Quantum mechanics model on a Kähler conifold

Stefano Bellucci,[1] Armen Nersessian,[2,3] and Armen Yeranyan[2]
[1]*INFN-Laboratori Nazionali di Frascati, P.O. Box 13, I-00044, Italy*
[2]*Yerevan State University, Alex Manoogian St., 1, Yerevan 375025, Armenia*
[3]*Yerevan Physics Institute, Alikhanian Brothers St., 2, Yerevan 375036, Armenia*
(Received 16 January 2004; published 17 August 2004)

We propose an exactly solvable model of the quantum oscillator on the class of Kähler spaces (with conic singularities), connected with two-dimensional complex projective spaces. Its energy spectrum is nondegenerate in the orbital quantum number, when the space has nonconstant curvature. We reduce the model to a three-dimensional system interacting with the Dirac monopole. Owing to noncommutativity of the reduction and quantization procedures, the Hamiltonian of the reduced system gets nontrivial quantum corrections. We transform the reduced system into a MIC-Kepler-like one and find that quantum corrections arise only in its energy and coupling constant. We present the exact spectrum of the generalized MIC-Kepler system. The one-(complex) dimensional analog of the suggested model is formulated on the Riemann surface over the complex projective plane and could be interpreted as a system with fractional spin.

DOI: 10.1103/PhysRevD.70.045006 PACS number(s): 03.65.−w, 02.30.Ik, 14.80.Hv

I. INTRODUCTION

The quantum oscillator ranks as a most important system of quantum mechanics, due to the existence of an overcomplete set of hidden symmetries which form a linear algebra. The hidden symmetries provide the oscillator with unique properties, e.g., a degenerate quantum-mechanical energy spectrum, the separability of variables and exact solvability in a several coordinate system. This allows one to preserve the exact solvability even after some deformation of the potential, or, at least, to simplify the perturbative calculations. These features make the oscillator to be a relevant system in a wide class of problems in theoretical physics, including string/field theory, gravity, and condensed matter. On the other hand, most problems in modern theoretical physics deal with higher-dimensional $(d>3)$ curved spaces. However, the quantum oscillator is generalized for spheres and hyperboloids only [1], which have a constant curvature and no singularities. For $d>2$ these spaces have no Kähler structure; consequently, the corresponding oscillators have a bad behavior with respect to supersymmetrization and the inclusion of a constant magnetic field. Moreover, in (super)string/brane and (super)gravity theories Kähler spaces [2] and spaces with conic singularities (conifolds), including Kähler conifolds, are of special importance (see, e.g., Ref. [3] and references therein).

In this note we propose a model of the four-dimensional quantum oscillator on the (ν,ϵ) parametric family of Kähler conifolds related to the nonsingular cases of the complex projective space $\mathbb{C}P^2$ (when $\nu=1$ and $\epsilon=1$) and its noncompact version, i.e., the four-dimensional Lobacewski space $\mathcal{L}_2$ (for $\nu=1$ and $\epsilon=-1$). The Kähler structure is defined by the potential

$$K=\frac{r_0^2}{2\epsilon}\log[1+\epsilon(z\bar{z})^{\nu}],\quad \nu>0,\quad \epsilon=\pm1, \tag{1}$$

so that the corresponding metric is given by the expression

$$g_{a\bar{b}}=\frac{\nu r_0^2(z\bar{z})^{\nu-1}}{2[1+\epsilon(z\bar{z})^{\nu}]}\left(\delta_{a\bar{b}}-\frac{1-\nu+\epsilon(z\bar{z})^{\nu}}{z\bar{z}[1+\epsilon(z\bar{z})^{\nu}]}\bar{z}^a z^b\right). \tag{2}$$

The scalar curvature takes the form

$$R=-\frac{4}{\nu r_0^2}\frac{\nu-1-\epsilon(2\nu+1)(z\bar{z})^{\nu}}{z\bar{z}}. \tag{3}$$

We choose the following oscillator potential:

$$V_{osc}=\omega^2 g^{\bar{a}b}\partial_{\bar{a}}K\partial_b K=\frac{\omega^2 r_0^2}{2}(z\bar{z})^{\nu}. \tag{4}$$

The exact classical solvability of the model was established in Refs. [4,5]. The potential (4) is distinguished also with its respect to supersymmetrization and inclusion of a constant magnetic field [4].

The system is described by the Schrödinger equation

$$\hat{\mathcal{H}}\Psi=E\Psi,\quad \hat{\mathcal{H}}=-\hbar^2 g^{a\bar{b}}\partial_a\partial_{\bar{b}}+V_{osc}, \tag{5}$$

where the metric and the potential are given by expressions (2) and (4), respectively. It is invariant under $U(2)$ rotations defined by the operators

$$2\hat{J}_0=z\partial-\overline{z\partial},\quad 2\hat{\mathbf{J}}=z\boldsymbol{\sigma}\partial-\overline{\partial\boldsymbol{\sigma}z},$$
$$[J_0,\mathbf{J}]=0,\quad [J_i,J_k]=\imath\epsilon_{ikl}J_l, \tag{6}$$

where $\boldsymbol{\sigma}$ are standard Pauli matrices, and $i,k,l=1,2,3$. Here and further below we use the notation $\partial_a=\partial/\partial z^a$, $\partial_{\bar{a}}=\partial/\partial\bar{z}^a$.

We find that the system has remarkable properties. It is exactly solvable: We find its *energy spectrum* and construct *a complete basis of wave functions*. On nonconstant curvature spaces the spectrum is nondegenerate in the orbital quantum number. Moreover, even on constant curvature spaces the spectrum is nondegenerate in the eigenvalue of the operator $\hat{J}_0$. Reducing the quantum Hamiltonian on the three-dimensional (ν,ϵ) parametric space we find that it gets a correction, with respect to the quantized three-dimensional Hamiltonian reduced from four dimensions classically. In other words, the reduction and quantization are noncommutative operations in the proposed model. The reduced system is specified by the presence of a Dirac monopole field. In the particular case $\nu=4$ it has no singularity, as its configuration space is a three-dimensional sphere/pseudosphere (two-sheet hyperboloid). In this case the four-dimensional potential reduces to the potential of the oscillator on a (pseudo)sphere. The reduced oscillator could be converted into another exactly solvable system on the $(4\nu,-1)$-parametric three-dimensional space. For $\nu=1$ it coincides with the MIC-Kepler system (i.e., the superintegrable generalization of the Coulomb system with the Dirac monopole [6]) on the two-sheet hyperboloid [7]. Hence, the latter system could be viewed as a generalization of the MIC-Kepler[1] system on nonconstant curvature conifolds. Quantum corrections change the only value of the coupling constant and the energy, so that one could get the energy spectrum and the wave functions of the MIC-Kepler-like system from the oscillator ones! The transformation to the Coulomb-like system does not have a mere academic interest. Being related to the Hopf map, it has numerous applications in physics. The newest one is the higher-dimensional quantum Hall effect [9]. Note that the obtained results could be straightforwardly extended to higher dimensions.

II. WAVE FUNCTIONS AND SPECTRUM

The Schrödinger equation (5) could be separated in the spherical coordinates

$$z^1=x^{1/\nu}\cos(\beta/2)\exp[(\imath/2)(\alpha+\gamma)],$$
$$z^2=-\imath x^{1/\nu}\sin(\beta/2)\exp[-(\imath/2)(\alpha-\gamma)], \tag{7}$$

upon the following choice of the wave function:

$$\Psi=\psi(x)D^j_{m,s}(\alpha,\beta,\gamma). \tag{8}$$

Here $\alpha\in[0,2\pi)$, $\beta\in[0,\pi]$, and $\gamma\in[0,4\pi)$, and x is a dimensionless radial coordinate taking values in the interval $[0,\infty)$ for $\epsilon=+1$, and in $[0,1]$ for $\epsilon=-1$. In the Wigner function $D^j_{m,s}(\alpha,\beta,\gamma)$ j, m denote orbital and azimuthal quantum numbers, respectively, while s is the eigenvalue of the operator $\hat{J}_0$

$$\hat{J}_0\Psi=s\Psi, \tag{9}$$

$$\hat{\mathbf{J}}^2\Psi=j(j+1)\Psi, \quad \hat{J}_3\Psi=m\Psi, \tag{10}$$

$$m,s=-j,-j+1,\dots,j-1,j \quad j=0,1/2,1,\dots \tag{11}$$

The volume element reads

$$dV_{(4)}=\frac{\nu^2r_0^4}{32}\frac{x^3}{(1+x^2)^3}\sin\beta dx d\alpha d\beta d\gamma. \tag{12}$$

The radial Schrödinger equation looks as follows:

$$\frac{d^2\psi}{dx^2}+\frac{3+\epsilon x^2}{1+\epsilon x^2}\frac{1}{x}\frac{d\psi}{dx}+\left[\frac{2r_0^2E+\epsilon\omega^2r_0^4}{\hbar^2(1+\epsilon x^2)^2}-\frac{4\nu j(j+1)+4(1-\nu)s^2}{\nu^2x^2(1+\epsilon x^2)}-\frac{\epsilon\delta^2}{1+\epsilon x^2}\right]\psi=0, \tag{13}$$

where

$$\delta^2=4\frac{s^2}{\nu^2}+\frac{\omega^2r_0^4}{\hbar^2}. \tag{14}$$

Making the further substitution

$$x=\tan\theta, \quad \text{for} \quad \epsilon=1,$$
$$x=\tanh\theta, \quad \text{for} \quad \epsilon=-1, \tag{15}$$

we shall get the regular wave functions, which form a complete orthonormal basis,

$$\psi=\begin{cases} C\sin^{j_1-1}\theta\cos^{\delta}\theta\, {}_2F_1(-n,n+\delta+j_1+1;j_1+1;\sin^2\theta), & \text{for} \quad \epsilon=1\\ C\sinh^{j_1-1}\theta\cosh^{-\delta+2n}\theta\, {}_2F_1(-n,-n+\delta,j_1+1,\tanh^2\theta), & \text{for} \quad \epsilon=-1\end{cases} \tag{16}$$

and the energy spectrum

$$E_{n,j,s}=\frac{\epsilon\hbar^2}{2r_0^2}\left(2n+j_1+\epsilon\sqrt{\frac{4s^2}{\nu^2}+\frac{\omega^2r_0^4}{\hbar^2}+1}\right)^2-\frac{2\epsilon\hbar^2}{r_0^2}-\frac{\omega^2r_0^2}{2\epsilon}. \tag{17}$$

Here

$$j_1^2=\frac{4j(j+1)}{\nu}+1-\frac{4(\nu-1)s^2}{\nu^2}, \tag{18}$$

whereas

$$n=\begin{cases}0,1,\dots,\infty & \text{for} \quad \epsilon=1\\ 0,1,\dots,n^{max}=[\delta/2-j-1], & \text{for} \quad \epsilon=-1\end{cases} \tag{19}$$

is the radial quantum number. The normalization constants are defined by the expression

[1]The MIC (McIntosh-Cisneros)-Kepler integrable system was constructed by Zwanziger and rediscovered by McIntosh-Cisneros [6].

$$\frac{\nu^2 r_0^4 \pi^2 n!\Gamma^2(j_1+1)}{4(2j+1)\Gamma(n+j_1+1)}C^2=\begin{cases}(2n+j_1+1+\delta)\Gamma(n+j_1+1+\delta)/\Gamma(n+1+\delta), & \text{for } \epsilon=1,\\ (\delta-2n-j_1-1)\Gamma(\delta-n)/\Gamma(\delta-n-j_1), & \text{for } \epsilon=-1.\end{cases} \quad (20)$$

It is seen that for $\nu\neq 1$ the energy spectrum is degenerate in the azimuthal quantum number only. The explicit dependence of the spectrum on the orbital quantum number j is the quantum-mechanical reflection of the unclosedness of classical trajectories [5]. On constant curvature spaces, where $\nu=1$, the spectrum depends on s and $N\equiv 2n+2j$, i.e., it is degenerate in the orbital quantum number j. This degeneracy is due to the existence of a hidden symmetry given by the operators

$$\mathbf{I}=\hbar^2 J\boldsymbol{\sigma}J^\dagger+\bar{z}\boldsymbol{\sigma}z,\quad J_a=\imath\partial_a+\imath\bar{z}^a(\overline{z\,\partial}). \quad (21)$$

In the flat limit ($r_0^2\to\infty$, $\theta\to 0$ and $r_0^2\theta=\text{const}$), we get the correct formula for the four-dimensional oscillator energy spectrum

$$E_N=\hbar\omega(N+2),\quad N=2n+2j=0,1,2,\dots, \quad (22)$$

i.e., $N=2n+2j$ becomes the "principal" quantum number. Notice that in the nonconstant curvature case, $\nu\neq 1$ the infiniteness/finiteness of the energy spectrum is not straightforwardly correlated with the curvature, as opposed to the case of constant curvature spaces [compare (3) and (19)].

The two-dimensional counterpart of our model has a single complex coordinate z. Performing the transformation $w=z^\nu$ we get the Hamiltonian and the angular momentum operator on the Riemann surface over $\mathbb{C}P^1$ (for $\epsilon=1$) or Lobacewski plane $\mathcal{L}$ (for $\epsilon=-1$):

$$\mathcal{H}=-\hbar^2(1+\epsilon w\bar{w})^2\partial_w\partial_{\bar{w}}+\omega^2 r_0^2 w\bar{w},$$

$$2J=\nu(w\partial_w-\bar{w}\partial_{\bar{w}}), \quad (23)$$

where $\arg w\in[0,2\pi\nu)$. The energy of the system is given by expression (17), where s/ν and j_1 are replaced by $\tilde{j}=j/\nu$. For integer values of ν we could make a reduction by the Z_ν group and get a family of oscillators specified by the fractional spin $k=1/\nu,2/\nu,\dots,(1-1/\nu)$ (see Ref. [8]). It this case $\tilde{j}=k,k+1,\dots$ gets the meaning of the orbital quantum number on the complex projective plane. The arising of fractional spins can be interpreted as a consequence of the presence of a magnetic flux tube (see Ref. [8]). As opposed to the higher dimensional case, the spectrum is nondegenerate for any ν, which reflects the absence of hidden symmetry of the system.

III. REDUCTION AND COULOMBLIKE SYSTEMS

There is a well-known Kustaanheimo-Stiefel (KS) transformation relating the four-dimensional oscillator with the three-dimensional Coulomb (and MIC-Kepler [6]) system. A similar transformation of the oscillator on the four-dimensional sphere and (two-sheet) hyperboloid yields the MIC-Kepler system on the three-dimensional hyperboloid [7]. The quantum KS transformation includes a reduction of the Schrödinger equation for the four-dimensional oscillator by the $\hat{J}_0$ operator, with a subsequent transformation to the Schrödinger equation of the MIC-Kepler system.

In Ref. [4] we applied a similar procedure to the classical oscillator on $\mathbb{C}P^2$ and found that, as in the (pseudo)spherical case, it yields the MIC-Kepler system on the three-dimensional hyperboloid.

Let us extend the KS transformation to the proposed model. We begin by considering the reduction of the system to three dimensions. For this purpose we consider Eq. (9) as a constraint and choose the functions below as coordinates of the reduced system:

$$\mathbf{x}=(z\bar{z})^{\nu/2-1}z\boldsymbol{\sigma}\bar{z},\quad [\hat{J}_0,\mathbf{x}]=0,$$
$$x_3=x\cos\beta,\quad x_2+ix_1=x\sin\beta e^{i\alpha}. \quad (24)$$

The wave function of the reduced system is related with the initial one as follows:

$$\Psi_{(3)}(x,\alpha,\beta)=\sqrt{x/(1+\epsilon x^2)}e^{-is\gamma}\Psi. \quad (25)$$

It is convenient to pass to the following coordinates:

$$\mathbf{y}=\left(\frac{\sqrt{1+\epsilon x^2}-1}{\epsilon x}\right)^{2/\nu}\frac{\mathbf{x}}{x}\Rightarrow x=\frac{2y^{\sqrt{\nu}/2}}{1-\epsilon y^{\sqrt{\nu}}}, \quad (26)$$

where the metric of the reduced space takes a conformally flat form

$$ds^2_{\nu,\epsilon,r_0}=\frac{2\nu r_0^2 y^{\sqrt{\nu}-2}(d\mathbf{y})^2}{(1+\epsilon y^{\sqrt{\nu}})^2}. \quad (27)$$

Thus, we arrive at the reduced Schrödinger equation

$$\hat{\mathcal{H}}_{\text{red}}(\mathbf{y},\boldsymbol{\pi})\Psi_{(3)}=E\Psi_{(3)},\quad \hat{\mathcal{H}}_{\text{red}}=\hat{\mathcal{H}}^0_{\text{red}}+\hbar^2\Lambda^1, \quad (28)$$

where

$$\hat{\mathcal{H}}^0_{\text{red}}=\frac{1}{\sqrt{g}}\hat{\pi}_i\sqrt{g}g^{ij}\hat{\pi}_j+s^2\frac{(1+\epsilon y^{\sqrt{\nu}})^4}{2\nu^2 r_0^2 y^{\sqrt{\nu}}(1-\epsilon y^{\sqrt{\nu}})^2}+\frac{2\omega^2 r_0^2 y^{\sqrt{\nu}}}{(1-\epsilon y^{\sqrt{\nu}})^2} \quad (29)$$

and

$$\Lambda^1=\frac{1}{8r_0^2}\left[\frac{4y^{\sqrt{\nu}}}{(1-\epsilon y^{\sqrt{\nu}})^2}-\frac{3(1-\epsilon y^{\sqrt{\nu}})^2}{4y^{\sqrt{\nu}}}+10\epsilon\right]. \quad (30)$$

Here $\hat{\boldsymbol{\pi}}$ is the momentum operator in the Dirac monopole field

$$\hat{\boldsymbol{\pi}}=-\imath\hbar\frac{\partial}{\partial\mathbf{y}}-s\mathbf{A}(\mathbf{y}),\quad [\hat{\pi}_i,\hat{\pi}_j]=\hbar s\epsilon_{ijk}\frac{y^k}{y^3}. \tag{31}$$

The energy spectrum is given by the same formula (17) as in the four-dimensional case, with the only difference that $s=0,\pm 1/2,1,\ldots$ becomes a fixed parameter (i.e., the "monopole number"). Hence, instead of (11) one has

$$j=|s|,|s|+1,\ldots,\quad m=-j,-j+1,\ldots,j-1,j. \tag{32}$$

Thus, we got a rather remarkable result: Reducing the quantum Hamiltonian yields a different outcome from quantizing the reduced classical Hamiltonian. In the special case $\nu=4$ we get the system on the three-dimensional sphere $(\epsilon=1)$ or pseudosphere (two-sheet hyperboloid) $(\epsilon=-1)$. In this case the reduced oscillator potential coincides with the potential of the oscillator on the sphere (hyperboloid) [1], so that in the absence of monopoles, $s=0$, $\mathcal{H}^0_{\rm red}$ coincides with the Hamiltonian of the three-dimensional Higgs oscillator. Comparing the spectrum of the latter system [10] with that constructed above (for $s=0$), one can see that they coincide only in the semiclassical limit. This is due to the quantum correction $\hbar^2\Lambda_1$.

Now, we can complete the KS transformation, dividing the both sides of the Schrödinger equation (28) by $r_0^2x^2$, and going to the wave function $\Psi_{(3)}=x^{-1/2}\Psi_{(3)}$. As a result, we shall get the Schrödinger equation of a MIC-Kepler-like system on a three-dimensional conifold with the metric $ds^2_{\nu_1,-1,R_0}$ given by Eq. (27), where $\nu_1=4\nu$, $R_0=r_0^2$,

$$\hat{\mathcal{H}}_{MIC}\tilde{\Psi}_{(3)}=\mathcal{E}\tilde{\Psi}_{(3)}, \tag{33}$$

where the Hamiltonian $\hat{\mathcal{H}}_{MIC}$ is of the form

$$\hat{\mathcal{H}}_{MIC}=\frac{1}{\sqrt{g}}\hat{\pi}_i\sqrt{g}g^{ij}\hat{\pi}_j+2s^2\hbar^2\frac{(1-y^{\sqrt{\nu_1}})^2}{\nu_1^2R_0^2y^{\sqrt{\nu_1}}}-\frac{\gamma}{R_0}\frac{1+y^{\sqrt{\nu_1}}}{2y^{\sqrt{\nu_1}}}, \tag{34}$$

while the energy and the coupling constant are given by the expressions

$$\gamma=\frac{E_{\rm osc}}{2}+\frac{\epsilon\hbar^2}{r_0^2}\left(1-\frac{s^2}{\nu^2}\right),$$

$$-2\mathcal{E}=\omega^2+\frac{\epsilon E_{osc}}{r_0^2}+\frac{\hbar^2}{r_0^4}\left(1+2\frac{s^2}{\nu^2}\right). \tag{35}$$

So, as opposed to the reduction procedure, where nontrivial quantum corrections arise, the KS transformation of the four-dimensional oscillator yields the MIC-Kepler-like system on the three-dimensional conifold, where the quantum corrections have an impact on the coupling constant and the energy only.

Using the expressions (16), we could convert the energy spectrum of the oscillator into that of the MIC-Kepler system

$$\mathcal{E}=-\frac{2(\gamma-\epsilon\hbar^2(2n+j_1+1)^2/(4R_0))^2}{\hbar^2(2n+j_1+1)^2}-\frac{\epsilon\gamma}{R_0}+\frac{\hbar^2}{2R_0^2}, \tag{36}$$

where j_1 is defined by the expression (18). We obtained an exactly solvable generalization of the MIC-Kepler system on a class of three-dimensional spaces having conic singularities and nonconstant curvature. In the case of constant curvature, the system is degenerate in the orbital quantum number j; otherwise it has an explicit dependence on j.

In our consideration we have used the common construction of the quantum Hamiltonian on the curved configuration space, where the kinetic energy term is replaced by the Laplace operator,

$$g^{ij}p_i\pi_j\to\frac{1}{\sqrt{g}}\partial_i\sqrt{g}g^{ij}\partial_j,$$

which guarantees that the Hamiltonian is Hermitean and reparametrization-invariant.

In fact this definition assumes that the following realization of the momenta operators, $\hat{\pi}_i=-\hbar\partial_i$, which is non-Hermitean in the case of a nonconstant metric. In Ref. [11] the *a priori* Hermitean realization of the momenta operators $\hat{\pi}_i=-\hbar(\partial_i-\frac{1}{4}\partial_i\log\det g)$ was suggested, together with a slightly different definition of the observables quadratic on momenta (and, consequently, of the Hamiltonian) respecting the reduction procedure. Upon this definition of momenta, both the initial and the reduced quantum Hamiltonian will get quantum corrections, with respect to the Hamiltonian as we defined it in our conventions. It seems to be interesting to compare the spectra and the properties of the systems under consideration in both approaches.

IV. SUMMARY AND CONCLUSION

Let us summarize our results. We proposed an exactly solvable model of a quantum oscillator on two- (complex) dimensional complex projective space (in its compact and noncompact versions), as well as on the related nonconstant curvature Kähler spaces with conic singularities, parameterized by $\nu>0$ and $\epsilon=\pm 1$. We reduced the oscillators to those on the (ν,ϵ) parametric family of three-dimensional nonconstant curvature conifolds related with the three-dimensional (pseudo)sphere. The reduced systems are specified by the presence of a Dirac monopole. During the reduction the quantum Hamiltonian gets additional corrections with respect to the quantized Hamiltonian, reduced from four-dimensional space at the classical level. Then we transformed the reduced (ν,ϵ) oscillator in the MIC-Kepler-like system on the $(4\nu,-1)$-conifold and get its exact energy spectrum. Opposite to the reduced oscillator, in the final system quantum corrections affect the energy and coupling constant only. The two-dimensional counterpart of the suggested model corresponds to a system with fractional spin on the complex projective plane. Its spectrum is nondegenerate for any ν.

It appears that the proposed system preserves its exact solvability after inclusion of a *constant magnetic field*. It is also a distinguished system with respect to supersymmetrization, as it admits a *nonstandard* $\mathcal{N}=4$ supersymmetric extension, which respects the inclusion of a constant magnetic field (cf. Ref. [4]). We are planning to discuss such matters in detail elsewhere.

ACKNOWLEDGMENTS

We are indebted to R. Avagyan and L. Mardoyan for useful conversations. Special thanks to L. Mardoyan for help in calculating the normalization constants (16). The work of S.B. was supported in part by the European Community's Human Potential Program under contract HPRN-CT-2000-00131 Quantum Spacetime, the INTAS-00-00254 grant and the NATO Collaborative Linkage Grant PST.CLG.979389. The work of A.N. was supported by grants INTAS 00-00262 and ANSEF PS81.

[1] P.W. Higgs, J. Phys. A **12**, 309 (1979); H.I. Leemon, *ibid.* **12**, 489 (1979).
[2] B. Zumino, Phys. Lett. **87B**, 203 (1979); M.J. Bowick and S.G. Rajeev, Phys. Rev. Lett. **58**, 535 (1987); **58**, 1158(E) (1987).
[3] D.V. Fursaev and S.N. Solodukhin, Phys. Rev. D **52**, 2133 (1995); A. Strominger, Nucl. Phys. **B451**, 96 (1995); C. Mayer and T. Mohaupt, Class. Quantum Grav. **21**, 1879 (2004).
[4] S. Bellucci and A. Nersessian, Phys. Rev. D **67**, 065013 (2003).
[5] A. Nersessian and A. Yeranyan, J. Phys. A **37**, 2791 (2004).
[6] D. Zwanziger, Phys. Rev. **176**, 1480 (1968); H.V. McIntosh and A. Cisneros, J. Math. Phys. **11**, 896 (1970).
[7] A. Nersessian and G. Pogosyan, Phys. Rev. A **63**, 020103(R) (2001).
[8] A. Nersessian, V. Ter-Antonyan, and M.M. Tsulaia, Mod. Phys. Lett. A **11**, 1605 (1996).
[9] S.C. Zhang and J.P. Hu, Science **294**, 823 (2001); D. Karabali and V.P. Nair, Nucl. Phys. **B641**, 533 (2002); B.A. Bernevig, J.P. Hu, N. Toumbas, and S.C. Zhang, Phys. Rev. Lett. **91**, 236803 (2003).
[10] E.G. Kalnins, W. Miller, Jr., and G.S. Pogosyan, J. Math. Phys. **41**, 2629 (2000).
[11] D. McMullan and J. Paterson, J. Math. Phys. **30**, 477 (1989); **30**, 487 (1989).