Two-loop hard-thermal-loop thermodynamics with quarks

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We calculate the quark contribution to the free energy of a hot quark-gluon plasma to two-loop order using hard-thermal-loop (HTL) perturbation theory. All ultraviolet divergences can be absorbed into renormalizations of the vacuum energy and the HTL quark and gluon mass parameters. The quark and gluon HTL mass parameters are determined self-consistently by a variational prescription. Combining the quark contribution with the two-loop HTL perturbation theory free energy for pure glue we obtain the total two-loop QCD free energy. Comparisons are made with lattice estimates of the free energy for $N_f=2$ and with exact numerical results obtained in the large-N_f limit.

I. INTRODUCTION

The current generation of relativistic heavy-ion collision experiments should exceed the energy density necessary for the formation of a quark-gluon plasma. It is therefore necessary to have a quantitative theoretical framework which can be used to calculate the properties of a quark-gluon plasma. The usual line of reasoning is that since QCD is asymptotically free, its running coupling constant α_s becomes weaker as the temperature increases and therefore the behavior of hadronic matter at sufficiently high temperature should be calculable using perturbative methods. Unfortunately, a straightforward perturbative expansion in powers of α_s does not seem to be of any quantitative use even at temperatures many orders of magnitude higher than those achievable in heavy-ion collisions.

The problem can be seen by looking at the perturbative expansion of the free energy $\mathcal{F}$ of a quark-gluon plasma, whose weak-coupling expansion has been calculated completely through order $\alpha_s^{5/2}$ [1–3]:

$$\mathcal{F} = -\frac{8\pi^2}{45}T^4\left[\mathcal{F}_0 + \mathcal{F}_2\frac{\alpha_s}{\pi} + \mathcal{F}_3\left(\frac{\alpha_s}{\pi}\right)^{3/2} + \mathcal{F}_4\left(\frac{\alpha_s}{\pi}\right)^2 \right.$$
$$\left. + \mathcal{F}_5\left(\frac{\alpha_s}{\pi}\right)^{5/2} + O(\alpha_s^3\log\alpha_s)\right], \tag{1}$$

with

$$\mathcal{F}_0 = 1 + \frac{21}{32}N_f, \tag{2}$$

$$\mathcal{F}_2 = -\frac{15}{4}\left(1 + \frac{5}{12}N_f\right), \tag{3}$$

$$\mathcal{F}_3 = 30\left(1 + \frac{1}{6}N_f\right)^{3/2}, \tag{4}$$

$$\mathcal{F}_4 = 237.2 + 15.97N_f - 0.413N_f^2$$
$$+ \frac{135}{2}\left(1 + \frac{1}{6}N_f\right)\log\left[\frac{\alpha_s}{\pi}\left(1 + \frac{1}{6}N_f\right)\right]$$
$$- \frac{165}{8}\left(1 + \frac{5}{12}N_f\right)\left(1 - \frac{2}{33}N_f\right)\log\frac{\mu}{2\pi T}, \tag{5}$$

$$\mathcal{F}_5 = -\left(1 + \frac{1}{6}N_f\right)^{1/2}[799.2 + 21.96N_f + 1.926N_f^2]$$
$$+ \frac{495}{2}\left(1 + \frac{1}{6}N_f\right)\left(1 - \frac{2}{33}N_f\right)\log\frac{\mu}{2\pi T}, \tag{6}$$

where μ is the renormalization scale, $\alpha_s = \alpha_s(\mu)$ is the running coupling constant in the modified minimal subtraction ($\overline{\text{MS}}$) scheme, and we have set $N_c = 3$. The coefficient of $\alpha_s^3\log\alpha_s$ has recently been computed [4]; however, since there are unknown perturbative and non-perturbative contributions at $O(\alpha_s^3)$, we do not include terms higher than $O(\alpha_s^{5/2})$ in Eq. (1).

In Fig. 1, the free energy with $N_f = 2$ is shown as a function of the temperature $T/\Lambda_{\overline{\text{MS}}}$. In the plot we have scaled the free energy by the free energy of an ideal gas of quarks and gluons which for arbitrary N_c and N_f is

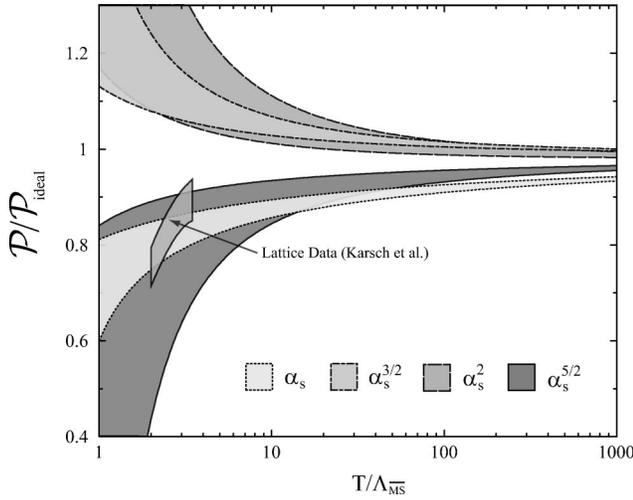*Present address: Nordita, Blegdamsvej 17, DK-2100 Copenhagen, Denmark.

FIG. 1. The perturbative free energy of QCD for $N_f = 2$ massless quarks as a function of $T/\Lambda_{(\overline{MS})}$. The weak-coupling expansions through orders α_s, $\alpha_s^{3/2}$, α_s^2, and $\alpha_s^{5/2}$ are shown as bands that correspond to varying the renormalization scale μ by a factor of 2 around $2\pi T$. Also shown is a lattice estimate by Karsch et al. [7] for the free energy. The band indicates the estimated systematic error of their result which is reported as $(15\pm5)\%$.

$$\mathcal{F}_{\text{ideal}} = -\frac{\pi^2}{45} T^4 \left(N_c^2 - 1 + \frac{7}{4} N_c N_f \right). \qquad (7)$$

The weak-coupling expansions through orders α_s, $\alpha_s^{3/2}$, α_s^2, and $\alpha_s^{5/2}$ are shown as bands that correspond to varying the renormalization scale, μ, by a factor of 2 around the central value $\mu = 2\pi T$. As successive terms in the weak-coupling expansion are added, the predictions change wildly and the sensitivity to the renormalization scale grows. It is clear that a reorganization of the perturbation series is essential if perturbative calculations are to be of any quantitative use at temperatures accessible in heavy-ion collisions.

The free energy can also be calculated nonperturbatively using lattice gauge theory [5]. The thermodynamic functions for pure-glue QCD have been calculated with high precision by Boyd et al. [6]. There have also been calculations which include dynamical quarks [7,8]. In Fig. 1 we have included the latest lattice estimate of Karsch et al. [7] for the free energy for $N_f = 2$ flavors of light quarks. The band indicates the estimated systematic error of their result which is reported as $(15\pm5)\%$. Note that the quarks in the simulations do have non-zero masses and that extrapolation to zero quark mass would require significant computing time. As a result of the difficulty associated with the inclusion of light and massless dynamical quarks on the lattice, it is therefore desirable to have analytic methods which can be used to estimate the thermodynamic functions.

The only rigorous method available for reorganizing perturbation theory in thermal QCD is *dimensional reduction* to an effective 3-dimensional field theory [9,10]. The coefficients of the terms in the effective Lagrangian are calculated using perturbation theory, but calculations within the effective field theory are carried out nonperturbatively using lattice gauge theory. Dimensional reduction has the same limitations as ordinary lattice gauge theory: it can be applied only to static quantities and only at small baryon number densities. Unlike in ordinary lattice gauge theory, light dynamical quarks do not require any additional computer power, because they only enter through the perturbatively calculated coefficients in the effective Lagrangian. This method has been applied to the Debye screening mass for QCD [10] as well as the pressure [9].

There are some proposals for reorganizing perturbation theory in QCD that are essentially just mathematical manipulations of the weak coupling expansion. The methods include *Padé approximates* [11], *Borel resummation* [12], and *self-similar approximates* [13]. These methods are used to construct more stable sequences of successive approximations that agree with the weak-coupling expansion when expanded in powers of α_s. These methods can only be applied to quantities for which several orders in the weak-coupling expansion are known, so they are limited in practice to the thermodynamic functions.

One promising approach for reorganizing perturbation theory in thermal QCD is to use a variational framework. The free energy $\mathcal{F}$ is expressed as the variational minimum of a thermodynamic potential $\Omega(T, \alpha_s; m^2)$ that depends on one or more variational parameters that we denote collectively by m^2:

$$\mathcal{F}(T, \alpha_s) = \Omega(T, \alpha_s; m^2)|_{\partial\Omega/\partial m^2 = 0}. \qquad (8)$$

A particularly compelling variational formulation is the Φ-*derivable approximation*, in which the complete propagator is used as an infinite set of variational parameters [14]. The Φ-derivable thermodynamic potential Ω is the two-particle-irreducible (2PI) effective action, the sum of all diagrams that are 2PI with respect to the complete propagator [15]. The n-loop Φ-derivable approximations, in which Ω is the sum of 2PI diagrams with up to n loops, form a systematically improvable sequence of variational approximations. Until recently, Φ-derivable approximations have proved to be intractable for relativistic field theories except for simple cases in which the self-energy is momentum independent. However, there has been some recent progress in solving the 3-loop Φ-derivable approximation for scalar field theories. Braaten and Petitgirard have developed an analytic method for solving the 3-loop Φ-derivable approximation for the massless ϕ^4 field theory [16]. van Hees and Knoll have developed numerical methods for solving the 3-loop Φ-derivable approximation for the massive ϕ^4 field theory [17]. They also investigated renormalization issues associated with the Φ-derivable approximation. These issues have recently been studied in detail by Blaizot, Iancu, and Reinosa [18].

The application of the Φ-derivable approximation to QCD was first discussed by Freedman and McLerran [19]. One problem with this approach is that the thermodynamic potential Ω is gauge dependent, and so are the resulting thermodynamic functions. The gauge dependence is the same order in α_s as the truncation error when evaluated off the stationary point and twice the order in α_s when evaluated at the stationary point [20]. However, the most serious problem

is that even the application of 2-loop Φ-derivable approximation to gauge theories has proved to be intractable.

The 2-loop Φ-derivable approximation for QCD has been used as the starting point for *hard-thermal-loop (HTL) resummations* of the entropy by Blaizot, Iancu, and Rebhan [21] and of the pressure by Peshier [22]. The thermodynamic potential $\Omega_{2\text{-loop}}$ is a functional of the complete gluon propagator $D_{\mu\nu}(P)$. However, in order to make the problem tractable the authors in Refs. [21] and [22] were forced to make a variational ansatz for the exact gluon propagator which they took as the HTL gluon propagator in the infrared and with a next-to-leading order corrected mass in the ultraviolet with an aribitrary momentum scale separating the two momentum regions. Using this ansatz they were able to calculate the QCD thermodynamic functions; however, a first-principles calculation of the corrections to their results for gauge theories would require the inclusion of exact vertices as well as exact propagators thus making the problem intractable.

The difficulties in calculating quantities using Φ-derivable approximations in gauge theories motivates the use of simpler variational approximations. One such strategy that involves a single variational parameter m has been called *optimized perturbation theory* [23], *variational perturbation theory* [24], or the *linear δ expansion* [25]. This strategy was applied to the thermodynamics of the massless ϕ^4 field theory by Karsch, Patkos, and Petreczky under the name *screened perturbation theory* [26]. The method has also been applied to spontaneously broken field theories at finite temperature [27]. The calculations of the thermodynamics of the massless ϕ^4 field theory using screened perturbation theory have been extended to 3 loops [28]. The calculations can be greatly simplified by using a double expansion in powers of the coupling constant and m/T [29].

HTL perturbation theory (HTLpt) is an adaptation of this strategy to thermal QCD [30]. The exactly solvable theory used as the starting point is one whose propagators are the HTL quark and gluon propagators. The variational mass parameters m_D and m_q are identified with the Debye screening mass and the induced quark mass. The one-loop free energy in HTLpt was calculated for QCD in Ref. [30] and for QCD with massless quarks in Ref. [31]. At this order, the parameters m_D and m_q could not be determined variationally, so their perturbative limits were used. The resulting thermodynamic functions had errors of order α_s, but the terms of order $\alpha_s^{3/2}$ associated with Debye screening were correct. A two-loop calculation is required to reduce the errors to order α_s^2.

In a previous paper we calculated the thermodynamic functions of pure-glue QCD to next-to-leading order in HTLpt [32]. In that paper we showed that it was possible to renormalize the resulting expressions for the thermodynamic potential at next-to-leading order using only vacuum and mass counterterms and we also showed that the corrections to the thermodynamic functions in going from leading-order to next-to-leading order were small down to temperatures on the order of $10\Lambda_{\overline{MS}}$. In this paper we calculate the thermo-

dynamic functions of QCD to next-to-leading order in HTLpt including the contributions from quark and quark-gluon interaction diagrams.

We begin with a brief summary of HTL perturbation theory including quarks in Sec. II. In Sec. III, we give the expressions for the one-loop and two-loop diagrams for the thermodynamic potential. In Sec. IV, we reduce those diagrams to scalar sum integrals. We are unable to compute those sum integrals exactly, so in Sec. V we evaluate them by treating m_D and m_q as $O(g)$ quantities and expand them in m_D/T and m_q/T, keeping all terms that contribute up to $O(g^5)$. The diagrams are combined in Sec. VII to obtain the final result for the two-loop thermodynamic potential up to $O(g^5)$. In Sec. VIII, we present our numerical results for the free energy of QCD at leading and next-to-leading order in HTLpt. In Sec. IX we evaluate the free energy in the large N_f limit where exact numerical results have been obtained [33,34].

There are several appendixes that contain technical details of the calculations. In Appendix A, we give the Feynman rules for HTL perturbation theory in Minkowski space to facilitate the application of this formalism to signatures of the quark-gluon plasma. The most difficult aspect of these calculations was the evaluation of the sum integrals obtained from the expansion in m_D/T and m_q/T. We give the results for these sum integrals in Appendix B. The evaluation of some difficult thermal integrals that were required to obtain the sum integrals is described in Appendix C.

II. HTL PERTURBATION THEORY

The Lagrangian density that generates the perturbative expansion for QCD can be expressed in the form

$$\mathcal{L}_{\text{QCD}} = -\frac{1}{2}\text{Tr}(G_{\mu\nu}G^{\mu\nu}) + i\bar{\psi}\gamma^\mu D_\mu\psi + \mathcal{L}_{\text{gf}} + \mathcal{L}_{\text{ghost}}$$
$$+ \Delta\mathcal{L}_{\text{QCD}}. \tag{9}$$

The gauge potential is $A_\mu = A_\mu^a t^a$, with generators t^a of the fundamental representation of $SU(N_c)$ normalized so that $\text{Tr}\, t^a t^b = \delta^{ab}/2$. The field strength tensor is $G_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - ig[A_\mu, A_\nu]$. In the quark term there is an implicit sum over the N_f quark flavors and $D_\mu = \partial_\mu - igA_\mu$ is the covariant derivative for the fundamental representation. The ghost term $\mathcal{L}_{\text{ghost}}$ depends on the choice of the gauge-fixing term $\mathcal{L}_{\text{gf}}$. Two choices for the gauge-fixing term that depend on an arbitrary gauge parameter ξ are the general covariant gauge and the general Coulomb gauge:

$$\mathcal{L}_{\text{gf}} = -\frac{1}{\xi}\text{Tr}[(\partial^\mu A_\mu)^2] \quad \text{covariant} \tag{10}$$

$$= -\frac{1}{\xi}\text{Tr}[(\nabla\cdot\mathbf{A})^2] \quad \text{Coulomb}. \tag{11}$$

It is also convenient to introduce various invariants associated with the representations of the $SU(N_c)$ gauge group. Denoting the generators of the adjoint representation as

$(F^a)^{bc} = -if^{abc}$ and generators of the fundamental representation as T^a we define the following group theory factors:

$$[F^c F^c]^{ab} = f^{adc} f^{bdc} = c_A \delta^{ab},$$

$$\mathrm{Tr}\, F^a F^b = s_A \delta^{ab},$$

$$\delta^{aa} = d_A,$$

$$[T^a T^a]_{ij} = c_F \delta_{ij},$$

$$\mathrm{Tr}\, T^a T^b = s_F \delta^{ab},$$

$$\delta^{ii} = d_F = s_F d_A / c_F. \tag{12}$$

With the standard normalization,

$$c_A = s_A = N_c,$$

$$d_A = N_c^2 - 1,$$

$$c_F = (N_c^2 - 1)/(2N_c),$$

$$s_F = N_f/2,$$

$$d_F = N_c N_f. \tag{13}$$

The perturbative expansion in powers of g generates ultraviolet divergences. The renormalizability of perturbative QCD guarantees that all divergences in physical quantities can be removed by renormalization of the coupling constant $\alpha_s = g^2/4\pi$. There is no need for wave function renormalization, because physical quantities are independent of the normalization of the field. There is also no need for renormalization of the gauge parameter, because physical quantities are independent of the gauge parameter.

Hard-thermal-loop perturbation theory is a reorganization of the perturbation series for thermal QCD. The Lagrangian density is written as

$$\mathcal{L} = (\mathcal{L}_{\mathrm{QCD}} + \mathcal{L}_{\mathrm{HTL}})|_{g \to \sqrt{\delta} g} + \Delta \mathcal{L}_{\mathrm{HTL}}. \tag{14}$$

The HTL improvement term is

$$\mathcal{L}_{\mathrm{HTL}} = -\frac{1}{2}(1 - \delta) m_D^2 \mathrm{Tr}\left(G_{\mu\alpha} \left\langle \frac{y^\alpha y^\beta}{(y \cdot D)^2} \right\rangle_y G^\mu_\beta \right)$$

$$+ (1 - \delta) i m_q^2 \bar{\psi} \gamma^\mu \left\langle \frac{y^\mu}{y \cdot D} \right\rangle_y \psi, \tag{15}$$

where in the first term D_μ is the covariant derivative in the adjoint representation, in the second term D_μ is the covariant derivative in the fundamental representation, $y^\mu = (1, \hat{\mathbf{y}})$ is a light-like four-vector, and $\langle \cdots \rangle_y$ represents the average over the directions of $\hat{\mathbf{y}}$. The term (15) has the form of the effective Lagrangian that would be induced by a rotationally invariant ensemble of colored sources with infinitely high momentum. The parameter m_D can be identified with the Debye screening mass and the parameter m_q can be identified as the

induced finite temperature quark mass. HTLpt is defined by treating δ as a formal expansion parameter.

The HTL perturbation expansion generates ultraviolet divergences. In QCD perturbation theory, renormalizability constrains the ultraviolet divergences to have a form that can be canceled by the counterterm Lagrangian $\Delta\mathcal{L}_{\mathrm{QCD}}$. We will demonstrate that renormalized perturbation theory can be implemented by including a counterterm Lagrangian $\Delta\mathcal{L}_{\mathrm{HTL}}$ among the interaction terms in Eq. (14). There is no proof that the HTL perturbation expansion is renormalizable, so the general structure of the ultraviolet divergences is not known; however, it was shown in our previous paper [32] that it was possible to renormalize the next-to-leading order HTLpt prediction for the free energy of pure-glue QCD using only a vacuum counterterm and Debye mass counterterm. Here we show that when quarks are included it is also possible to renormalize the resulting expressions using only vacuum, Debye mass, and quark mass counterterms.

The leading term in the delta expansion of the vacuum energy, $\mathcal{E}_0$, counterterm $\Delta\mathcal{E}_0$ was deduced in Ref. [30] by calculating the free energy to leading order in δ. The $\mathcal{E}_0$ counterterm $\Delta\mathcal{E}_0$ must therefore have the form

$$\Delta\mathcal{E}_0 = \left(\frac{d_A}{128\pi^2 \epsilon} + O(\delta\alpha_s) \right) (1 - \delta)^2 m_D^4. \tag{16}$$

To calculate the free energy to next-to-leading order in δ, we need the counterterm $\Delta\mathcal{E}_0$ to order δ and the counterterms Δm_D^2 and Δm_q^2 to order δ. We will show that there is a nontrivial cancellation of the ultraviolet divergences if the mass counterterms have the form

$$\Delta m_D^2 = -\frac{\alpha_s}{3\pi\epsilon}\left[\frac{11}{4} c_A - s_F \right] m_D^2, \tag{17}$$

$$\Delta m_q^2 = -\frac{\alpha_s}{3\pi\epsilon}\left[\frac{9}{8} \frac{d_A}{c_A} \right] m_q^2. \tag{18}$$

Physical observables are calculated in HTLpt by expanding them in powers of δ, truncating at some specified order, and then setting $\delta = 1$. This defines a reorganization of the perturbation series in which the effects of the m_D^2 and m_q^2 terms in Eq. (15) are included to all orders but then systematically subtracted out at higher orders in perturbation theory by the δm_D^2 and δm_q^2 terms in Eq. (15). If we set $\delta = 1$, the Lagrangian (14) reduces to the QCD Lagrangian (9). If the expansion in δ could be calculated to all orders, all dependence on m_D and m_q should disappear when we set $\delta = 1$. However, any truncation of the expansion in δ produces results that depend on m_D and m_q. Some prescription is required to determine m_D and m_q as a function of T and α_s.
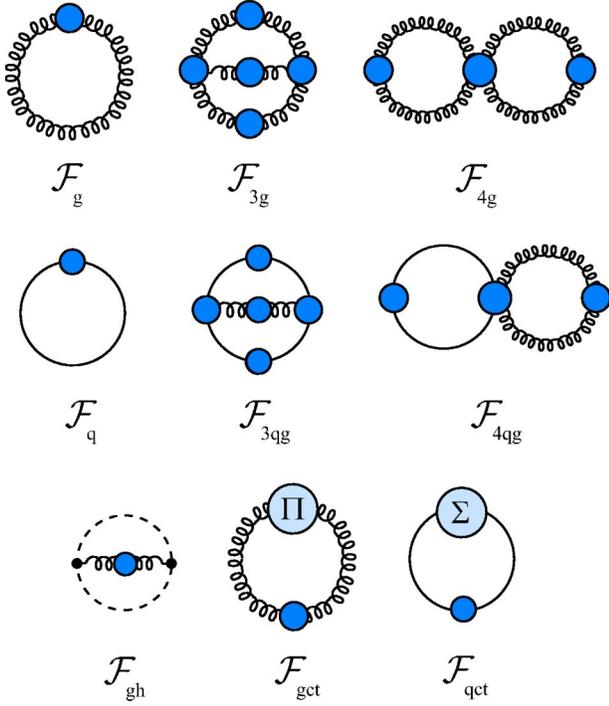
FIG. 2. Diagrams contributing through NLO in HTLpt. Shaded circles indicate dressed HTL propagators and vertices.

We choose to treat both as variational parameters that should be determined by minimizing the free energy. If we denote the free energy truncated at some order in δ by $\Omega(T, \alpha_s, m_D, m_q, \delta)$, our prescription is

$$\frac{\partial}{\partial m_D}\Omega(T, \alpha_s, m_D, m_q, \delta=1)=0,$$

$$\frac{\partial}{\partial m_q}\Omega(T, \alpha_s, m_D, m_q, \delta=1)=0. \quad (19)$$

Since $\Omega(T, \alpha_s, m_D, m_q, \delta=1)$ is a function of the variational parameters m_D and m_q, we will refer to it as the *thermodynamic potential*. We will refer to the variational equations (19) as the *gap equations*. The free energy $\mathcal{F}$ is obtained by evaluating the thermodynamic potential at the solution to the gap equations (19). Other thermodynamic functions can then be obtained by taking appropriate derivatives of $\mathcal{F}$ with respect to T.

III. DIAGRAMS FOR THE THERMODYNAMIC POTENTIAL

The thermodynamic potential at leading order in HTL perturbation theory (see Fig. 2) for an $SU(N_c)$ gauge theory with N_f massless quarks is

$$\Omega_{\mathrm{LO}}=d_A\mathcal{F}_g+d_F\mathcal{F}_q+\Delta_0\mathcal{E}_0, \quad (20)$$

where $\mathcal{F}_g$ is the contribution from each of the color states of the gluon:

$$\mathcal{F}_g=-\frac{1}{2}\sum\!\!\!\!\!\!\!\int_P \{(d-1)\log[-\Delta_T(P)]+\log\Delta_L(P)\}. \quad (21)$$

The transverse and longitudinal HTL propagators $\Delta_T(P)$ and $\Delta_L(P)$ are given in Eqs. (A51) and (A52). The quark contribution is

$$\mathcal{F}_q=-\sum\!\!\!\!\!\!\!\int_{\{P\}}\log\det[\not{P}-\Sigma(P)], \quad (22)$$

where $\Sigma(P)$ is the HTL fermion self-energy. The leading order counterterm $\Delta_0\mathcal{E}_0$ was determined in Ref. [30]:

$$\Delta_0\mathcal{E}_0=\frac{d_A}{128\pi^2\epsilon}m_D^4. \quad (23)$$

The thermodynamic potential at next-to-leading order in HTL perturbation theory can be written as

$$\Omega_{\mathrm{NLO}}=\Omega_{\mathrm{LO}}+d_A[\mathcal{F}_{3g}+\mathcal{F}_{4g}+\mathcal{F}_{gh}+\mathcal{F}_{gct}]$$
$$+d_As_F[\mathcal{F}_{3qg}+\mathcal{F}_{4qg}]+d_f\mathcal{F}_{qct}+\Delta_1\mathcal{E}_0$$
$$+\Delta_1m_D^2\frac{\partial}{\partial m_D^2}\Omega_{\mathrm{LO}}+\Delta_1m_q^2\frac{\partial}{\partial m_q^2}\Omega_{\mathrm{LO}}, \quad (24)$$

where $\Delta_1\mathcal{E}_0$, $\Delta_1m_D^2$, and $\Delta_1m_q^2$ are the terms of order δ in the vacuum energy density and mass counterterms. The contributions from the two-loop diagrams with the three-gluon and four-gluon vertices are

$$\mathcal{F}_{3g}=\frac{c_A}{12}g^2\sum\!\!\!\!\!\!\!\int_{PQ}\Gamma^{\mu\lambda\rho}(P,Q,R)\Gamma^{\nu\sigma\tau}(P,Q,R)\Delta^{\mu\nu}(P)$$
$$\times\Delta^{\lambda\sigma}(Q)\Delta^{\rho\tau}(R), \quad (25)$$

$$\mathcal{F}_{4g}=\frac{c_A}{8}g^2\sum\!\!\!\!\!\!\!\int_{PQ}\Gamma^{\mu\nu,\lambda\sigma}(P,-P,Q,-Q)$$
$$\times\Delta^{\mu\nu}(P)\Delta^{\lambda\sigma}(Q), \quad (26)$$

where $R=-Q-P$.

The contribution from the ghost diagram depends on the choice of gauge. The expressions in the covariant and Coulomb gauges are

$$\mathcal{F}_{gh} = \frac{c_A}{2} g^2 \sum_{PQ} \frac{1}{Q^2} \frac{1}{R^2} Q^\mu R^\nu \Delta^{\mu\nu}(P) \quad \text{covariant} \qquad (27)$$

$$= \frac{c_A}{2} g^2 \sum_{PQ} \frac{1}{q^2} \frac{1}{r^2} (Q^\mu - Q \cdot n n^\mu)(R^\nu - R \cdot n n^\nu)$$

$$\times \Delta^{\mu\nu}(P) \quad \text{Coulomb.} \qquad (28)$$

The contribution from the HTL gluon counterterm diagram is

$$\mathcal{F}_{gct} = \frac{1}{2} \sum_P \Pi^{\mu\nu}(P)\Delta^{\mu\nu}(P). \qquad (29)$$

The contributions from the two-loop diagrams with the quark-gluon three and four vertices are given by

$$\mathcal{F}_{3qg} = \frac{1}{2} g^2 \sum_{P\{Q\}} \mathrm{Tr}[\Gamma^\mu(P,Q,R)S(Q)\Gamma^\nu(P,Q,R)S(R)]$$

$$\times \Delta^{\mu\nu}(P), \qquad (30)$$

$$\mathcal{F}_{4qg} = \frac{1}{2} g^2 \sum_{P\{Q\}} \mathrm{Tr}[\Gamma^{\mu\nu}(P,-P,Q,Q)S(Q)]\Delta^{\mu\nu}(P), \qquad (31)$$

where Tr implies taking the trace over γ matrices. The contribution from the HTL quark counterterm is

$$\mathcal{F}_{qct} = -\sum_{\{P\}} \mathrm{Tr}[\Sigma(P)S(P)]. \qquad (32)$$

Provided that HTL perturbation theory is renormalizable, the ultraviolet divergences at any order in δ can be canceled by renormalizations of the vacuum energy density $\mathcal{E}_0$, the HTL mass parameters m_D^2 and m_q^2, and the coupling constant α_s. Renormalization of the coupling constant does not enter until order δ^2. We will calculate the thermodynamic potential as a double expansion in powers of m_D/T and m_q/T, and g including all terms through fifth order. The $\delta\alpha_s$ term in $\Delta\mathcal{E}_0$ does not contribute until sixth order in this expansion, so the term of order δ in $\Delta\mathcal{E}_0$ can be obtained simply by expanding Eq. (23) to first order in δ:

$$\Delta_1 \mathcal{E}_0 = -\frac{d_A}{64\pi^2\epsilon} m_D^4. \qquad (33)$$

The remaining ultraviolet divergences must be removed by renormalization of the mass parameters m_D and m_q. We will show below that, at order δ, all remaining divergences can be removed by the quark and Debye mass counterterms. This provides nontrivial evidence for the renormalizability of HTL perturbation theory at this order in δ.

The sum of the 3-gluon, 4-gluon, and ghost contributions is gauge invariant. By using the Ward identities, one can easily show that the sum of these three diagrams is independent of the gauge parameter ξ. With more effort, one can show the equivalence of the covariant gauge expression with $\xi=0$ and the Coulomb gauge expression with $\xi=0$. In a similar manner, it can be shown that the sum of Eqs. (30) and (31) is independent of ξ within the class of covariant and Coloumb gauges, as well as the equivalence of the two with $\xi=0$.

IV. REDUCTION TO SCALAR SUM INTEGRALS

The first step in calculating the quark contribution to the free energy is to reduce the sum of the diagrams to scalar sum integrals. The leading-order quark contribution can be rewritten as

$$\mathcal{F}_q = -2\sum_{\{P\}} \log P^2 - 2\sum_{\{P\}} \log\left[\frac{A_S^2 - A_0^2}{P^2}\right], \qquad (34)$$

where

$$A_0(P) = iP_0 - \frac{m_q^2}{iP_0} \mathcal{T}_P, \qquad (35)$$

$$A_S(P) = |\mathbf{p}| + \frac{m_q^2}{|\mathbf{p}|}[1 - \mathcal{T}_P]. \qquad (36)$$

The HTL quark counterterm can be rewritten as

$$\mathcal{F}_{qct} = -4\sum_{\{P\}} \frac{P^2 + m_q^2}{A_S^2 - A_0^2}. \qquad (37)$$

We proceed to simplify the sum of Eqs. (30) and (31) in Landau gauge. Using the Ward identities (A45) and (A48) the sum of Eqs. (30) and (31) becomes

$$\mathcal{F}_{3qg+4qg} = \frac{1}{2} g^2 \sum_{P\{Q\}} \{\Delta_X(P)\mathrm{Tr}[\Gamma^{00}S(Q)] - \Delta_T(P)\mathrm{Tr}[\Gamma^\mu S(Q)\Gamma^\mu S(R')] + \Delta_X(P)\mathrm{Tr}[\Gamma^0 S(Q)\Gamma^0 S(R')]\}, \qquad (38)$$

where S is the quark propagator, Δ_T is the transverse gluon propagator, Δ_X is a combination of the transverse and longitudinal gluon propagators defined in (A27), and $R' = Q - P$.

Performing the traces of γ matrices gives

$$\mathcal{F}_{3qg+4qg} = -g^2 \sum_{P\{Q\}} \frac{1}{A_S^2(Q)-A_0^2(Q)} \left\{ 2(d-1)\Delta_T(P) \frac{\hat{\mathbf{q}}\cdot\hat{\mathbf{r}}A_S(Q)A_S(R)-A_0(Q)A_0(R)}{A_S^2(R)-A_0^2(R)} \right.$$

$$-2\Delta_X(P)\frac{A_0(Q)A_0(R)+A_S(Q)A_S(R)\hat{\mathbf{q}}\cdot\hat{\mathbf{r}}}{A_S^2(R)-A_0^2(R)} - 4m_q^2\Delta_X(P)\left\langle \frac{A_0(Q)-A_s(Q)\hat{\mathbf{q}}\cdot\hat{\mathbf{y}}}{(P\cdot Y)^2-(Q\cdot Y)^2}\frac{1}{(Q\cdot Y)}\right\rangle_{\hat{\mathbf{y}}}$$

$$+\frac{8m_q^2\Delta_T(P)}{A_S^2(R)-A_0^2(R)}\left\langle \frac{[A_0(Q)-A_S(Q)\hat{\mathbf{q}}\cdot\hat{\mathbf{y}}][A_0(R)-A_S(R)\hat{\mathbf{r}}\cdot\hat{\mathbf{y}}]}{(Q\cdot Y)(R\cdot Y)}\right\rangle_{\hat{\mathbf{y}}}$$

$$\left.+\frac{4m_q^2\Delta_X(P)}{A_S^2(R)-A_0^2(R)}\left\langle \frac{2A_0(R)A_S(Q)\hat{\mathbf{q}}\cdot\hat{\mathbf{y}}-A_0(Q)A_0(R)-A_S(Q)A_S(R)\hat{\mathbf{q}}\cdot\hat{\mathbf{r}}}{(Q\cdot Y)(R\cdot Y)}\right\rangle_{\hat{\mathbf{y}}} \right\}+O(g^2m_q^4), \tag{39}$$

where A_0 and A_S are defined in Eqs. (A40) and (A41), respectively.

V. HIGH-TEMPERATURE EXPANSION

The free energy has been reduced to scalar sum integrals. If we tried to evaluate the 2-loop HTL free energy exactly, there are terms that could at best be reduced to 5-dimensional integrals which would have to be evaluated numerically. We will therefore evaluate the sum integrals approximately by expanding them in powers of m_D/T and m_q/T. We will carry out the expansion to high enough order to include all terms through order g^5 if m_D and m_q are taken to be of order g.

The free energy can be divided into contributions from hard and soft momenta. We proceed to calculate the hard-hard and hard-soft contributions. There is no soft-soft contribution since one of the momenta in the loop is always fermionic and therefore hard.

A. One-loop sum integrals

The one-loop sum integrals include the leading quark contribution (22) and the HTL quark counterterm (32). The leading order free energy must be expanded to order m_q^4 to include all terms through order g^5 if m_q is taken to be of order g.

1. Hard contributions

The hard contribution from the LO gluon term (21) was given in [32] and reads

$$\mathcal{F}_g^{(h)} = -\frac{\pi^2}{45}T^4 + \frac{1}{24}\left[1+\left(2+2\frac{\zeta'(-1)}{\zeta(-1)}\right)\epsilon\right]\left(\frac{\mu}{4\pi T}\right)^{2\epsilon}m_D^2T^2$$

$$-\frac{1}{128\pi^2}\left(\frac{1}{\epsilon}-7+2\gamma+\frac{2\pi^2}{3}\right)\left(\frac{\mu}{4\pi T}\right)^{2\epsilon}m_D^4. \tag{40}$$

The hard contribution from the HTL counterterm (29) was given in [32] and reads

$$\mathcal{F}_{gct}^{(h)} = -\frac{1}{24}m_D^2T^2 + \frac{1}{64\pi^2}\left(\frac{1}{\epsilon}-7+2\gamma+\frac{2\pi^2}{3}\right)\left(\frac{\mu}{4\pi T}\right)^{2\epsilon}m_D^4. \tag{41}$$

The sum integrals over P involve two momentum scales m_q and T. Since $P_0=(2n+1)\pi T$, the momentum is always hard. We can therefore expand in powers of m_q^2. To second order in m_q^2, we obtain

$$\mathcal{F}_q^{(h)} = -2\sum_{\{P\}}\log P^2 - 4m_q^2\sum_{\{P\}}\frac{1}{P^2}$$

$$+2m_q^4\sum_{\{P\}}\left[\frac{2}{P^4}-\frac{1}{p^2P^2}+\frac{2}{p^2P^2}\mathcal{T}_P-\frac{1}{p^2P_0^2}(\mathcal{T}_P)^2\right]. \tag{42}$$

Note that the function $\mathcal{T}_P$ cancels from the m_q^2 term. The values of the sum integrals are given in Appendix B. Inserting those expressions, the hard quark contributions to the free energy reduce to

$$\mathcal{F}_q^{(h)} = -\frac{7\pi^2}{180}T^4 + \frac{1}{6}\left[1+\left(2-2\log 2+2\frac{\zeta'(-1)}{\zeta(-1)}\right)\epsilon\right]$$

$$\times\left(\frac{\mu}{4\pi T}\right)^{2\epsilon}m_q^2T^2+\frac{1}{12\pi^2}(\pi^2-6)m_q^4. \tag{43}$$

Note that this contribution is finite and so the leading order counterterm $\Delta_0\mathcal{E}_0$ is the same as in the pure-glue case. The HTL quark counterterm is given in Eq. (37). Expanding this term to second order in m_q^2 yields

$$\mathcal{F}_{qct}^{(h)} = 4m_q^2\sum_{\{P\}}\frac{1}{P^2}-4m_q^4\sum_{\{P\}}\left[\frac{2}{P^4}-\frac{1}{p^2P^2}\right.$$

$$\left.+\frac{2}{p^2P^2}\mathcal{T}_P-\frac{1}{p^2P_0^2}(\mathcal{T}_P)^2\right]. \tag{44}$$

The values of the sum integrals are given in Appendix B. Inserting those expressions, the hard contributions to the HTL quark counterterm reduce to

$$\mathcal{F}_{qct}^{(h)} = -\frac{1}{6}m_q^2T^2 - \frac{1}{6\pi^2}(\pi^2-6)m_q^4. \tag{45}$$

Note that the first term in Eq. (45) cancels the order-ϵ^0 term in coefficient of m_q^2 in Eq. (43).

2. Soft contributions

The soft contribution comes from the $P_0 = 0$ term in the sum integral. At soft momentum $P = (0, \mathbf{p})$, the HTL self-energy functions reduce to $\Pi_T(P) = 0$ and $\Pi_L(P) = m_D^2$. The transverse term vanishes in dimensional regularization because there is no momentum scale in the integral over $\mathbf{p}$. Thus the soft contribution comes from the longitudinal term only.

The soft contribution to the leading order free energy (21) was given in Ref. [32] and reads

$$\mathcal{F}_g^{(s)} = -\frac{1}{12\pi}\left[1 + \frac{8}{3}\epsilon\right]\left(\frac{\mu}{2m_D}\right)^{2\epsilon} m_D^3 T. \tag{46}$$

The soft contribution to the HTL gluon counterterm (29) was given in Ref. [32] and reads

$$\mathcal{F}_{gct}^{(s)} = \frac{1}{8\pi} m_D^3 T. \tag{47}$$

There is no soft contribution from the leading-order quark term (34) or from the HTL quark counterterm (37).

B. Two-loop sum integrals

The sum of the two-loop sum integrals is given in Eq. (38). Since these integrals have an explicit factor of g^2, we

need only to expand the sum integrals to order $m_q^2 m_D / T^3$ and m_D^3 / T^3 to include all terms through order g^5.

The sum integrals involve two momentum scales m_q, m_D and T. In order to expand them in powers of these scales, we separate them into contributions from hard loop momenta and soft loop momenta. This gives two separate regions which we will denote (hh) and (hs). In the (hh) region, all three momenta P, Q, and R are hard. In the (hs) region, two of the three momenta are hard and the other soft.

1. Contributions from the (hh) region

For hard momenta, the self-energies are suppressed by m_D^2 / T^2 or m_q^2 / T^2 relative to the propagators, so we can expand in powers of Π_T, Π_L, and Σ.

The (hh) contribution from Eqs. (25)–(27) was given in Ref. [32] and reads

$$\mathcal{F}_{3g+4g+gh}^{(hh)} = \frac{\pi^2}{12}\frac{c_A\alpha_s}{3\pi}T^4$$
$$-\frac{7}{96}\left[\frac{1}{\epsilon} + 4.621\right]\frac{c_A\alpha_s}{3\pi}\left(\frac{\mu}{4\pi T}\right)^{4\epsilon} m_D^2 T^2. \tag{48}$$

The (hh) contribution from Eqs. (30) and (31) can be written as

$$\begin{aligned}
\mathcal{F}_{3qg+4qg}^{(hh)} &= (d-1)g^2\left[\sum\!\!\!\!\!\!\!\int_{\{PQ\}}\frac{1}{P^2Q^2} - \sum\!\!\!\!\!\!\!\int_{P\{Q\}}\frac{2}{P^2Q^2}\right] + 2m_D^2 g^2\sum\!\!\!\!\!\!\!\int_{P\{Q\}}\left[\frac{1}{p^2P^2Q^2}\mathcal{T}_P + \frac{1}{(P^2)^2Q^2} - \frac{d-2}{d-1}\frac{1}{p^2P^2Q^2}\right] \\
&+ m_D^2 g^2\sum\!\!\!\!\!\!\!\int_{\{PQ\}}\left[\frac{d+1}{d-1}\frac{1}{P^2Q^2r^2} - \frac{4d}{d-1}\frac{q^2}{P^2Q^2r^4} - \frac{2d}{d-1}\frac{P\cdot Q}{P^2Q^2r^4}\right]\mathcal{T}_R \\
&+ m_D^2 g^2\sum\!\!\!\!\!\!\!\int_{\{PQ\}}\left[\frac{3-d}{d-1}\frac{1}{P^2Q^2R^2} + \frac{2d}{d-1}\frac{P\cdot Q}{P^2Q^2r^4} - \frac{d+2}{d-1}\frac{1}{P^2Q^2r^2} + \frac{4d}{d-1}\frac{q^2}{P^2Q^2r^4} - \frac{4}{d-1}\frac{q^2}{P^2Q^2r^2R^2}\right] \\
&+ 2m_q^2 g^2(d-1)\sum\!\!\!\!\!\!\!\int_{\{PQ\}}\left[\frac{1}{P^2Q_0^2Q^2} + \frac{p^2-r^2}{P^2q^2Q_0^2R^2}\right]\mathcal{T}_Q + 2m_q^2 g^2(d-1)\sum\!\!\!\!\!\!\!\int_{P\{Q\}}\left[\frac{2}{P^2(Q^2)^2} - \frac{1}{P^2Q_0^2Q^2}\mathcal{T}_Q\right] \\
&+ 2m_q^2 g^2(d-1)\sum\!\!\!\!\!\!\!\int_{\{PQ\}}\left[\frac{d+3}{d-1}\frac{1}{P^2Q^2R^2} - \frac{2}{P^2(Q^2)^2} + \frac{r^2-p^2}{q^2P^2Q^2R^2}\right].
\end{aligned} \tag{49}$$

Using the expressions for the sum integrals in Appendix B, this reduces to

$$\mathcal{F}_{3qg+4qg} = \frac{5\pi^2}{72}\frac{\alpha_s}{\pi}T^4 - \frac{1}{72}\left[\frac{1}{\epsilon} + 1.2963\right]\frac{\alpha_s}{\pi}\left(\frac{\mu}{4\pi T}\right)^{4\epsilon}m_D^2 T^2 + \frac{1}{8}\left[\frac{1}{\epsilon} + 8.96751\right]\frac{\alpha_s}{\pi}\left(\frac{\mu}{4\pi T}\right)^{4\epsilon}m_q^2 T^2. \tag{50}$$

2. (hs) contribution

In the (hs) region, the momentum P is soft. The momenta Q and R are always hard. The function that multiplies the soft propagator $\Delta_T(0,\mathbf{p})$ or $\Delta_X(0,\mathbf{p})$ can be expanded in pow-

ers of the soft momentum $\mathbf{p}$. In the case of $\Delta_T(0,\mathbf{p})$, the resulting integrals over $\mathbf{p}$ have no scale and they vanish in dimensional regularization. The integration measure $\int_\mathbf{p}$ scales like m_D^3, the soft propagator $\Delta_X(0,\mathbf{p})$ scales like

$1/m_D^2$, and every power of p in the numerator scales like m_D.

The (hs) contribution from Eqs. (25)–(27) was given in Ref. [32] and reads

$$\mathcal{F}_{3g+4g+gh}^{(hs)} = -\frac{\pi}{2}\frac{c_A \alpha_s}{3\pi} m_D T^3 - \frac{11}{32\pi}\left(\frac{1}{\epsilon} + \frac{27}{11} + 2\gamma\right)$$

$$\times \frac{c_A \alpha_s}{3\pi}\left(\frac{\mu}{4\pi T}\right)^{2\epsilon}\left(\frac{\mu}{2m_D}\right)^{2\epsilon} m_D^3 T. \quad (51)$$

The only terms that contribute through order $g^2 m_D^3 T$ and $m_q^2 m_D g^3 T$ from Eqs. (30) and (31) are

$$\mathcal{F}_{3qg+4qg}^{(hs)} = g^2 T \int_{\mathbf{p}} \frac{1}{p^2 + m_D^2} \sum\!\!\!\!\!\!\int_{\{Q\}}\left[\frac{2}{Q^2} - \frac{4q^2}{(Q^2)^2}\right]$$

$$+ 2m_D^2 g^2 T \int_{\mathbf{p}} \frac{1}{p^2 + m_D^2} \sum\!\!\!\!\!\!\int_{\{Q\}}\left[\frac{1}{(Q^2)^2}\right.$$

$$\left. - (3+d)\frac{2}{d}\frac{q^2}{(Q^2)^3} + \frac{8}{d}\frac{q^4}{(Q^2)^4}\right]$$

$$- 4m_q^2 g^2 T \int_{\mathbf{p}} \frac{1}{p^2 + m_D^2} \sum\!\!\!\!\!\!\int_{\{Q\}}\left[\frac{3}{(Q^2)^2} - \frac{4q^2}{(Q^2)^3}\right.$$

$$\left. - \frac{4}{(Q^2)^2}\mathcal{T}_Q - \frac{2}{Q^2}\left\langle\frac{1}{(Q\cdot Y)^2}\right\rangle_{\hat{\mathbf{y}}}\right]. \quad (52)$$

In the terms that are already of order $g^2 m_D^3 T$, we can set $R = -Q$. In the terms of order $g^2 m_D T^3$, we must expand the integrand to second order in $\mathbf{p}$. After averaging over angles of $\mathbf{p}$, the linear terms in $\mathbf{p}$ vanish and quadratic terms of the form $p^i p^j$ are replaced by $p^2 \delta^{ij}/d$. We can set $p^2 = -m_D^2$, because any factor proportional to $p^2 + m_D^2$ will cancel the denominator of the integral over $\mathbf{p}$, leaving an integral with no scale. This gives

$$\mathcal{F}_{3qg+4qg}^{(hs)} = -\frac{1}{6}\alpha_s m_D T^3 + \frac{\alpha_s}{24\pi^2}\left[\frac{1}{\epsilon} + 1 + 2\gamma + 4\log 2\right]$$

$$\times \left(\frac{\mu}{4\pi T}\right)^{2\epsilon}\left(\frac{\mu}{2m_D}\right)^{2\epsilon} m_D^3 T - \frac{\alpha_s}{2\pi^2} m_q^2 m_D T. \quad (53)$$

3. (ss) contributions

The (ss) contribution from Eqs. (25)–(27) was given in Ref. [32] and reads

$$\mathcal{F}_{3g+4g+gh}^{(ss)} = \frac{3}{16}\left[\frac{1}{\epsilon} + 3\right]\frac{c_A \alpha_s}{3\pi}\left(\frac{\mu}{2m_D}\right)^{4\epsilon} m_D^2 T^2. \quad (54)$$

There is no (ss) contribution from the diagrams involving fermions.

VI. HTL-IMPROVED THERMODYNAMICS

The free energy at second order in HTL perturbation theory defines a function $\Omega(T, \alpha_s, m_D, m_q, \delta = 1)$. We will refer to this function as the thermodynamic potential. To obtain the free energy $F(T)$ as a function of the temperature, we need to specify a prescription for the mass parameter m_D as a function of T and α_s.

VII. THERMODYNAMIC POTENTIAL

In this section, we calculate the thermodynamic potential $\Omega(T, \alpha_s, m_D, m_q, \delta = 1)$ explicitly, first to leading order in the δ expansion and then to next-to-leading order.

A. Leading order

The complete expression for the leading order thermodynamic potential is the sum of the contributions from 1-loop diagrams and the leading term (23) in the vacuum energy counterterm. The contributions from the 1-loop diagrams, including all terms through order g^5, is the sum of Eqs. (40), (43), and (46),

$$\Omega_{\text{1-loop}} = -d_A \frac{\pi^2 T^4}{45}\left\{1 + \frac{7}{4}\frac{d_F}{d_A} - \frac{15}{2}\hat{m}_D^2 - 30\frac{d_F}{d_A}\hat{m}_q^2 + 30\hat{m}_D^3\right.$$

$$+ \frac{45}{8}\left(\frac{1}{\epsilon} + 2\log\frac{\hat{\mu}}{2} - 7 + 2\gamma + \frac{2\pi^2}{3}\right)\hat{m}_D^4$$

$$\left. - 60\frac{d_F}{d_A}(\pi^2 - 6)\hat{m}_q^4\right\}, \quad (55)$$

where $\hat{m}_D$, $\hat{m}_q$ and $\hat{\mu}$ are dimensionless variables:

$$\hat{m}_D = \frac{m_D}{2\pi T}, \quad (56)$$

$$\hat{m}_q = \frac{m_q}{2\pi T}, \quad (57)$$

$$\hat{\mu} = \frac{\mu}{2\pi T}. \quad (58)$$

Adding the counterterm (23), we obtain the thermodynamic potential at leading order in the delta expansion:

$$\Omega_{\text{LO}} = -d_A \frac{\pi^2 T^4}{45}\left\{1 + \frac{7}{4}\frac{d_F}{d_A} - \frac{15}{2}\hat{m}_D^2 - 30\frac{d_F}{d_A}\hat{m}_q^2 + 30\hat{m}_D^3\right.$$

$$\left. + \frac{45}{4}\left(\log\frac{\hat{\mu}}{2} - \frac{7}{2} + \gamma + \frac{\pi^2}{3}\right)\hat{m}_D^4 - 60\frac{d_F}{d_A}(\pi^2 - 6)\hat{m}_q^4\right\}. \quad (59)$$

B. Next-to-leading order

The complete expression for the next-to-leading order correction to the thermodynamic potential is the sum of the contributions from all 2-loop diagrams, the quark and gluon counterterms, and renormalization counterterms. The contribution from

the 2-loop diagrams, including all terms though order g^5, is the sum of Eqs. (48), (50), (51), (53), and (54) multiplied by the appropriate group structure constants listed in Eq. (24):

$$\Omega_{\text{2-loop}} = -d_A \frac{\pi^2 T^4}{45} \frac{\alpha_s}{\pi} \left\{ -\frac{5}{4}\left(c_A + \frac{5}{2}s_F\right) + 15(c_A + s_F)\hat{m}_D - \frac{55}{8}\left[\left(c_A - \frac{4}{11}s_F\right)\left(\frac{1}{\epsilon} + 4\log\frac{\hat{\mu}}{2}\right)\right. \right.$$

$$- c_A\left(\frac{72}{11}\log\hat{m}_D - 1.96869\right) - 0.4714 s_F \right]\hat{m}_D^2 - \frac{45}{2}s_F\left[\frac{1}{\epsilon} + 4\log\frac{\hat{\mu}}{2} + 8.96751\right]\hat{m}_q^2 + 180 s_F \hat{m}_D \hat{m}_q^2$$

$$+ \frac{165}{4}\left[\left(c_A - \frac{4}{11}s_F\right)\left(\frac{1}{\epsilon} + 4\log\frac{\hat{\mu}}{2} - 2\log\hat{m}_D\right) + c_A\left(\frac{27}{11} + 2\gamma\right) - \frac{4}{11}s_F(1 + 2\gamma + 4\log 2)\right]\hat{m}_D^3 \right\}. \quad (60)$$

The HTL gluon counterterm is the sum of Eqs. (41) and (47):

$$\Omega_{\text{gct}} = -d_A \frac{\pi^2 T^4}{45}\left\{\frac{15}{2}\hat{m}_D^2 - 45\hat{m}_D^3 - \frac{45}{4}\left(\frac{1}{\epsilon} + 2\log\frac{\hat{\mu}}{2} - 7 + 2\gamma + \frac{2\pi^2}{3}\right)\hat{m}_D^4\right\}. \quad (61)$$

The HTL quark counterterm is given by Eq. (45):

$$\Omega_{\text{qct}} = -d_F \frac{\pi^2 T^4}{45}\{30\hat{m}_q^2 + 120(\pi^2 - 6)\hat{m}_q^4\}. \quad (62)$$

The ultraviolet divergences that remain after these 3 terms are added can be removed by renormalization of the vacuum energy density $\mathcal{E}_0$ and the HTL mass parameters m_D and m_q. The renormalization contributions at first order in δ are

$$\Delta\Omega = \Delta_1 \mathcal{E}_0 + \Delta_1 m_D^2 \frac{\partial}{\partial m_D^2}\Omega_{\text{LO}} + \Delta_1 m_q^2 \frac{\partial}{\partial m_q^2}\Omega_{\text{LO}}. \quad (63)$$

Using the results listed in Eqs. (17), (18), and (33) the complete contribution from the counterterm at first order in δ is

$$\Delta\Omega = -d_A \frac{\pi^2 T^4}{45}\left\{\frac{45}{4\epsilon}\hat{m}_D^4 + \frac{\alpha_s}{\pi}\left[\frac{55}{8}\left(c_A - \frac{4}{11}s_F\right)\left(\frac{1}{\epsilon} + 2\log\frac{\hat{\mu}}{2} + 2\frac{\zeta'(-1)}{\zeta(-1)} + 2\right)\hat{m}_D^2 \right.\right.$$

$$- \frac{165}{4}\left(c_A - \frac{4}{11}s_F\right)\left(\frac{1}{\epsilon} + 2\log\frac{\hat{\mu}}{2} - 2\log\hat{m}_D + 2\right)\hat{m}_D^3 + \frac{45}{2}s_F\left(\frac{1}{\epsilon} + 2 + 2\log\frac{\hat{\mu}}{2} - 2\log 2 + 2\frac{\zeta'(-1)}{\zeta(-1)}\right)\hat{m}_q^2\right]\right\}. \quad (64)$$

Adding the contributions from the two-loop diagrams in Eq. (60), the HTL gluon and quark counterterms in Eqs. (61) and (62), the contribution from vacuum and mass renormalizations in Eq. (64), and the leading order thermodynamic potential in Eq. (59) we obtain the complete expression for the QCD thermodynamic potential at next-to-leading order (NLO) in HTLpt:

$$\Omega_{\text{NLO}} = -d_A \frac{\pi^2 T^4}{45}\left\{1 + \frac{7}{4}\frac{d_F}{d_A} - 15\hat{m}_D^3 - \frac{45}{4}\left(\log\frac{\hat{\mu}}{2} - \frac{7}{2} + \gamma + \frac{\pi^2}{3}\right)\hat{m}_D^4 + 60\frac{d_F}{d_A}(\pi^2 - 6)\hat{m}_q^4 + \frac{\alpha_s}{\pi}\left[-\frac{5}{4}\left(c_A + \frac{5}{2}s_F\right)\right.\right.$$

$$+ 15(c_A + s_F)\hat{m}_D - \frac{55}{4}\left\{c_A\left(\log\frac{\hat{\mu}}{2} - \frac{36}{11}\log\hat{m}_D - 2.001\right) - \frac{4}{11}s_F\left(\log\frac{\hat{\mu}}{2} - 2.337\right)\right\}\hat{m}_D^2 - 45 s_F\left(\log\frac{\hat{\mu}}{2} + 2.192\right)\hat{m}_q^2$$

$$+ \frac{165}{2}\left\{c_A\left(\log\frac{\hat{\mu}}{2} + \frac{5}{22} + \gamma\right) - \frac{4}{11}s_F\left(\log\frac{\hat{\mu}}{2} - \frac{1}{2} + \gamma + 2\log 2\right)\right\}\hat{m}_D^3 + 180 s_F \hat{m}_D \hat{m}_q^2\right]\right\}. \quad (65)$$

C. Gap equation

The quark and gluon mass parameters, m_q and m_D, are determined variationally by requiring that the derivative of Ω_{NLO} with respect to each parameter taken holding the other constant vanish:

$$\frac{\partial}{\partial m_q}\Omega_{\text{NLO}}(T, \alpha_s, m_D, m_q) = 0, \quad (66)$$

$$\frac{\partial}{\partial m_D}\Omega_{\text{NLO}}(T, \alpha_s, m_D, m_q) = 0. \quad (67)$$

The first equation above results in the following equation for m_q:

$$8\frac{d_F}{d_A}(\pi^2-6)\hat{m}_q^2=\frac{\alpha_s s_F}{\pi}\left[3\left(\log\frac{\hat{\mu}}{2}+2.192\right)-12\hat{m}_D\right].$$

(68)

In the limit of small α_s the above gap equation does not go to the perturbative limit for the quark mass which is $\hat{m}_{q,\text{pert}}^2=c_F\alpha_s/8\pi$. The fact that m_q does not go to the perturbative value in the small α_s limit is due to the fact that the perturbative limit of the quark gap equation results from terms which are $O(\alpha_s^2)$ and these terms are not included completely at NLO in HTLpt. One might hope that going to the next order in HTLpt would cure this problem; however, this will in fact not happen since the fermion sector is infrared safe and therefore only even powers of $\hat{m}_q$ will appear at each order. At NNLO all terms contributing at $O(\alpha_s^2)$ at NLO will be replaced by explicit powers of α_s and all $\hat{m}_q$ dependence will be pushed up to $O(\alpha_s^3)$. This behavior will persist at all orders in HTLpt so that at any order the weak-coupling limit of the gap equation quark mass will be scale dependent. In order to circumvent this problem we can consider other possible prescriptions for choosing $\hat{m}_q$ which include requiring that $\hat{m}_q$ be equal to its perturbative value for all α_s or requiring that $\hat{m}_q$ be proportional to $\hat{m}_D$ with the proportionality constant fixed in the weak-coupling limit.

Performing the derivative with respect to m_D while holding m_q fixed results in the following gap equation for m_D:

$$45\hat{m}_D^2\left[1+\left(\log\frac{\hat{\mu}}{2}-\frac{7}{2}+\gamma+\frac{\pi^2}{3}\right)\hat{m}_D\right]$$

$$=\frac{\alpha}{\pi}\left\{15(c_A+s_F)-\frac{55}{2}\left[c_A\left(\log\frac{\hat{\mu}}{2}-\frac{36}{11}\log\hat{m}_D\right.\right.\right.$$

$$\left.\left.-3.637\right)-\frac{4}{11}s_F\left(\log\frac{\hat{\mu}}{2}-2.337\right)\right]\hat{m}_D$$

$$+\frac{495}{2}\left[c_A\left(\log\frac{\hat{\mu}}{2}+\frac{5}{22}+\gamma\right)\right.$$

$$\left.\left.-\frac{4}{11}s_F\left(\log\frac{\hat{\mu}}{2}-\frac{1}{2}+\gamma+2\log 2\right)\right]\hat{m}_D^2+180s_F\hat{m}_q^2\right\}.$$

(69)

The last term in Eq. (69) proportional to $\hat{m}_q^2$ can be written in terms of $\hat{m}_D$ using Eq. (68). In Fig. 3 we plot the solutions to the gap equations for m_D and m_q for $N_c=3$ and $N_f=2$. The solution for m_D goes to the perturbative value in the limit of small α_s, decreases below the perturbative value as α_s increases, and becomes larger than the perturbative value at $\alpha_s\sim0.11$. The solution for m_q does not go to the perturbative value in the limit of small α_s and is instead scale dependent even at lowest order as discussed above. As α_s increases the
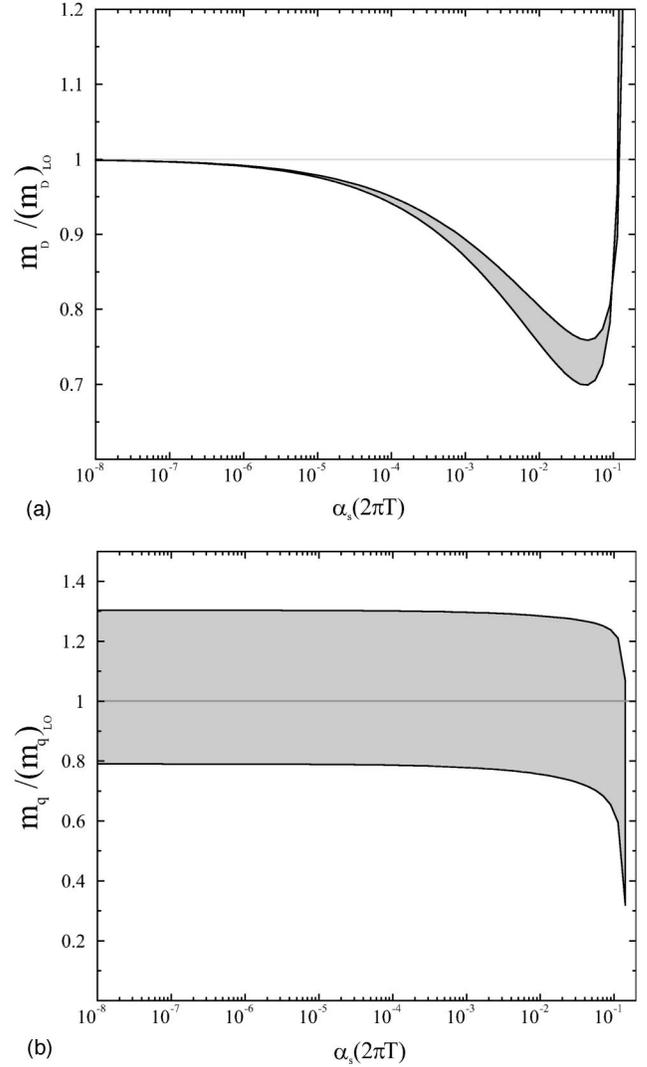

(a)

(b)

FIG. 3. Numerical solution of gap equations for (a) m_D, Eq. (69), and (b) m_q, Eq. (68), as a function of $\alpha_s(2\pi T)$ for $N_c=3$ and $N_f=3$. The shaded band corresponds to varying the renormalization scale μ by a factor of 2 around $\mu=2\pi T$.

value of m_q remains very flat regardless of the scale, changing significantly only near $\alpha_s\sim0.10$.

VIII. FREE ENERGY

The free energy is obtained by evaluating the leading and next-to-leading order thermodynamic potentials, Eqs. (59) and (65), at the solution to the gap equations (68) and (69). In Fig. 4 we plot the leading and next-to-leading order HTLpt predictions for the free energy of QCD with $N_c=3$ and $N_f=2$. We have studied the alternative prescriptions for the quark mass discussed in the previous section and find that the NLO free energy obtained using these prescriptions is numerically indistinguishable from that obtained using the quark gap equations. As can be seen from this figure the corrections in going from LO to NLO are small over the entire temperature range, especially when compared to convergence of the perturbative result. Additionally, the scale
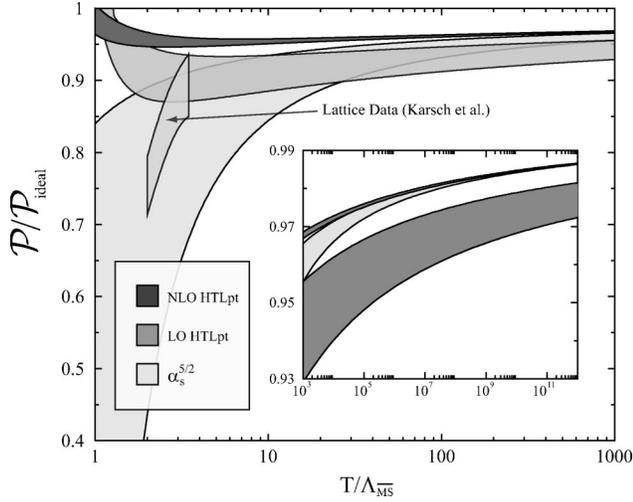
FIG. 4. LO and NLO HTLpt predictions for the free energy of QCD with $N_c = 3$ and $N_f = 2$ together with the perturbative prediction accurate to g^5. The shaded bands correspond to varying the renormalization scale μ by a factor of 2 around $\mu = 2\pi T$. Also shown is a lattice estimate by Karsch et al. [7] for the free energy. The band indicates the estimated systematic error of their result which is reported as $(15 \pm 5)\%$.

FIG. 5. NLO HTLpt prediction for the free energy of QCD with $N_c = 3$ and $N_f = \{0, 2, 4\}$. The shaded bands correspond to varying the renormalization scale μ by a factor of 2 around $\mu = 2\pi T$.

variation of the NLO HTLpt result for the free energy is much smaller than the LO showing that the partial resummation of the scale dependent logarithms reduces the scale variation of the final results significantly.

However, as was the case in pure-glue QCD [32], the results seem to lie significantly above the lattice data which are available below $5\Lambda_{\overline{\text{MS}}}$. There are several reasons for why HTLpt might fail to describe the lattice data in this temperature range. One is that the hard modes are not resummed properly within HTLpt and that a description using a Φ-derivable approach which explicitly separates the hard and soft modes as done in Ref. [21] is better. A second is that HTLpt discards some important physics like topological modes or the Z_N symmetry of QCD near the phase transition.

A third possibility is that the expansion in m_D/T and m_q/T breaks down at these temperatures. Numerically, $m_D/T \sim 1.2$ and $m_q/T \sim 0.5$ at $5\Lambda_{\overline{\text{MS}}}$ and $m_D/T \sim 1.6$ and $m_q/T \sim 0.6$ at $2\Lambda_{\overline{\text{MS}}}$, which casts doubt on the applicability of the expansion in this temperature range. However, in the case of pure glue we have been able to compare the LO HTLpt result expanded to $O(\hat{m}_D^6)$ with the nontruncated LO expression which is accurate to all orders in $\hat{m}_D$ and find that the expansions converge very rapidly. Numerically we find that at $m_D/T = 5$ truncations of the LO order result accurate to $\hat{m}_D^4$ and $\hat{m}_D^6$ reproduce the exact result to 5% and 0.2%, respectively. There have also been studies of the convergence of the mass expansions of the three-loop free energy for a massless scalar field theory using screened perturbation theory [29] and the Φ-derivable approach [16] which demonstrated that mass expansions also converge very rapidly at NLO and NNLO. This gives us some confidence that the truncated NLO solutions are numerically reliable.

We have also shown in Fig. 5 the NLO HTLpt results for the pressure for $N_f = \{0, 2, 4\}$ as a function of $T/\Lambda_{\overline{\text{MS}}}$. From
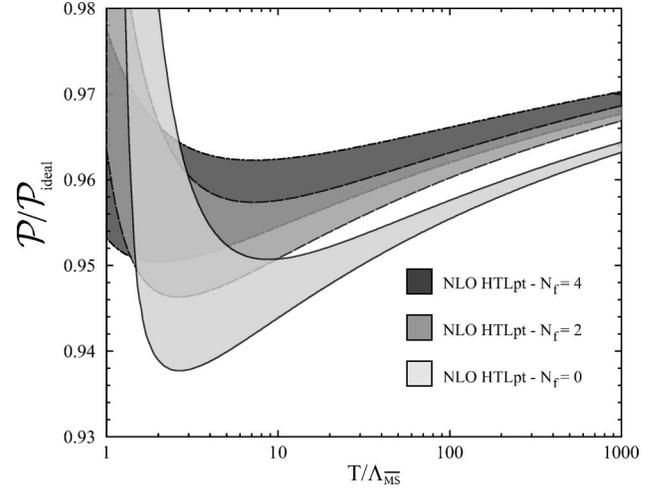
this figure we see that within HTLpt the addition of fermions increases the pressure at fixed $T/\Lambda_{\overline{\text{MS}}}$. Note that $\Lambda_{\overline{\text{MS}}}$ depends on N_f so that when plotted in terms of an absolute scale the various curves shown in this figure will be slightly shifted with respect to one another.

IX. LARGE N_f

In the limit that N_f is taken large while holding $g^2 N_f$ fixed it is possible to solve for the $O(N_f^0)$ contribution to the free energy exactly [33,34]. In Fig. 6 we plot the NLO HTLpt prediction for the $O(N_f^0)$ contribution to the free energy along with the numerical result of Ref. [34] and the perturbative prediction which is accurate to $O(\alpha_s^{5/2})$. In Fig. 7 we plot the NLO HTLpt prediction for m_D in the large N_f limit and the exact numerical result [35]. The HTLpt predictions for both the free energy and the Debye mass seem to diverge from the exact result around $g_{\text{eff}} \sim 2$ regardless of the scale which is chosen; however, for both quantities, choosing the scale to be $\mu = \mu_{\text{DR}} = \pi e^{-\gamma} T$ seems to provide a reasonable reproduction of the exact results. This result is comparable to the performance of the Φ-derivable approach in the large N_f limit [36].

X. CONCLUSIONS

In this paper we have extended our previous HTLpt calculation of the thermodynamic functions in pure-glue QCD to include the contribution of N_f massless quarks. We have presented results for the leading- and next-to-leading-order HTLpt predictions for the QCD free energy for arbitrary N_f. Using the NLO HTLpt expression for the thermodynamic potential we were able to find variational solutions for both the quark and gluon mass parameters, allowing a first-principles prediction of the QCD free energy. As in the case of pure glue we find that the NLO HTLpt prediction lies significantly above the available lattice data below $5\Lambda_{\overline{\text{MS}}}$; however, the problem of oscillation of successive approxi-
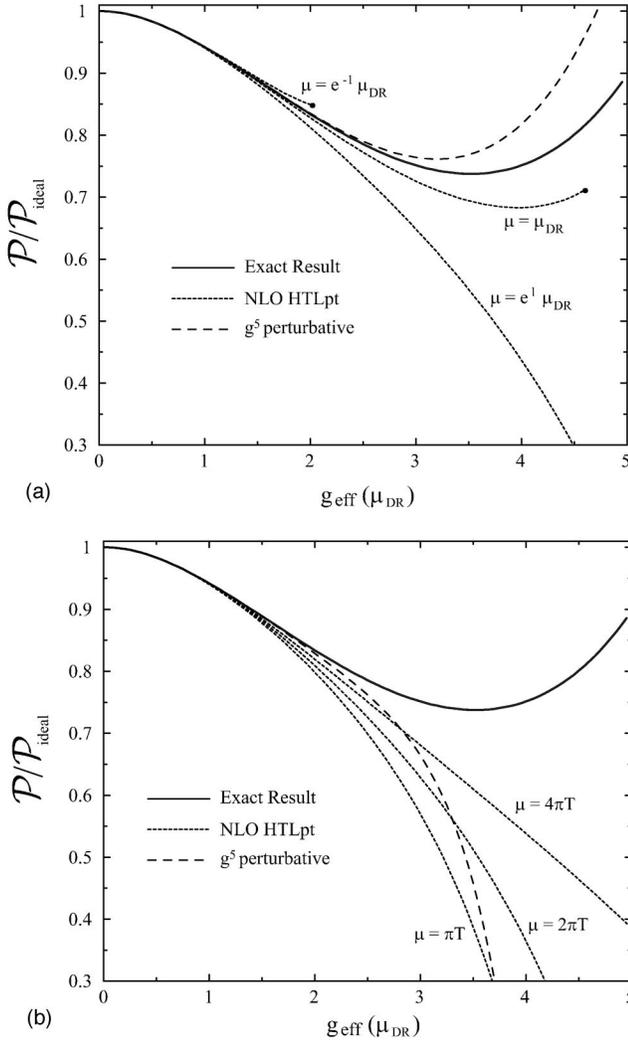
(a)

(b)

FIG. 6. NLO HTLpt prediction for the $O(N_f^0)$ contribution to the free energy, the numerical result of Ref. [34], and the perturbative prediction accurate to g^5 as a function of $g_{\rm eff}(\mu_{\rm DR})$ $= \sqrt{s_f}g(\mu_{\rm DR}) = 2\pi\sqrt{s_f}\alpha_s(\mu_{\rm DR})$ where $\mu_{\rm DR} = \pi e^{-\gamma}T$. Dots indicate the point at which there is no longer a real-valued solution to the gap equation for m_D. In (a) the renormalization scale μ is varied by a factor of e around $\mu_{\rm DR}$. In (b) the renormalization scale μ is varied by a factor of 2 around $2\pi T$. In both (a) and (b) the perturbative g^5 result is evaluated at the central scale.

mations and large scale dependence of the perturbative results is eliminated by using this reorganization.

The failure of HTLpt to describe the lattice data in this region could be attributed to the failure of the expansion performed in $\hat{m}_D$ and $\hat{m}_q$; however, a study of the convergence of the truncated LO expressions to a numerical evaluation of the exact LO expression shows that these expansions converge very rapidly. Therefore, we are steered towards the conclusion that a systematic description of QCD thermodynamics using HTLpt is not appropriate below $5\Lambda_{\overline{\rm MS}}$. The Φ-derivable approach seems to agree better with the lattice data in this range, so perhaps HTLpt is not resumming the hard modes properly and an explicit separation of hard and soft scales is required. However, we should point out that
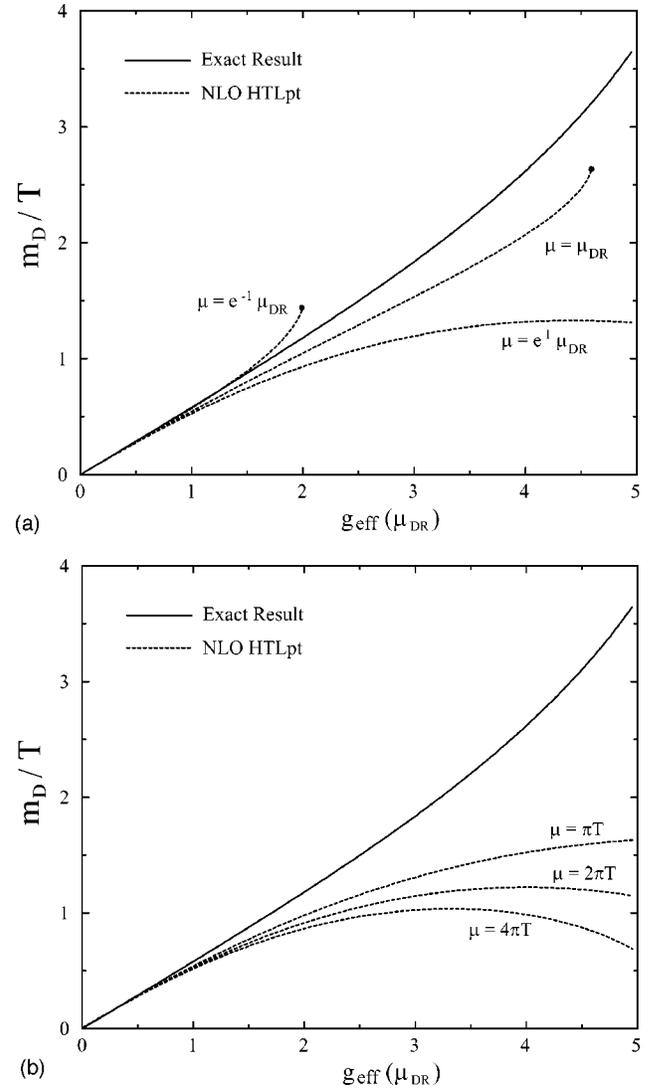
FIG. 7. NLO HTLpt prediction for m_D in the large N_f limit and the exact numerical result [35] as a function of $g_{\rm eff}(\mu_{\rm DR})$ $= \sqrt{s_f}g(\mu_{\rm DR}) = 2\pi\sqrt{s_f}\alpha_s(\mu_{\rm DR})$ where $\mu_{\rm DR} = \pi e^{-\gamma}T$. In (a) the renormalization scale μ is varied by a factor of e around $\mu_{\rm DR}$. In (b) the renormalization scale μ is varied by a factor of 2 around $2\pi T$.

some authors believe that a description of QCD thermodynamics near the phase transition in terms of Polyakov loops is necessary [37].

We have also compared the NLO HTLpt free energy and Debye mass with exact results which are available in the large N_f limit. This comparison shows that, in the large N_f limit, NLO HTLpt agrees with the exact result only out to $g_{\rm eff} \sim 2$ and has large scale dependence after this point. The large scale dependence is not surprising given the fact that in the large N_f limit the running of the coupling constant is driven by the presence of the Landau singularity and even the exact results are sensitive to this beyond $g_{\rm eff} \sim 5$. The poor performance of NLO HTLpt, however, is comparable to recent large N_f predictions within the Φ-derivable approach. The failure of both approaches to agree better with the exact result for large values of $g_{\rm eff}$ is an indication that a descrip-

tion of strongly coupled QCD thermodynamics solely in terms of HTL quasiparticles is perhaps inappropriate. However, it is possible that the physics of large-N_f QCD is so different from that of QCD with a small number of flavors that it cannot serve as a definitive testing ground for the applicability of the quasiparticle approach to the physical case [38].

APPENDIX A: HTL FEYNMAN RULES

In this appendix, we present Feynman rules for HTL perturbation theory in QCD. We give explicit expressions for the propagators and for the quark-gluon 3 and 4 vertices. The Feynman rules are given in Minkowski space to facilitate applications to real-time processes. A Minkowski momentum is denoted $p=(p_0,\mathbf{p})$, and the inner product is $p \cdot q = p_0 q_0 - \mathbf{p} \cdot \mathbf{q}$. The vector that specifies the thermal rest frame is $n = (1,\mathbf{0})$.

1. Gluon self-energy

The HTL gluon self-energy tensor for a gluon of momentum p is

$$\Pi^{\mu\nu}(p) = m_D^2 [\mathcal{T}^{\mu\nu}(p,-p) - n^\mu n^\nu]. \tag{A1}$$

The tensor $\mathcal{T}^{\mu\nu}(p,q)$, which is defined only for momenta that satisfy $p+q=0$, is

$$\mathcal{T}^{\mu\nu}(p,-p) = \left\langle y^\mu y^\nu \frac{p \cdot n}{p \cdot y} \right\rangle_{\hat{\mathbf{y}}}. \tag{A2}$$

The angular brackets indicate averaging over the spatial directions of the light-like vector $y = (1,\hat{\mathbf{y}})$. The tensor $\mathcal{T}^{\mu\nu}$ is symmetric in μ and ν and satisfies the "Ward identity"

$$p_\mu \mathcal{T}^{\mu\nu}(p,-p) = p \cdot n n^\nu. \tag{A3}$$

The self-energy tensor $\Pi^{\mu\nu}$ is therefore also symmetric in μ and ν and satisfies

$$p_\mu \Pi^{\mu\nu}(p) = 0, \tag{A4}$$

$$g_{\mu\nu} \Pi^{\mu\nu}(p) = -m_D^2. \tag{A5}$$

The gluon self-energy tensor can be expressed in terms of two scalar functions, the transverse and longitudinal self-energies Π_T and Π_L, defined by

$$\Pi_T(p) = \frac{1}{d-1} (\delta^{ij} - \hat{p}^i \hat{p}^j) \Pi^{ij}(p), \tag{A6}$$

$$\Pi_L(p) = -\Pi^{00}(p), \tag{A7}$$

where $\hat{\mathbf{p}}$ is the unit vector in the direction of $\mathbf{p}$. In terms of these functions, the self-energy tensor is

$$\Pi^{\mu\nu}(p) = -\Pi_T(p) T_p^{\mu\nu} - \frac{1}{n_p^2} \Pi_L(p) L_p^{\mu\nu}, \tag{A8}$$

where the tensors T_p and L_p are

$$T_p^{\mu\nu} = g^{\mu\nu} - \frac{p^\mu p^\nu}{p^2} - \frac{n_p^\mu n_p^\nu}{n_p^2}, \tag{A9}$$

$$L_p^{\mu\nu} = \frac{n_p^\mu n_p^\nu}{n_p^2}. \tag{A10}$$

The four-vector n_p^μ is

$$n_p^\mu = n^\mu - \frac{n \cdot p}{p^2} p^\mu \tag{A11}$$

and satisfies $p \cdot n_p = 0$ and $n_p^2 = 1 - (n \cdot p)^2/p^2$. Equation (A5) reduces to the identity

$$(d-1)\Pi_T(p) + \frac{1}{n_p^2}\Pi_L(p) = m_D^2. \tag{A12}$$

We can express both self-energy functions in terms of the function $\mathcal{T}^{00}$ defined by Eq. (A2):

$$\Pi_T(p) = \frac{m_D^2}{(d-1)n_p^2}[\mathcal{T}^{00}(p,-p) - 1 + n_p^2], \tag{A13}$$

$$\Pi_L(p) = m_D^2[1 - \mathcal{T}^{00}(p,-p)]. \tag{A14}$$

In the tensor $\mathcal{T}^{\mu\nu}(p,-p)$ defined in Eq. (A2), the angular brackets indicate the angular average over the unit vector $\hat{\mathbf{y}}$. In almost all previous work, the angular average in Eq. (A2) has been taken in $d=3$ dimensions. For consistency of higher order radiative corrections, it is essential to take the angular average in $d=3-2\epsilon$ dimensions and analytically continue to $d=3$ only after all poles in ϵ have been canceled. Expressing the angular average as an integral over the cosine of an angle, the expression for the 00 component of the tensor is

$$\mathcal{T}^{00}(p,-p) = \frac{w(\epsilon)}{2} \int_{-1}^1 dc (1-c^2)^{-\epsilon} \frac{p_0}{p_0 - |\mathbf{p}|c}, \tag{A15}$$

where the weight function $w(\epsilon)$ is

$$w(\epsilon) = \frac{\Gamma(2-2\epsilon)}{\Gamma^2(1-\epsilon)} 2^{2\epsilon} = \frac{\Gamma\left(\frac{3}{2}-\epsilon\right)}{\Gamma\left(\frac{3}{2}\right)\Gamma(1-\epsilon)}. \tag{A16}$$

The integral in Eq. (A15) must be defined so that it is analytic at $p_0 = \infty$. It then has a branch cut running from $p_0 = -|\mathbf{p}|$ to $p_0 = +|\mathbf{p}|$. If we take the limit $\epsilon \to 0$, it reduces to

$$\mathcal{T}^{00}(p, -p) = \frac{p_0}{2|\mathbf{p}|} \log \frac{p_0 + |\mathbf{p}|}{p_0 - |\mathbf{p}|}, \qquad (A17)$$

which is the expression that appears in the usual HTL self-energy functions.

The Feynman rule for the gluon propagator is

$$i\,\delta^{ab} \Delta_{\mu\nu}(p), \qquad (A18)$$

where the gluon propagator tensor $\Delta_{\mu\nu}$ depends on the choice of gauge fixing. We consider two possibilities that introduce an arbitrary gauge parameter ξ: general covariant gauge and general Coulomb gauge. In both cases, the inverse propagator reduces in the limit $\xi \to \infty$ to

$$\Delta_\infty^{-1}(p)^{\mu\nu} = -p^2 g^{\mu\nu} + p^\mu p^\nu - \Pi^{\mu\nu}(p). \qquad (A19)$$

This can also be written

$$\Delta_\infty^{-1}(p)^{\mu\nu} = -\frac{1}{\Delta_T(p)} T_p^{\mu\nu} + \frac{1}{n_p^2 \Delta_L(p)} L_p^{\mu\nu}, \qquad (A20)$$

where Δ_T and Δ_L are the transverse and longitudinal propagators:

$$\Delta_T(p) = \frac{1}{p^2 - \Pi_T(p)}, \qquad (A21)$$

$$\Delta_L(p) = \frac{1}{-n_p^2 p^2 + \Pi_L(p)}. \qquad (A22)$$

The inverse propagator for general ξ is

$$\Delta^{-1}(p)^{\mu\nu} = \Delta_\infty^{-1}(p)^{\mu\nu} - \frac{1}{\xi} p^\mu p^\nu \quad \text{covariant} \quad (A23)$$

$$= \Delta_\infty^{-1}(p)^{\mu\nu} - \frac{1}{\xi}(p^\mu - p \cdot n n^\mu)$$

$$\times (p^\nu - p \cdot n n^\nu) \quad \text{Coulomb.} \qquad (A24)$$

The propagators obtained by inverting the tensors in Eqs. (A24) and (A23) are

$$\Delta^{\mu\nu}(p) = -\Delta_T(p) T_p^{\mu\nu} + \Delta_L(p) n_p^\mu n_p^\nu - \xi \frac{p^\mu p^\nu}{(p^2)^2} \quad \text{covariant}$$
$$\qquad (A25)$$

$$= -\Delta_T(p) T_p^{\mu\nu} + \Delta_L(p) n^\mu n^\nu - \xi \frac{p^\mu p^\nu}{(n_p^2 p^2)^2}$$

$$\text{Coulomb.} \qquad (A26)$$

It is convenient to define the following combination of propagators:

$$\Delta_X(p) = \Delta_L(p) + \frac{1}{n_p^2} \Delta_T(p). \qquad (A27)$$

Using Eqs. (A12), (A21), and (A22), it can be expressed in the alternative form

$$\Delta_X(p) = [m_D^2 - d\Pi_T(p)] \Delta_L(p) \Delta_T(p), \qquad (A28)$$

which shows that it vanishes in the limit $m_D \to 0$. In the covariant gauge, the propagator tensor can be written

$$\Delta^{\mu\nu}(p) = [-\Delta_T(p) g^{\mu\nu} + \Delta_X(p) n^\mu n^\nu]$$

$$\qquad - \frac{n \cdot p}{p^2} \Delta_X(p)(p^\mu n^\nu + n^\mu p^\nu)$$

$$\qquad + \left[\Delta_T(p) + \frac{(n \cdot p)^2}{p^2} \Delta_X(p) - \frac{\xi}{p^2} \right] \frac{p^\mu p^\nu}{p^2}. \qquad (A29)$$

This decomposition of the propagator into three terms has proved to be particularly convenient for explicit calculations. For example, the first term satisfies the identity

$$[-\Delta_T(p) g_{\mu\nu} + \Delta_X(p) n_\mu n_\nu] \Delta_\infty^{-1}(p)^{\nu\lambda}$$

$$= g_\mu{}^\lambda - \frac{p_\mu p^\lambda}{p^2} + \frac{n \cdot p}{n_p^2 p^2} \frac{\Delta_X(p)}{\Delta_L(p)} p_\mu n_p^\lambda. \qquad (A30)$$

2. Quark self-energy

The HTL self-energy of a quark with momentum p is given by

$$\Sigma(P) = m_q^2 \mathcal{T}(p), \qquad (A31)$$

where

$$\mathcal{T}^\mu(p) = \left\langle \frac{y^\mu}{p \cdot y} \right\rangle_{\hat{\mathbf{y}}}. \qquad (A32)$$

Expressing the angular average as an integral over the cosine of an angle, the expression is

$$\mathcal{T}^\mu(p) = \frac{w(\epsilon)}{2} \int_{-1}^1 dc\,(1-c^2)^{-\epsilon} \frac{y^\mu}{p_0 - |\mathbf{p}|c}. \qquad (A33)$$

The integral in Eq. (A33) must be defined so that it is analytic at $p_0 = \infty$. It then has a branch cut running from $p_0 = -|\mathbf{p}|$ to $p_0 = +|\mathbf{p}|$. In three dimensions, this reduces to

$$\Sigma(P) = \frac{m_q^2}{2|\mathbf{p}|} \gamma_0 \log \frac{p_0 + |\mathbf{p}|}{p_0 - |\mathbf{p}|}$$

$$\qquad + \frac{m_q^2}{|\mathbf{p}|} \gamma \cdot \hat{\mathbf{p}} \left(1 - \frac{p_0}{2|\mathbf{p}|} \log \frac{p_0 + |\mathbf{p}|}{p_0 - |\mathbf{p}|} \right). \qquad (A34)$$

3. Quark propagator

The Feynman rule for the quark propagator is

$$i\delta^{ab}S(p). \tag{A35}$$

The quark propagator can be written as

$$S(p)=\frac{1}{\not{p}-\Sigma(p)}, \tag{A36}$$

where the quark self-energy is given by Eq. (A31). The inverse quark propagator can be written as

$$S^{-1}(p)=\not{p}-\Sigma(p). \tag{A37}$$

This can be written as

$$S^{-1}(p)=\mathcal{A}(p), \tag{A38}$$

where we have organized $A_0(p)$ and $A_S(p)$ into

$$A_\mu(p)=(A_0(p),A_S(p)\hat{\mathbf{p}}). \tag{A39}$$

The functions $A_0(p)$ and $A_S(p)$ are defined

$$A_0(p)=p_0-\frac{m_q^2}{p_0}\mathcal{T}_p, \tag{A40}$$

$$A_S(p)=|\mathbf{p}|+\frac{m_q^2}{|\mathbf{p}|}[1-\mathcal{T}_p]. \tag{A41}$$

4. Quark-gluon vertex

The quark-gluon vertex with outgoing gluon momentum p, incoming fermion momentum q, and outgoing quark momentum r, Lorentz index μ and color index a is

$$\Gamma_a^\mu(p,q,r)=gt_a[\gamma^\mu-m_q^2\widetilde{\mathcal{T}}^\mu(p,q,r)]. \tag{A42}$$

The tensor in the HTL correction term is only defined for $p-q+r=0$:

$$\widetilde{\mathcal{T}}^\mu(p,q,r)=\left\langle y^\mu\left(\frac{\not{y}}{q\cdot y r\cdot y}\right)\right\rangle_{\hat{\mathbf{y}}}. \tag{A43}$$

This tensor is even under the permutation of q and r. It satisfies the "Ward identity"

$$p_\mu\widetilde{\mathcal{T}}^\mu(p,q,r)=\widetilde{\mathcal{T}}^\mu(q)-\widetilde{\mathcal{T}}^\mu(r). \tag{A44}$$

The quark-gluon vertex therefore satisfies the Ward identity

$$p_\mu\Gamma^\mu(p,q,r)=S^{-1}(q)-S^{-1}(r). \tag{A45}$$

5. Quark-gluon four-vertex

We define the quark-gluon four-point vertex with outgoing gluon momenta p and q, incoming fermion momentum r, and outgoing fermion momentum s. Generally this vertex has both adjoint and fundamental indices; however, for this cal-

culation we will only need the quark-gluon four-point vertex traced over the adjoint color indices. In this case,

$$\delta^{ab}\Gamma_{abij}^{\mu\nu}(p,q,r,s)=-g^2m_q^2c_F\delta_{ij}\widetilde{\mathcal{T}}^{\mu\nu}(p,q,r,s)$$

$$\equiv g^2c_F\delta_{ij}\Gamma^{\mu\nu}, \tag{A46}$$

where $c_F=(N_c^2-1)/(2N_c)$. There is no tree-level term. The tensor in the HTL correction term is only defined for $p+q-r+s=0$:

$$\widetilde{\mathcal{T}}^{\mu\nu}(p,q,r,s)=\left\langle y^\mu y^\nu\left(\frac{1}{r\cdot y}+\frac{1}{s\cdot y}\right)\right.$$

$$\left.\times\frac{\not{y}}{[(r-p)\cdot y][(s+p)\cdot y]}\right\rangle. \tag{A47}$$

This tensor is symmetric in μ and ν and is traceless. It satisfies the Ward identity

$$p_\mu\Gamma^{\mu\nu}(p,q,r,s)=\Gamma^\nu(q,r-p,s)-\Gamma^\nu(q,r,s+p). \tag{A48}$$

6. HTL quark counterterm

The Feynman rule for the insertion of an HTL quark counterterm into a quark propagator is

$$i\delta^{ab}\Sigma(p), \tag{A49}$$

where $\Sigma(p)$ is the HTL quark self-energy given in Eq. (A39).

7. Imaginary-time formalism

In the imaginary-time formalism, Minkoswski energies have discrete imaginary values $p_0=i(2\pi nT)$ and integrals over Minkowski space are replaced by sum integrals over Euclidean vectors $(2\pi nT,\mathbf{p})$. We will use the notation $P=(P_0,\mathbf{p})$ for Euclidean momenta. The magnitude of the spatial momentum will be denoted $p=|\mathbf{p}|$, and should not be confused with a Minkowski vector. The inner product of two Euclidean vectors is $P\cdot Q=P_0Q_0+\mathbf{p}\cdot\mathbf{q}$. The vector that specifies the thermal rest frame remains $n=(1,\mathbf{0})$.

The Feynman rules for Minkowski space given above can be easily adapted to Euclidean space. The Euclidean tensor in a given Feynman rule is obtained from the corresponding Minkowski tensor with raised indices by replacing each Minkowski energy p_0 by iP_0, where P_0 is the corresponding Euclidean energy, and multiplying by $-i$ for every 0 index. This prescription transforms $p=(p_0,\mathbf{p})$ into $P=(P_0,\mathbf{p})$, $g^{\mu\nu}$ into $-\delta^{\mu\nu}$, and $p\cdot q$ into $-P\cdot Q$. The effect on the HTL tensors defined in Eqs. (A2), (A43), and (A47) is equivalent to substituting $p\cdot n\rightarrow-P\cdot N$ where $N=(-i,\mathbf{0})$, $p\cdot y\rightarrow-P\cdot Y$ where $Y=(-i,\hat{\mathbf{y}})$, and $y^\mu\rightarrow Y^\mu$. For example, the Euclidean tensor corresponding to Eq. (A2) is

$$\mathcal{T}^{\mu\nu}(P,-P)=\left\langle Y^\mu Y^\nu\frac{P\cdot N}{P\cdot Y}\right\rangle. \tag{A50}$$

The average is taken over the directions of the unit vector $\hat{\mathbf{y}}$.

Alternatively, one can calculate a diagram by using the Feynman rules for Minkowski momenta, reducing the expressions for diagrams to scalars, and then make the appropriate substitutions, such as $p^2 \to -P^2$, $p \cdot q \to -P \cdot Q$, and $n \cdot p \to i n \cdot P$. For example, the propagator functions (A21) and (A22) become

$$\Delta_T(P) = \frac{-1}{P^2 + \Pi_T(P)}, \tag{A51}$$

$$\Delta_L(P) = \frac{1}{p^2 + \Pi_L(P)}. \tag{A52}$$

The expressions for the HTL self-energy functions $\Pi_T(P)$ and $\Pi_L(P)$ are given by Eqs. (A13) and (A14) with n_p^2 replaced by $n_P^2 = p^2/P^2$ and $\mathcal{T}^{00}(p, -p)$ replaced by

$$\mathcal{T}_P = \frac{w(\epsilon)}{2} \int_{-1}^{1} dc (1 - c^2)^{-\epsilon} \frac{iP_0}{iP_0 - pc}. \tag{A53}$$

Note that this function differs by a sign from the 00 component of the Euclidean tensor corresponding to Eq. (A2):

$$\mathcal{T}^{00}(P, -P) = -\mathcal{T}^{00}(p, -p)\big|_{p_0 \to iP_0} = -\mathcal{T}_P. \tag{A54}$$

A more convenient form for calculating sum integrals that involve the function $\mathcal{T}_P$ is

$$\mathcal{T}_P = \left\langle \frac{P_0^2}{P_0^2 + p^2 c^2} \right\rangle_c, \tag{A55}$$

where the angular brackets represent an average over c defined by

$$\langle f(c) \rangle_c \equiv w(\epsilon) \int_0^1 dc (1 - c^2)^{-\epsilon} f(c) \tag{A56}$$

and $w(\epsilon)$ is given in Eq. (A16).

APPENDIX B: SUM INTEGRALS

In the imaginary-time formalism for thermal field theory, the 4-momentum $P = (P_0, \mathbf{p})$ is Euclidean with $P^2 = P_0^2 + \mathbf{p}^2$. The Euclidean energy p_0 has discrete values: $P_0 = 2n\pi T$ for bosons and $P_0 = (2n+1)\pi T$ for fermions, where n is an integer. Loop diagrams involve sums over P_0 and integrals over $\mathbf{p}$. With dimensional regularization, the integral is generalized to $d = 3 - 2\epsilon$ spatial dimensions. We define the dimensionally regularized sum integral by

$$\sum_P \!\!\!\!\!\!\!\int \equiv \left(\frac{e^\gamma \mu^2}{4\pi} \right)^\epsilon T \sum_{P_0 = 2n\pi T} \int \frac{d^{3-2\epsilon} p}{(2\pi)^{3-2\epsilon}}, \tag{B1}$$

$$\sum_{\{P\}} \!\!\!\!\!\!\!\int \equiv \left(\frac{e^\gamma \mu^2}{4\pi} \right)^\epsilon T \sum_{P_0 = (2n+1)\pi T} \int \frac{d^{3-2\epsilon} p}{(2\pi)^{3-2\epsilon}}, \tag{B2}$$

where $3 - 2\epsilon$ is the dimension of space and μ is an arbitrary momentum scale. The factor $(e^\gamma / 4\pi)^\epsilon$ is introduced so that, after minimal subtraction of the poles in ϵ due to ultraviolet divergences, μ coincides with the renormalization scale of the $\overline{\text{MS}}$ renormalization scheme.

1. One-loop sum integrals

The simple one-loop sum integrals required in our calculations can be derived from the formulas

$$\sum_P \!\!\!\!\!\!\!\int \frac{p^{2m}}{(P^2)^n} = \left(\frac{\mu}{4\pi T} \right)^{2\epsilon} \frac{2\Gamma(\frac{3}{2} + m - \epsilon)\Gamma(n - \frac{3}{2} - m + \epsilon)}{\Gamma(n)\Gamma(2 - 2\epsilon)}$$
$$\times \Gamma(1 - \epsilon)\zeta(2n - 2m - 3 + 2\epsilon)$$
$$\times e^{\epsilon\gamma} T^{4 + 2m - 2n} (2\pi)^{1 + 2m - 2n}, \tag{B3}$$

$$\sum_{\{P\}} \!\!\!\!\!\!\!\int \frac{p^{2m}}{(P^2)^n} = (2^{2n - 2m - d} - 1) \sum_P \!\!\!\!\!\!\!\int \frac{p^{2m}}{(P^2)^n}. \tag{B4}$$

The specific bosonic one-loop sum integrals needed are

$$\sum_P \!\!\!\!\!\!\!\int \frac{1}{P^2} = \frac{T^2}{12} \left(\frac{\mu}{4\pi T} \right)^{2\epsilon} \left[1 + \left(2 + 2\frac{\zeta'(-1)}{\zeta(-1)} \right) \epsilon \right.$$
$$\left. + \left(4 + \frac{\pi^2}{4} + 4\frac{\zeta'(-1)}{\zeta(-1)} + 2\frac{\zeta''(-1)}{\zeta(-1)} \right) \epsilon^2 \right], \tag{B5}$$

$$\sum_P \!\!\!\!\!\!\!\int \frac{p^2}{(P^2)^2} = \frac{1}{8} T^2, \tag{B6}$$

$$\sum_P \!\!\!\!\!\!\!\int \frac{1}{(P^2)^2} = \frac{1}{(4\pi)^2} \left(\frac{\mu}{4\pi T} \right)^{2\epsilon} \left[\frac{1}{\epsilon} + 2\gamma + \left(\frac{\pi^2}{4} - 4\gamma_1 \right) \epsilon \right], \tag{B7}$$

$$\sum_P \!\!\!\!\!\!\!\int \frac{1}{p^2 P^2} = \frac{1}{(4\pi)^2} \left(\frac{\mu}{4\pi T} \right)^{2\epsilon} 2 \left[\frac{1}{\epsilon} + 2\gamma + 2 \right.$$
$$\left. + \left(4 + 4\gamma + \frac{\pi^2}{4} - 4\gamma_1 \right) \epsilon \right]. \tag{B8}$$

The specific fermionic one-loop sum integrals needed are

$$\sum_{\{P\}} \!\!\!\!\!\!\!\int \log P^2 = \frac{7\pi^2}{360} T^4, \tag{B9}$$

$$\sum_{\{P\}} \frac{1}{P^2} = -\frac{T^2}{24}\left(\frac{\mu}{4\pi T}\right)^{2\epsilon}\left[1+\left(2-2\log 2\right.\right.$$
$$+2\frac{\zeta'(-1)}{\zeta(-1)}\right)\epsilon$$
$$+\left(4+\frac{\pi^2}{4}-4\log 2-2\log^2 2\right.$$
$$+4(1-\log 2)\frac{\zeta'(-1)}{\zeta(-1)}+2\frac{\zeta''(-1)}{\zeta(-1)}\right)\epsilon^2\bigg],$$
$$\text{(B10)}$$

$$\sum_{\{P\}} \frac{1}{(P^2)^2} = \frac{1}{(4\pi)^2}\left(\frac{\mu}{4\pi T}\right)^{2\epsilon}\left[\frac{1}{\epsilon}+2\gamma+4\log 2\right],$$
$$\text{(B11)}$$

$$\sum_{\{P\}} \frac{p^2}{(P^2)^2} = -\frac{T^2}{16}\left(\frac{\mu}{4\pi T}\right)^{2\epsilon}$$
$$\times\left[1+\left(\frac{4}{3}-2\log 2+2\frac{\zeta'(-1)}{\zeta(-1)}\right)\epsilon\right],$$
$$\text{(B12)}$$

$$\sum_{\{P\}} \frac{p^2}{(P^2)^3} = \frac{1}{(4\pi)^2}\left(\frac{\mu}{4\pi T}\right)^{2\epsilon}\frac{3}{4}\left[\frac{1}{\epsilon}+2\gamma-\frac{2}{3}\right.$$
$$+4\log 2\bigg], \quad \text{(B13)}$$

$$\sum_{\{P\}} \frac{p^4}{(P^2)^3} = -\frac{5T^2}{64}\left(\frac{\mu}{4\pi T}\right)^{2\epsilon}$$
$$\times\left[1+\left(\frac{14}{15}-2\log 2+2\frac{\zeta'(-1)}{\zeta(-1)}\right)\epsilon\right],$$
$$\text{(B14)}$$

$$\sum_{\{P\}} \frac{p^4}{(P^2)^4} = \frac{1}{(4\pi)^2}\left(\frac{\mu}{4\pi T}\right)^{2\epsilon}\frac{5}{8}\left[\frac{1}{\epsilon}+2\gamma-\frac{16}{15}\right.$$
$$+4\log 2\bigg], \quad \text{(B15)}$$

$$\sum_{\{P\}} \frac{1}{p^2P^2} = \frac{1}{(4\pi)^2}\left(\frac{\mu}{4\pi T}\right)^{2\epsilon}2\left[\frac{1}{\epsilon}+2+2\gamma+4\log 2\right.$$
$$+\left(4+8\log 2+4\log^2 2\right.$$
$$+4\gamma(1+2\log 2)+\frac{\pi^2}{4}-4\gamma_1\bigg)\epsilon\bigg]. \text{ (B16)}$$

The errors are all of one order higher in ϵ than the smallest term shown. The number γ_1 is the first Stieltjes gamma constant defined by the equation

$$\zeta(1+z) = \frac{1}{z}+\gamma-\gamma_1 z+O(z^2). \quad \text{(B17)}$$

2. One-loop HTL sum integrals

We also need some more difficult one-loop sum integrals that involve the HTL function defined in Eq. (A33).

The specific bosonic sum integrals needed are

$$\sum_{P}\frac{1}{p^4}\mathcal{T}_P = \frac{1}{(4\pi)^2}\left(\frac{\mu}{4\pi T}\right)^{2\epsilon}(-1)\left[\frac{1}{\epsilon}+2\gamma+2\log 2\right],$$
$$\text{(B18)}$$

$$\sum_{P}\frac{1}{p^2P^2}\mathcal{T}_P = \frac{1}{(4\pi)^2}\left(\frac{\mu}{4\pi T}\right)^{2\epsilon}$$
$$\times\left[2\log 2\left(\frac{1}{\epsilon}+2\gamma\right)+2\log^2 2+\frac{\pi^2}{3}\right],$$
$$\text{(B19)}$$

$$\sum_{P}\frac{1}{(P^2)^2}\mathcal{T}_P = \frac{1}{(4\pi)^2}\left(\frac{\mu}{4\pi T}\right)^{2\epsilon}\frac{1}{2}\left[\frac{1}{\epsilon}+2\gamma+1\right].$$
$$\text{(B20)}$$

The specific fermionic sum integrals needed are

$$\sum_{\{P\}}\frac{1}{(P^2)^2}\mathcal{T}_P = \frac{1}{(4\pi)^2}\left(\frac{\mu}{4\pi T}\right)^{2\epsilon}\frac{1}{2}\left[\frac{1}{\epsilon}+2\gamma+1\right.$$
$$+4\log 2\bigg], \quad \text{(B21)}$$

$$\sum_{\{P\}}\frac{1}{p^2P^2}\mathcal{T}_P = \frac{2}{(4\pi)^2}\left(\frac{\mu}{4\pi T}\right)^{2\epsilon}\left[\log 2\left(\frac{1}{\epsilon}+2\gamma\right)\right.$$
$$+5\log^2 2+\frac{\pi^2}{6}\bigg], \quad \text{(B22)}$$

$$\sum_{\{P\}}\frac{1}{P^2P_0^2}\mathcal{T}_P = \frac{1}{(4\pi)^2}\left(\frac{\mu}{4\pi T}\right)^{2\epsilon}\left[\frac{1}{\epsilon^2}+2(\gamma\right.$$
$$+2\log 2)\frac{1}{\epsilon}+\frac{\pi^2}{4}+4\log^2 2$$
$$+8\gamma\log 2-4\gamma_1\bigg], \quad \text{(B23)}$$

$$\sum_{\{P\}}\frac{1}{p^2P_0^2}(\mathcal{T}_P)^2 = \frac{4}{(4\pi)^2}\left(\frac{\mu}{4\pi T}\right)^{2\epsilon}\left[\log 2\left(\frac{1}{\epsilon}+2\gamma\right)\right.$$
$$+5\log^2 2\bigg], \quad \text{(B24)}$$

$$\sum_{\{P\}}\frac{1}{P^2}\left\langle\frac{1}{(Q\cdot Y)^2}\right\rangle_c = \frac{1}{(4\pi)^2}\left(\frac{\mu}{4\pi T}\right)^{2\epsilon}(-1)\left[\frac{1}{\epsilon}-1+2\gamma\right.$$
$$+4\log 2\bigg]. \quad \text{(B25)}$$

The errors are all of order ϵ

It is straightforward to calculate the sum integrals (B21)–(B24) using the representation (A55) of the function $\mathcal{T}_P$. For example, the sum integral (B18) can be written

$$\sum\!\!\!\!\!\!\!\int_P \frac{1}{p^4}\mathcal{T}_P = \sum\!\!\!\!\!\!\!\int_P \frac{1}{p^4}\left\langle \frac{P_0^2}{P_0^2+p^2c^2} \right\rangle_c, \tag{B26}$$

where the angular brackets denote an average over c as defined in Eq. (A56):

$$\sum\!\!\!\!\!\!\!\int_P \frac{1}{p^4}\mathcal{T}_P = \sum\!\!\!\!\!\!\!\int_P \frac{1}{p^4}\left[1 - \left\langle \frac{p^2c^2}{P_0^2+p^2c^2} \right\rangle_c \right]. \tag{B27}$$

The first term in the square brackets vanishes with dimensional regularization, while after rescaling the momentum by $\mathbf{p}\to\mathbf{p}/c$, the second term reads

$$\sum\!\!\!\!\!\!\!\int_P \frac{1}{p^4}\mathcal{T}_P = -\langle c^{1+2\epsilon}\rangle_c \sum\!\!\!\!\!\!\!\int_P \frac{1}{p^2P^2}. \tag{B28}$$

Evaluating the average over c, using the expression (B8) for the sum integral, and expanding in powers of ϵ, we obtain the result (B18). Following the same strategy, all the sum integrals (B21)–(B24) can be reduced to linear combinations of simple sum integrals with coefficients that are averages over c. The only difficult integral is the double average over c that arises from Eq. (B24):

$$\left\langle \frac{c_1^{1+2\epsilon}-c_2^{1+2\epsilon}}{c_1^2-c_2^2} \right\rangle_{c_1,c_2} = 2\log 2 + 2(\log^2 2 - 2\log 2)\epsilon. \tag{B29}$$

3. Simple two-loop sum integrals

The simple two-loop sum integrals that are needed are

$$\sum\!\!\!\!\!\!\!\int_{\{PQ\}} \frac{1}{P^2Q^2R^2} = 0, \tag{B30}$$

$$\sum\!\!\!\!\!\!\!\int_{\{PQ\}} \frac{1}{P^2Q^2r^2} = \frac{T^2}{(4\pi)^2}\left(\frac{\mu}{4\pi T}\right)^{4\epsilon}\left(-\frac{1}{6}\right)$$
$$\times\left[\frac{1}{\epsilon}+4-2\log 2+4\frac{\zeta'(-1)}{\zeta(-1)}\right], \tag{B31}$$

$$\sum\!\!\!\!\!\!\!\int_{\{PQ\}} \frac{q^2}{P^2Q^2r^4} = \frac{T^2}{(4\pi)^2}\left(\frac{\mu}{4\pi T}\right)^{4\epsilon}\left(-\frac{1}{12}\right)\left[\frac{1}{\epsilon}+\frac{11}{3}\right.$$
$$\left.+2\gamma-2\log 2+2\frac{\zeta'(-1)}{\zeta(-1)}\right], \tag{B32}$$

$$\sum\!\!\!\!\!\!\!\int_{\{PQ\}} \frac{q^2}{P^2Q^2r^2R^2} = \frac{T^2}{(4\pi)^2}\left(\frac{\mu}{4\pi T}\right)^{4\epsilon}\left(-\frac{1}{72}\right)\left[\frac{1}{\epsilon}-7.002\right], \tag{B33}$$

$$\sum\!\!\!\!\!\!\!\int_{\{PQ\}} \frac{P\cdot Q}{P^2Q^2r^4} = \frac{T^2}{(4\pi)^2}\left(\frac{\mu}{4\pi T}\right)^{4\epsilon}\left(-\frac{1}{36}\right)$$
$$\times\left[1-6\gamma+6\frac{\zeta'(-1)}{\zeta(-1)}\right], \tag{B34}$$

$$\sum\!\!\!\!\!\!\!\int_{\{PQ\}} \frac{p^2}{q^2P^2Q^2R^2} = \frac{T^2}{(4\pi)^2}\left(\frac{\mu}{4\pi T}\right)^{4\epsilon}\left(\frac{5}{72}\right)\left[\frac{1}{\epsilon}+9.5667\right], \tag{B35}$$

$$\sum\!\!\!\!\!\!\!\int_{\{PQ\}} \frac{r^2}{q^2P^2Q^2R^2} = \frac{T^2}{(4\pi)^2}\left(\frac{\mu}{4\pi T}\right)^{4\epsilon}\left(-\frac{1}{18}\right)$$
$$\times\left[\frac{1}{\epsilon}+8.1420\right], \tag{B36}$$

where $R=-(P+Q)$ and $r=|\mathbf{p}+\mathbf{q}|$. The corrections are all of order ϵ. To motivate the integration formula we will use to evaluate the two-loop sum integrals, we first present the analogous integration formula for one-loop sum integrals. In a one-loop sum integral, the sum over P_0 can be replaced by a contour integral in $p_0=-iP_0$:

$$\sum\!\!\!\!\!\!\!\int_P F(P) = \lim_{\eta\to 0^+}\int \frac{dp_0}{2\pi i}\int_{\mathbf{p}}[F(-ip_0,\mathbf{p})$$
$$-F(0,\mathbf{p})]e^{\eta p_0}n(p_0), \tag{B37}$$

where $n(p_0)=1/(e^{\beta p_0}-1)$ is the Bose-Einstein thermal distribution and the contour runs from $-\infty$ to $+\infty$ above the real axis and from $+\infty$ to $-\infty$ below the real axis. This formula can be expressed in a more convenient form by collapsing the contour onto the real axis and separating out those terms with the exponential convergence factor $n(|p_0|)$. The remaining terms run along contours from $-\infty\pm i\varepsilon$ to 0 and have the convergence factor $e^{\eta p_0}$. This allows the contours to be deformed so that they run from 0 to $\pm i\infty$ along the imaginary p_0 axis, which corresponds to real values of $P_0=-ip_0$. Assuming that $F(-ip_0,\mathbf{p})$ is a real function of p_0, i.e. that it satisfies $F(-ip_0^*,\mathbf{p})=F(-ip_0,\mathbf{p})^*$, the resulting formula for the sum integral is

$$\sum\!\!\!\!\!\!\!\int_P F(P) = \int_P F(P) + \int_p \epsilon(p_0)n(|p_0|)2\,\mathrm{Im}\,F(-ip_0+\varepsilon,\mathbf{p}), \tag{B38}$$

where $\epsilon(p_0)$ is the sign of p_0. The first integral on the right side is over the $(d+1)$-dimensional Euclidean vector $P=(P_0,\mathbf{p})$ and the second is over the $(d+1)$-dimensional Minkowskian vector $p=(p_0,\mathbf{p})$.

The two-loop sum integrals can be evaluated by using a generalization of the one-loop formula (B38):

$$\sum\!\!\!\!\!\!\!\int_{\{PQ\}} F(P)G(Q)H(R) = \int_{PQ} F(P)G(Q)H(R) - \int_p \epsilon(p_0)n_F(|p_0|)2\,\text{Im}\,F(-ip_0+\varepsilon,\mathbf{p})\,\text{Re}\int_Q G(Q)H(R)\bigg|_{P_0=-ip_0+\varepsilon}$$

$$- \int_p \epsilon(p_0)n_F(|p_0|)2\,\text{Im}\,G(-ip_0+\varepsilon,\mathbf{p})\,\text{Re}\int_Q H(Q)F(R)\bigg|_{P_0=-ip_0+\varepsilon}$$

$$+ \int_p \epsilon(p_0)n_B(|p_0|)2\,\text{Im}\,H(-ip_0+\varepsilon,\mathbf{p})\,\text{Re}\int_Q F(Q)G(R)\bigg|_{P_0=-ip_0+\varepsilon}$$

$$+ \int_p \epsilon(p_0)n_F(|p_0|)2\,\text{Im}\,F(-ip_0+\varepsilon,\mathbf{p})\int_q \epsilon(q_0)n_F(|q_0|)2\,\text{Im}\,G(-iq_0+\varepsilon,\mathbf{q})\,\text{Re}\,H(R)|_{R_0=i(p_0+q_0)+\varepsilon}$$

$$- \int_p \epsilon(p_0)n_F(|p_0|)2\,\text{Im}\,G(-ip_0+\varepsilon,\mathbf{p})\int_q \epsilon(q_0)n_B(|q_0|)2\,\text{Im}\,H(-iq_0+\varepsilon,\mathbf{q})\,\text{Re}\,F(R)\bigg|_{R_0=i(p_0+q_0)+\varepsilon}$$

$$- \int_p \epsilon(p_0)n_B(|p_0|)2\,\text{Im}\,H(-ip_0+\varepsilon,\mathbf{p})\int_q \epsilon(q_0)n_F(|q_0|)2\,\text{Im}\,F(-iq_0+\varepsilon,\mathbf{q})\,\text{Re}\,G(R)\bigg|_{R_0=i(p_0+q_0)+\varepsilon}. \tag{B39}$$

This formula can be derived in 3 steps. First, express the sum over P_0 as the sum of two contour integrals over p_0, one that encloses the real axis $\text{Im}\,p_0=0$ and another that encloses the line $\text{Im}\,p_0=-\text{Im}\,q_0$. Second, express the sum over q_0 as a contour integral that encloses the real-q_0 axis. The resulting terms can be combined into the expression (B39). The integrals of the imaginary parts that enter into our calculation can be reduced to

$$\int_p \epsilon(p_0)n(|p_0|)2\,\text{Im}\,\frac{1}{P^2}\bigg|_{P_0=-ip_0+\varepsilon} f(-ip_0+\varepsilon,\mathbf{p})$$

$$= \int_\mathbf{p} \frac{n(p)}{p}\frac{1}{2}\sum_\pm f(\pm ip+\varepsilon,\mathbf{p}), \tag{B40}$$

$$\int_p \epsilon(p_0)n(|p_0|)2\,\text{Im}\,\mathcal{T}_P\bigg|_{P_0=-ip_0+\varepsilon} f(-ip_0+\varepsilon,\mathbf{p})$$

$$= -\int_\mathbf{p} p\,n(p)\frac{1}{2}\sum_\pm \langle c^{-3+2\epsilon}f(\pm ip+\varepsilon,\mathbf{p}/c)\rangle_c. \tag{B41}$$

The latter equation is obtained by inserting the expression (A55) for $\mathcal{T}_P$, using Eq. (B40), and then making the change of variable $\mathbf{p}\to\mathbf{p}/c$ to put the thermal integral into a standard form.

As a simple illustration, we apply the formula (B39) to the sum integral (B31). The nonvanishing terms are

$$\sum\!\!\!\!\!\!\!\int_{\{PQ\}} \frac{1}{P^2Q^2r^2} = -2\int_p n_F(|p_0|)2\pi\delta(p_0^2-p^2)\int_Q \frac{1}{Q^2r^2}$$

$$+ \int_p n_F(|p_0|)2\pi\delta(p_0^2-p^2)$$

$$\times \int_q n_F(|q_0|)2\pi\delta(q_0^2-q^2)\frac{1}{r^2}. \tag{B42}$$

The delta functions can be used to evaluate the integrals over p_0 and q_0. The integral over Q is given in Eq. (C111) up to corrections of order ϵ. This reduces the sum integral to

$$\sum\!\!\!\!\!\!\!\int_{\{PQ\}} \frac{1}{P^2Q^2r^2} = -\frac{4}{(4\pi)^2}\bigg[\frac{1}{\epsilon}+4$$

$$-2\log 2\bigg]\mu^{2\epsilon}\int_\mathbf{p} \frac{n_F(p)}{p}p^{-2\epsilon}$$

$$+ \int_\mathbf{pq} \frac{n_F(p)n_F(q)}{pq}\frac{1}{r^2}. \tag{B43}$$

The momentum integrals are evaluated in Eqs. (C3) and (C4). Keeping all terms that contribute through order ϵ^0, we get the result (B31). The sum integral (B32) can be evaluated in the same way:

$$\sum\!\!\!\!\!\!\!\int_{\{PQ\}} \frac{q^2}{P^2Q^2r^4} = -\frac{2}{(4\pi)^2}\bigg[\frac{1}{\epsilon}-2\log 2\bigg]\mu^{2\epsilon}\int_\mathbf{p} \frac{n_F(p)}{p}p^{-2\epsilon}$$

$$+ \int_\mathbf{pq} \frac{n_F(p)n_F(q)}{pq}\frac{q^2}{r^4}. \tag{B44}$$

The sum integral (B34) can be reduced to a linear combination of Eqs. (B31) and (B32) by expressing the numerator in the form $P \cdot Q = P_0 Q_0 + (r^2 - p^2 - q^2)/2$ and noting that the $P_0 Q_0$ term vanishes upon summing over P_0 or Q_0.

The sum integral (B33) is a little more difficult. After applying the formula (B39) and using the delta functions to integrate over p_0, q_0, and r_0, it can be reduced to

$$\sum\!\!\!\!\!\!\!\int_{\{PQ\}} \frac{q^2}{P^2 Q^2 r^2 R^2} = \int_{\mathbf{p}} \frac{n_B(p)}{p} \int_Q \frac{q^2}{p^2 Q^2 R^2} \bigg|_{P=-ip}$$

$$- \int_{\mathbf{p}} \frac{n_F(p)}{p} \int_Q \frac{1}{Q^2 R^2} \left(\frac{q^2}{r^2} + \frac{p^2}{q^2} \right) \bigg|_{P=-ip}$$

$$+ \int_{\mathbf{pq}} \frac{n_F(p) n_F(q)}{pq} \frac{p^2}{r^2} \frac{r^2 - p^2 - q^2}{\Delta(p,q,r)}$$

$$- \int_{\mathbf{pq}} \frac{n_F(p) n_B(q)}{pq} \left(\frac{p^2}{q^2} \right.$$

$$\left. + \frac{r^2}{q^2} \right) \frac{r^2 - p^2 - q^2}{\Delta(p,q,r)}, \tag{B45}$$

where $\Delta(p,q,r)$ is the triangle function that is negative when p, q, and r are the lengths of 3 sides of a triangle:

$$\Delta(p,q,r) = p^4 + q^4 + r^4 - 2(p^2 q^2 + q^2 r^2 + r^2 p^2). \tag{B46}$$

After using Eqs. (C117)–(C119) to integrate over Q, the first term on the right side of Eq. (B45) is evaluated using Eq. (C3). The 2-loop thermal integrals on the right side of Eq. (B45) are given in Eqs. (C8)–(C11). Adding together all the terms, we get the final result (B33). The sum integrals (B35) and (B36) are evaluated in a similar manner.

4. Two-loop HTL sum integrals

We also need some more difficult two-loop sum integrals that involve the functions $\mathcal{T}_P$ defined in Eq. (A33):

$$\sum\!\!\!\!\!\!\!\int_{\{PQ\}} \frac{1}{P^2 Q^2 r^2} \mathcal{T}_R = \frac{T^2}{(4\pi)^2} \left(\frac{\mu}{4\pi T} \right)^{4\epsilon} \left(-\frac{1}{48} \right)$$

$$\times \left[\frac{1}{\epsilon^2} + \left(2 + 12 \log 2 + 4 \frac{\zeta'(-1)}{\zeta(-1)} \right) \right.$$

$$\left. \times \frac{1}{\epsilon} + 136.362 \right], \tag{B47}$$

$$\sum\!\!\!\!\!\!\!\int_{\{PQ\}} \frac{q^2}{P^2 Q^2 r^4} \mathcal{T}_R = \frac{T^2}{(4\pi)^2} \left(\frac{\mu}{4\pi T} \right)^{4\epsilon} \left(-\frac{1}{576} \right) \left[\frac{1}{\epsilon^2} + \left(\frac{26}{3} \right. \right.$$

$$\left. + 52 \log 2 + 4 \frac{\zeta'(-1)}{\zeta(-1)} \right) \frac{1}{\epsilon}$$

$$\left. + 446.438 \right], \tag{B48}$$

$$\sum\!\!\!\!\!\!\!\int_{\{PQ\}} \frac{P \cdot Q}{P^2 Q^2 r^4} \mathcal{T}_R = \frac{T^2}{(4\pi)^2} \left(\frac{\mu}{4\pi T} \right)^{4\epsilon} \left(-\frac{1}{96} \right) \left[\frac{1}{\epsilon^2} \right.$$

$$\left. + \left(4 \log 2 + 4 \frac{\zeta'(-1)}{\zeta(-1)} \right) \frac{1}{\epsilon} + 69.174 \right], \tag{B49}$$

$$\sum\!\!\!\!\!\!\!\int_{\{PQ\}} \frac{r^2 - p^2}{P^2 q^2 Q_0^2 R^2} \mathcal{T}_Q = -\frac{T^2}{(4\pi)^2} \left(\frac{\mu}{4\pi T} \right)^{4\epsilon} \frac{1}{8} \left[\frac{1}{\epsilon^2} + \left(2 + 2\gamma \right. \right.$$

$$\left. + \frac{10}{3} \log 2 + 2 \frac{\zeta'(-1)}{\zeta(-1)} \right) \frac{1}{\epsilon}$$

$$\left. + 46.8757 \right]. \tag{B50}$$

The errors are all of order ϵ. To calculate the sum integral (B47), we begin by using the representation (A55) of the function $\mathcal{T}_R$:

$$\sum\!\!\!\!\!\!\!\int_{\{PQ\}} \frac{1}{P^2 Q^2 r^2} \mathcal{T}_R = \sum\!\!\!\!\!\!\!\int_{\{PQ\}} \frac{1}{P^2 Q^2 r^2}$$

$$- \sum\!\!\!\!\!\!\!\int_{\{PQ\}} \frac{1}{P^2 Q^2} \left\langle \frac{c^2}{R_0^2 + r^2 c^2} \right\rangle_c. \tag{B51}$$

The first sum integral on the right hand side is given by Eq. (B31). To evaluate the second sum integral, we apply the sum integral formula (B39):

$$\sum\!\!\!\!\!\!\!\int_{\{PQ\}} \frac{1}{P^2 Q^2 (R_0^2 + r^2 c^2)}$$

$$= -\int_{\mathbf{p}} \frac{n_F(p)}{p} 2 \, \text{Re} \int_Q \frac{1}{Q^2 (R_0^2 + r^2 c^2)} \bigg|_{P_0 = -ip + \varepsilon}$$

$$+ c^{-3+2\epsilon} \int_{\mathbf{p}} \frac{n_B(p)}{p} \int_Q \frac{1}{Q^2 R^2} \bigg|_{P \to (-ip, \mathbf{p}/c)}$$

$$+ \int_{\mathbf{pq}} \frac{n_F(p) n_F(q)}{pq} \, \text{Re} \frac{r^2 c^2 - p^2 - q^2}{\Delta(p+i\varepsilon, q, rc)}$$

$$- 2 c^{-3+2\epsilon} \int_{\mathbf{pq}} \frac{n_F(p) n_B(q)}{pq} \, \text{Re} \frac{r_c^2 - p^2 - q^2}{\Delta(p+i\varepsilon, q, r_c)}, \tag{B52}$$

where $r_c = |\mathbf{p} + \mathbf{q}/c|$. In the terms on the right side with a single thermal integral, the appropriate averages over c of the integrals over Q are given in Eqs. (C115) and (C123).

The subsequent integrals over $\mathbf{p}$ are special cases of Eqs. (C3) and (C4):

$$\int_{\mathbf{p}} n_B(p)p^{-1-2\epsilon}=2^{8\epsilon}\frac{(1)_{-4\epsilon}\left(\frac{1}{2}\right)_{2\epsilon}}{(1)_{-2\epsilon}\left(\frac{3}{2}\right)_{-\epsilon}}\frac{\zeta(-1+4\epsilon)}{\zeta(-1)}$$

$$\times(e^\gamma\mu^2)^\epsilon(4\pi T)^{-4\epsilon}\frac{T^2}{12}, \qquad (B53)$$

$$\int_{\mathbf{p}} n_F(p)p^{-1-2\epsilon}=[1-2^{-1+4\epsilon}]\int_{\mathbf{p}} n_B(p)p^{-1-2\epsilon}. \qquad (B54)$$

This yields

$$-2\int_{\mathbf{p}}\frac{n_F(p)}{p}\mathrm{Re}\int_Q\frac{1}{Q^2}\left\langle\frac{c^2}{R_0^2+r^2c^2}\right\rangle_c\bigg|_{P_0=-ip+\varepsilon}$$

$$+\int_{\mathbf{p}}\frac{n_B(p)}{p}\left\langle c^{-1+2\epsilon}\int_Q\frac{1}{Q^2R^2}\bigg|_{P\to(-ip,\mathbf{p}/c)}\right\rangle_c$$

$$=\frac{T^2}{(4\pi)^2}\left(\frac{\mu}{4\pi T}\right)^{4\epsilon}\frac{1}{48}\bigg[\frac{1}{\epsilon^2}-\left(6-12\log 2\right.$$

$$\left.-4\frac{\zeta'(-1)}{\zeta(-1)}\right)\frac{1}{\epsilon}+70.122\bigg]. \qquad (B55)$$

For the two terms in Eq. (B51) with a double thermal integral, the averages weighted by c^2 are given in Eqs. (C17) and (C21). Adding them to Eq. (B55), the final result is

$$\sum\!\!\!\!\!\!\!\int_{\{PQ\}}\frac{1}{P^2Q^2}\left\langle\frac{c^2}{R_0^2+r^2c^2}\right\rangle_c=\frac{T^2}{(4\pi)^2}\left(\frac{\mu}{4\pi T}\right)^{4\epsilon}\left(\frac{1}{48}\right)\bigg[\frac{1}{\epsilon^2}-\bigg(6$$

$$-12\log 2-4\frac{\zeta'(-1)}{\zeta(-1)}\bigg)\frac{1}{\epsilon}$$

$$+51.9307\bigg]. \qquad (B56)$$

Inserting this into Eq. (B51), we obtain the final result (B47).

The sum integral (B48) is evaluated in a similar way to Eq. (B47). Using the representation (A55) for $\mathcal{T}_R$, we get

$$\sum\!\!\!\!\!\!\!\int_{\{PQ\}}\frac{q^2}{P^2Q^2r^4}\mathcal{T}_R=\sum\!\!\!\!\!\!\!\int_{\{PQ\}}\frac{q^2}{P^2Q^2r^4}$$

$$-\sum\!\!\!\!\!\!\!\int_{\{PQ\}}\frac{q^2}{P^2Q^2r^2}\left\langle\frac{c^2}{R_0^2+r^2c^2}\right\rangle_c. \qquad (B57)$$

The first sum integral on the right hand side is given by Eq. (B32). To evaluate the second sum integral, we apply the sum integral formula (B39):

$$\sum\!\!\!\!\!\!\!\int_{\{PQ\}}\frac{q^2}{P^2Q^2r^2(R_0^2+r^2c^2)}$$

$$=-\int_{\mathbf{p}}\frac{n_F(p)}{p}\mathrm{Re}\int_Q\frac{p^2+q^2}{Q^2r^2(R_0^2+r^2c^2)}\bigg|_{P_0=-ip+\varepsilon}+c^{-1+2\epsilon}\int_{\mathbf{p}}\frac{n_B(p)}{p}p^{-2}\int_Q\frac{q^2}{Q^2R^2}\bigg|_{P\to(-ip,\mathbf{p}/c)}$$

$$+\int_{\mathbf{pq}}\frac{n_F(p)n_F(q)}{pq}\frac{q^2}{r^2}\mathrm{Re}\frac{r^2c^2-p^2-q^2}{\Delta(p+i\varepsilon,q,rc)}-c^{-1+2\epsilon}\int_{\mathbf{pq}}\frac{n_F(p)n_B(q)}{pq}\frac{p^2+r_c^2}{q^2}\mathrm{Re}\frac{r_c^2-p^2-q^2}{\Delta(p+i\varepsilon,q,r_c)}. \qquad (B58)$$

In the terms on the right side with a single thermal integral, the weighted averages over c of the integrals over Q are given in Eqs. (C121), (C126), and (C127): After using Eq. (B54) to evaluate the thermal integral, we obtain

$$-\int_{\mathbf{p}}\frac{n_F(p)}{p}\mathrm{Re}\int_Q\frac{p^2+q^2}{Q^2r^2}\left\langle\frac{c^2}{R_0^2+r^2c^2}\right\rangle_c\bigg|_{P_0=-ip+\varepsilon}+\int_{\mathbf{p}}\frac{n_B(p)}{p}\frac{1}{p^2}\left\langle c^{1+2\epsilon}\int_Q\frac{q^2}{Q^2R^2}\bigg|_{P\to(-ip,\mathbf{p}/c)}\right\rangle_c$$

$$=\frac{T^2}{(4\pi)^2}\left(\frac{\mu}{4\pi T}\right)^{4\epsilon}\left(\frac{1}{576}\right)\bigg[\frac{1}{\epsilon^2}-\left(\frac{34}{3}-36\log 2-4\frac{\zeta'(-1)}{\zeta(-1)}\right)\frac{1}{\epsilon}+229.354\bigg]. \qquad (B59)$$

For the two terms in Eq. (B58) with a double thermal integral, the averages weighted by c^2 are given in Eqs. (C19), (C23), and (C24). Adding them to Eq. (B59), the final result is

$$\sum\!\!\!\!\!\!\!\int_{\{PQ\}}\frac{q^2}{P^2Q^2r^2}\left\langle\frac{c^2}{R_0^2+r^2c^2}\right\rangle_c=\frac{T^2}{(4\pi)^2}\left(\frac{\mu}{4\pi T}\right)^{4\epsilon}\left(\frac{1}{576}\right)\bigg[\frac{1}{\epsilon^2}-\left(\frac{118}{3}-52\log 2-4\frac{\zeta'(-1)}{\zeta(-1)}\right)\frac{1}{\epsilon}+91.002\bigg]. \qquad (B60)$$

To evaluate Eq. (B49), we use the expression (A55) for $\mathcal{T}_R$ and the identity $P\cdot Q=(R^2-P^2-Q^2)/2$ to write it in the form

$$\sum\!\!\!\!\!\!\!\!\!\!\!\int_{\{PQ\}}\frac{P\cdot Q}{P^2Q^2r^4}\mathcal{T}_R=\sum\!\!\!\!\!\!\!\!\!\!\!\int_{\{PQ\}}\frac{P\cdot Q}{P^2Q^2r^4}-\sum\!\!\!\!\!\!\!\!\!\!\!\int_{\{P\}}\frac{1}{P^2}\sum\!\!\!\!\!\!\!\!\!\!\!\int_{R}\frac{1}{r^4}\mathcal{T}_R-\frac{1}{2}\langle c^2\rangle_c\sum\!\!\!\!\!\!\!\!\!\!\!\int_{\{PQ\}}\frac{1}{P^2Q^2r^2}$$

$$-\frac{1}{2}\sum\!\!\!\!\!\!\!\!\!\!\!\int_{\{PQ\}}\frac{1}{P^2Q^2}\left\langle\frac{c^2(1-c^2)}{R_0^2+r^2c^2}\right\rangle_c. \tag{B61}$$

The sum integrals in the first 3 terms on the right side of Eq. (B61) are given in Eqs. (B10), (B18), (B31), and (B34). The last sum integral before the average weighted by c is given in Eq. (B52). The average weighted by c^2 is given in Eq. (B56). The average weighted by c^4 can be computed in the same way. In the integrand of the single thermal integral, the weighted averages over c of the integrals over Q are given in Eqs. (C116) and (C125): After using Eq. (B54) to evaluate the thermal integral, we obtain

$$-2\int_{\mathbf{p}}\frac{n_F(p)}{p}\operatorname{Re}\int_Q\frac{1}{Q^2}\left\langle\frac{c^4}{R_0^2+r^2c^2}\right\rangle_c\Bigg|_{P_0=-ip+\varepsilon}+\int_{\mathbf{p}}\frac{n_B(p)}{p}\left\langle c^{1+2\epsilon}\int_Q\frac{1}{Q^2R^2}\Bigg|_{P\to(-ip,\mathbf{p}/c)}\right\rangle_c$$

$$=\frac{T^2}{(4\pi)^2}\left(\frac{\mu}{4\pi T}\right)^{4\epsilon}\left[-\left(\frac{7}{72}-\frac{1}{6}\log 2\right)\frac{1}{\epsilon}+0.2150\right]. \tag{B62}$$

For the two terms with a double thermal integral, the averages weighted by c^4 are given in Eqs. (C18) and (C22). Adding them to Eq. (B62), we obtain

$$\sum\!\!\!\!\!\!\!\!\!\!\!\int_{\{PQ\}}\frac{1}{P^2Q^2}\left\langle\frac{c^4}{R_0^2+r^2c^2}\right\rangle_c=\frac{T^2}{(4\pi)^2}\left(\frac{\mu}{4\pi T}\right)^{4\epsilon}\left[-\left(\frac{7}{72}-\frac{1}{6}\log 2\right)\frac{1}{\epsilon}+0.1359\right]. \tag{B63}$$

We finally need to evaluate Eq. (B50). Applying Eq. (B39) gives

$$\sum\!\!\!\!\!\!\!\!\!\!\!\int_{\{PQ\}}\frac{r^2-p^2}{P^2q^2Q_0^2R^2}\mathcal{T}_Q=\left[\int_{\mathbf{p}}\frac{n_B(p)}{p}+\int_{\mathbf{p}}\frac{n_F(p)}{p}\right]\operatorname{Re}\int_Q\left\langle\frac{p^2-q^2}{Q^2r^2(R_0^2+r^2c^2)}\right\rangle_c\Bigg|_{P_0=-ip}$$

$$+\int_{\mathbf{pq}}\frac{n_F(p)n_F(q)}{pq}\operatorname{Re}\left\langle\frac{r_c^2-p^2}{q^2}\frac{r_c^2-p^2-q^2}{\Delta(p+i\epsilon,q,r_c)}c^{-1+2\epsilon}\right\rangle_c$$

$$+\int_{\mathbf{pq}}\frac{n_B(p)n_F(q)}{pq}\operatorname{Re}\left\langle\frac{r_c^2-p^2}{q^2}\frac{r_c^2-p^2-q^2}{\Delta(p+i\epsilon,q,r_c)}c^{-1+2\epsilon}\right\rangle_c$$

$$+\int_{\mathbf{pq}}\frac{n_F(p)n_B(q)}{pq}\operatorname{Re}\left\langle\frac{p^2-q^2}{r^2}\frac{r^2c^2-p^2-q^2}{\Delta(p+i\epsilon,q,rc)}\right\rangle_c. \tag{B64}$$

In the terms on the right side, with a single thermal factor, the weighted average is given in Eq. (C128). After using Eq. (B54) to evaluate the thermal integral, we obtain

$$\left[\int_{\mathbf{p}}\frac{n_B(p)}{p}+\int_{\mathbf{p}}\frac{n_F(p)}{p}\right]\int_Q\left\langle\frac{p^2-q^2}{Q^2r^2(R_0^2+r^2c^2)}\right\rangle_c=\frac{T^2}{(4\pi)^2}\left(\frac{\pi^2}{24}\right). \tag{B65}$$

The terms with two thermal factors are given in Eqs. (C20), (C25) and (C26). Adding them to Eq. (B65), we finally obtain Eq. (B50).

APPENDIX C: INTEGRALS

Dimensional regularization can be used to regularize both the ultraviolet divergences and infrared divergences in 3-dimensional integrals over momenta. The spatial dimension is generalized to $d=3-2\epsilon$ dimensions. Integrals are evaluated at a value of d for which they converge and then analytically continued to $d=3$. We use the integration measure

$$\int_{\mathbf{p}}\equiv\left(\frac{e^\gamma\mu^2}{4\pi}\right)^\epsilon\int\frac{d^{3-2\epsilon}p}{(2\pi)^{3-2\epsilon}}. \tag{C1}$$

1. Three-dimensional integrals

We require one integral that does not involve the Bose-Einstein distribution function. The momentum scale in this integral is set by the mass $m = m_D$. The one-loop integral is

$$\int_{\mathbf{p}} \frac{1}{p^2 + m^2} = -\frac{m}{4\pi} \left(\frac{\mu}{2m}\right)^{2\epsilon} [1 + 2\epsilon]. \tag{C2}$$

The error is one order higher in ϵ than the smallest term shown.

2. Thermal integrals

The thermal integrals involve the Fermi-Dirac distribution $n_F(p) = 1/(e^{\beta p} + 1)$. The one-loop integrals can all be obtained from the general formula

$$\int_{\mathbf{p}} \frac{n_F(p)}{p} p^{2\alpha} = (1 - 2^{-1-2\alpha+2\epsilon}) \frac{\zeta(2+2\alpha-2\epsilon)}{4\pi^2} \frac{\Gamma(2+2\alpha-2\epsilon)\Gamma(\frac{1}{2})}{\Gamma(\frac{3}{2}-\epsilon)} (e^{\gamma}\mu^2)^{\epsilon} T^{2+2\alpha-2\epsilon}. \tag{C3}$$

The simple two-loop thermal integrals are

$$\int_{\mathbf{pq}} \frac{n_F(p)n_F(q)}{pq} \frac{1}{r^2} = \frac{T^2}{(4\pi)^2} \frac{1}{3} [1 - \log 2], \tag{C4}$$

$$\int_{\mathbf{pq}} \frac{n_F(p)n_F(q)}{pq} \frac{q^2}{r^4} = \frac{T^2}{(4\pi)^2} \left(\frac{\mu}{4\pi T}\right)^{4\epsilon} \left(-\frac{1}{36}\right) \left[5 + 6\gamma + 6\log 2 - 6\frac{\zeta'(-1)}{\zeta(-1)} + 3.076\epsilon\right], \tag{C5}$$

$$\int_{\mathbf{pq}} \frac{n_B(p)n_F(q)}{pq} \frac{p^2}{r^4} = \frac{T^2}{(4\pi)^2} \left(\frac{\mu}{4\pi T}\right)^{4\epsilon} \left(-\frac{1}{36}\right) \left[7 - 6\gamma - 18\log 2 + 6\frac{\zeta'(-1)}{\zeta(-1)} + 29.508\epsilon\right], \tag{C6}$$

$$\int_{\mathbf{pq}} \frac{n_B(p)n_F(q)}{pq} \frac{q^2}{r^4} = \frac{T^2}{(4\pi)^2} \left(\frac{\mu}{4\pi T}\right)^{4\epsilon} \left(\frac{1}{18}\right) \left[1 - 6\gamma - 12\log 2 + 6\frac{\zeta'(-1)}{\zeta(-1)} + 31.134\epsilon\right]. \tag{C7}$$

We also need some more complicated 2-loop thermal integrals that involve the triangle function defined in Eq. (B46):

$$\int_{\mathbf{pq}} \frac{n_F(p)}{p} \frac{n_F(q)}{q} \frac{r^4}{q^2\Delta(p,q,r)} = \frac{T^2}{(4\pi)^2} \left(\frac{\mu}{4\pi T}\right)^{4\epsilon} \left(-\frac{7}{96}\right) \left[\frac{1}{\epsilon^2} + \left(\frac{22}{7} + 2\gamma + 2\log 2 + 2\frac{\zeta'(-1)}{\zeta(-1)} - \frac{7}{20}\zeta(3)\right) \frac{1}{\epsilon} + 47.2406\right], \tag{C8}$$

$$\int_{\mathbf{pq}} \frac{n_F(p)}{p} \frac{n_F(q)}{q} \frac{r^2}{\Delta(p,q,r)} = \frac{T^2}{(4\pi)^2} \left(\frac{\mu}{4\pi T}\right)^{4\epsilon} \left(-\frac{1}{48}\right) \left[\frac{1}{\epsilon^2} + 2\left(1 + \gamma + \log 2 + \frac{\zeta'(-1)}{\zeta(-1)}\right)\frac{1}{\epsilon} + 4 + 4\gamma + \frac{\pi^2}{2} + 4\gamma\log 2 \right.$$
$$\left. - 6\log^2 2 + 4\log 2 - 4\gamma_1 + 4(1 + \gamma + \log 2)\frac{\zeta'(-1)}{\zeta(-1)} + 2\frac{\zeta''(-1)}{\zeta(-1)}\right], \tag{C9}$$

$$\int_{\mathbf{pq}} \frac{n_F(p)}{p} \frac{n_F(q)}{q} \frac{p^4}{q^2\Delta(p,q,r)} = \frac{T^2}{(4\pi)^2} \left(\frac{\mu}{4\pi T}\right)^{4\epsilon} \left(\frac{49\zeta(3)}{1920}\right) \left[\frac{1}{\epsilon} + 2 + 2\log 2 + 2\frac{\zeta'(-3)}{\zeta(-3)} + 2\frac{\zeta'(3)}{\zeta(3)}\right], \tag{C10}$$

$$\int_{\mathbf{pq}} \frac{n_F(p)}{p} \frac{n_F(q)}{q} \frac{p^2(p^2+q^2)}{r^2\Delta(p,q,r)} = \frac{T^2}{(4\pi)^2} \left(\frac{\mu}{4\pi T}\right)^{4\epsilon} \left(-\frac{1}{96}\right) \left[\frac{1}{\epsilon^2} + \left(\frac{26}{3} + 10\gamma - 6\frac{\zeta'(-1)}{\zeta(-1)} + 10\log 2\right)\frac{1}{\epsilon} + 41.1586\right], \tag{C11}$$

$$\int_{\mathbf{pq}} \frac{n_F(p)}{p} \frac{n_F(q)}{q} \frac{p^2}{\Delta(p,q,r)} = \frac{T^2}{(4\pi)^2} \left(\frac{\mu}{4\pi T}\right)^{4\epsilon} \left(-\frac{1}{96}\right) \left[\frac{1}{\epsilon^2} + 2\left(1 + \gamma + \log 2 + \frac{\zeta'(-1)}{\zeta(-1)}\right)\frac{1}{\epsilon} + 37.0573\right], \tag{C12}$$

$$\int_{\mathbf{pq}} \frac{n_F(p)}{p} \frac{n_B(q)}{q} \frac{p^2}{\Delta(p,q,r)} = \frac{T^2}{(4\pi)^2} \left(\frac{\mu}{4\pi T}\right)^{4\epsilon} \left(\frac{1}{96}\right) \left[\frac{1}{\epsilon^2} + 2\left(1 + \gamma - \log 2 + \frac{\zeta'(-1)}{\zeta(-1)}\right)\frac{1}{\epsilon} + 19.2257\right],$$
(C13)

$$\int_{\mathbf{pq}} \frac{n_F(p)}{p} \frac{n_B(q)}{q} \frac{p^4}{q^2\Delta(p,q,r)} = \frac{T^2}{(4\pi)^2} \left(\frac{\mu}{4\pi T}\right)^{4\epsilon} \left(-\frac{7\zeta(3)}{1920}\right) \left[\frac{1}{\epsilon} + 2 - \frac{2}{7}\log 2 + 2\frac{\zeta'(-3)}{\zeta(-3)} + 2\frac{\zeta'(3)}{\zeta(3)}\right],$$
(C14)

$$\int_{\mathbf{pq}} \frac{n_F(p)}{p} \frac{n_B(q)}{q} \frac{r^4}{q^2\Delta(p,q,r)} = \frac{T^2}{(4\pi)^2} \left(\frac{\mu}{4\pi T}\right)^{4\epsilon} \left(\frac{1}{24}\right) \left[\frac{1}{\epsilon^2} + \left(4 + 2\gamma - 5\log 2 - \frac{7\zeta(3)}{80} + 2\frac{\zeta'(-1)}{\zeta(-1)}\right)\frac{1}{\epsilon} + 18.1551\right],$$
(C15)

$$\int_{\mathbf{pq}} \frac{n_F(p)}{p} \frac{n_B(q)}{q} \frac{r^2}{\Delta(p,q,r)} = \frac{T^2}{(4\pi)^2} \left(\frac{\mu}{4\pi T}\right)^{4\epsilon} \left(-\frac{1}{96}\right) \left[\frac{1}{\epsilon^2} + 2\left(1 + \gamma + 5\log 2 + \frac{\zeta'(-1)}{\zeta(-1)}\right)\frac{1}{\epsilon} + 84.2513\right].$$
(C16)

The most difficult thermal integrals to evaluate involve both the triangle function and the HTL average defined in Eq. (A56). There are 2 sets of these integrals. The first set is

$$\int_{\mathbf{pq}} \frac{n_F(p)n_F(q)}{pq} \mathrm{Re}\left\langle c^2 \frac{r^2c^2-p^2-q^2}{\Delta(p+i\varepsilon,q,rc)}\right\rangle_c = \frac{T^2}{(4\pi)^2}[1.458\times10^{-2}],$$
(C17)

$$\int_{\mathbf{pq}} \frac{n_F(p)n_F(q)}{pq} \mathrm{Re}\left\langle c^4 \frac{r^2c^2-p^2-q^2}{\Delta(p+i\varepsilon,q,rc)}\right\rangle_c = \frac{T^2}{(4\pi)^2}[1.7715\times10^{-2}],$$
(C18)

$$\int_{\mathbf{pq}} \frac{n_F(p)n_F(q)}{pq} \mathrm{Re}\left\langle \frac{q^2}{r^2}c^2 \frac{r^2c^2-p^2-q^2}{\Delta(p+i\varepsilon,q,rc)}\right\rangle_c = \frac{T^2}{(4\pi)^2}[-1.1578\times10^{-2}],$$
(C19)

$$\int_{\mathbf{pq}} \frac{n_B(p)}{p} \frac{n_F(q)}{q} \mathrm{Re}\left\langle \frac{p^2-q^2}{r^2} \frac{r^2c^2-p^2-q^2}{\Delta(p+i\epsilon,q,rc)}\right\rangle_c = \frac{T^2}{(4\pi)^2}[0.17811].$$
(C20)

The second set of these integrals involve the variable $r_c = |\mathbf{p} + \mathbf{q}/c|$:

$$\int_{\mathbf{pq}} \frac{n_F(p)n_B(q)}{pq} \mathrm{Re}\left\langle c^{-1+2\epsilon} \frac{r_c^2-p^2-q^2}{\Delta(p+i\varepsilon,q,r_c)}\right\rangle_c = \frac{T^2}{(4\pi)^2}[0.19678],$$
(C21)

$$\int_{\mathbf{pq}} \frac{n_F(p)n_B(q)}{pq} \mathrm{Re}\left\langle c^{1+2\epsilon} \frac{r_c^2-p^2-q^2}{\Delta(p+i\varepsilon,q,r_c)}\right\rangle_c = \frac{T^2}{(4\pi)^2}[4.8368\times10^{-2}],$$
(C22)

$$\int_{\mathbf{pq}} \frac{n_F(p)n_B(q)}{pq} \frac{p^2}{q^2} \mathrm{Re}\left\langle c^{1+2\epsilon} \frac{r_c^2-p^2-q^2}{\Delta(p+i\varepsilon,q,r_c)}\right\rangle_c = \frac{T^2}{(4\pi)^2}\left(\frac{\mu}{4\pi T}\right)^{4\epsilon} \frac{1}{96}\left[\frac{1}{\epsilon} + 7.7702\right],$$
(C23)

$$\int_{\mathbf{pq}} \frac{n_F(p)n_B(q)}{pq} \mathrm{Re}\left\langle c^{1+2\epsilon} \frac{r_c^2}{q^2} \frac{r_c^2-p^2-q^2}{\Delta(p+i\varepsilon,q,r_c)}\right\rangle_c = \frac{T^2}{(4\pi)^2}\left(\frac{\mu}{4\pi T}\right)^{4\epsilon} \frac{11-8\log 2}{288}\left[\frac{1}{\epsilon} + 7.79693\right],$$
(C24)

$$\int_{\mathbf{pq}} \frac{n_F(p)}{p} \frac{n_F(p)}{q} \mathrm{Re}\left\langle \frac{r_c^2-p^2}{q^2} \frac{r_c^2-p^2-q^2}{\Delta(p+i\epsilon,q,r_c)} c^{-1+2\epsilon}\right\rangle_c = -\frac{T^2}{(4\pi)^2}\left(\frac{1}{24}\right) \left[\frac{1}{\epsilon^2} + \left(2 + 2\gamma + 2\log 2 + 2\frac{\zeta'(-1)}{\zeta(-1)}\right)\frac{1}{\epsilon} + 40.316\right],$$
(C25)

$$\int_{\mathbf{pq}} \frac{n_B(p)}{p} \frac{n_F(p)}{q} \mathrm{Re}\left\langle \frac{r_c^2-p^2}{q^2} \frac{r_c^2-p^2-q^2}{\Delta(p+i\epsilon,q,r_c)} c^{-1+2\epsilon}\right\rangle_c = -\frac{T^2}{(4\pi)^2}\left(\frac{1}{12}\right) \left[\frac{1}{\epsilon^2} + \left(2 + 2\gamma + 4\log 2 + 2\frac{\zeta'(-1)}{\zeta(-1)}\right)\frac{1}{\epsilon} + 52.953\right].$$
(C26)

The simplest way to evaluate integrals like Eqs. (C4)–(C7) whose integrands factor into separate functions of p, q, and r is to Fourier transform to coordinate space where they reduce to an integral over a single coordinate $\mathbf{R}$:

$$\int_{\mathbf{pq}} f(p)g(q)h(r) = \int_{\mathbf{R}} \tilde{f}(R)\tilde{g}(R)\tilde{h}(R). \tag{C27}$$

The Fourier transform is

$$\tilde{f}(R) = \int_{\mathbf{p}} e^{i\mathbf{p}\cdot\mathbf{R}} f(p), \tag{C28}$$

and the dimensionally regularized coordinate integral is

$$\int_{\mathbf{R}} = \left(\frac{e^{\gamma}\mu^2}{4\pi}\right)^{-\epsilon} \int d^{3-2\epsilon}R. \tag{C29}$$

The Fourier transforms we need are

$$\int_{\mathbf{p}} p^{2\alpha} e^{i\mathbf{p}\cdot\mathbf{R}} = \frac{1}{8\pi} \frac{\Gamma(\frac{3}{2}+\alpha-\epsilon)}{\Gamma(\frac{1}{2})\Gamma(-\alpha)} (e^{\gamma}\mu^2)^{\epsilon} \left(\frac{2}{R}\right)^{3+2\alpha-2\epsilon}, \tag{C30}$$

$$\int_{\mathbf{p}} \frac{n(p)}{p} p^{2\alpha} e^{i\mathbf{p}\cdot\mathbf{R}} = \frac{1}{4\pi} \frac{1}{\Gamma(\frac{1}{2})} (e^{\gamma}\mu^2)^{\epsilon} \left(\frac{2}{R}\right)^{1/2-\epsilon} \int_0^{\infty} dp\, p^{2\alpha+1/2-\epsilon} n(p) J_{1/2-\epsilon}(pR). \tag{C31}$$

If α is an even integer, the Fourier transform (C31) is particularly simple in the limit $d \to 3$:

$$\int_{\mathbf{p}} \frac{n_F(p)}{p} e^{i\mathbf{p}\cdot\mathbf{R}} = \frac{T}{4\pi R} \left(\frac{1}{x} - \operatorname{csch} x\right), \tag{C32}$$

$$\int_{\mathbf{p}} \frac{n_F(p)}{p} p^2 e^{i\mathbf{p}\cdot\mathbf{R}} = \frac{\pi T^3}{2R} \left(\operatorname{csch}^3 x + \frac{1}{2}\operatorname{csch} x - \frac{1}{x^3}\right), \tag{C33}$$

where $x = \pi RT$.

We can use these simple expressions only if the integral over the coordinate $\mathbf{R}$ in Eq. (C27) converges for $d=3$. Otherwise, we must first make subtractions on the integrand to make the integral convergent.

The integrals (C4)–(C7) can be evaluated directly by applying the Fourier transform formula (C27) in the limit $\epsilon \to 0$. The integrals (C8)–(C10) can be evaluated by first averaging over angles. The triangle function can be expressed as

$$\Delta(p,q,r) = -4p^2 q^2 (1 - \cos^2\theta), \tag{C34}$$

where θ is the angle between $\mathbf{p}$ and $\mathbf{q}$. For example, the angle average for Eq. (C8) is

$$\left\langle \frac{r^4}{\Delta(p,q,r)} \right\rangle_{\hat{\mathbf{p}}\cdot\hat{\mathbf{q}}} = -\frac{w(\epsilon)}{8} \int_{-1}^{+1} dx (1-x^2)^{-1-\epsilon} \frac{(p^2+q^2+2pqx)^2}{p^2 q^2}. \tag{C35}$$

After integrating over x and inserting the result into Eq. (C8), the integral reduces to

$$\int_{\mathbf{pq}} \frac{n_F(p)}{p} \frac{n_F(q)}{q} \frac{r^4}{q^2\Delta(p,q,r)} = \int_{\mathbf{pq}} \frac{n_F(p)}{p} \frac{n_F(q)}{q} \left(\frac{1-2\epsilon}{8\epsilon} \frac{p^2}{q^4} + \frac{7-6\epsilon}{8\epsilon} \frac{1}{q^2}\right). \tag{C36}$$

The integrals over $\mathbf{p}$ and $\mathbf{q}$ factor into separate integrals that can be evaluated using Eq. (C3). After averaging over angles, the integrals (C9) and (C10) reduce to

$$\int_{\mathbf{pq}} \frac{n_F(p)}{p} \frac{n_F(q)}{q} \frac{r^2}{\Delta(p,q,r)} = \frac{1-2\epsilon}{4\epsilon} \int_{\mathbf{p}} \frac{n_F(p)}{p} \int_{\mathbf{q}} \frac{n_F(q)}{q} \frac{1}{q^2}, \tag{C37}$$

$$\int_{\mathbf{pq}} \frac{n_F(p)}{p} \frac{n_F(q)}{q} \frac{p^4}{q^2\Delta(p,q,r)} = \frac{1-2\epsilon}{8\epsilon} \int_{\mathbf{p}} \frac{n_F(p)}{p} p^2 \int_{\mathbf{q}} \frac{n_F(q)}{q} \frac{1}{q^4}. \tag{C38}$$

The integral (C11) can be evaluated by using the identity

$$\left\langle \frac{p^2+q^2}{r^2\Delta(p,q,r)} \right\rangle_{\hat{\mathbf{p}}\cdot\hat{\mathbf{q}}} = \frac{1}{2\epsilon}\left\langle \frac{1}{r^4} \right\rangle_{\hat{\mathbf{p}}\cdot\hat{\mathbf{q}}} + \frac{1-2\epsilon}{8\epsilon}\frac{1}{p^2q^2}. \tag{C39}$$

The identity can be proved by expressing the angular averages in terms of integrals over the cosine of the angle between $\mathbf{p}$ and $\mathbf{q}$ as in Eq. (C35), and then integrating by parts. Inserting the identity (C39) into Eq. (C11), the integral reduces to

$$\int_{\mathbf{pq}} \frac{n_F(p)}{p}\frac{n_F(q)}{q}\frac{p^2(p^2+q^2)}{r^2\Delta(p,q,r)} = \frac{1}{2\epsilon}\int_{\mathbf{pq}} \frac{n_F(p)}{p}\frac{n_F(q)}{q}\frac{p^2}{r^4} + \frac{1-2\epsilon}{8\epsilon}\int_{\mathbf{p}} \frac{n_F(p)}{p}\int_{\mathbf{q}} \frac{n_F(q)}{q}\frac{1}{q^2}. \tag{C40}$$

The integral in the first term on the right is given in Eq. (C5), while the second term can be evaluated using Eq. (C3).

The integral (C17) can be evaluated directly in three dimensions by first averaging over c and x, and then integrating the resulting functions numerically over p and q.

To evaluate the weighted averages over c of the thermal integrals in Eqs. (C18)–(C20), we first isolate the divergent parts, which come from the region $p-q\to0$. We write the product of thermal functions in the form

$$n_F(p)n_F(q) = \left(n_F(p)n_F(q) - \frac{s^2n_F^2(s)}{pq} \right) + \frac{s^2n_F^2(s)}{pq}, \tag{C41}$$

where $s=(p+q)/2$. In the difference term, the HTL average over c and the angular average over $x=\hat{\mathbf{p}}\cdot\hat{\mathbf{q}}$ can be calculated in three dimensions:

$$\mathrm{Re}\left\langle c^4\frac{r^2c^2-p^2-q^2}{\Delta(p+i\varepsilon,q,rc)} \right\rangle_{c,x} = \frac{2(p^2+q^2)}{3(p^2-q^2)^2} + \frac{1}{12pq}\log\frac{p+q}{|p-q|} - \frac{(3p^2+q^2)(p^2+3q^2)}{6(p^2-q^2)^3}\log(p/q), \tag{C42}$$

$$\mathrm{Re}\left\langle c^2\frac{q^2}{r^2}\frac{r^2c^2-p^2-q^2}{\Delta(p+i\varepsilon,q,rc)} \right\rangle_{c,x} = \frac{q^2}{3(p^2-q^2)^2}\left[2 - \frac{1}{2}\log\frac{|p^2-q^2|}{pq} - \frac{p^2+q^2}{4pq}\log\frac{p+q}{|p-q|} - \frac{p^2+q^2}{p^2-q^2}\log(p/q) \right], \tag{C43}$$

$$\mathrm{Re}\left\langle \frac{p^2-q^2}{r^2}\frac{r^2c^2-p^2-q^2}{\Delta(p+i\epsilon,q,rc)} \right\rangle_{c,x} = \frac{1}{4pq(p^2-q^2)}\left[-(p^2+q^2)\log\frac{p+q}{|p-q|} - 2pq\log\frac{|p^2-q^2|}{pq} \right]. \tag{C44}$$

The remaining 2-dimensional integral over p and q can then be evaluated numerically:

$$\int_{\mathbf{pq}} \left(\frac{n_F(p)n_F(q)}{pq} - \frac{s^2n_F^2(s)}{p^2q^2} \right)\mathrm{Re}\left\langle c^4\frac{r^2c^2-p^2-q^2}{\Delta(p+i\varepsilon,q,rc)} \right\rangle_c = \frac{T^2}{(4\pi)^2}[8.980\times10^{-3}], \tag{C45}$$

$$\int_{\mathbf{pq}} \left(\frac{n_F(p)n_F(q)}{pq} - \frac{s^2n_F^2(s)}{p^2q^2} \right)\frac{q^2}{r^2}\mathrm{Re}\left\langle c^2\frac{r^2c^2-p^2-q^2}{\Delta(p+i\varepsilon,q,rc)} \right\rangle_c = \frac{T^2}{(4\pi)^2}[7.792\times10^{-3}], \tag{C46}$$

$$\int_{\mathbf{pq}} \left(\frac{n_B(p)}{p}\frac{n_F(q)}{q} - \frac{s^2n_B(s)n_F(s)}{p^2q^2} \right)\mathrm{Re}\left\langle \frac{p^2-q^2}{r^2}\frac{r^2c^2-p^2-q^2}{\Delta(p+i\epsilon,q,rc)} \right\rangle_c = \frac{T^2}{(4\pi)^2}[0.17811]. \tag{C47}$$

The integrals involving the $n_F^2(s)$ term in Eq. (C41) are divergent, so the HTL average over c and the angular average over $x=\hat{\mathbf{p}}\cdot\hat{\mathbf{q}}$ must be calculated in $3-2\epsilon$ dimensions. The first step in the calculation of the $n^2(s)$ term is to change variables from $\mathbf{p}$ and $\mathbf{q}$ to $s=(p+q)/2$, $\beta=4pq/(p+q)^2$, and $x=\hat{\mathbf{p}}\cdot\hat{\mathbf{q}}$:

$$\int_{\mathbf{pq}} \frac{s^2n_F^2(s)}{p^2q^2}f(p,q,r) = \frac{64}{(4\pi)^4}\left[(e^\gamma\mu^2)^\epsilon\frac{\Gamma(\frac{3}{2})}{\Gamma(\frac{3}{2}-\epsilon)} \right]^2\int_0^\infty ds\,s^{1-4\epsilon}n_F^2(s)s^2\int_0^1 d\beta\,\beta^{-2\epsilon}(1-\beta)^{-1/2}$$

$$\times\langle f(s_+,s_-,r)+f(s_-,s_+,r)\rangle_x, \tag{C48}$$

where $s_\pm=s[1\pm\sqrt{1-\beta}]$ and $r=s[4-2\beta(1-x)]^{1/2}$. The 2 terms inside the average over x come from the regions $p>q$ and $p<q$, respectively. The integral over s is easily evaluated:

$$\int_0^\infty ds\, s^{1-4\epsilon} n_F^2(s) = \Gamma(2-4\epsilon)[-(1-2^{4\epsilon})\zeta(1-4\epsilon) + (1-2^{-1+4\epsilon})\zeta(2-4\epsilon)]T^{2-4\epsilon}, \tag{C49}$$

$$\int_0^\infty ds\, s^{1-4\epsilon} n_F(s) n_B(s) = 2^{-2+4\epsilon}\Gamma(2-4\epsilon)\zeta(2-4\epsilon)T^{2-4\epsilon}. \tag{C50}$$

It remains only to evaluate the averages over c and x and the integral over β.

The first step in the calculation of the $n_F^2(s)$ term of Eq. (C18) is to decompose the integrand into 2 terms:

$$\frac{r^2c^2 - p^2 - q^2}{\Delta(p+i\varepsilon, q, rc)} = -\frac{1}{2}\sum_\pm \frac{1}{(p+i\varepsilon\pm q)^2 - r^2c^2}. \tag{C51}$$

The weighted averages over c gives a hypergeometric function

$$\left\langle \frac{c^4}{(p+i\varepsilon\pm q)^2 - r^2c^2} \right\rangle_c = \frac{3}{(3-2\epsilon)(5-2\epsilon)}\frac{1}{(p+i\varepsilon\pm q)^2} {}_2F\left(\begin{array}{c}\frac{5}{2},1\\\frac{7}{2}-\epsilon\end{array}\middle| \frac{r^2}{(p+i\varepsilon\pm q)^2}\right). \tag{C52}$$

In the $+q$ case of Eq. (C52), the $i\varepsilon$ prescription is unnecessary. The argument of the hypergeometric function can be written $1-\beta y$, where $y = (1-x)/2$. After using a transformation formula to change the argument to βy, we can evaluate the angular average over x to obtain hypergeometric functions with argument β. For example, the average over x of Eq. (C52) is

$$\left\langle F\left(\begin{array}{c}\frac{5}{2},1\\\frac{7}{2}-\epsilon\end{array}\middle| \frac{r^2}{(p+q)^2}\right)\right\rangle_x = -\frac{5-2\epsilon}{2\epsilon}\left[F\left(\begin{array}{c}1-\epsilon,\frac{5}{2},1\\2-2\epsilon,1+\epsilon\end{array}\middle| \beta\right) - \frac{(1)_\epsilon(1)_{-2\epsilon}(2)_{-2\epsilon}(\frac{5}{2})_{-\epsilon}}{(1)_{-\epsilon}(2)_{-3\epsilon}}\beta^{-\epsilon}F\left(\begin{array}{c}1-2\epsilon,\frac{5}{2}-\epsilon\\2-3\epsilon\end{array}\middle| \beta\right)\right], \tag{C53}$$

where $(a)_b$ is Pochhammer's symbol which is defined in Eq. (C146). Integrating over β, we obtain hypergeometric functions with argument 1:

$$s^2\int_0^1 d\beta\,\beta^{-2\epsilon}(1-\beta)^{-1/2}\left\langle\frac{c^4}{(p+q)^2 - r^2c^2}\right\rangle_{c,x} = -\frac{1}{4\epsilon}\frac{(1)_\epsilon(2)_{-2\epsilon}}{(1)_{-\epsilon}}\left[\frac{(1)_{-2\epsilon}(1)_{-\epsilon}}{(\frac{5}{2})_{-2\epsilon}(2)_{-2\epsilon}(1)_\epsilon}F\left(\begin{array}{c}1-2\epsilon,1-\epsilon,\frac{5}{2},1\\\frac{5}{2}-2\epsilon,2-2\epsilon,1+\epsilon\end{array}\middle| 1\right)\right.$$
$$\left. -\frac{(1)_{-3\epsilon}(1)_{-2\epsilon}(\frac{5}{2})_{-\epsilon}}{(\frac{5}{2})_{-3\epsilon}(2)_{-3\epsilon}}F\left(\begin{array}{c}1-3\epsilon,1-2\epsilon,\frac{5}{2}-\epsilon\\\frac{5}{2}-3\epsilon,2-3\epsilon\end{array}\middle| 1\right)\right]. \tag{C54}$$

Expanding in powers of ϵ, we obtain

$$s^2\int_0^1 d\beta\,\beta^{-2\epsilon}(1-\beta)^{-1/2}\left\langle\frac{c^4}{(p+q)^2 - r^2c^2}\right\rangle_{c,x} = \frac{\pi^2}{72}(1 + 10.8408\epsilon). \tag{C55}$$

In the $-q$ case of Eq. (C52), the argument of the hypergeometric functions can be written $(1-\beta y)/(1-\beta\pm i\varepsilon)$, where $y = (1-x)/2$ and the prescriptions $+i\varepsilon$ and $-i\varepsilon$ correspond to the regions $p>q$ and $p<q$, respectively. These regions correspond to the two terms inside the average over x in Eq. (C48). In order to obtain an analytic result in terms of hypergeometric functions, it is necessary to integrate over β before averaging over x. The integrals over β can be evaluated by first using a transformation formula to change the argument of the hypergeometric function to $-\beta(1-y)/(1-\beta)$ and then using the integration formula (C153) to obtain hypergeometric functions with arguments y or $1-y$:

$$\int_0^1 d\beta \beta^{-2\epsilon}(1-\beta)^{-3/2}F\left(\begin{array}{c}\frac{3}{2},1\\\frac{5}{2}-\epsilon\end{array}\middle|\frac{1-\beta y}{1-\beta+i\varepsilon}\right)$$

$$=\frac{3-2\epsilon}{\epsilon}\frac{(1)_{-2\epsilon}}{(\frac{1}{2})_{-2\epsilon}}F\left(\begin{array}{c}1-2\epsilon,1\\1+\epsilon\end{array}\middle|1-y\right)$$

$$-\frac{3-2\epsilon}{\epsilon}\frac{(1)_{\epsilon}}{(\frac{1}{2})_{\epsilon}}(1-y)^{-1/2}F\left(\begin{array}{c}\frac{1}{2}-2\epsilon,1\\\frac{1}{2}+\epsilon\end{array}\middle|1-y\right)$$

$$+\frac{3}{2\epsilon(1-3\epsilon)}e^{i\pi\epsilon}(1)_{\epsilon}(\tfrac{5}{2})_{-\epsilon}(1-y)^{-\epsilon}F$$

$$\times\left(\begin{array}{c}1-3\epsilon,\frac{3}{2}-\epsilon\\2-3\epsilon\end{array}\middle|y\right). \tag{C56}$$

$$s^2\int_0^1 d\beta \beta^{-2\epsilon}(1-\beta)^{-1/2}\left\langle\frac{c^2}{(p+i\varepsilon-q)^2-r^2c^2}\right\rangle_{c,x}$$

$$=\frac{1}{4\epsilon}\frac{(1)_{-2\epsilon}}{(\frac{1}{2})_{-2\epsilon}}F\left(\begin{array}{c}1-\epsilon,1-2\epsilon,1\\2-2\epsilon,1+\epsilon\end{array}\middle|1\right)$$

$$-\frac{1}{2\epsilon}\frac{(2)_{-2\epsilon}(1)_{\epsilon}(\frac{1}{2})_{-\epsilon}}{(1)_{-\epsilon}(\frac{1}{2})_{\epsilon}(\frac{3}{2})_{-2\epsilon}}F\left(\begin{array}{c}\frac{1}{2}-\epsilon,\frac{1}{2}-2\epsilon,1\\\frac{3}{2}-2\epsilon,\frac{1}{2}+\epsilon\end{array}\middle|1\right)$$

$$+\frac{1}{8\epsilon(1-3\epsilon)}e^{i\pi\epsilon}\frac{(2)_{-2\epsilon}(1)_{-2\epsilon}(1)_{\epsilon}(\frac{3}{2})_{-\epsilon}}{(1)_{-\epsilon}(2)_{-3\epsilon}}F$$

$$\times\left(\begin{array}{c}1-\epsilon,1-3\epsilon,\frac{3}{2}-\epsilon\\2-3\epsilon,2-3\epsilon\end{array}\middle|1\right). \tag{C57}$$

After averaging over x, we obtain hypergeometric functions with argument 1:

Expanding in powers of ϵ and then taking the real parts, we obtain

$$\operatorname{Re} s^2\int_0^1 d\beta \beta^{-2\epsilon}(1-\beta)^{-1/2}\left\langle\frac{c^4}{(p+i\varepsilon-q)^2-r^2c^2}\right\rangle_{c,x}=-\frac{12+\pi^2}{72}(1+1.10518\epsilon). \tag{C58}$$

To evaluate the subtraction in the integral (C46), we use the identity $q^2=(r^2+q^2-p^2-2\mathbf{p}\cdot\mathbf{q})/2$. The integral with q^2-p^2 in the numerator is purely imaginary. Thus the real part of the integral can be expressed as

$$\int_{\mathbf{pq}}\frac{s^2n^2(s)}{p^2q^2}\frac{q^2}{r^2}\operatorname{Re}\left\langle c^2\frac{r^2c^2-p^2-q^2}{\Delta(p+i\varepsilon,q,rc)}\right\rangle_c=\int_{\mathbf{pq}}\frac{s^2n^2(s)}{p^2q^2}\left(\frac{1}{2}-\frac{\mathbf{p}\cdot\mathbf{q}}{r^2}\right)\operatorname{Re}\left\langle c^2\frac{r^2c^2-p^2-q^2}{\Delta(p+i\varepsilon,q,rc)}\right\rangle_c. \tag{C59}$$

The first term in Eq. (C59) is decomposed into 2 terms:

$$\frac{r^2c^2-p^2-q^2}{\Delta(p+i\varepsilon,q,rc)}=-\frac{1}{2}\sum_{\pm}\frac{1}{(p+i\varepsilon\pm q)^2-r^2c^2}. \tag{C60}$$

The weighted averages over c give hypergeometric functions

$$\left\langle\frac{c^2}{(p+i\varepsilon\pm q)^2-r^2c^2}\right\rangle_c=\frac{1}{3-2\epsilon}\frac{1}{(p+i\varepsilon\pm q)^2}F\left(\begin{array}{c}\frac{3}{2},1\\\frac{5}{2}-\epsilon\end{array}\middle|\frac{r^2}{(p+i\varepsilon\pm q)^2}\right), \tag{C61}$$

$$\left\langle\frac{c^4}{(p+i\varepsilon\pm q)^2-r^2c^2}\right\rangle_c=\frac{3}{(3-2\epsilon)(5-2\epsilon)}\frac{1}{(p+i\varepsilon\pm q)^2}F\left(\begin{array}{c}\frac{5}{2},1\\\frac{7}{2}-\epsilon\end{array}\middle|\frac{r^2}{(p+i\varepsilon\pm q)^2}\right). \tag{C62}$$

In the $+q$ case of Eq. (C61), the $i\varepsilon$ prescription is unnecessary. The argument of the hypergeometric function can be written

$1 - \beta y$, where $y = (1 - x)/2$. After using a transformation formula to change the argument to βy, we can evaluate the angular average over x to obtain hypergeometric functions with argument β. For example, the average over x of Eq. (C61) is

$$\left\langle F\left(\begin{array}{c} \frac{3}{2},1 \\ \frac{5}{2} - \epsilon \end{array}\middle| \frac{r^2}{(p+q)^2}\right)\right\rangle_x = -\frac{3-2\epsilon}{2\epsilon}\left[F\left(\begin{array}{c} 1-\epsilon,\frac{3}{2},1 \\ 2-2\epsilon,1+\epsilon \end{array}\middle| \beta\right) - \frac{(1)_\epsilon(1)_{-2\epsilon}(2)_{-2\epsilon}(\frac{3}{2})_{-\epsilon}}{(1)_{-\epsilon}(2)_{-3\epsilon}}\beta^{-\epsilon}F\left(\begin{array}{c} 1-2\epsilon,\frac{3}{2}-\epsilon \\ 2-3\epsilon \end{array}\middle| \beta\right)\right], \quad (C63)$$

where $(a)_b$ is Pochhammer's symbol which is defined in Eq. (C146). Integrating over β, we obtain hypergeometric functions with argument 1:

$$s^2\int_0^1 d\beta\beta^{-2\epsilon}(1-\beta)^{-1/2}\left\langle\frac{c^2}{(p+q)^2-r^2c^2}\right\rangle_{c,x}$$

$$= -\frac{1}{4\epsilon}\frac{(1)_\epsilon(2)_{-2\epsilon}}{(1)_{-\epsilon}}\left[\frac{(1)_{-2\epsilon}(1)_{-\epsilon}}{(\frac{3}{2})_{-2\epsilon}(2)_{-2\epsilon}(1)_\epsilon}F\left(\begin{array}{c} 1-2\epsilon,1-\epsilon,\frac{3}{2},1 \\ \frac{3}{2}-2\epsilon,2-2\epsilon,1+\epsilon \end{array}\middle| 1\right)\right.$$

$$\left.-\frac{(1)_{-3\epsilon}(1)_{-2\epsilon}(\frac{3}{2})_{-\epsilon}}{(\frac{3}{2})_{-3\epsilon}(2)_{-3\epsilon}}F\left(\begin{array}{c} 1-3\epsilon,1-2\epsilon,\frac{3}{2}-\epsilon \\ \frac{3}{2}-3\epsilon,2-3\epsilon \end{array}\middle| 1\right)\right]. \quad (C64)$$

Expanding in powers of ϵ, we obtain

$$s^2\int_0^1 d\beta\beta^{-2\epsilon}(1-\beta)^{-1/2}\left\langle\frac{c^2}{(p+q)^2-r^2c^2}\right\rangle_{c,x} = \frac{\pi^2}{24}+O(\epsilon). \quad (C65)$$

In the $-q$ case of Eq. (C61), the argument of the hypergeometric functions can be written $(1-\beta y)/(1-\beta\pm i\varepsilon)$, where $y = (1-x)/2$ and the prescriptions $+i\varepsilon$ and $-i\varepsilon$ correspond to the regions $p>q$ and $p<q$, respectively. These regions correspond to the two terms inside the average over x in Eq. (C48). In order to obtain an analytic result in terms of hypergeometric functions, it is necessary to integrate over β before averaging over x. The integrals over β can be evaluated by first using a transformation formula to change the argument of the hypergeometric function to $-\beta(1-y)/(1-\beta)$ and then using the integration formula (C153) to obtain hypergeometric functions with arguments y or $1-y$:

$$\int_0^1 d\beta\beta^{-2\epsilon}(1-\beta)^{-3/2}F\left(\begin{array}{c} \frac{3}{2},1 \\ \frac{5}{2}-\epsilon \end{array}\middle| \frac{1-\beta y}{1-\beta+i\varepsilon}\right)$$

$$= \frac{3-2\epsilon}{\epsilon}\frac{(1)_{-2\epsilon}}{(\frac{1}{2})_{-2\epsilon}}F\left(\begin{array}{c} 1-2\epsilon,1 \\ 1+\epsilon \end{array}\middle| 1-y\right) - \frac{3-2\epsilon}{\epsilon}\frac{(1)_\epsilon}{(\frac{1}{2})_\epsilon}(1-y)^{-1/2}F\left(\begin{array}{c} \frac{1}{2}-2\epsilon,1 \\ \frac{1}{2}+\epsilon \end{array}\middle| 1-y\right)$$

$$+ \frac{3}{2\epsilon(1-3\epsilon)}e^{i\pi\epsilon}(1)_\epsilon(\frac{5}{2})_{-\epsilon}(1-y)^{-\epsilon}F\left(\begin{array}{c} 1-3\epsilon,\frac{3}{2}-\epsilon \\ 2-3\epsilon \end{array}\middle| y\right). \quad (C66)$$

After averaging over x, we obtain hypergeometric functions with argument 1:

$$s^2 \int_0^1 d\beta \beta^{-2\epsilon}(1-\beta)^{-1/2}\left\langle \frac{c^2}{(p+i\varepsilon-q)^2-r^2c^2}\right\rangle_{c,x}$$

$$= \frac{1}{4\epsilon}\frac{(1)_{-2\epsilon}}{(\frac{1}{2})_{-2\epsilon}}F\left(\begin{matrix}1-\epsilon,1-2\epsilon,1\\2-2\epsilon,1+\epsilon\end{matrix}\middle|1\right) - \frac{1}{2\epsilon}\frac{(2)_{-2\epsilon}(1)_\epsilon(\frac{1}{2})_{-\epsilon}}{(1)_{-\epsilon}(\frac{1}{2})_\epsilon(\frac{3}{2})_{-2\epsilon}}F\left(\begin{matrix}\frac{1}{2}-\epsilon,\frac{1}{2}-2\epsilon,1\\\frac{3}{2}-2\epsilon,\frac{1}{2}+\epsilon\end{matrix}\middle|1\right)$$

$$+\frac{1}{8\epsilon(1-3\epsilon)}e^{i\pi\epsilon}\frac{(2)_{-2\epsilon}(1)_{-2\epsilon}(1)_\epsilon(\frac{3}{2})_{-\epsilon}}{(1)_{-\epsilon}(2)_{-3\epsilon}}F\left(\begin{matrix}1-\epsilon,1-3\epsilon,\frac{3}{2}-\epsilon\\2-3\epsilon,2-3\epsilon\end{matrix}\middle|1\right). \tag{C67}$$

Expanding in powers of ϵ and then taking the real parts, we obtain

$$\text{Re}\, s^2 \int_0^1 d\beta \beta^{-2\epsilon}(1-\beta)^{-1/2}\left\langle \frac{c^2}{(p+i\varepsilon-q)^2-r^2c^2}\right\rangle_{c,x} = -\frac{\pi^2}{24}+O(\epsilon). \tag{C68}$$

Inserting the sum of the integrals (C65) and (C68) into the thermal integral (C48), we obtain

$$\int_{\mathbf{pq}}\frac{s^2 n_F^2(s)}{p^2 q^2}\text{Re}\left\langle c^2 \frac{r^2c^2-p^2-q^2}{\Delta(p+i\varepsilon,q,rc)}\right\rangle_c = O(\epsilon). \tag{C69}$$

It remains only to evaluate the integral in Eq. (C59) with $\mathbf{p}\cdot\mathbf{q}$ in the numerator. We begin by using the identity

$$\left\langle c^2\frac{\mathbf{p}\cdot\mathbf{q}}{r^2}\frac{r^2c^2-p^2-q^2}{\Delta(p+i\varepsilon,q,rc)}\right\rangle_{c,x} = -\frac{p^2+q^2}{(p^2-q^2+i\varepsilon)^2}\langle c^2\rangle_c\left\langle\frac{\mathbf{p}\cdot\mathbf{q}}{r^2}\right\rangle_x - \frac{1}{2}\sum_{\pm}\frac{1}{(p+i\varepsilon\pm q)^2}\left\langle\frac{\mathbf{p}\cdot\mathbf{q}c^4}{(p+i\varepsilon\pm q)^2-r^2c^2}\right\rangle_{c,x}. \tag{C70}$$

In the first term on the right side, the average over c is a simple multiplicative factor: $\langle c^2\rangle_c = 1/(3-2\epsilon)$. The average over x gives hypergeometric functions of argument β:

$$\left\langle\frac{\mathbf{p}\cdot\mathbf{q}}{r^2}\right\rangle_x = \frac{1}{8}\beta\left[F\left(\begin{matrix}1-\epsilon,1\\3-2\epsilon\end{matrix}\middle|\beta\right) - F\left(\begin{matrix}2-\epsilon,1\\3-2\epsilon\end{matrix}\middle|\beta\right)\right]. \tag{C71}$$

The integral over β gives hypergeometric functions of argument 1:

$$s^2 \int_0^1 d\beta \beta^{-2\epsilon}(1-\beta)^{-1/2}\frac{p^2+q^2}{(p^2-q^2)^2}\left\langle\frac{\mathbf{p}\cdot\mathbf{q}}{r^2}\right\rangle_x = -\frac{1}{8}\frac{(2)_{-2\epsilon}}{(\frac{3}{2})_{-2\epsilon}}\left[F\left(\begin{matrix}2-2\epsilon,1-\epsilon,1\\\frac{3}{2}-2\epsilon,3-2\epsilon\end{matrix}\middle|1\right) - F\left(\begin{matrix}2-2\epsilon,2-\epsilon,1\\\frac{3}{2}-2\epsilon,3-2\epsilon\end{matrix}\middle|1\right)\right]$$

$$+ \frac{1}{12}\frac{(3)_{-2\epsilon}}{(\frac{5}{2})_{-2\epsilon}}\left[F\left(\begin{matrix}1-\epsilon,1\\\frac{5}{2}-2\epsilon\end{matrix}\middle|1\right) - F\left(\begin{matrix}2-\epsilon,1\\\frac{5}{2}-2\epsilon\end{matrix}\middle|1\right)\right]. \tag{C72}$$

Expanding in powers of ϵ, we obtain

$$s^2 \int_0^1 d\beta \beta^{-2\epsilon}(1-\beta)^{-1/2}\frac{p^2+q^2}{(p^2-q^2)^2}\left\langle\frac{\mathbf{p}\cdot\mathbf{q}}{r^2}\right\rangle_x = -\frac{\pi^2}{16}+O(\epsilon). \tag{C73}$$

In the second term of Eq. (C70), the average over c is given by Eq. (C62). In the $+q$ term, the average over $x=\hat{\mathbf{p}}\cdot\hat{\mathbf{q}}$ is

$$\left\langle xF\left(\begin{matrix}1,\frac{5}{2}\\\frac{7}{2}-\epsilon\end{matrix}\middle|\frac{r^2}{(p+q)^2}\right)\right\rangle_x = \frac{5-2\epsilon}{4\epsilon}\left[F\left(\begin{matrix}2-\epsilon,1,\frac{5}{2}\\3-2\epsilon,1+\epsilon\end{matrix}\middle|\beta\right)-F\left(\begin{matrix}1-\epsilon,1,\frac{5}{2}\\3-2\epsilon,1+\epsilon\end{matrix}\middle|\beta\right)\right]$$

$$+\frac{5}{4\epsilon}\frac{(1)_\epsilon(1)_{-2\epsilon}(3)_{-2\epsilon}(\frac{7}{2})_{-\epsilon}}{(1)_{-\epsilon}(3)_{-3\epsilon}}\beta^{-\epsilon}\left[F\left(\begin{matrix}1-2\epsilon,\frac{5}{2}-\epsilon\\3-3\epsilon\end{matrix}\middle|\beta\right)-\frac{1-2\epsilon}{1-\epsilon}F\left(\begin{matrix}2-2\epsilon,\frac{5}{2}-\epsilon\\3-3\epsilon\end{matrix}\middle|\beta\right)\right].$$

(C74)

Integrating over β, we obtain hypergeometric functions of argument 1:

$$\int_0^1 d\beta\,\beta^{-2\epsilon}(1-\beta)^{-1/2}\left\langle\frac{\mathbf{p}\cdot\mathbf{q}c^4}{(p+q)^2-r^2c^2}\right\rangle_{c,x}$$

$$=\frac{1}{4\epsilon(3-2\epsilon)}\frac{(2)_{-2\epsilon}}{(\frac{5}{2})_{-2\epsilon}}\left[F\left(\begin{matrix}2-2\epsilon,2-\epsilon,1,\frac{5}{2}\\\frac{5}{2}-2\epsilon,3-2\epsilon,1+\epsilon\end{matrix}\middle|1\right)-F\left(\begin{matrix}2-2\epsilon,1-\epsilon,1,\frac{5}{2}\\\frac{5}{2}-2\epsilon,3-2\epsilon,1+\epsilon\end{matrix}\middle|1\right)\right]+\frac{1}{6\epsilon(2-3\epsilon)}$$

$$\times\frac{(1)_\epsilon(1)_{-2\epsilon}(3)_{-2\epsilon}(\frac{3}{2})_{-\epsilon}}{(1)_{-\epsilon}(\frac{5}{2})_{-3\epsilon}}\left[F\left(\begin{matrix}2-3\epsilon,1-2\epsilon,\frac{5}{2}-\epsilon\\\frac{5}{2}-3\epsilon,3-3\epsilon\end{matrix}\middle|1\right)-\frac{1-2\epsilon}{1-\epsilon}F\left(\begin{matrix}2-3\epsilon,2-2\epsilon,\frac{5}{2}-\epsilon\\\frac{5}{2}-3\epsilon,3-3\epsilon\end{matrix}\middle|1\right)\right].\qquad\text{(C75)}$$

Expanding in powers of ϵ, we obtain

$$\int_0^1 d\beta\,\beta^{-2\epsilon}(1-\beta)^{-1/2}\left\langle\frac{\mathbf{p}\cdot\mathbf{q}c^4}{(p+q)^2-r^2c^2}\right\rangle_{c,x}=\frac{\pi^2-6}{18}.\qquad\text{(C76)}$$

In the $-q$ term in the integral of the second term of Eq. (C70), we integrate over β before averaging over x. The integral over β can be expressed in terms of hypergeometric functions of type $_2F_1$:

$$s^2\int_0^1 d\beta\,\beta^{-2\epsilon}(1-\beta)^{-1/2}\frac{4\mathbf{p}\cdot\mathbf{q}}{(p-q)^2}\left\langle\frac{c^4}{(p+i\varepsilon-q)^2-r^2c^2}\right\rangle_c$$

$$=-\frac{1}{2(3-2\epsilon)\epsilon}\frac{(2)_{-2\epsilon}}{(\frac{5}{2})_{-2\epsilon}}(1-2y)F\left(\begin{matrix}2-2\epsilon,1\\1+\epsilon\end{matrix}\middle|1-y\right)-\frac{1}{4(3-2\epsilon)\epsilon}\frac{(1)_\epsilon}{(-\frac{1}{2})_\epsilon}(1-2y)(1-y)^{-3/2}F\left(\begin{matrix}\frac{1}{2}-2\epsilon,1\\-\frac{1}{2}+\epsilon\end{matrix}\middle|1-y\right)$$

$$+\frac{1}{8(2-3\epsilon)\epsilon}e^{\mp i\pi\epsilon}(1)_\epsilon(\frac{3}{2})_{-\epsilon}(1-2y)(1-y)^{-\epsilon}F\left(\begin{matrix}2-3\epsilon,\frac{5}{2}-\epsilon\\3-3\epsilon\end{matrix}\middle|y\right).\qquad\text{(C77)}$$

The phase in the last term is $e^{-i\pi\epsilon}$ for the $f(s_+,s_-,r)$ term of Eq. (C48), which comes from the $p>q$ region of the integral, and $e^{i\pi\epsilon}$ for the $f(s_-,s_+,r)$ term, which comes from the $p<q$ region. The average over $x=\hat{\mathbf{p}}\cdot\hat{\mathbf{q}}$ can be expressed in terms of hypergeometric functions of type $_3F_2$ evaluated at 1:

$$s^2 \int_0^1 d\beta \beta^{-2\epsilon} (1-\beta)^{-1/2} \left\langle \frac{4\mathbf{p}\cdot\mathbf{q}}{(p-q)^2} \frac{c^4}{(p+i\varepsilon-q)^2 - r^2 c^2} \right\rangle_{c,x}$$

$$= \frac{1}{4(3-2\epsilon)\epsilon} \frac{(2)_{-2\epsilon}}{(\frac{1}{2})_{-2\epsilon}} \left[F\left(\begin{matrix} 1-\epsilon, 2-2\epsilon, 1 \\ 3-2\epsilon, 1+\epsilon \end{matrix} \middle| 1 \right) - F\left(\begin{matrix} 2-\epsilon, 2-2\epsilon, 1 \\ 3-2\epsilon, 1+\epsilon \end{matrix} \middle| 1 \right) \right] - \frac{1}{(3-2\epsilon)\epsilon} \frac{(1)_\epsilon (3)_{-2\epsilon} (-\frac{1}{2})_{-\epsilon}}{(1)_{-\epsilon} (-\frac{1}{2})_\epsilon (\frac{3}{2})_{-2\epsilon}}$$

$$\times \left[F\left(\begin{matrix} -\frac{1}{2}-\epsilon, \frac{1}{2}-2\epsilon, 1 \\ \frac{3}{2}-2\epsilon, -\frac{1}{2}+\epsilon \end{matrix} \middle| 1 \right) + \frac{1+2\epsilon}{2(1-\epsilon)} F\left(\begin{matrix} \frac{1}{2}-\epsilon, \frac{1}{2}-2\epsilon, 1 \\ \frac{3}{2}-2\epsilon, -\frac{1}{2}+\epsilon \end{matrix} \middle| 1 \right) \right]$$

$$+ \frac{1}{16(2-3\epsilon)\epsilon} e^{\mp i\pi\epsilon} \frac{(1)_\epsilon (2)_{-2\epsilon} (2)_{-2\epsilon} (\frac{3}{2})_{-\epsilon}}{(1)_{-\epsilon} (3)_{-3\epsilon}}$$

$$\times \left[F\left(\begin{matrix} 1-\epsilon, 2-3\epsilon, \frac{5}{2}-\epsilon \\ 3-3\epsilon, 3-3\epsilon \end{matrix} \middle| 1 \right) - \frac{1-\epsilon}{1-2\epsilon} F\left(\begin{matrix} 2-\epsilon, 2-3\epsilon, \frac{5}{2}-\epsilon \\ 3-3\epsilon, 3-3\epsilon \end{matrix} \middle| 1 \right) \right]. \tag{C78}$$

The expansion of the real part of the integral in powers of ϵ is

$$s^2 \int_0^1 d\beta \beta^{-2\epsilon} (1-\beta)^{-1/2} \mathrm{Re}\left\langle \frac{4\mathbf{p}\cdot\mathbf{q}}{(p-q)^2} \frac{c^4}{(p+i\varepsilon-q)^2 - r^2 c^2} \right\rangle_{c,x} = \frac{9-\pi^2}{18} + O(\epsilon). \tag{C79}$$

Inserting Eqs. (C73), (C76), and (C79) into the thermal integral of Eq. (C70), we obtain

$$\int_{\mathbf{pq}} \frac{s^2 n_F^2(s)}{p^2 q^2} \frac{\mathbf{p}\cdot\mathbf{q}}{r^2} \mathrm{Re}\left\langle c^2 \frac{r^2 c^2 - p^2 - q^2}{\Delta(p+i\varepsilon, q, rc)} \right\rangle_c = \frac{T^2}{(4\pi)^2} \frac{\pi^2 - 1}{6\pi^2} \left[\frac{\pi^2}{12} - \log 2 \right]. \tag{C80}$$

Inserting this along with Eq. (C69) into Eq. (C59), we obtain

$$\int_{\mathbf{pq}} \frac{s^2 n_F^2(s)}{p^2 r^2} \mathrm{Re}\left\langle c^2 \frac{r^2 c^2 - p^2 - q^2}{\Delta(p+i\varepsilon, q, rc)} \right\rangle_c = \frac{T^2}{(4\pi)^2} \frac{1-\pi^2}{6\pi^2} \left[\frac{\pi^2}{12} - \log 2 \right]. \tag{C81}$$

Adding this integral to the subtracted integral in Eq. (C46), we obtain the final result in Eq. (C19). The subtracted integral appearing in Eq. (C47) vanishes due to antisymmetry of the integrand. Thus the final result (C20) is given by Eq. (C47).

The integrals (C21) and (C22) can be computed directly in three dimensions, as described above. The integrals (C23)–(C26) are divergent and require subtractions to remove the divergences. We first isolate the divergent part which come from the region $q \to 0$. We need one subtraction

$$n_B(q) = \left(n_B(q) - \frac{T}{q} + \frac{1}{2} \right) + \frac{T}{q} - \frac{1}{2}. \tag{C82}$$

In the integral (C24), it is convenient to first use the identity $r_c^2 = p^2 + 2\mathbf{p}\cdot\mathbf{q}/c + q^2/c^2$ to expand it into 3 integrals, two of which are Eqs. (C21) and (C23). In the third integral, the subtraction (C82) is needed to remove the divergences.

For the convergent terms, the HTL average over c and the angular average over $x = \hat{\mathbf{p}}\cdot\hat{\mathbf{q}}$ can be calculated in three dimensions:

$$\mathrm{Re}\left\langle c \frac{r_c^2 - p^2 - q^2}{\Delta(p+i\varepsilon, q, r_c)} \right\rangle_{c,x} = \frac{1}{6(4p^2 - q^2)} + \frac{q^2(4p^2 + 3q^2)}{3(4p^2 - q^2)^3} \log\frac{2p}{q} + \frac{(p+q)(4p^2 + 2pq + q^2)}{12pq(2p+q)^3} \log\frac{p+q}{p}$$

$$- \frac{(p-q)(4p^2 - 2pq + q^2)}{12pq(2p-q)^3} \log\frac{|p-q|}{p}, \tag{C83}$$

$$\mathrm{Re}\left\langle \hat{\mathbf{p}}\cdot\hat{\mathbf{q}}\frac{r_c^2-p^2-q^2}{\Delta(p+i\varepsilon,q,r_c)}\right\rangle_{c,x} = \frac{1}{6pq} - \frac{q(12p^2-q^2)}{6p(4p^2-q^2)^2}\log\frac{4p}{q} + \frac{(p+q)(2p^2-2pq-q^2)}{12p^2q(2p+q)^2}\log\frac{p+q}{4p}$$
$$+ \frac{(p-q)(2p^2+2pq-q^2)}{12p^2q(2p-q)^2}\log\frac{|p-q|}{4p},$$
(C84)

$$\mathrm{Re}\left\langle \frac{r_c^2-p^2}{q^2}\frac{r_c^2-p^2-q^2}{\Delta(p+i\epsilon,q,r_c)}c^{-1} - \frac{1}{q^2}c^{-1} + \frac{\log 2}{q^2}\right\rangle_{c,x}$$
$$= \frac{1}{4pq^2}\left[q\log\frac{p+q}{|p-q|} + p\log\frac{|p^2-q^2|}{p^2}\right].$$
(C85)

The remaining 2-dimensional integral over p and q can then be evaluated numerically:

$$\int_{\mathbf{pq}}\frac{n_F(p)}{p}\left(\frac{n_B(q)}{q} - \frac{T}{q^2} + \frac{1}{2q}\right)\frac{p^2}{q^2}\mathrm{Re}\left\langle c\frac{r_c^2-p^2-q^2}{\Delta(p+i\varepsilon,q,r_c)}\right\rangle_c = \frac{T^2}{(4\pi)^2}[1.480\times10^{-2}],$$
(C86)

$$\int_{\mathbf{pq}}\frac{n_F(p)}{p}\left(\frac{n_B(q)}{q} - \frac{T}{q^2} + \frac{1}{2q}\right)\frac{\mathbf{p}\cdot\mathbf{q}}{q^2}\mathrm{Re}\left\langle \frac{r_c^2-p^2-q^2}{\Delta(p+i\varepsilon,q,r_c)}\right\rangle_c = \frac{T^2}{(4\pi)^2}[-2.832\times10^{-3}],$$
(C87)

$$\int_{\mathbf{pq}}\frac{n_F(p)}{p}\frac{n_F(q)}{q}\mathrm{Re}\left\langle \frac{r_c^2-p^2}{q^2}\frac{r_c^2-p^2-q^2}{\Delta(p+i\epsilon,q,r_c)}c^{-1} - \frac{1}{q^2}c^{-1} + \frac{\log 2}{q^2}\right\rangle_c = \frac{T^2}{(4\pi)^2}[4.134\times10^{-2}],$$
(C88)

$$\int_{\mathbf{pq}}\frac{n_B(p)}{p}\frac{n_F(q)}{q}\mathrm{Re}\left\langle \frac{r_c^2-p^2}{q^2}\frac{r_c^2-p^2-q^2}{\Delta(p+i\epsilon,q,r_c)}c^{-1} - \frac{1}{q^2}c^{-1} + \frac{\log 2}{q^2}\right\rangle_c = \frac{T^2}{(4\pi)^2}[2.530\times10^{-1}].$$
(C89)

The integrals involving the terms subtracted from $n(q)$ in Eq. (C82) are divergent, so the HTL average over c and the angular average over $x=\hat{\mathbf{p}}\cdot\hat{\mathbf{q}}$ must be calculated in $3-2\epsilon$ dimensions. The first step in the calculation of the subtracted terms is to replace the average over c of the integral over q by an average over c and x:

$$\int_{\mathbf{q}}\frac{1}{q^n}\left\langle f(c)\frac{r_c^2-p^2-q^2}{\Delta(p+i\varepsilon,q,r_c)}\right\rangle_c = (-1)^{n-1}\frac{1}{8\pi^2\epsilon}\frac{(1)_{2\epsilon}(1)_{-2\epsilon}}{(\frac{3}{2})_{-\epsilon}}(e^\gamma\mu^2)^\epsilon(2p)^{1-n-2\epsilon}$$
$$\times\left\langle f(c)c^{3-n-2\epsilon}(1-c^2)^{n-2+2\epsilon}\sum_{\pm}(x\mp c-i\varepsilon)^{1-n-2\epsilon}\right\rangle_{c,x}.$$
(C90)

The integral over p can now be evaluated easily using either Eq. (B54) or

$$\int_{\mathbf{p}}n_F(p)p^{-2-2\epsilon} = \frac{1}{2\pi^2}\frac{(1)_{-4\epsilon}}{(\frac{3}{2})_{-\epsilon}}(1-2^{4\epsilon})\zeta(1-4\epsilon)(e^\gamma\mu^2)^\epsilon T^{1-4\epsilon}.$$
(C91)

It remains only to calculate the averages over c and x. The averages over x give $_2F_1$ hypergeometric functions with argument $[(1\mp c)/2-i\varepsilon]^{-1}$:

$$\langle (x \mp c - i\varepsilon)^{-n-2\epsilon} \rangle_x = (1 \mp c)^{-n-2\epsilon} F\left(\begin{matrix} 1-\epsilon, n+2\epsilon \\ 2-2\epsilon \end{matrix} \middle| [(1 \mp c)/2 - i\varepsilon]^{-1} \right), \tag{C92}$$

$$\langle x(x \mp c - i\varepsilon)^{-n-2\epsilon} \rangle_x = \frac{1}{2}(1 \mp c)^{-n-2\epsilon}\left[F\left(\begin{matrix} 1-\epsilon, n+2\epsilon \\ 3-2\epsilon \end{matrix} \middle| [(1 \mp c)/2 - i\varepsilon]^{-1} \right) \right.$$
$$\left. - F\left(\begin{matrix} 2-\epsilon, n+2\epsilon \\ 3-2\epsilon \end{matrix} \middle| [(1 \mp c)/2 - i\varepsilon]^{-1} \right) \right]. \tag{C93}$$

Using a transformation formula, the arguments can be changed to $(1 \mp c)/2 - i\varepsilon$. If the expressions (C92) and (C93) are averaged over c with a weight that is an even function of c, the $+$ and $-$ terms combine to give ${}_3F_2$ hypergeometric functions with argument 1. For example,

$$\left\langle (1-c^2)^{2\epsilon} \sum_{\pm} (x \mp c - i\varepsilon)^{-1-2\epsilon} \right\rangle_{c,x} = \frac{1}{3\epsilon} \frac{(2)_{-2\epsilon}(1)_{\epsilon}(\frac{3}{2})_{-\epsilon}}{(1)_{-\epsilon}(1)_{-\epsilon}} \left\{ -e^{-i\pi\epsilon} \frac{(1)_{3\epsilon}(1)_{-2\epsilon}}{(1)_{2\epsilon}(2)_{-\epsilon}} F\left(\begin{matrix} 1-2\epsilon, 1-\epsilon, \epsilon \\ 2-\epsilon, 1-3\epsilon \end{matrix} \middle| 1 \right) \right.$$
$$\left. + e^{i2\pi\epsilon} \frac{(1)_{-3\epsilon}(1)_{\epsilon}}{(1)_{-4\epsilon}(2)_{2\epsilon}} F\left(\begin{matrix} 1+\epsilon, 1+2\epsilon, 4\epsilon \\ 2+2\epsilon, 1+3\epsilon \end{matrix} \middle| 1 \right) \right\}. \tag{C94}$$

Upon expanding the hypergeometric functions in powers of ϵ and taking the real parts, we obtain

$$\mathrm{Re}\left\langle (1-c^2)^{2\epsilon} \sum_{\pm} (x \mp c - i\varepsilon)^{-1-2\epsilon} \right\rangle_{c,x} = \pi^2[-\epsilon + 2(1-\log 2)\epsilon^2], \tag{C95}$$

$$\mathrm{Re}\left\langle c^2(1-c^2)^{2\epsilon} \sum_{\pm} (x \mp c - i\varepsilon)^{-1-2\epsilon} \right\rangle_{c,x} = \pi^2\left[-\frac{1}{3}\epsilon + \frac{2}{9}(2-3\log 2)\epsilon^2 \right], \tag{C96}$$

$$\mathrm{Re}\left\langle (1-c^2)^{2+2\epsilon} \sum_{\pm} (x \mp c - i\varepsilon)^{-3-2\epsilon} \right\rangle_{c,x} = \pi^2\left[-\frac{8}{3}\epsilon^2 \right], \tag{C97}$$

$$\mathrm{Re}\left\langle x(1-c^2)^{1+2\epsilon} \sum_{\pm} (x \mp c - i\varepsilon)^{-2-2\epsilon} \right\rangle_{c,x} = \pi^2\left[-\frac{2}{3}\epsilon + \frac{2}{9}(1-6\log 2)\epsilon^2 \right]. \tag{C98}$$

If the expressions (C92) and (C93) are averaged over c with a weight that is an odd function of c, they reduce to integrals of ${}_2F_1$ hypergeometric functions with argument y. For example,

$$\left\langle c(1-c^2)^{1+2\epsilon} \sum_{\pm} (x \mp c - i\varepsilon)^{-2-2\epsilon} \right\rangle_{c,x} = \frac{(2)_{-2\epsilon}(\frac{3}{2})_{-\epsilon}}{(1)_{-\epsilon}(1)_{-\epsilon}} \left\{ -2e^{-i\pi\epsilon} \frac{(1)_{3\epsilon}}{(2)_{2\epsilon}} \int_0^1 dy\, y^{-2\epsilon}(1-y)^{1+\epsilon}|1-2y| F\left(\begin{matrix} 1-\epsilon, \epsilon \\ -3\epsilon \end{matrix} \middle| y \right) \right.$$
$$\left. - \frac{8}{3(1+3\epsilon)} e^{2i\pi\epsilon} \frac{(1)_{-3\epsilon}}{(1)_{-4\epsilon}} \int_0^1 dy\, y^{1+\epsilon}(1-y)^{1+\epsilon}|1-2y| F\left(\begin{matrix} 2+2\epsilon, 1+4\epsilon \\ 2+3\epsilon \end{matrix} \middle| y \right) \right\}. \tag{C99}$$

The resulting expansions for the real parts of the averages over c and x are

$$\mathrm{Re}\left\langle c(1-c^2)^{1+2\epsilon} \sum_{\pm} (x \mp c - i\varepsilon)^{-2-2\epsilon} \right\rangle_{c,x} = -1 + \frac{14(1-\log 2)}{3}\epsilon, \tag{C100}$$

$$\mathrm{Re}\left\langle xc(1-c^2)^{2\epsilon} \sum_{\pm} (x \mp c - i\varepsilon)^{-1-2\epsilon} \right\rangle_{c,x} = \frac{2(1-\log 2)}{3} + \left(\frac{4}{9} + \frac{8}{9}\log 2 - \frac{4}{3}\log^2 2 + \frac{\pi^2}{18} \right)\epsilon. \tag{C101}$$

Multiplying each of these expansions by the appropriate factors from the integral over q in Eq. (C90) and the integral over p in Eq. (C91) or (B54), we obtain

$$\int_{\mathbf{pq}} \frac{n_F(p)}{p} \frac{p^2}{q^3} \mathrm{Re}\left\langle c^{1+2\epsilon} \frac{r_c^2-p^2-q^2}{\Delta(p+i\epsilon,q,r_c)}\right\rangle_c = \frac{T^2}{(4\pi)^2}\left(\frac{\mu}{4\pi T}\right)^{4\epsilon}\left(-\frac{1}{48}\right)\left[\frac{1}{\epsilon}-\frac{2}{3}-\frac{4}{3}\log 2+4\frac{\zeta'(-1)}{\zeta(-1)}\right],$$ (C102)

$$\int_{\mathbf{pq}} \frac{n_F(p)}{p} \frac{p^2}{q^4} \mathrm{Re}\left\langle c^{1+2\epsilon} \frac{r_c^2-p^2-q^2}{\Delta(p+i\epsilon,q,r_c)}\right\rangle_c = O(\epsilon),$$ (C103)

$$\int_{\mathbf{pq}} \frac{n_F(p)}{p} \frac{\mathbf{p}\cdot\mathbf{q}}{q^3} \mathrm{Re}\left\langle c^{2\epsilon} \frac{r_c^2-p^2-q^2}{\Delta(p+i\epsilon,q,r_c)}\right\rangle_c = \frac{T^2}{(4\pi)^2}\left(\frac{\mu}{4\pi T}\right)^{4\epsilon}\left(-\frac{1}{36}\right)\left[(1-\log 2)\left(\frac{1}{\epsilon}+\frac{14}{3}-4\log 2+4\frac{\zeta'(-1)}{\zeta(-1)}\right)\right.$$
$$\left. +\frac{\pi^2}{12}\right],$$ (C104)

$$\int_{\mathbf{pq}} \frac{n_F(p)}{p} \frac{\mathbf{p}\cdot\mathbf{q}}{q^4} \mathrm{Re}\left\langle c^{2\epsilon} \frac{r_c^2-p^2-q^2}{\Delta(p+i\epsilon,q,r_c)}\right\rangle_c = \frac{T}{(4\pi)^2}\left(-\frac{1}{6}\log 2\right).$$ (C105)

Adding Eq. (C103) to the subtracted integral (C86) we obtain the final result in Eq. (C23). Combining Eq. (C87) with Eqs. (C104) and (C105), we obtain

$$\int_{\mathbf{pq}} \frac{n_F(p)n_B(q)}{pq} \frac{\mathbf{p}\cdot\mathbf{q}}{q^2} \mathrm{Re}\left\langle c^{2\epsilon} \frac{r_c^2-p^2-q^2}{\Delta(p+i\epsilon,q,r_c)}\right\rangle_c = \frac{T^2}{(4\pi)^2}\left(\frac{\mu}{4\pi T}\right)^{4\epsilon}\left(\frac{1-\log 2}{72}\right)\left[\frac{1}{\epsilon}-15.2566\right].$$ (C106)

The integral (C24) is obtained from Eqs. (C21), (C23) and (C106). Finally consider Eqs. (C25) and (C26). In order to evaluate them we need two subtractions for each integral:

$$\int_{\mathbf{pq}} \frac{n_F(p)}{p} \frac{n_F(p)}{q} \frac{1}{q^2}\langle c^{2\epsilon}\rangle_c = \frac{T^2}{(4\pi)^2}\left(\frac{\mu}{4\pi T}\right)^{4\epsilon}\left(-\frac{1}{12}\right)\left[\frac{1}{\epsilon}+2+2\log 2+2\gamma+2\frac{\zeta'(-1)}{\zeta(-1)}\right],$$ (C107)

$$\int_{\mathbf{pq}} \frac{n_F(p)}{p} \frac{n_F(q)}{q} \frac{1}{q^2}\langle c^{-1+2\epsilon}\rangle_c = \frac{T^2}{(4\pi)^2}\left(\frac{\mu}{4\pi T}\right)^{4\epsilon}\left(-\frac{1}{24}\right)\left[\frac{1}{\epsilon^2}+\left(2+2\gamma+4\log 2+2\frac{\zeta'(-1)}{\zeta(-1)}\right)\frac{1}{\epsilon}+53.1064\right],$$ (C108)

$$\int_{\mathbf{pq}} \frac{n_B(p)}{p} \frac{n_F(q)}{q} \frac{1}{q^2}\langle c^{2\epsilon}\rangle_c = \frac{T^2}{(4\pi)^2}\left(\frac{\mu}{4\pi T}\right)^{4\epsilon}\left(-\frac{1}{6}\right)\left[\frac{1}{\epsilon}+2+4\log 2+2\gamma+2\frac{\zeta'(-1)}{\zeta(-1)}\right],$$ (C109)

$$\int_{\mathbf{pq}} \frac{n_B(p)}{p} \frac{n_F(q)}{q} \frac{1}{q^2}\langle c^{-1+2\epsilon}\rangle_c = \frac{T^2}{(4\pi)^2}\left(\frac{\mu}{4\pi T}\right)^{4\epsilon}\left(-\frac{1}{12}\right)\left[\frac{1}{\epsilon^2}+\left(2+2\gamma+6\log 2+2\frac{\zeta'(-1)}{\zeta(-1)}\right)\frac{1}{\epsilon}+69.7097\right].$$ (C110)

The subtractions can be evaluated directly in three dimensions and the results are given in Eqs. (C88) and (C89) The integrals (C25) and (C26) are then given by the sum of the difference terms (C88) and (C89) and the subtraction terms (C107)–(C110).

3. Four-dimensional integrals

In the sum-integral formula (B39), the second term on the right side involves an integral over 4-dimensional Euclidean momenta. The integrands are functions of the integration variable Q and $R=-(P+Q)$. The simplest integrals to evaluate are those whose integrands are independent of P_0:

$$\int_Q \frac{1}{Q^2 r^2} = \frac{1}{(4\pi)^2}\mu^{2\epsilon}p^{-2\epsilon}2\left[\frac{1}{\epsilon}+4-2\log 2\right],$$ (C111)

$$\int_Q \frac{q^2}{Q^2 r^4} = \frac{1}{(4\pi)^2}\mu^{2\epsilon}p^{-2\epsilon}2\left[\frac{1}{\epsilon}+1-2\log 2\right],$$ (C112)

$$\int_Q \frac{1}{Q^2 r^4} = \frac{1}{(4\pi)^2}\mu^{2\epsilon}p^{-2-2\epsilon}(-2)[1+(-2-2\log 2)\epsilon].$$ (C113)

Another simple integral that is needed depends only on $P^2 = P_0^2 + p^2$:

$$\int_Q \frac{1}{Q^2 R^2} = \frac{1}{(4\pi)^2} (e^\gamma \mu^2)^\epsilon (P^2)^{-\epsilon} \frac{1}{\epsilon} \frac{(1)_\epsilon (1)_{-\epsilon} (1)_{-\epsilon}}{(2)_{-2\epsilon}}, \tag{C114}$$

where $(a)_b$ is Pochhammer's symbol which is defined in Eq. (C146). We need the following weighted averages over c of this function evaluated at $P = (-ip, \mathbf{p}/c)$:

$$\left\langle c^{-1+2\epsilon} \int_Q \frac{1}{Q^2 R^2} \Big|_{P \to (-ip, \mathbf{p}/c)} \right\rangle_c = \frac{1}{(4\pi)^2} \mu^{2\epsilon} p^{-2\epsilon} \frac{1}{4} \left[\frac{1}{\epsilon^2} + \frac{2 \log 2}{\epsilon} + 2 \log^2 2 + \frac{3\pi^2}{4} \right], \tag{C115}$$

$$\left\langle c^{1+2\epsilon} \int_Q \frac{1}{Q^2 R^2} \Big|_{P \to (-ip, \mathbf{p}/c)} \right\rangle_c = \frac{1}{(4\pi)^2} \mu^{2\epsilon} p^{-2\epsilon} \frac{1}{2} \left[\frac{1}{\epsilon} + 2 \log 2 \right]. \tag{C116}$$

The remaining integrals are functions of P_0 that must be analytically continued to the point $P_0 = -ip + \varepsilon$. Several of these integrals are straightforward to evaluate:

$$\int_Q \frac{q^2}{Q^2 R^2} \Big|_{P_0 = -ip} = 0, \tag{C117}$$

$$\int_Q \frac{q^2}{Q^2 r^2 R^2} \Big|_{P_0 = -ip} = \frac{1}{(4\pi)^2} \mu^{2\epsilon} p^{-2\epsilon} (-1) \left[\frac{1}{\epsilon^2} + \frac{1 - 2 \log 2}{\epsilon} + 10 - 2 \log 2 + 2 \log^2 2 - \frac{7\pi^2}{12} \right], \tag{C118}$$

$$\int_Q \frac{1}{Q^2 r^2 R^2} \Big|_{P_0 = -ip} = \frac{1}{(4\pi)^2} \mu^{2\epsilon} p^{-2-2\epsilon} \left[\frac{1}{\epsilon} - 2 - 2 \log 2 \right]. \tag{C119}$$

We also need a weighted average over c of the integral in Eq. (C117) evaluated at $P = (-ip, \mathbf{p}/c)$. The integral itself is

$$\int_Q \frac{q^2}{Q^2 R^2} \Big|_{P \to (-ip, \mathbf{p}/c)} = \frac{1}{(4\pi)^2} (e^\gamma \mu^2)^\epsilon p^{2-2\epsilon} \frac{(1)_\epsilon}{\epsilon} \frac{1}{4} \frac{(1)_{-\epsilon} (1)_{-\epsilon}}{(2)_{-2\epsilon}} \left(\frac{1}{3 - 2\epsilon} + c^2 \right) c^{-2+2\epsilon} (1 - c^2)^{-\epsilon}. \tag{C120}$$

The weighted averages are

$$\left\langle c^{1+2\epsilon} \int_Q \frac{q^2}{Q^2 R^2} \Big|_{P \to (-ip, \mathbf{p}/c)} \right\rangle = \frac{1}{(4\pi)^2} \mu^{2\epsilon} p^{2-2\epsilon} \frac{1}{48} \left[\frac{1}{\epsilon^2} + \frac{2(10 + 3 \log 2)}{3\epsilon} + \frac{4}{9} + \frac{40}{3} \log 2 + 2 \log^2 2 + \frac{3\pi^2}{4} \right], \tag{C121}$$

$$\left\langle c^{-1+2\epsilon} \int_Q \frac{q^2}{Q^2 R^2} \Big|_{P \to (-ip, \mathbf{p}/c)} \right\rangle_c = \frac{1}{(4\pi)^2} \mu^{2\epsilon} p^{2-2\epsilon} \frac{1}{16} \left[\frac{1}{\epsilon^2} + \frac{2 \log 2}{\epsilon} + 2 \log^2 2 + \frac{3\pi^2}{4} \right]. \tag{C122}$$

The most difficult 4-dimensional integrals to evaluate involve an HTL average of an integral with denominator $R_0^2 + r^2 c^2$:

$$\text{Re} \int_Q \frac{1}{Q^2} \left\langle \frac{c^2}{R_0^2 + r^2 c^2} \right\rangle_c = \frac{1}{(4\pi)^2} \mu^{2\epsilon} p^{-2\epsilon} \left[\frac{2 - 2 \log 2}{\epsilon} + 8 - 4 \log 2 + 4 \log^2 2 - \frac{\pi^2}{2} \right], \tag{C123}$$

$$\text{Re} \int_Q \frac{1}{Q^2} \left\langle \frac{c^2(1 - c^2)}{R_0^2 + r^2 c^2} \right\rangle_c \frac{1}{(4\pi)^2} \mu^{2\epsilon} p^{-2\epsilon} = \frac{1}{3} \left[\frac{1}{\epsilon} + \frac{20}{3} - 6 \log 2 \right], \tag{C124}$$

$$\text{Re} \int_Q \frac{1}{Q^2} \left\langle \frac{c^4}{R_0^2 + r^2 c^2} \right\rangle_c = \frac{1}{(4\pi)^2} \mu^{2\epsilon} p^{-2\epsilon} \left[\frac{5 - 6 \log 2}{3\epsilon} + \frac{52}{9} - 2 \log 2 + 4 \log^2 2 - \frac{\pi^2}{2} \right], \tag{C125}$$

$$\text{Re} \int_Q \frac{1}{Q^2 r^2} \left\langle \frac{c^2}{R_0^2 + r^2 c^2} \right\rangle_c = \frac{1}{(4\pi)^2} \mu^{2\epsilon} p^{-2-2\epsilon} \left(-\frac{1}{4} \right) \left[\frac{1}{\epsilon} + \frac{4}{3} + \frac{2}{3} \log 2 \right], \tag{C126}$$

$$\mathrm{Re}\int_Q \frac{q^2}{Q^2 r^2}\left\langle\frac{c^2}{R_0^2+r^2 c^2}\right\rangle_c = \frac{1}{(4\pi)^2}\mu^{2\epsilon}p^{-2\epsilon}\left[\frac{13-16\log 2}{12\epsilon}+\frac{29}{9}-\frac{19}{18}\log 2+\frac{8}{3}\log^2 2-\frac{4}{9}\pi^2\right],$$

(C127)

$$\left\langle\int_Q\frac{q^2-p^2}{Q^2 r^2(R_0^2+r^2 c^2)}\right\rangle_c = \frac{1}{(4\pi)^2}\mu^{2\epsilon}p^{-2\epsilon}\left[-\frac{\pi^2}{3}\right].$$

(C128)

The analytic continuation to $P_0=-ip+\varepsilon$ is implied in these integrals and in all the 4-dimensional integrals in the remainder of this subsection.

We proceed to describe the evaluation of the integrals (C123) and (C125). The integral over Q_0 can be evaluated by introducing a Feynman parameter to combine Q^2 and $R_0^2+r^2 c^2$ into a single denominator:

$$\int_Q\frac{1}{Q^2(R_0^2+r^2 c^2)} = \frac{1}{4}\int_0^1 dx\int_{\mathbf{r}}[(1-x+xc^2)r^2+2(1-x)\mathbf{r}\cdot\mathbf{p}+(1-x)^2 p^2-i\varepsilon]^{-3/2},$$

(C129)

where we have carried out the analytic continuation to $P_0=-ip+\varepsilon$. Integrating over $\mathbf{r}$ and then over the Feynman parameter, we get a $_2F_1$ hypergeometric function with argument $1-c^2$:

$$\int_Q\frac{1}{Q^2(R_0^2+r^2 c^2)} = \frac{1}{(4\pi)^2}(e^\gamma\mu^2)^\epsilon p^{-2\epsilon}\frac{(1)_\epsilon}{\epsilon}e^{i\pi\epsilon}\frac{(1)_{-2\epsilon}(1)_{-\epsilon}}{(2)_{-3\epsilon}}(1-c^2)^{-\epsilon}F\left(\begin{array}{c}\frac{3}{2}-2\epsilon,1-\epsilon\\ 2-3\epsilon\end{array}\bigg|1-c^2\right).$$

(C130)

The subsequent weighted averages over c give $_3F_2$ hypergeometric functions with argument 1:

$$\int_Q\frac{1}{Q^2}\left\langle\frac{c^2}{R_0^2+r^2 c^2}\right\rangle_c = \frac{1}{(4\pi)^2}(e^\gamma\mu^2)^\epsilon p^{-2\epsilon}\frac{(1)_\epsilon}{\epsilon}\frac{1}{3}e^{i\pi\epsilon}\frac{(\frac{3}{2})_{-\epsilon}(1)_{-2\epsilon}(1)_{-2\epsilon}}{(\frac{5}{2})_{-2\epsilon}(2)_{-3\epsilon}}F\left(\begin{array}{c}1-2\epsilon,\frac{3}{2}-2\epsilon,1-\epsilon\\ \frac{5}{2}-2\epsilon,2-3\epsilon\end{array}\bigg|1\right),$$

(C131)

$$\int_Q\frac{1}{Q^2}\left\langle\frac{c^2(1-c^2)}{R_0^2+r^2 c^2}\right\rangle_c = \frac{1}{(4\pi)^2}(e^\gamma\mu^2)^\epsilon p^{-2\epsilon}\frac{(1)_\epsilon}{\epsilon}\frac{2}{15}e^{i\pi\epsilon}\frac{(\frac{3}{2})_{-\epsilon}(1)_{-2\epsilon}(2)_{-2\epsilon}}{(\frac{7}{2})_{-2\epsilon}(2)_{-3\epsilon}}F\left(\begin{array}{c}2-2\epsilon,\frac{3}{2}-2\epsilon,1-\epsilon\\ \frac{7}{2}-2\epsilon,2-3\epsilon\end{array}\bigg|1\right).$$

(C132)

After expanding in powers of ϵ, the real part is Eq. (C125).

The integral (C126) has a factor of $1/r^2$ in the integrand. After using Eq. (C129), it is convenient to use a second Feynman parameter to combine $(1-x+xc^2)r^2$ with the other denominator before integrating over $\mathbf{r}$:

$$\int_Q\frac{1}{Q^2 r^2(R_0^2+r^2 c^2)} = \frac{3}{8}\int_0^1 dx(1-x+xc^2)\int_0^1 dy\, y^{1/2}\int_{\mathbf{r}}[(1-x+xc^2)r^2+2y(1-x)\mathbf{r}\cdot\mathbf{p}+y(1-x)^2 p^2-i\varepsilon]^{-5/2}.$$

(C133)

After integrating over $\mathbf{r}$ and then y, we obtain $_2F_1$ hypergeometric functions with arguments $x(1-c^2)$. The integral over x gives a $_2F_1$ hypergeometric function with argument $1-c^2$:

$$\int_Q\frac{1}{Q^2 r^2(R_0^2+r^2 c^2)}$$

$$= \frac{1}{(4\pi)^2}(e^\gamma\mu^2)^\epsilon p^{-2-2\epsilon}\frac{(1)_\epsilon}{\epsilon}\left\{\frac{(-\frac{1}{2})_{-\epsilon}(1)_{-\epsilon}}{(\frac{1}{2})_{-2\epsilon}}-\frac{3}{2(1+2\epsilon)}e^{i\pi\epsilon}\frac{(1)_{-2\epsilon}(1)_{-\epsilon}}{(1)_{-3\epsilon}}(1-c^2)^{-\epsilon}F\left(\begin{array}{c}\frac{1}{2}-2\epsilon,-\epsilon\\ -3\epsilon\end{array}\bigg|1-c^2\right)\right\}.$$

(C134)

After averaging over c, we get hypergeometric functions with argument 1:

$$\int_Q \frac{1}{Q^2 r^2} \left\langle \frac{c^2}{R_0^2 + r^2 c^2} \right\rangle_c = \frac{1}{(4\pi)^2} (e^\gamma \mu^2)^\epsilon p^{-2-2\epsilon} \frac{(1)_\epsilon}{\epsilon} \left\{ \frac{1}{3-2\epsilon} \frac{(-\frac{1}{2})_{-\epsilon}(1)_{-\epsilon}}{(\frac{1}{2})_{-2\epsilon}} \right.$$

$$\left. - \frac{1}{2} e^{i\pi\epsilon} \frac{(-\frac{1}{2})_{-\epsilon}(1)_{-2\epsilon}(2)_{-2\epsilon}}{(\frac{5}{2})_{-2\epsilon}(1)_{-3\epsilon}} F\left(\begin{array}{c} 1-2\epsilon, \frac{1}{2}-2\epsilon, -\epsilon \\ \frac{5}{2}-2\epsilon, -3\epsilon \end{array} \middle| 1 \right) \right\}, \tag{C135}$$

$$\int_Q \frac{1}{Q^2 r^2} \left\langle \frac{1}{R_0^2 + r^2 c^2} \right\rangle_c = \frac{1}{(4\pi)^2} (e^\gamma \mu^2)^\epsilon p^{-2-2\epsilon} \frac{(1)_\epsilon}{\epsilon} \left\{ \frac{(-\frac{1}{2})_{-\epsilon}(\frac{3}{2})_{-\epsilon}}{(\frac{1}{2})_{-2\epsilon}} \right.$$

$$\left. - \frac{1}{2} e^{i\pi\epsilon} \frac{(1)^2_{-2\epsilon}(1)_{-\epsilon}}{(\frac{3}{2})_{-2\epsilon}(1)_{-3\epsilon}} F\left(\begin{array}{c} 1-2\epsilon, \frac{1}{2}-2\epsilon, -\epsilon \\ \frac{3}{2}-2\epsilon, -3\epsilon \end{array} \middle| 1 \right) \right\}. \tag{C136}$$

After expanding in powers of ϵ the real part is Eq. (C126).

To evaluate the integral (C28), it is convenient to first express it as the sum of 3 integrals by expanding the factor of q^2 in the numerator as $q^2 = p^2 + 2\mathbf{p} \cdot \mathbf{r} + r^2$:

$$\int_Q \frac{q^2}{Q^2 r^2 (R_0^2 + r^2 c^2)} = \int_Q \left(\frac{p^2}{r^2} + 2\frac{\mathbf{p} \cdot \mathbf{r}}{r^2} + 1 \right) \frac{1}{Q^2 (R_0^2 + r^2 c^2)}. \tag{C137}$$

To evaluate the integral with $\mathbf{p} \cdot \mathbf{r}$ in the numerator, we first combine the denominators using Feynman parameters as in Eq. (C133). After integrating over $\mathbf{r}$ and then y, we obtain $_2F_1$ hypergeometric functions with arguments $x(1-c^2)$. The integral over x gives $_2F_1$ hypergeometric functions with arguments $1-c^2$:

$$\int_Q \frac{\mathbf{p} \cdot \mathbf{r}}{Q^2 r^2 (R_0^2 + r^2 c^2)} = \frac{1}{(4\pi)^2} (e^\gamma \mu^2)^\epsilon p^{-2\epsilon} \frac{(1)_\epsilon}{2\epsilon^2} \left\{ -\frac{(\frac{3}{2})_{-\epsilon}(1)_{-\epsilon}}{(\frac{3}{2})_{-2\epsilon}} + e^{i\pi\epsilon} \frac{(1)_{-2\epsilon}(1)_{-\epsilon}}{(1)_{-3\epsilon}} (1-c^2)^{-\epsilon} F\left(\begin{array}{c} \frac{3}{2}-2\epsilon, -\epsilon \\ 1-3\epsilon \end{array} \middle| 1-c^2 \right) \right\}.$$

$$\tag{C138}$$

After averaging over c, we get a hypergeometric function with argument 1:

$$\int_Q \frac{\mathbf{p} \cdot \mathbf{r}}{Q^2 r^2} \left\langle \frac{c^2}{R_0^2 + r^2 c^2} \right\rangle_c = \frac{1}{(4\pi)^2} (e^\gamma \mu^2)^\epsilon p^{-2\epsilon} \frac{(1)_\epsilon}{2\epsilon^2} \left\{ -\frac{1}{3-2\epsilon} \frac{(\frac{3}{2})_{-\epsilon}(1)_{-\epsilon}}{(\frac{3}{2})_{-2\epsilon}} \right.$$

$$\left. + \frac{1}{3} e^{i\pi\epsilon} \frac{(\frac{3}{2})_{-\epsilon}(1)_{-2\epsilon}(1)_{-2\epsilon}}{(\frac{5}{2})_{-2\epsilon}(1)_{-3\epsilon}} F\left(\begin{array}{c} 1-2\epsilon, \frac{3}{2}-2\epsilon, -\epsilon \\ \frac{5}{2}-2\epsilon, 1-3\epsilon \end{array} \middle| 1 \right) \right\}. \tag{C139}$$

After expanding in powers of ϵ, the real part is

$$\mathrm{Re}\int_Q \frac{\mathbf{p}\cdot\mathbf{r}}{Q^2 r^2}\left\langle \frac{c^2}{R_0^2+r^2 c^2}\right\rangle_c = \frac{1}{(4\pi)^2}\mu^{2\epsilon}p^{-2\epsilon}\left[\frac{-1+\log 2}{3\epsilon}-\frac{20}{9}+\frac{14}{9}\log 2-\frac{2}{3}\log^2 2+\frac{\pi^2}{36}\right]. \tag{C140}$$

Combining this with Eqs. (C123) and (C125), we obtain the integral (C127).

To evaluate the integral (C128), we first express the numerator as a sum of two integrals whose averages have been calculated:

$$\left\langle \int_Q \frac{q^2-p^2}{Q^2 r^2(R_0^2+r^2 c^2)}\right\rangle_x = \left\langle \int_Q \frac{2\mathbf{p}\cdot\mathbf{r}+r^2}{Q^2 r^2(R_0^2+r^2 c^2)}\right\rangle_x = \frac{1}{(4\pi)^2}(e^\gamma\mu^2)^\epsilon p^{-2\epsilon}\frac{(1)_\epsilon}{\epsilon}\left\{-\frac{1}{\epsilon}\frac{(\frac{3}{2})_{-\epsilon}(1)_{-\epsilon}}{(\frac{3}{2})_{-2\epsilon}}+e^{i\pi\epsilon}\frac{(1)_{-\epsilon}(1)_{-2\epsilon}}{(1)_{-3\epsilon}}\right.$$

$$\times\frac{1}{\epsilon}(1-c^2)^{-\epsilon}F\left(\begin{array}{c}-\epsilon,\frac{3}{2}-2\epsilon\\1-3\epsilon\end{array}\bigg|1-c^2\right)+e^{i\pi\epsilon}\frac{(1)_{-\epsilon}(1)_{-2\epsilon}}{(2)_{-3\epsilon}}(1-c^2)^{-\epsilon}F\left(\begin{array}{c}1-\epsilon,\frac{3}{2}-2\epsilon\\2-3\epsilon\end{array}\bigg|1-c^2\right)\right\}. \tag{C141}$$

The two hypergeometric functions are now combined into a single hypergeometric functions, which yields

$$\left\langle \int_Q \frac{2\mathbf{p}\cdot\mathbf{r}+r^2}{Q^2 r^2(R_0^2+r^2 c^2)}\right\rangle_x$$

$$=\frac{1}{(4\pi)^2}(e^\gamma\mu^2)^\epsilon p^{-2\epsilon}\frac{(1)_\epsilon}{\epsilon^2}\left\{-\frac{(\frac{3}{2})_{-\epsilon}(1)_{-\epsilon}}{(\frac{3}{2})_{-2\epsilon}}+e^{i\pi\epsilon}\frac{(1)_{-\epsilon}(2)_{-2\epsilon}}{(2)_{-3\epsilon}}(1-c^2)^{-\epsilon}F\left(\begin{array}{c}-\epsilon,\frac{3}{2}-2\epsilon\\2-3\epsilon\end{array}\bigg|1-c^2\right)\right\}. \tag{C142}$$

Averaging over c yields

$$\left\langle \int_Q \frac{2\mathbf{p}\cdot\mathbf{r}+r^2}{Q^2 r^2(R_0^2+r^2 c^2)}\right\rangle_{c,x}$$

$$=\frac{1}{(4\pi)^2}(e^\gamma\mu^2)^\epsilon p^{-2\epsilon}\frac{1}{\epsilon^2}\frac{(1)_\epsilon(1)_{-\epsilon}(\frac{3}{2})_{-\epsilon}}{(\frac{3}{2})_{-2\epsilon}}$$

$$\times\left[-1+e^{i\pi\epsilon}\frac{(1)_{-2\epsilon}}{(1)^2_{-\epsilon}}\right]. \tag{C143}$$

Expansion in powers of ϵ, yields Eq. (C128).

4. Hypergeometric functions

The generalized hypergeometric function of type $_pF_q$ is an analytic function of one variable with $p+q$ parameters. In our case, the parameters are functions of ϵ, so the list of parameters sometimes gets lengthy and the standard notation for these functions becomes cumbersome. We therefore introduce a more concise notation

$$F\left(\begin{array}{c}\alpha_1,\alpha_2,\ldots,\alpha_p\\\beta_1,\ldots,\beta_q\end{array}\bigg|z\right)$$

$$\equiv {}_pF_q(\alpha_1,\alpha_2,\ldots,\alpha_p;\beta_1,\ldots,\beta_q;z). \tag{C144}$$

The generalized hypergeometric function has a power series representation

$$F\left(\begin{array}{c}\alpha_1,\alpha_2,\ldots,\alpha_p\\\beta_1,\ldots,\beta_q\end{array}\bigg|z\right)=\sum_{n=0}^\infty \frac{(\alpha_1)_n(\alpha_2)_n\cdots(\alpha_p)_n}{(\beta_1)_n\cdots(\beta_q)_n n!}z^n, \tag{C145}$$

where $(a)_b$ is Pochhammer's symbol:

$$(a)_b=\frac{\Gamma(a+b)}{\Gamma(a)}. \tag{C146}$$

The power series converges for $|z|<1$. For $z=1$, it converges if $\mathrm{Re}\, s>0$, where

$$s=\sum_{i=1}^{p-1}\beta_i-\sum_{i=1}^{p}\alpha_i. \tag{C147}$$

The hypergeometric function of type $_{p+1}F_{q+1}$ has an integral representation in terms of the hypergeometric function of type $_{p}F_{q}$:

$$\int_0^1 dt\, t^{\nu-1}(1-t)^{\mu-1} F\left(\begin{array}{c}\alpha_1,\alpha_2,\ldots,\alpha_p\\\beta_1,\ldots,\beta_q\end{array}\middle| tz\right)$$

$$= \frac{\Gamma(\mu)\Gamma(\nu)}{\Gamma(\mu+\nu)} F\left(\begin{array}{c}\alpha_1,\alpha_2,\ldots,\alpha_p,\nu\\\beta_1,\ldots,\beta_q,\mu+\nu\end{array}\middle| z\right). \quad \text{(C148)}$$

If a hypergeometric function has an upper and a lower parameter that are equal, both parameters can be deleted:

$$F\left(\begin{array}{c}\alpha_1,\alpha_2,\ldots,\alpha_p,\nu\\\beta_1,\ldots,\beta_q,\nu\end{array}\middle| z\right) = F\left(\begin{array}{c}\alpha_1,\alpha_2,\ldots,\alpha_p\\\beta_1,\ldots,\beta_q\end{array}\middle| z\right). \quad \text{(C149)}$$

The simplest hypergeometric function is the one of type $_1F_0$. It can be expressed in an analytic form:

$$_1F_0(\alpha;;z) = (1-z)^{-\alpha}. \quad \text{(C150)}$$

The next simplest hypergeometric functions are those of type $_2F_1$. They satisfy transformation formulas that allow an $_2F_1$ with argument z to be expressed in terms of an $_2F_1$ with argument $z/(z-1)$ or as a sum of two $_2F_1$'s with arguments $1-z$ or $1/z$ or $1/(1-z)$. The hypergeometric functions of type $_2F_1$ with argument $z=1$ can be evaluated analytically in terms of gamma functions:

$$F\left(\begin{array}{c}\alpha_1,\alpha_2\\\beta_1\end{array}\middle| 1\right) = \frac{\Gamma(\beta_1)\Gamma(\beta_1-\alpha_1-\alpha_2)}{\Gamma(\beta_1-\alpha_1)\Gamma(\beta_1-\alpha_2)}. \quad \text{(C151)}$$

The hypergeometric function of type $_3F_2$ with argument $z=1$ can be expressed as a $_3F_2$ with argument $z=1$ and different parameters [39]:

$$F\left(\begin{array}{c}\alpha_1,\alpha_2,\alpha_3\\\beta_1,\beta_2\end{array}\middle| 1\right)$$

$$= \frac{\Gamma(\beta_1)\Gamma(\beta_2)\Gamma(s)}{\Gamma(\alpha_1+s)\Gamma(\alpha_2+s)\Gamma(\alpha_3)} F\left(\begin{array}{c}\beta_1-\alpha_3,\beta_2-\alpha_3,s\\\alpha_1+s,\alpha_2+s\end{array}\middle| 1\right), \quad \text{(C152)}$$

where $s = \beta_1+\beta_2-\alpha_1-\alpha_2-\alpha_3$. If all the parameters of a $_3F_2$ are integers and half-odd integers, this identity can be used to obtain equal numbers of half-odd integers among the upper and lower parameters. If the parameters of a $_3F_2$ reduce to integers and half-odd integers in the limit $\epsilon\to0$, the use of this identity simplifies the expansion of the hypergeometric functions in powers of ϵ.

The most important integration formulas involving $_2F_1$ hypergeometric functions is Eq. (C148) with $p=2$ and $q=1$. Another useful integration formula is

$$\int_0^1 dt\, t^{\nu-1}(1-t)^{\mu-1} F\left(\begin{array}{c}\alpha_1,\alpha_2\\\beta_1\end{array}\middle| \frac{t}{1-t}z\right)$$

$$= \frac{\Gamma(\mu)\Gamma(\nu)}{\Gamma(\mu+\nu)} F\left(\begin{array}{c}\alpha_1,\alpha_2,\nu\\\beta_1,1-\mu\end{array}\middle| -z\right)$$

$$+ \frac{\Gamma(\alpha_1+\mu)\Gamma(\alpha_2+\mu)\Gamma(\beta_1)\Gamma(-\mu)}{\Gamma(\alpha_1)\Gamma(\alpha_2)\Gamma(\beta_1+\mu)}(-z)^{\mu}$$

$$\times F\left(\begin{array}{c}\alpha_1+\mu,\alpha_2+\mu,\nu+\mu\\\beta_1+\mu,1+\mu\end{array}\middle| -z\right). \quad \text{(C153)}$$

This is derived by first inserting the integral representation for $_2F_1$ in Eq. (C148) with integration variable t' and then evaluating the integral over t to get a $_2F_1$ with argument $1+t'z$. After using a transformation formula to change the argument to $-t'z$, the remaining integrals over t' are evaluated using Eq. (C148) to get $_3F_2$'s with arguments $-z$.

For the calculation of two-loop thermal integrals involving HTL averages, we require the expansion in powers of ϵ for hypergeometric functions of type $_pF_{p-1}$ with argument 1 and parameters that are linear in ϵ. If the power series representation (C145) of the hypergeometric function is convergent at $z=1$ for $\epsilon=0$, this can be accomplished simply by expanding the summand in powers of ϵ and then evaluating the sums. If the power series is divergent, we must make subtractions on the sum before expanding in powers of ϵ. The convergence properties of the power series at $z=1$ is determined by the variable s defined in Eq. (C147). If $s>0$, the power series converges. If $s\to0$ in the limit $\epsilon\to0$, only one subtraction is necessary to make the sum convergent:

$$F\left(\begin{array}{c}\alpha_1,\alpha_2,\ldots,\alpha_p\\\beta_1,\ldots,\beta_{p-1}\end{array}\middle| 1\right) = \frac{\Gamma(\beta_1)\cdots\Gamma(\beta_{p-1})}{\Gamma(\alpha_1)\Gamma(\alpha_2)\cdots\Gamma(\alpha_p)}\zeta(s+1)$$

$$+ \sum_{n=0}^{\infty}\left(\frac{(\alpha_1)_n(\alpha_2)_n\cdots(\alpha_p)_n}{(\beta_1)_n\cdots(\beta_q)_n n!}\right.$$

$$- \frac{\Gamma(\beta_1)\cdots\Gamma(\beta_{p-1})}{\Gamma(\alpha_1)\Gamma(\alpha_2)\cdots\Gamma(\alpha_p)}(n$$

$$\left. +1)^{-s-1}\right). \quad \text{(C154)}$$

If $s\to-1$ in the limit $\epsilon\to0$, two subtractions are necessary to make the sum convergent:

$$F\left(\begin{array}{c}\alpha_1,\alpha_2,\ldots,\alpha_p\\\beta_1,\ldots,\beta_{p-1}\end{array}\middle| 1\right)$$

$$= \frac{\Gamma(\beta_1)\cdots\Gamma(\beta_{p-1})}{\Gamma(\alpha_1)\Gamma(\alpha_2)\cdots\Gamma(\alpha_p)}[\zeta(s+1)+t\zeta(s+2)]$$

$$+ \sum_{n=0}^{\infty}\left(\frac{(\alpha_1)_n(\alpha_2)_n\cdots(\alpha_p)_n}{(\beta_1)_n\cdots(\beta_q)_n n!}\right.$$

$$- \frac{\Gamma(\beta_1)\cdots\Gamma(\beta_{p-1})}{\Gamma(\alpha_1)\Gamma(\alpha_2)\cdots\Gamma(\alpha_p)}$$

$$\left.\times[(n+1)^{-s-1}+t(n+1)^{-s-2}]\right), \quad \text{(C155)}$$

where t is given by

$$t = \sum_{i=1}^{p} \frac{(\alpha_i - 1)(\alpha_i - 2)}{2} - \sum_{i=1}^{p-1} \frac{(\beta_i - 1)(\beta_i - 2)}{2}. \tag{C156}$$

The expansion of a $_pF_{p-1}$ hypergeometric function in powers of ϵ is particularly simple if in the limit $\epsilon \to 0$ all its parameters are integers or half-odd integers, with equal numbers of half-odd integers among the upper and lower parameters. If the power series representation for such a hypergeometric function is expanded in powers of ϵ, the terms in the summand will be rational functions of n, possibly multiplied by factors of the polylogarithm function $\psi(n+a)$ or its derivatives. The terms in the sums can often be simplified by using the obvious identity

$$\sum_{n=0}^{\infty} [f(n) - f(n+k)] = \sum_{i=0}^{k-1} f(i). \tag{C157}$$

The sums over n of rational functions of n can be evaluated by applying the partial fraction decomposition and then using identities such as

$$\sum_{n=0}^{\infty} \left(\frac{1}{n+a} - \frac{1}{n+b} \right) = \psi(b) - \psi(a), \tag{C158}$$

$$\sum_{n=0}^{\infty} \frac{1}{(n+a)^2} = \psi'(a). \tag{C159}$$

The sums of polygamma functions of $n+1$ or $n+\frac{1}{2}$ divided by $n+1$ or $n+\frac{1}{2}$ can be evaluated using

$$\sum_{n=0}^{\infty} \left(\frac{\psi(n+1)}{n+1} - \frac{\log(n+1)}{n+1} \right) = -\frac{1}{2}\gamma^2 - \frac{\pi^2}{12} - \gamma_1, \tag{C160}$$

$$\sum_{n=0}^{\infty} \left(\frac{\psi(n+1)}{n+\frac{1}{2}} - \frac{\log(n+1)}{n+1} \right) = -\frac{1}{2}(\gamma + 2\log 2)^2 + \frac{\pi^2}{12} - \gamma_1, \tag{C161}$$

$$\sum_{n=0}^{\infty} \left(\frac{\psi(n+\frac{1}{2})}{n+1} - \frac{\log(n+1)}{n+1} \right) = -\frac{1}{2}\gamma^2 - 4\log 2 + 2\log^2 2 - \frac{\pi^2}{12} - \gamma_1, \tag{C162}$$

$$\sum_{n=0}^{\infty} \left(\frac{\psi(n+\frac{1}{2})}{n+\frac{1}{2}} - \frac{\log(n+1)}{n+1} \right) = -\frac{1}{2}(\gamma + 2\log 2)^2 - \frac{\pi^2}{4} - \gamma_1, \tag{C163}$$

where γ_1 is Stieltje's first gamma constant defined in Eq. (B17). The sums of polygamma functions of $n+1$ or $n+\frac{1}{2}$ can be evaluated using

$$\sum_{n=0}^{\infty} \left(\psi(n+1) - \log(n+1) + \frac{1}{2(n+1)} \right) = \frac{1}{2} + \frac{1}{2}\gamma - \frac{1}{2}\log(2\pi), \tag{C164}$$

$$\sum_{n=0}^{\infty} \left(\psi(n+\tfrac{1}{2}) - \log(n+1) + \frac{1}{n+1} \right) = \frac{1}{2}\gamma - \log 2 - \frac{1}{2}\log(2\pi). \tag{C165}$$

We also need the expansions in ϵ of some integrals of $_2F_1$ hypergeometric functions of y that have a factor of $|1-2y|$. For example, the following 2 integrals are needed to obtain Eq. (C100):

$$\int_0^1 dy\, y^{-2\epsilon}(1-y)^{1+\epsilon}|1-2y|F\left(\begin{matrix} 1-\epsilon, \epsilon \\ -3\epsilon \end{matrix} \Big| y \right) = \frac{1}{6} + \left(\frac{2}{9} + \frac{4}{9}\log 2 \right)\epsilon, \tag{C166}$$

$$\int_0^1 dy\, y^{1+\epsilon}(1-y)^{1+\epsilon}|1-2y|F\left(\begin{matrix} 2+2\epsilon, 1+\epsilon \\ 2+3\epsilon \end{matrix} \Big| y \right) = \frac{1}{4} + \left(\frac{7}{12} + \frac{2}{3}\log 2 \right)\epsilon. \tag{C167}$$

These integrals can be evaluated by expressing them in the form

$$\int_0^1 dy\, y^{\nu-1}(1-y)^{\mu-1}|1-2y|F\left(\begin{matrix}\alpha_1,\alpha_2\\\beta_1\end{matrix}\bigg|y\right) = \int_0^1 dy\, y^{\nu-1}(1-y)^{\mu-1}(2y-1)F\left(\begin{matrix}\alpha_1,\alpha_2\\\beta_1\end{matrix}\bigg|y\right)$$

$$+ 2\int_0^{1/2} dy\, y^{\nu-1}(1-y)^{\mu-1}(1-2y)F\left(\begin{matrix}\alpha_1,\alpha_2\\\beta_1\end{matrix}\bigg|y\right). \tag{C168}$$

The evaluation of the first integral on the right side gives $_3F_2$ hypergeometric functions with argument 1. The integrals from 0 to $\frac{1}{2}$ can be evaluated by expanding the power series representation (C145) of the hypergeometric function in powers of ϵ. The resulting series can be summed analytically and then the integral over y can be evaluated.

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