i\pi z}),$$

$$q \equiv e^{2i\pi\tau} = \frac{1}{i}\sum_{n\in\mathbf{Z}} (-1)^n e^{i\pi\tau(n+1/2)^2} e^{(2n+1)i\pi z}, \tag{A62}$$

which has the properties

$$\vartheta_1'(0|\tau) = 2\pi\eta^3(\tau),$$

$$\vartheta_1'(0|\tau) \equiv \partial\vartheta_1(\nu|\tau)/\partial\nu|_{\nu=0},$$

$$\vartheta_1(\nu|\tau+1) = e^{i\pi/4}\vartheta_1(\nu|\tau),$$

$$\vartheta_1(\nu+1|\tau) = -\vartheta_1(\nu|\tau), \tag{A63}$$

$$\vartheta_1(\nu+\tau|\tau) = -e^{-i\pi\tau-2i\pi\nu}\vartheta_1(\nu|\tau)$$

$$\vartheta_1(-\nu/\tau|-1/\tau) = e^{i\pi/4}\tau^{1/2}\exp(i\pi\nu^2/\tau)\,\vartheta_1(\nu|\tau).$$

Our conventions for ϑ_1 are those of Ref. [3]. $\vartheta_1(z|\tau)$ above is equal to $\vartheta_1(\pi z|\tau)$ of Ref. [28], Eq. 8.180(2).

Using these properties one can show that

$$-\ln|\vartheta_1(\Delta_{\rho_2} - U\Delta_{\rho_1}|U)|^2 + 2\pi U_2 \Delta_{\rho_1}^2 = -\ln|\vartheta_1(\rho_2 - U\rho_1|U)|^2 + 2\pi U_2 \rho_1^2, \tag{A64}$$

where Δ_{ρ_i} is the fractional part of ρ_i defined as $\rho_i = [\rho_i] + \Delta_{\rho_i}$, $[\rho] \in \mathbf{Z}$, $0 \leqslant \Delta_{\rho_i} < 1$. Throughout the Appendix we used the Poisson resummation formula:

$$\sum_{n\in Z} e^{-\pi A(n+\sigma)^2} = \frac{1}{\sqrt{A}} \sum_{\tilde{n}\in Z} e^{-\pi A^{-1}\tilde{n}^2 + 2i\pi\tilde{n}\sigma}. \tag{A65}$$

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