Deriving formulations for numerical computation of binary neutron stars in quasicircular orbits

Masaru Shibata
Graduate School of Arts and Sciences, University of Tokyo, Komaba, Meguro, Tokyo 153-8902, Japan

Kōji Uryū
SISSA, Via Beirut 2/4, 34014 Trieste, Italy

John L. Friedman
Department of Physics, University of Wisconsin-Milwaukee, P.O. Box 413, Milwaukee, Wisconsin 53201, USA
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Two relations, the virial relation $M_{\rm ADM}=M_{\rm K}$ and the first law in the form $\delta M_{\rm ADM}=\Omega\,\delta J$, should be satisfied by a solution and a sequence of solutions describing binary compact objects in quasiequilibrium circular orbits. Here, $M_{\rm ADM}$, $M_{\rm K}$, J, and Ω are the Arnowitt-Deser-Misner (ADM) mass, Komar mass, angular momentum, and orbital angular velocity, respectively. δ denotes an Eulerian variation. These two conditions restrict the allowed formulations that we may adopt. First, we derive relations between $M_{\rm ADM}$ and $M_{\rm K}$ and between $\delta M_{\rm ADM}$ and $\Omega\,\delta J$ for general asymptotically flat spacetimes. Then, to obtain solutions that satisfy the virial relation and sequences of solutions that satisfy the first law at least approximately, we propose a formulation for computation of quasiequilibrium binary neutron stars in general relativity. In contrast to previous approaches in which a part of the Einstein equation is solved, in the new formulation, the full Einstein equation is solved with maximal slicing and in a transverse gauge for the conformal three-metric. Helical symmetry is imposed in the near zone, while in the distant zone, a waveless condition is assumed. We expect the solutions obtained in this formulation to be excellent quasiequilibria as well as initial data for numerical simulations of binary neutron star mergers.

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I. INTRODUCTION

A detailed theoretical understanding of evolution of close binary neutron stars is one of the most important goals in general relativity, since they are promising sources of gravitational waves for laser interferometric gravitational wave detectors such as LIGO, TAMA, GEO, and VIRGO [1,2]. For the inspiral phase in which the orbital separation a is much larger than the radius R of a neutron star, the orbital velocity is much smaller than the speed of light, and finite-size effects of neutron stars may be ignored. Thus, a post-Newtonian study together with the point particle approximation is appropriate [3]. For $a/R \lesssim 4$, however, the post-Newtonian and point particle approximations break down and numerical study is required to take into account the effect of tidal deformation of each star and full effects of general relativity. The procedure to be adopted for such close orbits up to the merger is (i) to compute a quasiequilibrium circular orbit at a distant orbit with $a \lesssim 4R$ and $a \gtrsim 10M$ for which the ratio of a radial approaching velocity to the orbital one will be small (less than 1%) [4], and then (ii) to perform a numerical relativity simulation adopting a distant quasiequilibrium with $a \gtrsim 10M$ as the initial condition [5,6]. In this paper, we focus on the formulation for computation of the quasiequilibrium in circular orbits.

So far, the quasiequilibrium states of binary neutron stars have been widely computed in the so-called conformal flatness approximation (or Isenberg-Wilson-Mathews formalism) [7–11], in which the conformal three-metric is assumed to be flat. The solution in this formulation satisfies the constraint equations of general relativity and, hence, it is fully general relativistic for the initial value problem. However, it is only an approximate *quasiequilibrium* solution for compact binaries, since conformally nonflat parts of the three-metric are not vanishing for the quasiequilibrium binaries. Thus, such approximation produces a systematic error of magnitude $\sim(M/a)^2$ for the solution of quasiequilibrium configurations [4]. The systematic error is also included in gravitational waves computed from the conformal flat data of the quasiequilibrium binary [4,12] and in the numerical results of fully general relativistic simulations started from initial conditions of the conformally flat quasiequilibria [5].

Formulations for computation of binary compact objects in quasiequilibrium circular orbits with a conformally nonflat three-metric have been proposed by several authors (e.g., Refs. [13–18,4]). A promising approach to this problem is to assume a helical Killing symmetry for the spacetime [17,18]. In this case, however, the solution contains standing gravitational waves in the whole spacetime and the averaged energy density of gravitational waves falls off as r^{-2} where r denotes a radial coordinate, resulting in an asymptotically nonflat spacetime. Thus, the solution obtained in such a formulation is not physical in the distant wave zone, although the solution in the near zone and in a local wave zone would describe a realistic spacetime of binary compact objects.

In this paper, we consider general relativistic formulations for computation of the quasiequilibrium circular orbits assuming that the spacetime is asymptotically flat. First, we require that the following two conditions should be satisfied for a solution and a sequence of the solutions of quasiequilibrium states:

(1) A quasiequilibrium solution that is stationary in the corotating frame should satisfy a virial relation associated with the equality

$$M_{\mathrm{ADM}}=M_{\mathrm{K}}, \tag{1.1}$$

of the Arnowitt-Deser-Misner (ADM) and Komar masses defined in Sec. III.

(2) Binary compact objects inspiral adiabatically as a result of gravitational wave emission, conserving baryon mass, entropy, and vorticity. Thus, along a sequence of quasiequilibrium solutions, the first law should be satisfied. Here, the first law is written in the form

$$\delta M_{\mathrm{ADM}}=\Omega\,\delta J, \tag{1.2}$$

where δM_{ADM} and δJ are infinitesimal differences of the ADM mass and angular momentum along a quasiequilibrium sequence, and Ω is an orbital angular velocity.

These two conditions are likely to be satisfied for binary neutron stars in nature. Thus, we should adopt a formulation that provides a solution and a sequence of the solutions that satisfy two conditions at least approximately.

Based on this motivation, in this paper, we first derive relations for the differences, $M_{\mathrm{ADM}}-M_{\mathrm{K}}$ and $\delta M_{\mathrm{ADM}}-\Omega\,\delta J$, in arbitrary asymptotic flat spacetimes. The condition that the differences vanish can be used to restrict formulations that we can adopt. Using these conditions, several possible candidates for the formulations emerge. Among them, we propose a formulation in which helical symmetry is imposed only in the near zone instead of in the whole spacetime. Specifically, we impose a mixed condition; a helical symmetry condition in the near zone and a waveless condition in the distant zone. To fix the gauge, we adopt the maximal slicing condition and a transverse gauge condition for the upper component of the conformal three-metric. In this case, all components of the Einstein equation reduce to elliptic equations as in the post-Newtonian approximation. This implies that no standing waves appear in the wave zone, although in the near zone, gravitational-wave-like components are present. We expect the solutions obtained in this formulation to be excellent quasiequilibria as well as initial data for numerical simulations of binary neutron star mergers.

The paper is organized as follows. In Sec. II, we describe the basic equations for quasiequilibria. In Sec. III, we derive a relation between M_{ADM} and M_{K} for arbitrary formulation and clarify the condition for the formulation that its solution satisfies the virial relation $M_{\mathrm{ADM}}=M_{\mathrm{K}}$. In Sec. IV, we derive a relation for the difference, $\delta M_{\mathrm{ADM}}-\Omega\,\delta J$, and clarify the conditions on the formulation for which a sequence of solutions satisfies the first law. In Sec. V, we propose formulations whose solutions and sequences of solutions approximately satisfy the virial relation and the first law. Section VI is devoted to a summary.

Throughout this paper, we use geometrical units with $G=1=c$. Spacetime indices are Greek, spatial indices Latin, and the metric signature is $-+++$. Readers familiar with abstract indices can regard indices early in the alphabet as abstract, while $i,j,k,\ldots$ are concrete, associated with a chart $\{x^i\}$. If S is a 2-surface in a 3-space Σ and ϵ_{abc} is the volume form on Σ associated with a 3-metric γ_{ab}, we write $dS_a=\epsilon_{abc}dS^{bc}$; for S a surface of constant r, $dS_a=\nabla_a r\sqrt{\gamma}d^2x$.

II. FORMULATION

A. 3+1 formalism

Let Σ_t be a family of spacelike hypersurfaces, labeled by a time function t. Let t^α be a vector field transverse to Σ_t for which $t^\alpha\nabla_\alpha t=1$, and denote by α and β^α a nonvanishing lapse and a shift vector, respectively, with

$$t^\alpha=\alpha n^\alpha+\beta^\alpha, \quad \beta^\alpha n_\alpha=0. \tag{2.1}$$

Then, in a chart $\{t,x^i\}$, we have $t^\alpha=\partial_t$, and the metric $g_{\alpha\beta}=\gamma_{\alpha\beta}-n_\alpha n_\beta$ has the form

$$ds^2=-\alpha^2dt^2+\gamma_{ij}(dx^i+\beta^i dt)(dx^j+\beta^j dt). \tag{2.2}$$

With a spatial covariant derivative D_a compatible with the spatial metric γ_{ab}, the extrinsic curvature of Σ_t is given by

$$K_{ab}=-\frac{1}{2}\pounds_n\gamma_{ab}=\frac{1}{2\alpha}(-\partial_t\gamma_{ab}+D_a\beta_b+D_b\beta_a), \tag{2.3}$$

where γ_{ab} and $\partial_t\gamma_{ab}$ are the pullbacks to Σ_t of $\gamma_{\alpha\beta}$ and $\pounds_t\gamma_{\alpha\beta}$.

In the canonical formulation of general relativity [19], γ_{ab}, π^{ab}, α and β^a are regarded as independent gravitational field variables, where π^{ab} is defined by

$$\pi^{ab}:=-(K^{ab}-\gamma^{ab}K)\sqrt{\gamma}. \tag{2.4}$$

A perfect fluid is described by a stress-energy tensor

$$T^{\alpha\beta}=(\epsilon+p)u^\alpha u^\beta+pg^{\alpha\beta}, \tag{2.5}$$

where u^α, p, and ϵ are the fluid four-velocity, pressure, and energy density, respectively. The pressure and the energy density are assumed to satisfy an equation of state of the form

$$p=p(\rho,s), \quad \epsilon=\epsilon(\rho,s), \tag{2.6}$$

where ρ is the baryon mass density and s the entropy per unit baryon mass.

In calculating the variation of the Lagrangian following Routh procedure, a perfect-fluid spacetime is specified by the canonical variables, the lapse and the shift, that together describe the metric, and by Lagrangian variables for the fluid,

$$Q(\lambda):=[\gamma_{ab}(\lambda),\pi^{ab}(\lambda),\alpha(\lambda),\beta^a(\lambda),u^\alpha(\lambda),\rho(\lambda),s(\lambda)].$$

The difference between two nearby solutions can be treated in either of two ways. Changes in the metric variables will be written as Eulerian variations, denoted by δ; the Eulerian change is the difference between corresponding quantities in the two solutions at a fixed point in spacetime. Changes in fluid variables will be written as Lagrangian variations. Introducing a Lagrangian displacement vector field ξ^α, one defines the Lagrangian change in any fluid variable as the change with respect to a frame dragged by ξ^α. Formally the Lagrangian change ΔQ in a quantity Q is then related to the Eulerian change δQ by

$$\Delta Q = \delta Q + \pounds_\xi Q. \tag{2.7}$$

The description of fluid perturbation in terms of a Lagrangian displacement ξ^α has a gauge freedom associated with a class of trivial displacements that yield no Eulerian change in the fluid variables. We use this freedom to choose a gauge in which $\xi^t := \xi^\alpha \nabla_\alpha t = 0$, following Refs. [20,21].

The Einstein-Hilbert action

$$S = \int \mathcal{L} d^4x, \tag{2.8}$$

with the Lagrangian density

$$\mathcal{L} = \left(\frac{1}{16\pi} {}^4R - \epsilon \right) \sqrt{-g}, \tag{2.9}$$

takes, in terms of Hamiltonian metric variables, the form

$$16\pi\mathcal{L} = \pi^{ab}\partial_t \gamma_{ab} - \alpha \mathcal{H}_G - \beta_a \mathcal{C}^a_G + D_a(-2D^a\alpha\sqrt{\gamma} - 2\beta^b \pi^a{}_b + \beta^a \pi) - \partial_t \pi - 16\pi\epsilon\sqrt{-g}, \tag{2.10}$$

where ${}^4R$ is the Ricci scalar,

$$\mathcal{H}_G := -2G^{\alpha\beta} n_\alpha n_\beta \sqrt{\gamma}, \quad \mathcal{C}^a_G := -2G^{\alpha\beta}\gamma_\alpha{}^a n_\beta \sqrt{\gamma}, \tag{2.11}$$

and $G^{\alpha\beta}$ is the Einstein tensor.

The variation in the Lagrangian density is given by

$$\begin{aligned}\delta\mathcal{L} = & -\rho T\sqrt{-g}\Delta s - \frac{h}{u^t}\Delta(\rho u^t \sqrt{-g}) + \frac{1}{16\pi}\Bigg[-\delta\alpha\mathcal{H} \\ & -\delta\beta^a \mathcal{C}_a + \delta\pi^{ab}\left\{ \partial_t \gamma_{ab} - D_a\beta_b - D_b\beta_a - \frac{2\alpha}{\sqrt{\gamma}}\left(\pi_{ab} - \frac{1}{2}\gamma_{ab}\pi \right) \right\} - \delta\gamma_{ab}(G^{ab} - 8\pi S^{ab})\alpha\sqrt{\gamma} \Bigg] \\ & - \xi_\alpha \nabla_\beta T^{\alpha\beta}\sqrt{-g} + D_a \tilde{\Theta}^a \sqrt{\gamma} - \frac{1}{16\pi}\partial_t(\delta\pi^{ab}\gamma_{ab}) \\ & + \partial_t(j_a \xi^a \sqrt{\gamma}),\end{aligned} \tag{2.12}$$

where T is the temperature, h is the enthalpy defined by $h := (\epsilon + p)/\rho$, and

$$G^{ab} := G^{\alpha\beta}\gamma_\alpha{}^a \gamma_\beta{}^b \quad \text{and} \quad S^{ab} := T^{\alpha\beta}\gamma_\alpha{}^a\gamma_\beta{}^b. \tag{2.13}$$

With definitions

$$\rho_H := T^{\alpha\beta} n_\alpha n_\beta \quad \text{and} \quad j^a := -T^{\alpha\beta}\gamma_\alpha{}^a n_\beta, \tag{2.14}$$

we set

$$\begin{aligned}\mathcal{H} &:= -2(G^{\alpha\beta} - 8\pi T^{\alpha\beta}) n_\alpha n_\beta \sqrt{\gamma} = \mathcal{H}_G + 16\pi\rho_H \\ &= -\left[R - \frac{1}{\gamma}\left(\pi_{ab}\pi^{ab} - \frac{1}{2}\pi^2 \right) - 16\pi\rho_H \right]\sqrt{\gamma},\end{aligned} \tag{2.15}$$

$$\begin{aligned}\mathcal{C}^a &:= -2(G^{\alpha\beta} - 8\pi T^{\alpha\beta})\gamma_\alpha{}^a n_\beta\sqrt{\gamma} = \mathcal{C}^a_G - 16\pi j^a\sqrt{\gamma} \\ &= -2(D_b\pi^{ab} + 8\pi j^a\sqrt{\gamma}),\end{aligned} \tag{2.16}$$

where R is the Ricci scalar with respect to γ_{ab}. The density $\tilde{\Theta}^a$ is the surface term of the Lagrangian density,

$$\begin{aligned}\tilde{\Theta}^a = \frac{1}{16\pi}\Bigg[& \frac{1}{\sqrt{\gamma}}\{ -2\delta(D^a\alpha\sqrt{\gamma}) + (\beta^a\gamma_{bc}\delta\pi^{bc} + \pi\delta\beta^a \\ & - 2\pi^a{}_b\delta\beta^b)\} + (\gamma^{ac}\gamma^{bd} - \gamma^{ab}\gamma^{cd})(\alpha D_b\delta\gamma_{cd} \\ & - D_b\alpha\,\delta\gamma_{cd}) \Bigg] + \alpha(\epsilon + p) q^a{}_b \xi^b - \beta^a j^b \xi_b,\end{aligned} \tag{2.17}$$

where $q^{ab} := (g^{\alpha\beta} + u^\alpha u^\beta)\gamma_\alpha{}^a\gamma_\beta{}^b$.

Independently varying the metric variables, $\{\delta\alpha, \delta\beta^a, \delta\gamma_{ab}, \delta\pi^{ab}\}$, gives the field equations,

$$\mathcal{H} = 0, \quad \mathcal{C}_a = 0, \quad \text{and} \quad G^{ab} - 8\pi S^{ab} = 0, \tag{2.18}$$

and the relation,

$$\partial_t \gamma_{ab} - D_a\beta_b - D_b\beta_a - \frac{2\alpha}{\sqrt{\gamma}}\left(\pi_{ab} - \frac{1}{2}\gamma_{ab}\pi \right) = 0. \tag{2.19}$$

Equation (2.19) is consistent with the definition of π^{ab} [cf. Eq. (2.4)].

When the field equations are satisfied, the Bianchi identity implies $\nabla_\beta T^{\alpha\beta} = 0$. The variation of the action with respect to the (spatial) Lagrangian displacement vector is the spatial projection of this relation, the relativistic Euler equation,

$$\gamma^a{}_\alpha \nabla_\beta T^{\alpha\beta} = 0. \tag{2.20}$$

For an isentropic fluid, conservation of baryon mass and entropy are given by

$$\pounds_u(\rho\sqrt{-g}) = 0 \quad \text{and} \quad \pounds_u s = 0. \tag{2.21}$$

Equations (2.21) and (2.20) together imply $\nabla_\beta T^{\alpha\beta} = 0$.

It is often convenient to rewrite the above set of basic equations in terms of the conformally related spatial metric $\tilde{\gamma}_{ab}$ and the trace-free part of the extrinsic curvature $\tilde{A}_{ab}$, defined by

$$\tilde{\gamma}_{ab} := \psi^{-4}\gamma_{ab}, \tag{2.22}$$

$$\tilde{A}_{ab} := \psi^{-4}\left(K_{ab} - \frac{1}{3}\gamma_{ab}K \right), \tag{2.23}$$

where ψ is a conformal factor and $K := K_{ab}\gamma^{ab}$. Here, we may impose the condition, $\tilde{\gamma} := \det(\tilde{\gamma}_{ab}) = \det(\eta_{ab}) =: \eta$, where η_{ab} is a flat 3-metric. In the following, indices of variables with a tilde, such as $\tilde{A}_{ab}$, $\tilde{A}^{ab}$, $\tilde{\beta}_a$, and $\tilde{\beta}^a$ $(=\beta^a)$, are raised and lowered by $\tilde{\gamma}_{ab}$ and $\tilde{\gamma}^{ab}$, respectively.

The Hamiltonian constraint $\mathcal{H}=0$ and the momentum constraint $\mathcal{C}_a=0$ are written in terms of K_{ab}:

$$R-K_{ab}K^{ab}+K^2=16\pi\rho_H, \tag{2.24}$$

$$D_bK^b{}_a-D_aK=8\pi j_a. \tag{2.25}$$

With the conformal transformation (2.22) and (2.23), Eqs. (2.24) and (2.25) are rewritten in the form

$$\tilde{\Delta}\psi=\frac{\psi}{8}\tilde{R}-2\pi\rho_H\psi^5-\frac{\psi^5}{8}\left(\tilde{A}_{ab}\tilde{A}^{ab}-\frac{2}{3}K^2\right), \tag{2.26}$$

$$\tilde{D}_b(\psi^6\tilde{A}^b{}_a)-\frac{2}{3}\psi^6\tilde{D}_aK=8\pi j_a\psi^6. \tag{2.27}$$

Here, $\tilde{R}$, $\tilde{D}_a$, and $\tilde{\Delta}$ are the Ricci scalar, the covariant derivative, and the Laplacian with respect to $\tilde{\gamma}_{ab}$, respectively.

The evolution equations for γ_{ab} and K_{ab} are

$$\partial_t\gamma_{ab}=-2\alpha K_{ab}+D_a\beta_b+D_b\beta_a, \tag{2.28}$$

$$\begin{aligned}\partial_tK_{ab}=&\alpha R_{ab}-D_aD_b\alpha+\alpha(KK_{ab}-2K_{ac}K_b{}^c)\\&+(D_b\beta^c)K_{ca}+(D_a\beta^c)K_{cb}+(D_cK_{ab})\beta^c\\&-8\pi\alpha\left[S_{ab}+\frac{1}{2}\gamma_{ab}(\rho_H-S_c{}^c)\right],\end{aligned} \tag{2.29}$$

where R_{ab} is the Ricci tensor with respect to γ_{ab}.

Contracting γ^{ab} with Eqs. (2.28) and (2.29) and using Eq. (2.24), one obtains

$$\partial_t\psi=\frac{\psi}{6}(-\alpha K+D_c\beta^c), \tag{2.30}$$

$$\partial_tK=\alpha K_{ab}K^{ab}-\Delta\alpha+4\pi\alpha(\rho_H+S_a{}^a)+\beta^a\partial_aK, \tag{2.31}$$

where $\Delta=D_aD^a$. To write the evolution equation of K in the form of Eq. (2.31), we use the Hamiltonian constraint equation (2.24).

In the following, we choose the maximal time slicing condition $K=0=\partial_tK$. With this condition, Eq. (2.31) reduces to an elliptic equation for α,

$$\Delta\alpha=\alpha\tilde{A}_{ab}\tilde{A}^{ab}+4\pi\alpha(\rho_H+S_a{}^a), \tag{2.32}$$

where we keep spatial indices abstract until fixing the spatial gauge condition. Using Eq. (2.26), this equation is rewritten

$$\tilde{\Delta}(\alpha\psi)=2\pi\alpha\psi^5(\rho_H+2S_a{}^a)+\frac{7}{8}\alpha\psi^5\tilde{A}_{ab}\tilde{A}^{ab}+\frac{\alpha\psi}{8}\tilde{R}. \tag{2.33}$$

Using Eqs. (2.28), (2.29), (2.30), and (2.31), the evolution equations for $\tilde{\gamma}_{ab}$ and $\tilde{A}_{ab}$ are

$$\partial_t\tilde{\gamma}_{ab}=-2\alpha\tilde{A}_{ab}+\tilde{D}_a\tilde{\beta}_b+\tilde{D}_b\tilde{\beta}_a-\frac{2}{3}\tilde{\gamma}_{ab}\tilde{D}_c\tilde{\beta}^c, \tag{2.34}$$

$$\begin{aligned}\partial_t\tilde{A}_{ab}=&\psi^{-4}\left[\alpha\left(R_{ab}-\frac{\psi^4}{3}\tilde{\gamma}_{ab}R\right)-\left(D_aD_b\alpha-\frac{\psi^4}{3}\tilde{\gamma}_{ab}D_cD^c\alpha\right)\right]-2\alpha\tilde{A}_{ac}\tilde{A}_b{}^c+\tilde{D}_a\beta^c\tilde{A}_{cb}\\&+\tilde{D}_b\beta^c\tilde{A}_{ca}-\frac{2}{3}\tilde{D}_c\beta^c\tilde{A}_{ab}+\beta^c\tilde{D}_c\tilde{A}_{ab}\\&-8\pi\alpha\left(\psi^{-4}S_{ab}-\frac{1}{3}\tilde{\gamma}_{ab}S_c{}^c\right).\end{aligned} \tag{2.35}$$

Now, R_{ab} is split,

$$R_{ab}=\tilde{R}_{ab}+R^{\psi}_{ab}, \tag{2.36}$$

where $\tilde{R}_{ab}$ is the Ricci tensor with respect to $\tilde{\gamma}_{ab}$ and

$$\begin{aligned}R^{\psi}_{ab}=&-\frac{2}{\psi}\tilde{D}_a\tilde{D}_b\psi-\frac{2}{\psi}\tilde{\gamma}_{ab}\tilde{\Delta}\psi+\frac{6}{\psi^2}\tilde{D}_a\psi\tilde{D}_b\psi\\&-\frac{2}{\psi^2}\tilde{\gamma}_{ab}\tilde{D}_c\psi\tilde{D}^c\psi.\end{aligned} \tag{2.37}$$

$\tilde{R}_{ab}$ is then written in the form

$$\begin{aligned}\tilde{R}_{ab}=&\frac{1}{2}[-\tilde{\gamma}^{cd}\overset{(0)}{D}_c\overset{(0)}{D}_d\tilde{\gamma}_{ab}-\overset{(0)}{D}_b(\tilde{\gamma}_{ac}F^c)-\overset{(0)}{D}_a(\tilde{\gamma}_{bc}F^c)\\&-(\overset{(0)}{D}_c\tilde{\gamma}_{bd})\overset{(0)}{D}_a\tilde{\gamma}^{cd}-(\overset{(0)}{D}_c\tilde{\gamma}_{ad})\overset{(0)}{D}_b\tilde{\gamma}^{cd}+2F^cC_{c,ab}\\&-2C^d_{cb}C^c_{ad}]\\=&\frac{1}{2}[-\overset{(0)}{\Delta}\tilde{h}_{ab}-\overset{(0)}{D}_b(\tilde{\gamma}_{ac}F^c)-\overset{(0)}{D}_a(\tilde{\gamma}_{bc}F^c)]+\tilde{R}^{\mathrm{NL}}_{ab},\end{aligned} \tag{2.38}$$

where $F^a:=\overset{(0)}{D}_b\tilde{\gamma}^{ab}$, $\overset{(0)}{D}_a$ is the covariant derivative associated with η_{ab}, and $\overset{(0)}{\Delta}=\eta^{cd}\overset{(0)}{D}_c\overset{(0)}{D}_d$. $\tilde{R}^{\mathrm{NL}}_{ab}$ is the collection of the nonlinear terms in $\tilde{h}_{ab}$ and defined by

$$\begin{aligned}\tilde{R}^{\mathrm{NL}}_{ab}:=&-\frac{1}{2}[\tilde{f}^{cd}\overset{(0)}{D}_c\overset{(0)}{D}_d\tilde{h}_{ab}+(\overset{(0)}{D}_c\tilde{h}_{bd})\overset{(0)}{D}_a\tilde{f}^{cd}\\&+(\overset{(0)}{D}_c\tilde{h}_{ad})\overset{(0)}{D}_b\tilde{f}^{cd}]+F^cC_{c,ab}-C^d_{cb}C^c_{ad}.\end{aligned} \tag{2.39}$$

Here, $\tilde{h}_{ab}$ and $\tilde{f}^{ab}$ are introduced, respectively, by

$$\tilde{h}_{ab}:=\tilde{\gamma}_{ab}-\eta_{ab}\quad\text{and}\quad\tilde{f}^{ab}:=\tilde{\gamma}^{ab}-\eta^{ab}. \tag{2.40}$$

C^c_{ab} and $C_{c,ab}$ are defined by

$$C^c_{ab} := \frac{\tilde{\gamma}^{cd}}{2}(\overset{(0)}{D}_a \tilde{h}_{bd} + \overset{(0)}{D}_b \tilde{h}_{ad} - \overset{(0)}{D}_d \tilde{h}_{ab})$$

and

$$C_{d,ab} := \tilde{\gamma}_{cd} C^c_{ab}, \tag{2.41}$$

and $C^c_{cd} = \overset{(0)}{D}_d(\sqrt{\tilde{\gamma}/\eta})/\sqrt{\tilde{\gamma}/\eta} = 0$, when $\tilde{\gamma} = \eta$.

B. Basic equations for quasiequilibria

Compact binary systems in quasiequilibrium circular orbits evolve toward merger due to gravitational radiation reaction. Since the emission time scale of gravitational waves is longer than the orbital period even just before the merger, we may expect that the fluid and field variables near the support of fluid source (or inside the light cylinder) in the frame rotating with the same angular velocity as the orbital motion are approximately unchanged along a direction of a helical vector

$$k^\alpha = t^\alpha + \Omega \phi^\alpha, \tag{2.42}$$

where ϕ^α is a spatial vector field that generates a family of closed circular curves on Σ_t, and Ω denotes the orbital angular velocity.

First, we derive hydrodynamic equations to describe binary neutron stars in quasiequilibrium circular orbits. The baryon mass conservation law Eq. (2.21) and the Euler equation (2.20) are written

$$£_{k+v}(\rho u^t \sqrt{-g}) = 0, \tag{2.43}$$

$$\gamma_a{}^\alpha £_{k+v}(h u_\alpha) + D_a\left(\frac{h}{u^t}\right) = 0. \tag{2.44}$$

Then, we impose the conditions that the Lie derivatives along k^α vanish:

$$£_k(\rho u^t \sqrt{-g}) = 0, \tag{2.45}$$

$$\gamma_a{}^\alpha £_k(h u_\alpha) = 0. \tag{2.46}$$

Here, a spatial velocity vector v^α is introduced by

$$u^\alpha = u^t(k^\alpha + v^\alpha). \tag{2.47}$$

From conditions (2.45) and (2.46), the relation $£_k(j_a\sqrt{\gamma}) = 0$ also follows. In the above, we assumed isentropic flow, which leads to the local first law of thermodynamics,

$$\frac{1}{\rho}\nabla_\alpha p = \nabla_\alpha h. \tag{2.48}$$

Equation (2.48) also implies that a one-parameter equation of state may be chosen. We thus have four independent variables for the fluid, three for the fluid velocity, and one thermodynamic variable, governed by four equations (2.43) and (2.44).

For corotational flow $u^\alpha = u^t k^\alpha$ (that is, $v^\alpha = 0$) or irrotational flow $h u_\alpha = \nabla_\alpha \Phi$, where Φ is a velocity potential, one can obtain a first integral of the Euler equation that is useful for computing quasiequilibrium configurations. For corotational flow, the velocity field becomes trivial and the first integral is the statement that the injection energy is constant in the fluid:

$$\frac{h}{u^t} = \text{const.} \tag{2.49}$$

For irrotational flow, one thermodynamic variable and a velocity potential are governed by two equations [22],

$$D_a\left[\frac{\alpha\rho}{h}(D^a\Phi - h u^t \omega^a)\right] = 0, \tag{2.50}$$

$$£_v \Phi + \frac{h}{u^t} = \text{const}, \tag{2.51}$$

derived from Eqs. (2.43) and (2.44), respectively. Here, the Lie derivative with respect to the spatial vector v^a, $£_v$, is defined on Σ_t with a relation

$$v^a = \gamma^a_\alpha\left(\frac{u^\alpha}{u^t} - \omega^\alpha\right) = \frac{1}{h u^t} D^a \Phi - \omega^a, \tag{2.52}$$

where ω^α is a rotational shift vector defined by $\omega^\alpha = \beta^\alpha + \Omega \phi^\alpha$.

Note that the symmetry of the spacetime with respect to the helical vector k^α has not been imposed yet. Thus, the Lie derivatives of the fluid quantities along k^α may not vanish, e.g., $£_k \rho \neq 0$, $£_k u_\alpha \neq 0$, $£_k \rho_H \neq 0$, and $£_k S_{ab} \neq 0$. The values of these quantities depend on the formulation for gravitational fields that we choose below; their magnitude measures the deviation from the helical symmetry.

We turn to the formulation for the gravitational fields of binary systems in quasicircular orbits. Here, we do not assume a global helical symmetry for the whole spacetime. First, we define $u^{ab} := \partial_t \tilde{\gamma}^{ab}$ and regard it as an input quantity: It is determined when we impose a certain condition between $\tilde{\gamma}^{ab}$ on two spatial hypersurfaces of infinitesimal time difference, following the concept of a thin sandwich formalism proposed by York [23]. In this section, we continue the calculation without fixing the condition for u^{ab} except for a requirement

$$u^{ab} = \partial_t \tilde{\gamma}^{ab} = O(r^{-3}), \tag{2.53}$$

in a far zone ($r \gg 2\pi\Omega^{-1}$). This condition guarantees the asymptotic flatness of the system on a slice Σ_t, but breaks the helical symmetry for $\tilde{\gamma}^{ab}$ in the far zone. The lower component of the time derivative is defined by

$$u_{ab} := -\tilde{\gamma}_{ac}\tilde{\gamma}_{bd} u^{cd}. \tag{2.54}$$

We also define

$$v_{ab} := \partial_t \hat{A}_{ab}, \tag{2.55}$$

where $\hat{A}_{ab} := \psi^6 \tilde{A}_{ab}$. Later, we will impose a condition on v_{ab}.

Bonazzola *et al.* [15] propose a "gravito-inelastic approximation" in which they set $u^{ab}=0$, while $\partial_t \hat{A}^{ab}$ is free. On the other hand, Schäfer and Gopakumar [16] propose a minimal no-radiation approximation, in which the transverse–trace-free part of $\partial_t \pi^{ab}$ is restricted. We determine the conditions in the more rigid way: Our conditions for u^{ab} and v_{ab} are determined from the requirement that the quasiequilibrium solution and its sequence satisfy the virial relation and first law at least approximately. This subject will be discussed in Secs. III and IV.

Equation (2.34) is regarded as the equation for determining $\tilde{A}_{ab}$, namely,

$$2\alpha \tilde{A}_{ab} = \tilde{D}_a \tilde{\beta}_b + \tilde{D}_b \tilde{\beta}_a - \frac{2}{3}\tilde{\gamma}_{ab}\tilde{D}_c \tilde{\beta}^c - u_{ab}. \tag{2.56}$$

Then, substituting Eq. (2.56) into Eq. (2.27), we obtain

$$\tilde{\Delta}\tilde{\beta}_a + \frac{1}{3}\tilde{D}_a\tilde{D}_b\tilde{\beta}^b + \tilde{R}_{ab}\tilde{\beta}^b + \tilde{D}^b \ln\left(\frac{\psi^6}{\alpha}\right)\left[\tilde{D}_b\tilde{\beta}_a + \tilde{D}_a\tilde{\beta}_b - \frac{2}{3}\tilde{\gamma}_{ab}\tilde{D}_c\tilde{\beta}^c\right] - \frac{\alpha}{\psi^6}\tilde{D}_c(\alpha^{-1}\psi^6\tilde{\gamma}^{bc}u_{ab}) = 16\pi\alpha j_a. \tag{2.57}$$

This elliptic equation determines $\tilde{\beta}_a$.

Regarding v_{ab} as an input quantity, the evolution equation for $\tilde{A}_{ab}$ may be rewritten as an elliptic equation for $\tilde{h}_{ab}$,

$$\overset{(0)}{\Delta}\tilde{h}_{ab} = 2\left[R^{\psi}_{ab} - \frac{1}{2}\overset{(0)}{D}_a(F^c\tilde{\gamma}_{cb}) - \frac{1}{2}\overset{(0)}{D}_b(F^c\tilde{\gamma}_{ca}) + R^{\rm NL}_{ab} - \frac{\psi^4}{3}\tilde{\gamma}_{ab}R - \frac{1}{\alpha}\left(D_aD_b\alpha - \frac{\psi^4}{3}\tilde{\gamma}_{ab}\Delta\alpha\right)\right] - 4\psi^4\tilde{A}_{ac}\tilde{A}_b{}^c + \frac{2\psi^4}{\alpha}\left(\tilde{D}_a\beta^c\tilde{A}_{cb} + \tilde{D}_b\beta^c\tilde{A}_{ac} - \frac{2}{3}\tilde{D}_c\beta^c\tilde{A}_{ab} + \beta^c\tilde{D}_c\tilde{A}_{ab}\right) - 16\pi\left(S_{ab} - \frac{\psi^4}{3}\tilde{\gamma}_{ab}S_c{}^c\right) - \frac{2\psi^4}{\alpha}\partial_t\tilde{A}_{ab}. \tag{2.58}$$

Together with the above equations for $\tilde{h}_{ab}$ and $\tilde{\beta}_a$, the elliptic equations (2.26) and (2.33) are solved for ψ and $\chi := \alpha\psi$, respectively.

To summarize, the Einstein equations are rewritten as four elliptic equations (2.26), (2.33), (2.57), and (2.58) for ψ, χ, β^a, and $\tilde{h}_{ab}$, with $u^{ab} = \partial_t \tilde{\gamma}^{ab}$ and $v_{ab} = \partial_t \hat{A}_{ab}$, which are regarded as input quantities that satisfy a certain ansatz. The asymptotic behavior of u^{ab} and v_{ab} is chosen to preclude standing waves in the far zone, $r \gg 2\pi\Omega^{-1}$. The simultaneous equations for the metric and the fluid on a slice Σ_t is then similar to the equations for the initial value problem since the equations for the metric variables are elliptic.

To solve the equations for $\tilde{h}_{ab}$, we need to fix the spatial gauge. The simplest choice is a transverse gauge [4] (or a generalized Dirac gauge in the terminology of Ref. [15]), satisfying

$$F^a = \overset{(0)}{D}_b \tilde{\gamma}^{ab} = 0. \tag{2.59}$$

With this choice, the equations for $\tilde{h}_{ij}$ and $\tilde{R}$ are significantly simplified, which results in the operator for the linear terms of $\tilde{h}_{ij}$ in its elliptic equation becoming the flat Laplacian. Furthermore, the behavior of the source terms of the elliptic equations for ψ and χ for $r \gg 2\pi\Omega^{-1}$ have suitable asymptotic behavior, because $F^i = O(r^{-3})$ and, hence, $\tilde{R} = O(r^{-4})$. Thus, the source terms of elliptic equations for ψ and χ are $O(r^{-4})$ in the present gauge. This implies that we can numerically solve these equations without serious difficulties.

As a result of the present choice of gauge conditions, the asymptotic behavior as $r \to \infty$ of the geometric variables is [24]

$$\psi = 1 + \frac{M_{\rm ADM}}{2r} + O(r^{-2}), \tag{2.60}$$

$$\alpha = 1 - \frac{M_{\rm K}}{r} + O(r^{-2}), \tag{2.61}$$

$$\beta^k = -\frac{1}{4r^2}(6Z_{kl}\hat{r}^l + 3Z_{ij}\hat{r}^i\hat{r}^j\hat{r}^k - Z_{ll}\hat{r}^k + 8Z^{\rm AS}_{kl}\hat{r}^l) + O(r^{-3}), \tag{2.62}$$

$$\tilde{h}_{ij} = O(r^{-1}), \quad \partial_k\tilde{h}_{ij} = O(r^{-2}), \quad \partial_k\partial_l\tilde{h}_{ij} = O(r^{-3}), \tag{2.63}$$

$$\tilde{A}_{ij} = -\frac{1}{4r^3}[6Z_{ij} - 2\delta_{ij}Z_{kk} - 6Z_{il}\hat{r}^l\hat{r}^j - 6Z_{jl}\hat{r}^l\hat{r}^i + 4\delta_{ij}Z_{kl}\hat{r}^k\hat{r}^l + (5Z_{kl}\hat{r}^k\hat{r}^l - Z_{ll})(\delta_{ij} - 3\hat{r}^i\hat{r}^j) - 12(Z^{\rm AS}_{ik}\hat{r}^k\hat{r}^j + Z^{\rm AS}_{jk}\hat{r}^k\hat{r}^i) + U_{ij}] + O(r^{-4}), \tag{2.64}$$

where $M_{\rm ADM}$ and $M_{\rm K}$ denote the ADM mass and Komar mass (see Sec. III), $\hat{r}^k = x^k/r$, and Z_{kl} and $Z^{\rm AS}_{kl}$ are time-dependent symmetric and antisymmetric moments, respectively. Equation (2.64) implies that the total linear momentum of the system is implicitly assumed to be zero, as

$$\int_\infty \tilde{A}_j{}^i dS_i = \int_\infty \psi^6 \tilde{A}_j{}^i dS_i = 0. \tag{2.65}$$

Here

$$\int_\infty := \lim_{r\to\infty}\int_{S_r}$$

with S_r a sphere of constant r. The antisymmetric moment can be related to the angular momentum of the system as

$$Z^{\rm AS}_{kl} = -J_j\epsilon_{jkl}, \tag{2.66}$$

where ϵ_{jkl} is the completely antisymmetric symbol. In the Newtonian limit,

$$Z_{kl} = \int \rho(v^k x^l + v^l x^k) d^3x = \frac{dI_{kl}}{dt}, \tag{2.67}$$

$$Z^{\rm AS}_{kl} = \int \rho(v^k x^l - v^l x^k) d^3x, \tag{2.68}$$

where I_{ij} is the quadrupole moment. U_{ij} in Eq. (2.64) is a symmetric trace-free moment determined by the asymptotic behavior of u_{ij}.

A solution to the simultaneous equations derived here satisfies the constraint equations of the Einstein equation. In this sense, it can be referred to as a fully general relativistic solution and can be used as an initial condition of the (3+1) numerical simulation. However, since we do not assume the helical symmetric relation $\pounds_k\gamma_{ab}\neq 0$, the solution does not satisfy the equation for quasiequilibrium exactly. As a result, $\pounds_k\psi$, $\pounds_k\alpha$, and $\pounds_k\beta^a$ would be slightly different from zero in general. Deviation of these quantities from zero can measure the violation of the helical symmetry. The magnitude of the deviation depends on our choice of u^{ab} and v_{ab}.

III. RELATIONS BETWEEN $M_{\rm ADM}$ AND $M_{\rm K}$

In this section, we derive the conditions needed for equality of the ADM mass $M_{\rm ADM}$ and the Komar mass $M_{\rm K}$. This equality is closely related to the virial relations as discussed in the Appendix.

A. Sufficient conditions for equality of $M_{\rm K}$ and $M_{\rm ADM}$

The Komar mass [25] is constructed from a vector ζ^α that approaches a timelike Killing vector of a flat asymptotic metric at spatial infinity. As presented in Ref. [26], $\zeta^\alpha = -\alpha^2\nabla^\alpha t$, and

$$M_{\rm K} = \frac{1}{8\pi}\int_\infty (\nabla^\beta\zeta^\alpha - \nabla^\alpha\zeta^\beta) n_\alpha dS_\beta. \tag{3.1}$$

With the metric written in the form (2.2), one uses the relations $n_\alpha = \alpha\nabla_\alpha t$ and $n^\beta\nabla_\beta n^\alpha = \alpha^{-1}\gamma^a_\alpha D_a\alpha$, to obtain

$$\nabla_\beta(\nabla^\beta\zeta^\alpha - \nabla^\alpha\zeta^\beta)n_\alpha = \alpha\nabla_\beta\{\gamma^\beta_\gamma[\nabla^\alpha(\alpha n^\gamma) - \nabla^\gamma(\alpha n^\alpha)]\nabla_\alpha t\}$$
$$= 2D_aD^a\alpha. \tag{3.2}$$

Using Eq. (3.2), we have

$$\frac{1}{8\pi}\int 2D_aD^a\alpha dV = \frac{1}{4\pi}\int_\infty D^a\alpha dS_a, \tag{3.3}$$

and thus, we reach the familiar form [27]

$$M_{\rm K} = \frac{1}{4\pi}\int_\infty D^a\alpha dS_a. \tag{3.4}$$

For a stationary spacetime, with ζ^α the asymptotically timelike Killing vector, Beig [28] and Ashtekar and Magnon-Ashtekar [29] prove the equality $M_{\rm K} = M_{\rm ADM}$. We obtain here more general asymptotic conditions sufficient for equality in the following way (patterned in part on Beig's work). Suppose that the metric has the form (2.2), with[1]

$$\gamma_{ij} = \eta_{ij} + h_{ij}, \tag{3.5}$$

$$h_{ij} = O(r^{-1}), \quad D_k h_{ij} = o(r^{-3/2}), \quad D_kD_l h_{ij} = o(r^{-2}), \tag{3.6}$$

and suppose that the lapse, shift, and extrinsic curvature satisfy

$$\alpha = 1 - \frac{\overset{1}{\alpha}}{r} + o(r^{-1}), \quad \partial_r\alpha = \frac{\overset{1}{\alpha}}{r^2} + o(r^{-2}),$$

$$\partial_r^2\alpha = -2\frac{\overset{1}{\alpha}}{r^3} + o(r^{-3}); \tag{3.7}$$

$$\beta^i = O(r^{-1}), \quad \partial_j\beta^i = o(r^{-3/2}); \tag{3.8}$$

$$\int_\infty d\hat\Omega K_{rr} = o(r^{-3}), \quad K_{ij} = o(r^{-3/2}),$$

$$\partial_k K_{ij} = o(r^{-2}), \tag{3.9}$$

where $\int d\hat\Omega$ denotes a surface integral. Then $M_{\rm K} = M_{\rm ADM}$. Note that the equality also follows with the slightly altered conditions, $h_{ij} = O(r^{-1-\epsilon})$ and $D_kD_lh_{ij} = o(r^{-2+\epsilon})$; and/or the conditions $\beta^i = O(r^{-1-\epsilon})$ and $\partial_k K_{ij} = o(r^{-2+\epsilon})$.

To prove this claim, it is useful to introduce $\delta\,{}^3G_{ij}$, the part of ${}^3G_{ij}$ linear in h_{ij}. The idea is to show that if $\delta\,{}^3G_{ij}$ is the asymptotically dominant part of ${}^3G_{ij}$, then

$$\int_\infty {}^3G^i_j x^j dS_i = -8\pi M_{\rm ADM}. \tag{3.10}$$

One then uses the field equation for ${}^3G_{ij}$ (the dynamical equation for K_{ij}) to show that this integral can be written in the form (3.4) of the Komar mass, when K_{ij}, $\pounds_\beta K_{ij}$, and $\partial_t K_{rr}$ fall off rapidly enough at spatial infinity.

Formally,

$$\delta\,{}^3G_{ij} = \frac{1}{2}(\overset{(0)}{D}_i\overset{(0)}{D}_k h^k_j + \overset{(0)}{D}_j\overset{(0)}{D}_k h^k_i - \overset{(0)}{\Delta} h_{ij} - \overset{(0)}{D}_i\overset{(0)}{D}_j h^k_k + \eta_{ij}\overset{(0)}{\Delta} h^k_k$$
$$- \eta_{ij}\overset{(0)}{D}_k\overset{(0)}{D}_l h^{kl}), \tag{3.11}$$

[1]The definition h_{ij} here is slightly different from $\tilde h_{ij}$ in Eq. (2.40) in Sec. II B.

where the index of h_{ij} is raised by the flat metric η^{ij}. The asymptotic behavior of ${}^3G_{ij}$ is given by

$$ {}^3G_{ij}-\delta^3G_{ij}=o(r^{-3}). \tag{3.12}$$

This is because each term in ${}^3G_{ij}$ involves either $D_kD_lh_{ij}$ or $D_kh_{ij}D_lh_{mn}$; then terms quadratic or higher order in h_{ij} fall off as rapidly as either $h_{mn}D_kD_lh_{ij}$ or $D_kh_{ij}D_lh_{mn}$.

From the linearized Bianchi identity, $\overset{(0)}{D}{}^j\delta^3G_{ij}=0$, we have

$$\int_\infty {}^3G^i{}_jx^jdS_i=\int_\infty \delta^3G^i{}_jx^jdS_i=\int_V \overset{(0)}{D}{}^i(\delta^3G_{ij}x^j)\sqrt{\eta}d^3x$$
$$=\int_V\delta^3G_{ij}\eta^{ij}\sqrt{\eta}d^3x. \tag{3.13}$$

Now $\delta^3G_{ij}\eta^{ij}\sqrt{\eta}$ is a simpler divergence,

$$\eta^{ij}\delta^3G_{ij}=-\frac{1}{2}\overset{(0)}{D}_i\overset{(0)}{D}_jh^{ij}+\frac{1}{2}\overset{(0)}{\Delta}h^k_k=\frac{1}{2}\overset{(0)}{D}_i(\overset{(0)}{D}{}^ih^k_k-\overset{(0)}{D}_jh^{ij}), \tag{3.14}$$

and we return to a surface integral,

$$\int_\infty {}^3G^i_jx^jdS_i=-\frac{1}{2}\int_\infty(\overset{(0)}{D}_jh^{ij}-\overset{(0)}{D}{}^ih^k_k)dS_i=:-8\pi M_{\rm ADM}. \tag{3.15}$$

As written, the integrations-by-parts of Eqs. (3.13) and (3.15) appear to assume an interior with no boundary, a restriction that eliminates typical hypersurfaces of black-hole spacetimes; and Beig explicitly takes $V=\mathbb{R}^3$. This restriction, however, is unnecessary. Because the steps from Eqs. (3.13) to (3.15) use no field equations, one can replace h_{ab} by any smooth symmetric tensor field that agrees with h_{ab} for r greater than some radius R_2 and that vanishes for r less than some smaller radius R_1 outside all interior boundary points of the hypersurface V. The verification of the identity (3.15) is then valid as written.

Because the equality $\int_\infty\delta^3G^i_jx^j\,dS_i=-\frac{1}{2}\int_\infty(\overset{(0)}{D}_j\,h^{ij}-\overset{(0)}{D}{}^ih^k_k)dS_i$ does not depend on the values of h_{ab} in the interior, the integrands should differ only by a two-dimensional divergence, allowing a derivation that uses no volume integral. We show this directly in the following:

Let I,J be indices on the 2-sphere, and let $\mathcal{D}$ be the two-dimensional covariant derivative operator associated with the metric on the unit 2-sphere. Write $h^I_I=\eta^{IJ}h_{IJ}$. The integrands are

$$\delta^3G^i_jx^j\,dS_i=\delta^3G_{rr}r^3d\Omega$$

and

$$-\frac{1}{2}(\overset{(0)}{D}_j\,h^{ij}-\overset{(0)}{D}{}^ih^k_k)dS_i=-\frac{1}{2}(\overset{(0)}{D}_j\,h^{rj}-\overset{(0)}{D}{}^rh^k_k-\frac{1}{r}\,h^I_I)r^2d\Omega.$$

The rr component of Eq. (3.11) gives δ^3G_{rr} as a sum of six terms that we will label I–VI. They can be rewritten in the following manner:

$$\mathrm{I+II}=\overset{(0)}{D}_r\overset{(0)}{D}_kh^k_r=\partial_r\left[\frac{1}{r^2}\partial_r(r^2h_{rr})-\frac{1}{r}h^I_I\right]+\partial_r\mathcal{D}_Ih^I_r.$$

$$\mathrm{III+V}=\frac{1}{2}(\delta^{ij}-\hat{r}^i\hat{r}^j)\overset{(0)}{\Delta}h_{ij}=\frac{1}{2}\partial_r^2h^I_I+\partial_r\left(\frac{1}{r}h^I_I\right)+\frac{2}{r^2}h_{rr}$$
$$+\frac{1}{2}\mathcal{D}_I\left(\mathcal{D}^Ih^J_J+\frac{4}{r}h^I_r\right).$$

$$\mathrm{IV}=-\frac{1}{2}\overset{(0)}{D}{}^2_rh^k_k=-\frac{1}{2}\partial^2_rh^k_k.=-\frac{1}{2}\partial_r\left[\frac{1}{r^2}\partial_r(r^2h_{rr})\right]$$
$$+\partial_r\left(\frac{1}{r}h_{rr}\right)-\frac{1}{2}\partial^2_rh^I_I.$$

$$\mathrm{VI}=-\frac{1}{2}\overset{(0)}{D}_k\overset{(0)}{D}_l\,h^{kl}=-\frac{1}{2}\partial_r\left[\frac{1}{r^2}\partial_r(r^2h_{rr})-\frac{1}{r}h^I_I\right]-\frac{1}{r}\partial_rh_{rr}$$
$$-\frac{2}{r^2}h_{rr}+\frac{1}{r^2}h^I_I-\frac{1}{2}\mathcal{D}_I\left[\mathcal{D}_Jh^{IJ}+\frac{6}{r}h^I_r\right]-\partial_r\mathcal{D}_Ih^I_r.$$

Adding the terms, we find

$$\delta G_{rr}=\mathrm{I}+\cdots+\mathrm{VI}=-\frac{1}{2r}(\overset{(0)}{D}_j\,h^{ij}-\overset{(0)}{D}{}^ih^k_k)\nabla_ir+\mathcal{D}_IV^I,$$

where $V^I=-(1/2)(\mathcal{D}_Jh^{IJ}-\mathcal{D}^Ih^J_J+(1/r)h^I_r)$. Because $r^3\mathcal{D}_IV^I=\mathcal{D}_I(r^3V^I)$, the integrands differ by a two-dimensional divergence, as claimed.

Next, from Eqs. (2.29) and (2.24), ${}^3G_{ij}$ has, outside the matter, the form

$${}^3G_{ij}=R_{ij}-\frac{1}{2}\gamma_{ij}R$$
$$=\frac{1}{\alpha}D_iD_j\alpha-\frac{1}{2}\gamma_{ij}\frac{1}{\alpha}\Delta\alpha+\frac{1}{\alpha}\left(\partial_tK_{ij}-\frac{1}{2}\gamma_{ij}\gamma^{kl}\partial_tK_{kl}\right)$$
$$+2K_{ik}K^k_j-K_{ij}K-\gamma_{ij}\left(K_{kl}K^{kl}-\frac{1}{2}K^2\right)$$
$$+\frac{1}{\alpha}\pounds_\beta K_{ij}-\frac{1}{2\alpha}\gamma_{ij}\gamma^{kl}\pounds_\beta K_{kl}. \tag{3.16}$$

Since the terms involving K_{ij} are $o(r^{-3})$, we have

$$\int_\infty {}^3G^i_jx^jdS_i=\int_\infty D_iD_j\alpha\hat{r}^i\hat{r}^jr^3d\hat{\Omega}. \tag{3.17}$$

Now $\partial_k\gamma_{ij}=O(r^{-2})$ and $\partial_i\alpha=O(r^{-2})$ imply

$$D_r{}^2\alpha=\partial_r{}^2\alpha+O(r^{-4}). \tag{3.18}$$

Finally, from Eq. (3.4) and the asymptotic form (3.7) of α, we have

$$M_{\rm K}=\frac{1}{4\pi}\int_{\infty}\overset{1}{\alpha}d\hat{\Omega}, \tag{3.19}$$

and

$$\int_{\infty}{}^{3}G^{i}_{j}x^{j}dS_{i}=\int_{\infty}D_{r}{}^{2}\alpha r^{3}d\hat{\Omega}=-8\pi M_{\rm K}, \tag{3.20}$$

whence

$$M_{\rm ADM}=M_{\rm K}. \tag{3.21}$$

The asymptotic behavior of Eqs. (2.60)–(2.64) does *not always* satisfy the conditions (3.9), because $\partial_t K_{ij}$ is $O(r^{-3})$, not $o(r^{-3})$.

B. Relation between $M_{\rm K}$ and $M_{\rm ADM}$ in the waveless approximation

For quasiequilibrium binary solutions that satisfy Eqs. (2.54)–(2.58), with $K=0$, a slightly different relation holds between $M_{\rm K}$ and $M_{\rm ADM}$. The asymptotic behavior given in Eqs. (2.63) and (2.60) implies

$$D_i h_{ij}=O(r^{-2}),\quad D_iD_jh_{ij}=O(r^{-3}). \tag{3.22}$$

For the asymptotic behavior of Eqs. (2.60)–(2.64), together with Eq. (3.22), Eq. (3.15) still holds and

$$\begin{aligned}\int_{\infty}{}^{3}G^{i}_{j}x^{j}dS_{i}&=-8\pi M_{\rm K}+\int_{\infty}\partial_t K_{rr}r^{3}d\hat{\Omega}\\&=-8\pi M_{\rm K}+\frac{1}{4}\int_{\infty}\partial_t(12Z_{rr}-U_{rr})d\hat{\Omega},\end{aligned} \tag{3.23}$$

or

$$M_{\rm ADM}=M_{\rm K}-\frac{d}{dt}\left(\frac{3}{2}Z_{rr}-\frac{1}{8}U_{rr}\right). \tag{3.24}$$

Using

$$\int_{\infty}\hat{r}^{i}\hat{r}^{j}d\hat{\Omega}=\frac{4\pi}{3}\delta_{ij}, \tag{3.25}$$

Equation (3.23) may be written in asymptotically Cartesian coordinates as

$$\begin{aligned}\int_{\infty}{}^{3}G^{i}_{j}x^{j}dS_{i}&=-8\pi M_{\rm K}+\int_{\infty}\partial_t K_{ij}\hat{r}^{i}\hat{r}^{j}r^{3}d\hat{\Omega}\\&=-8\pi M_{\rm K}+4\pi\frac{dZ_{kk}}{dt},\end{aligned} \tag{3.26}$$

whence

$$M_{\rm ADM}=M_{\rm K}-\frac{1}{2}\frac{dZ_{kk}}{dt}. \tag{3.27}$$

We note that a similar expression has been derived for the maximal slicing condition in the first post-Newtonian approximation [30]. The expression here is the fully general relativistic generalization.

In the presence of a timelike Killing vector, we may set

$$\partial_t\tilde{\gamma}_{ab}=0, \tag{3.28}$$

$$\partial_t(\psi^4\pi^{ab})=0, \tag{3.29}$$

$$\frac{d}{dt}Z_{kk}=0, \tag{3.30}$$

and, hence, the virial relation $M_{\rm ADM}=M_{\rm K}$ holds. This condition should be satisfied not only for axisymmetric equilibria but also for nonaxisymmetric ones such as general relativistic Dedekind solutions.

The integral of K_{ij} in the above expression may be computed further as

$$\begin{aligned}\int_{\infty}\partial_t K_{ij}\hat{r}^{i}\hat{r}^{j}r^{3}d\hat{\Omega}&=\frac{1}{8\pi}\frac{d}{dt}\int_{\infty}K_{a}{}^{b}x^{a}dS_{b}\\&=\frac{1}{8\pi}\frac{d}{dt}\int D_{b}(\pi_{a}{}^{b}x^{a})d^{3}x\\&=-\frac{d}{dt}\int x^{a}j_{a}dV\\&\quad+\frac{1}{16\pi}\frac{d}{dt}\int\pi^{ab}x^{c}\overset{(0)}{D}_{c}\gamma_{ab}d^{3}x,\end{aligned} \tag{3.31}$$

which yields

$$\begin{aligned}M_{\rm ADM}&=M_{\rm K}+\frac{d}{dt}\int x^{a}j_{a}dV-\frac{1}{16\pi}\int(\partial_t\pi^{ab}x^{c}\overset{(0)}{D}_{c}\gamma_{ab}\\&\quad+\pi^{ab}x^{c}\overset{(0)}{D}_{c}\partial_t\gamma_{ab})d^{3}x\\&=M_{\rm K}+\frac{d}{dt}\int x^{a}j_{a}dV\\&\quad-\frac{1}{16\pi}\int[\partial_t(\psi^4\pi^{ab})x^{c}\overset{(0)}{D}_{c}\tilde{\gamma}_{ab}\\&\quad+\psi^4\pi^{ab}x^{c}\overset{(0)}{D}_{c}\partial_t\tilde{\gamma}_{ab}]d^{3}x.\end{aligned} \tag{3.32}$$

In a gauge with $K=0$, Eq. (3.32) can be written as

$$\begin{aligned}M_{\rm ADM}&=M_{\rm K}+\frac{d}{dt}\int x^{a}j_{a}dV-\frac{1}{16\pi}\int[\partial_t\hat{A}_{ab}x^{c}\overset{(0)}{D}_{c}\tilde{\gamma}^{ab}\\&\quad+\hat{A}_{ab}x^{c}\overset{(0)}{D}_{c}\partial_t\tilde{\gamma}^{ab}]d^{3}x\\&=M_{\rm K}+\frac{d}{dt}\int x^{a}j_{a}dV-\frac{1}{16\pi}\int[v_{ab}x^{c}\overset{(0)}{D}_{c}\tilde{\gamma}^{ab}\\&\quad+\hat{A}_{ab}x^{c}\overset{(0)}{D}_{c}u^{ab}]d^{3}x,\end{aligned} \tag{3.33}$$

where a relation $\pi^{ab}x^c \overset{(0)}{D}_c \gamma_{ab}=\hat{A}_{ab}x^c \overset{(0)}{D}_c \tilde{\gamma}^{ab}$ is used. Since we always impose a condition $£_k(j_a\sqrt{\gamma})=0$, the second integral can be discarded because of the relation,

$$\frac{d}{dt}\int x^a j_a dV=\int \partial_t(x^a j_a\sqrt{\gamma})d^3x=\int £_k(x^a j_a\sqrt{\gamma})d^3x=0. \tag{3.34}$$

Thus, Eq. (3.33) is written as

$$M_{\rm ADM}=M_{\rm K}-\frac{1}{16\pi}\int (v_{ab}x^c \overset{(0)}{D}_c \tilde{\gamma}^{ab}+\hat{A}_{ab}x^c \overset{(0)}{D}_c u^{ab})d^3x. \tag{3.35}$$

If we assume that the spacetime is everywhere helically symmetric, $£_k g_{\alpha\beta}=0$, then it is not asymptotically flat, and hence the integrals in Eqs. (3.32) and (3.33), as well as $M_{\rm ADM}$, are not defined. Instead, we require helical symmetry only in the near zone and the local distant zone. In this case, in the local zone, we should set $£_k\tilde{\gamma}^{ab}=0$ and $£_k\hat{A}_{ab}=0$, which can be written as

$$u^{ab}=\partial_t\tilde{\gamma}^{ab}=-£_{\Omega\phi}\tilde{\gamma}^{ab}, \tag{3.36}$$

$$v_{ab}=\partial_t\hat{A}_{ab}=-£_{\Omega\phi}\hat{A}_{ab}, \tag{3.37}$$

while in the distant zone, $u^{ab}\to 0$ and $v_{ab}\to 0$ for $r\to\infty$. With these choices, dZ_{kk}/dt vanishes, and hence, the virial relation $M_{\rm ADM}=M_{\rm K}$ is satisfied. The virial relation is satisfied for the simple case $u^{ab}=v_{ab}=0$. Equation (3.35) reminds us that it is satisfied in the conformal flatness approximation $\tilde{\gamma}^{ab}=\eta^{ab}$, i.e., $u^{ab}=0$ [21].

IV. RELATION FOR $\delta M_{\rm ADM}$ AND δJ

A first law for binary systems with a helical Killing field k^α has been formulated as relating a change in a conserved charge Q, associated with a family of helically symmetric spacetimes, to the changes in the vorticity, baryon mass, and entropy of the fluid as well as in the area of black holes [21]. Also shown is that a relation $\delta Q=\delta M_{\rm ADM}-\Omega\,\delta J$ holds for asymptotically flat systems, where Ω is the orbital angular velocity of a binary system in circular orbits.

In inspiraling binary neutron stars, entropy, baryon mass, and vorticity are almost constant, and hence energy and angular momentum are dissipated only by gravitational radiation. Thus, the relation $dM_{\rm ADM}/dt=\Omega dJ/dt$ is satisfied. Then, we should require that, in a formalism for computing asymptotically flat binary equilibria, a first law $\delta M_{\rm ADM}=\Omega\,\delta J$ is satisfied. In this section, we present a heuristic way to derive a relation held between the variations of the ADM mass and the angular momentum without assuming any symmetry for perfect-fluid spacetimes, but requiring the field equations to be satisfied. Then, we identify sources for the violation of the first law. In the following calculation, no gauge condition is specified; and surface terms associated with black hole horizons are not included.

In contrast to our earlier paper [21], no helical symmetry is assumed. Instead, we assume only that the field equations derived in Sec. II are satisfied. The field equations are used to relate the Komar mass to the Lagrangian density and the other terms in the manner

$$\begin{aligned}M_{\rm K}&=\frac{1}{4\pi}\int_\infty D^a\alpha dS_a=\frac{1}{4\pi}\int D^aD_a\alpha dV\\&=2\int\left\{\rho u^t\sqrt{-g}hu_\alpha v^\alpha-\mathcal{L}-\frac{1}{16\pi}\gamma_{ab}\partial_t\pi^{ab}\right.\\&\quad\left.-\frac{1}{16\pi}D_a(2\pi^{ab}\beta_b-\pi\beta^a)\right\}d^3x,\end{aligned} \tag{4.1}$$

where the definition of v^α, slightly different from the previous section, is $u^\alpha=u^t(t^\alpha+v^\alpha)$ with $v^\alpha n_\alpha=0$. To compute the last equality, we first use a part of the Einstein equation,

$$\begin{aligned}&(G^{\alpha\beta}-8\pi T^{\alpha\beta})\left(\gamma_{\alpha\beta}+n_\alpha n_\beta+\frac{2}{\alpha}\beta_\alpha n_\beta\right)\alpha\sqrt{\gamma}\\&=\gamma_{ab}\partial_t\pi^{ab}+2D^aD_a\alpha\sqrt{\gamma}+16\pi\left(-\frac{1}{2}T_\alpha^{\ \alpha}-\epsilon\right)\sqrt{-g}\\&\quad-16\pi\rho u^t\sqrt{-g}hu_\alpha v^\alpha+D_a(2\pi^{ab}\beta_b-\pi\beta^a)=0.\end{aligned} \tag{4.2}$$

We then subtract the trace of the Einstein equation $(G^{\alpha\beta}-8\pi T^{\alpha\beta})g_{\alpha\beta}=-R-8\pi T_\alpha^{\ \alpha}=0$ to relate $M_{\rm K}$ to the Lagrangian density (2.9).

The angular momentum is defined by

$$\begin{aligned}J&=-\frac{1}{8\pi}\int_\infty \pi^a{}_b\phi^b dS_a=-\frac{1}{8\pi}\int D_a(\pi^a{}_b\phi^b)d^3x\\&=-\frac{1}{8\pi}\int(\phi^aD_b\pi^b{}_a+\pi^{ab}D_b\phi_a)d^3x\\&=\int\left(j_a\phi^a\sqrt{\gamma}-\frac{1}{16\pi}\pi^{ab}£_\phi\gamma_{ab}\right)d^3x,\end{aligned} \tag{4.3}$$

where the momentum constraint $\mathcal{C}^a=0$ is used in the last equality. The variations of $M_{\rm K}$ and J are computed following Ref. [21]. First, we take the variation of Eq. (4.1) for $M_{\rm K}$,

$$\begin{aligned}\delta M_{\rm K}&=2\int\left\{\Delta(\rho u^t\sqrt{-g}hu_\alpha v^\alpha)-\delta\mathcal{L}-\frac{1}{16\pi}\delta(\gamma_{ab}\partial_t\pi^{ab})\right.\\&\quad\left.-\frac{1}{16\pi}D_a\delta(2\pi^{ab}\beta_b-\pi\beta^a)\right\}d^3x,\end{aligned} \tag{4.4}$$

and substitute the following relation into the first term,

$$\begin{aligned}\Delta(\rho u^t\sqrt{-g}hu_\alpha v^\alpha)&=hu_\alpha v^\alpha\Delta(\rho u^t\sqrt{-g})\\&\quad+\rho u^t\sqrt{-g}v^\alpha\Delta(hu_\alpha)+j_a\sqrt{\gamma}\partial_t\xi^a,\end{aligned} \tag{4.5}$$

where we used the facts that a choice of $\xi^t=0$ implies $\Delta v^\alpha=-\Delta t^\alpha=£_t\xi^\alpha$ and that $£_t\xi^\alpha\nabla_\alpha t=0$ yields

$\rho u^t\sqrt{-g}hu_\alpha \pounds_t \xi^\alpha = j_a\sqrt{\gamma}\partial_t\xi^a$. To the second term, the variation in the Lagrangian density (2.12) is applied.

When all components of the Einstein equation and the Bianchi identity, Eqs. (2.18)–(2.21), are satisfied, $\delta M_{\rm K}$ becomes

$$\delta M_{\rm K} = 2\int\Bigg[\rho T\Delta s\sqrt{-g} + \left(\frac{h}{u^t} + hu_\alpha v^\alpha\right)\Delta(\rho u^t\sqrt{-g}) + v^\alpha\Delta(hu_\alpha)\rho u^t\sqrt{-g} - \xi^a\partial_t(j_a\sqrt{\gamma}) + \frac{1}{16\pi}(\delta\pi^{ab}\partial_t\gamma_{ab} - \delta\gamma_{ab}\partial_t\pi^{ab}) - D_a\left\{\tilde{\Theta}^a\sqrt{\gamma} + \frac{1}{16\pi}\delta(2\pi^{ab}\beta_b - \pi\beta^a)\right\}\Bigg]d^3x. \tag{4.6}$$

The contribution of the surface term of the above equation is given by

$$-2\int D_a\left[\tilde{\Theta}^a\sqrt{\gamma} + \frac{1}{16\pi}\delta(2\pi^{ab}\beta_b - \pi\beta^a)\right]d^3x = -2\int_\infty \tilde{\Theta}^a dS_a = \frac{1}{4\pi}\int_\infty D^a\delta\alpha dS_a - \frac{1}{8\pi}\int_\infty(\gamma^{ac}\gamma^{bd} - \gamma^{ab}\gamma^{cd})D_b\delta\gamma_{cd}dS_a = \delta M_{\rm K} - 2\delta M_{\rm ADM}. \tag{4.7}$$

Combining Eqs. (4.6) and (4.7), the variation in the ADM mass $M_{\rm ADM}$, instead of the Komar mass $M_{\rm K}$, is written

$$\delta M_{\rm ADM} = \int\left\{\rho T\Delta s\sqrt{-g} + \left(\frac{h}{u^t} + hu_\alpha v^\alpha\right)\Delta(\rho u^t\sqrt{-g}) + v^\alpha\Delta(hu_\alpha)\rho u^t\sqrt{-g} - \xi^a\partial_t(j_a\sqrt{\gamma}) + \frac{1}{16\pi}(\delta\pi^{ab}\partial_t\gamma_{ab} - \delta\gamma_{ab}\partial_t\pi^{ab})\right\}d^3x. \tag{4.8}$$

The variation in the angular momentum δJ, computed in a similar way from Eq. (4.3), is

$$\delta J = \int\left\{\Delta(j_a\phi^a\sqrt{\gamma}) - \frac{1}{16\pi}\delta(\pi^{ab}\pounds_\phi\gamma_{ab})\right\}d^3x = \int\Bigg\{hu_\alpha\phi^\alpha\Delta(\rho u^t\sqrt{-g}) + \phi^\alpha\Delta(hu_\alpha)\rho u^t\sqrt{-g} + \xi^a\pounds_\phi(j_a\sqrt{\gamma}) - \frac{1}{16\pi}(\delta\pi^{ab}\pounds_\phi\gamma_{ab} - \delta\gamma_{ab}\pounds_\phi\pi^{ab}) - \pounds_\phi\left(j_a\xi^a\sqrt{\gamma} + \frac{1}{16\pi}\pi^{ab}\delta\gamma_{ab}\right)\Bigg\}d^3x, \tag{4.9}$$

where

$$\Delta(j_a\phi^a\sqrt{\gamma}) = \Delta(\rho u^t\sqrt{-g})hu_\alpha\phi^\alpha + \rho u^t\sqrt{-g}\phi^\alpha\Delta(hu_\alpha) - j_a\sqrt{\gamma}\pounds_\phi\xi^a = \Delta(\rho u^t\sqrt{-g})hu_\alpha\phi^\alpha + \rho u^t\sqrt{-g}\phi^\alpha\Delta(hu_\alpha) + \xi^a\pounds_\phi(j_a\sqrt{\gamma}) - \pounds_\phi(j_a\xi^a\sqrt{\gamma}), \tag{4.10}$$

which results from relations $j_a\phi^a\sqrt{\gamma} = \rho u^t\sqrt{-g}hu_\alpha\phi^\alpha$, $\delta\phi^\alpha = 0$, and $\Delta\phi^\alpha = -\pounds_\phi\xi^\alpha$. The last term in the integral of Eq. (4.9) vanishes since it becomes a surface integral with a combination $\phi^a dS_a = 0$.

Finally, Eqs. (4.8) and (4.9) are combined to derive a relation

$$\delta M_{\rm ADM} - \Omega\delta J = \int\Bigg[\rho T\Delta s\sqrt{-g} + \left\{\frac{h}{u^t} + hu_\alpha(v^\alpha - \Omega\phi^\alpha)\right\}\Delta(\rho u^t\sqrt{-g}) + (v^\alpha - \Omega\phi^\alpha)\Delta(hu_\alpha)\rho u^t\sqrt{-g} - \xi^a\pounds_{t+\Omega\phi}(j_a\sqrt{\gamma}) + \frac{1}{16\pi}(\delta\pi^{ab}\pounds_{t+\Omega\phi}\gamma_{ab} - \delta\gamma_{ab}\pounds_{t+\Omega\phi}\pi^{ab})\Bigg]d^3x, \tag{4.11}$$

where $\pounds_{t+\Omega\phi} = \partial_t + \pounds_{\Omega\phi}$, which operates on the spatial quantities, is understood as a pullback of the Lie derivative along the helical vector field onto Σ_t. The first three terms of Eq. (4.11) are the same as those derived in our previous paper [21] except for the definition of the spatial velocity $v^\alpha = u^\alpha/u^t$ (in Ref. [21], we used the definition $v^\alpha = u^\alpha/u^t - \Omega\phi^\alpha$). The difference also changes the definition of the shift.

As discussed in Ref. [21], for an isentropic fluid, conservation of baryon mass, entropy, and vorticity become

$$\pounds_u(\rho\sqrt{-g}) = 0, \quad \pounds_u s = 0, \quad \text{and } \pounds_u\omega_{\alpha\beta} = 0. \tag{4.12}$$

These imply perturbed conservation laws,

$$\Delta(\rho u^t\sqrt{-g}) = 0, \quad \Delta s = 0, \quad \text{and } \Delta\omega_{\alpha\beta} = 0, \tag{4.13}$$

that will be almost satisfied during binary inspiral before the merger. Here, the relativistic vorticity $\omega_{\alpha\beta}$ is given by

$$\omega_{\alpha\beta} = q_\alpha{}^\gamma q_\beta{}^\delta[\nabla_\gamma(hu_\delta) - \nabla_\delta(hu_\gamma)] = \nabla_\alpha(hu_\beta) - \nabla_\beta(hu_\alpha). \tag{4.14}$$

The third term in Eq. (4.11) vanishes for (i) corotating binaries, flows with $v^\alpha = \Omega\phi^\alpha$, and (ii) irrotational binaries, potential flows with $hu_\alpha = \nabla_\alpha\Phi$. For the latter case, the third term in Eq. (4.11) with $\Delta(hu_\alpha) = \Delta\nabla_\alpha\Phi = \nabla_\alpha\Delta\Phi$ becomes

$$\begin{aligned}\int & (v^\alpha-\Omega\phi^\alpha)\Delta(hu_\alpha)\rho u^t\sqrt{-g}d^3x\\ &=\int (v^\alpha-\Omega\phi^\alpha)\nabla_\alpha\Delta\Phi\rho u^t\sqrt{-g}d^3x\\ &=\int [D_\alpha\{(v^\alpha-\Omega\phi^\alpha)\Delta\Phi\rho u^t\alpha\sqrt{\gamma}\}\\ &\quad+\{\pounds_k(\rho\sqrt{-g})-\pounds_u(\rho\sqrt{-g})\}\Delta\Phi]d^3x\\ &=\int [\pounds_k(\rho\sqrt{-g})-\pounds_u(\rho\sqrt{-g})]\Delta\Phi d^3x,\end{aligned} \tag{4.15}$$

where we assume $\rho=0$ for the distant zone to derive the last line. Then, together with Eqs. (4.12) and (4.13), Eq. (4.11) is rewritten as

$$\begin{aligned}\delta M_{\rm ADM}-\Omega\,\delta J=\int \Big\{ &\pounds_k(\rho\sqrt{-g})\Delta\Phi-\xi^a\pounds_k(j_a\sqrt{\gamma})\\ &+\frac{1}{16\pi}(\delta\pi^{ab}\pounds_k\gamma_{ab}-\delta\gamma_{ab}\pounds_k\pi^{ab})\Big\}d^3x.\end{aligned} \tag{4.16}$$

The form of the "first law" described in Eq. (4.11) or Eq. (4.16) is derived without relying on the helical symmetry of the spacetime and the fluid. Choosing the maximal slicing condition $K=0=\pi$, we may rewrite Eq. (4.16) using the trace-free part of the conjugate momentum $\hat{\pi}^{ab}=\pi^{ab}-\frac{1}{3}\gamma^{ab}\pi$ and the conformal metric $\tilde{\gamma}_{ab}=\psi^{-4}\gamma_{ab}$ as

$$\begin{aligned}\delta M_{\rm ADM}-\Omega\,\delta J=\int \Big[&\pounds_k(\rho\sqrt{-g})\Delta\Phi-\xi^a\pounds_k(j_a\sqrt{\gamma})\\ &+\frac{1}{16\pi}\{\delta(\psi^4\hat{\pi}^{ab})\pounds_k\tilde{\gamma}_{ab}\\ &-\delta\tilde{\gamma}_{ab}\pounds_k(\psi^4\hat{\pi}^{ab})\}\Big]d^3x\\ =\int\Big[&\pounds_k(\rho\sqrt{-g})\Delta\Phi-\xi^a\pounds_k(j_a\sqrt{\gamma})\\ &+\frac{1}{16\pi}(\delta\hat{A}_{ab}\pounds_k\tilde{\gamma}^{ab}-\delta\tilde{\gamma}^{ab}\pounds_k\hat{A}_{ab})\Big]d^3x.\end{aligned} \tag{4.17}$$

Here, the gauge choice $\pi=0$ implies $\delta\pi=0$ and $\pounds_k\pi=0$.

If one requires $\pounds_k\gamma_{ab}=0$ and $\pounds_k\pi^{ab}=0$ (or $\pounds_k\tilde{\gamma}^{ab}=0$ and $\pounds_k\hat{A}_{ab}=0$) in a gauge $\pi=0=K$, together with conditions for the fluid variables $\pounds_k(\rho\sqrt{-g})=0$ and $\pounds_k(j_a\sqrt{\gamma})=0$, Eq. (4.16) leads to the first law relation $\delta M_{\rm ADM}=\Omega\,\delta J$. However, these assumptions are equivalent to imposing helical symmetry on the whole spacetime and, hence, preclude asymptotic flatness; in other words, $M_{\rm ADM}$ and J are ill defined.

In a realistic system, the radiation reaction violates the Killing symmetry. In its presence, $\delta\gamma_{ab}$ and $\delta\pi^{ab}$ are determined by the radiation reaction, and these terms may be proportional to the violation of the helical symmetry near the source. Namely, we expect that the following relations hold:

$$(\pounds_k\gamma_{ab})\,\delta t=\delta\gamma_{ab}, \tag{4.18}$$

$$(\pounds_k\pi^{ab})\,\delta t=\delta\pi^{ab}, \tag{4.19}$$

or in the gauge $K=0$,

$$(\pounds_k\tilde{\gamma}^{ab})\,\delta t=\delta\tilde{\gamma}^{ab}, \tag{4.20}$$

$$(\pounds_k\hat{A}_{ab})\,\delta t=\delta\hat{A}_{ab}, \tag{4.21}$$

where δt is a radiation reaction time scale. In this case, the right-hand side of Eq. (4.16) vanishes with the symmetry for the fluid variables. This indicates that even with the slight violation of the helical symmetry due to the radiation reaction, a relation $\delta M_{\rm ADM}=\Omega\,\delta J$ may be well satisfied.

Finally, as shown in Ref. [21], $\delta M_{\rm ADM}=\Omega\,\delta J$ is exact in the conformal flatness approximation [Isenberg-Wilson-Mathews (IWM) formalism]. In this case, one needs to replace the Lagrangian density (2.9) by one that reproduces the field equations of the IWM formalism. One can derive such a Lagrangian density by substituting $\pi=0$ and $\tilde{\gamma}_{ab}=\eta_{ab}$ into Eq. (2.9). Then, assuming helical symmetry for the fluid and from the fact $\delta\tilde{\gamma}_{ab}=0$, the first law is shown to be satisfied. (See Ref. [21] for a description of the artificiality of this choice in a helically symmetric IWM framework.)

V. CANDIDATE FORMULATIONS FOR QUASIEQUILIBRIA

The condition $u^{ab}=O(r^{-3})$ is not compatible with helical symmetry in the whole spacetime. Thus, we propose to impose

$$u^{ab}=\begin{cases}-\pounds_{\Omega\phi}\tilde{\gamma}^{ab} & \text{for } r\le r_0,\\ 0 & \text{for } r\ge r_0,\end{cases} \tag{5.1}$$

where r_0 is an arbitrary radius. With this condition, the type of the field equation for $\tilde{h}_{ij}$ changes from Helmholtz-type to elliptic for $r\simeq r_0$. To make the equation be almost elliptic for numerical computation, it may be desirable to take r_0 within the light cylinder radius as $r_0\lesssim 2\pi/\Omega$. On the other hand, we can impose helical symmetry on $\hat{A}_{ab}$ without serious difficulty. In this case, helical symmetry is exact in the near zone and, as a result, the violation of the first law is given by

$$\delta M_{\rm ADM}-\Omega\,\delta J=\frac{1}{16\pi}\int_{r>r_0}(\delta\hat{A}_{ab})\pounds_{\Omega\phi}\tilde{\gamma}^{ab}d^3x. \tag{5.2}$$

Since $(\delta\hat{A}_{ab})\pounds_{\Omega\phi}\tilde{\gamma}^{ab}$ falls off as $O(r^{-4})$ and the integral is done only in the distant zone, the magnitude of the integral would be very small. Thus, even with the modified formulation, the first law would be satisfied approximately. Furthermore, the virial relation is satisfied in this formulation.

The condition for $\partial_t \hat{A}_{ab}$ may be changed to

$$v_{ab}=\begin{cases}-\pounds_{\Omega\phi}\hat{A}_{ab} & \text{for } r\leq r_0,\\ 0 & \text{for } r\geq r_0.\end{cases} \tag{5.3}$$

Then,

$$\delta M_{\rm ADM}-\Omega\,\delta J=\frac{1}{16\pi}\int_{r>r_0}[(\delta\hat{A}_{ab})\pounds_{\Omega\phi}\tilde{\gamma}^{ab}-(\pounds_{\Omega\phi}\hat{A}_{ab})\delta\tilde{\gamma}^{ab}]d^3x. \tag{5.4}$$

Even in this case, the magnitude of the violation of the first law would be small, and the virial relation holds. The merit in this approach is that the right-hand side of the elliptic equation for $\tilde{h}_{ab}$ falls off as $O(r^{-4})$. As a result, it is numerically easier to integrate the equation.

We also note that instead of using the step function, we may write

$$u^{ab}=-\pounds_{f(r)\Omega\phi}\tilde{\gamma}^{ab}, \tag{5.5}$$

$$v_{ab}=-\pounds_{f(r)\Omega\phi}\hat{A}_{ab}, \tag{5.6}$$

where $f(r)$ is a smooth function that satisfies the condition

$$f(r)=\begin{cases}1 & \text{for } r\ll r_0,\\ 0 & \text{for } r\gg r_0.\end{cases} \tag{5.7}$$

This choice is equivalent to taking a Killing vector of the form

$$k^\mu=\left(\frac{\partial}{\partial t}\right)^\mu+f(r)\Omega\left(\frac{\partial}{\partial\varphi}\right)^\mu. \tag{5.8}$$

This Killing vector is helical in the near zone and purely timelike for $r\rightarrow\infty$.

Finally we comment on other possible formulations. In the formulation with $u^{ab}=0=v_{ab}$, the virial relation is satisfied for a quasiequilibrium binary. However, the first law along quasiequilibrium sequences is not satisfied in general. The violation of the first law is written as

$$\delta M_{\rm ADM}-\Omega\,\delta J=\frac{\Omega}{16\pi}\int[(\delta\hat{A}_{ab})\pounds_{\phi}\tilde{\gamma}^{ab}-(\pounds_{\phi}\hat{A}_{ab})\delta\tilde{\gamma}^{ab}]d^3x$$
$$=-\frac{\Omega}{16\pi}\delta\int(\pounds_{\phi}\hat{A}_{ab})\tilde{\gamma}^{ab}d^3x. \tag{5.9}$$

VI. SUMMARY

Two relations, the virial relation $M_{\rm ADM}=M_{\rm K}$ and the first law $\delta M_{\rm ADM}=\Omega\,\delta J$, are regarded as guiding principles to develop a formalism for computing binary compact objects in quasiequilibrium circular orbits in general relativity. Deriving the explicit equations for $M_{\rm ADM}-M_{\rm K}$ and $\delta M_{\rm ADM}-\Omega\,\delta J$ on the assumption that the spacetime is asymptotically flat, it is shown that a solution and a sequence of the solutions computed in some formulations satisfy these two conditions at least approximately. We propose a formulation in which the full Einstein equation is solved with the maximal slicing and in a transverse gauge for the conformal three-metric. In the proposed formulation, the solution in the near zone is helically symmetric, but in the distant zone, it is asymptotically waveless.

So far, quasiequilibria of binary neutron stars have been computed using the conformal flatness approximation for the three-metric [9,11]. In this formulation, only five components of the Einstein equation are satisfied, and thus, the obtained numerical solutions for quasiequilibria involve a systematic error. Specifically, in a real solution of the quasiequilibrium circular orbit, the conformal nonflat part of the three-metric will be of order $(M/a)^2$, which can be ~ 0.1 near the neutron stars for close circular orbits of $a\lesssim 10M$ (e.g., Ref. [4]). This implies that to compute an accurate quasiequilibrium in circular orbits of error within, say, 1%, it will be necessary to take into account the conformal nonflat part of the three-metric. In the new formulations described here, such term is computed, and thus, more accurate solutions of quasiequilibria will be obtained. Currently, we are working in computation of binary neutron stars in quasiequilibrium circular orbits using these formulations. In a subsequent paper [31], we will present the numerical results. Such a numerical solution will also be used as an appropriate initial condition for simulations of binary neutron star mergers [5].

In this paper, we restrict our attention to the system in which no black hole exists. In the presence of black holes, we should carefully treat the surface terms at event horizons. The surface terms would modify the equations for the first law [21,33]. The formulation for computation of quasiequilibrium black hole binaries are left for the future [32].

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APPENDIX: EVOLUTION EQUATION FOR THE SCALAR MOMENT AND VIRIAL RELATION

In this section, we derive the virial relation by direct integration of the Euler equation. Thus, the virial relation we consider here is associated with an evolution equation for the scalar moment as in the Newtonian case. In the end, we confirm that the virial relation derived is equivalent to $M_{\rm K}=M_{\rm ADM}$.

In the following, we often refer to M_χ as a "Komar-like mass," which is defined by the asymptotic behavior of a function $\chi:=\alpha\psi$ at $r\rightarrow\infty$,

$$\chi\rightarrow 1-\frac{M_\chi}{2r}+O(r^{-2}). \tag{A1}$$

For simplicity, in the following calculation, we adopt a gauge in which $K=0$, $F^k=0$, and $\tilde{\gamma}=\eta$, and we carry out the calculations in Cartesian coordinates. We often need to evaluate surface integrals at $r\to\infty$. In the evaluation, we assume Eqs. (2.60)–(2.64) as well as (A1) the asymptotic behaviors of geometric variables. As a consequence, all the volume integrals that appear below are well defined, and furthermore, the surface integrals derived during the calculation can be safely discarded.

From the asymptotic behavior as $r\to\infty$, we can define M_χ and $M_{\rm ADM}$ using the surface integrals

$$M_\chi = \frac{1}{2\pi}\int_\infty \psi\partial_i\chi dS^i, \tag{A2}$$

$$M_{\rm ADM} = -\frac{1}{2\pi}\int_\infty \chi\partial_i\psi dS^i. \tag{A3}$$

Using Gauss's law, they can be rewritten in other forms,

$$M_\chi = \frac{1}{2\pi}\int (\psi\tilde{\Delta}\chi + \tilde{\gamma}^{ij}\partial_i\chi\partial_j\psi)d^3x, \tag{A4}$$

$$M_{\rm ADM} = -\frac{1}{2\pi}\int (\chi\tilde{\Delta}\psi + \tilde{\gamma}^{ij}\partial_i\psi\partial_j\chi)d^3x. \tag{A5}$$

The difference between $M_{\rm ADM}$ and M_χ is written in the form

$$M_\chi - M_{\rm ADM} = \int \bigg[2\chi\psi^5 S_k{}^k + \frac{3}{8\pi}\chi\psi^5\tilde{A}_i{}^j\tilde{A}_j{}^i + \frac{1}{8\pi}\chi\psi\tilde{R} + \frac{1}{\pi}\tilde{\gamma}^{ij}\partial_i\psi\partial_j\chi\bigg]d^3x. \tag{A6}$$

Here, using Eq. (2.34), we can derive an identity,

$$\begin{aligned}\int \alpha\psi^6\tilde{A}^i{}_j\tilde{A}^j{}_i d^3x &= \frac{1}{2}\int \psi^6\tilde{A}^i{}_j(\partial_i\beta^j + \tilde{\gamma}_{ik}\tilde{\gamma}^{jn}\partial_n\beta^k + \tilde{\gamma}^{jn}\beta^k\partial_k\tilde{\gamma}_{ni} - \tilde{\gamma}^{jk}u_{ik})d^3x \\ &= \int \bigg[\psi^6\tilde{A}^i{}_j\partial_i\beta^j + \frac{1}{2}\psi^6\tilde{A}^{jn}\beta^k\partial_k\tilde{\gamma}_{jn} - \frac{1}{2}\psi^6\tilde{A}^{ij}u_{ij}\bigg]d^3x \\ &= \int \bigg[-\{\partial_i(\psi^6\tilde{A}^i{}_j) - \psi^6\tilde{A}^l{}_k\tilde{\Gamma}^k_{jl}\}\beta^j + \frac{1}{2}\psi^6\tilde{A}_{ij}u^{ij}\bigg]d^3x \\ &= -\int \bigg[8\pi j_i\psi^6\beta^i - \frac{1}{2}\hat{A}_{ij}u^{ij}\bigg]d^3x,\end{aligned} \tag{A7}$$

where $\tilde{\Gamma}^k_{ij}$ denotes the Christoffel symbol with respect to $\tilde{\gamma}_{ij}$, and $\hat{A}_{ij}=\psi^6\tilde{A}_{ij}$. Thus, we obtain

$$M_\chi - M_{\rm ADM} = \int \bigg[\psi^6 j_l(2v^l - \beta^l) + 6\alpha\psi^6 P + \frac{3}{16\pi}\hat{A}_{ij}u^{ij} + \frac{1}{8\pi}\chi\psi\tilde{R} + \frac{1}{\pi}\tilde{\gamma}^{ij}(\partial_i\psi)\partial_j\chi\bigg]d^3x, \tag{A8}$$

where we use the relation

$$\alpha S_k{}^k = \frac{j_k u_l\gamma^{kl}}{u^t} + 3\alpha P = j_k(v^k+\beta^k) + 3\alpha P. \tag{A9}$$

(Note that v^k here is defined by $v^k=u^k/u^t$.) In stationary spacetimes, the relation $M_{\rm ADM}=M_\chi$ [28,29] and $u_{ij}=\partial_t\tilde{\gamma}_{ij}=0$ should hold. Thus, we get the virial relation as

$$\int \bigg[\psi^6 j_l(2v^l-\beta^l) + 6\alpha\psi^6 P + \frac{1}{8\pi}\chi\psi\tilde{R} + \frac{1}{\pi}\tilde{\gamma}^{ij}(\partial_i\psi)\partial_j\chi\bigg]d^3x = 0. \tag{A10}$$

In quasiequilibrium binaries, $u_{ij}\neq 0$ in general. Thus, the virial relation, $M_{\rm ADM}=M_\chi$, is written as

$$\int \bigg[\psi^6 j_l(2v^l-\beta^l) + 6\alpha\psi^6 P + \frac{3}{16\pi}\hat{A}_{ij}u^{ij} + \frac{1}{8\pi}\chi\psi\tilde{R} + \frac{1}{\pi}\tilde{\gamma}^{ij}(\partial_i\psi)\partial_j\chi\bigg]d^3x = 0. \tag{A11}$$

As in the Newtonian case, we can derive the general relativistic virial relation from the evolution equation for the scalar moment. First, we write the general relativistic Euler equation $\gamma^\nu_k\nabla_\mu T^\mu{}_\nu=0$ in the form

$$\partial_t(j_k\psi^6) + \partial_j(j_k\psi^6 v^j) + \partial_k(\alpha\psi^6 P) + \rho_H\psi^5\partial_k\chi - (\rho_H + 2S_l{}^l)\chi\psi^4\partial_k\psi - \psi^6 j_l\partial_k\beta^l + \frac{1}{2}\chi\psi S_{ij}\partial_k\tilde{\gamma}^{ij} = 0. \tag{A12}$$

Equation (A12) is a fully general relativistic expression, and no simplification is done. Taking an inner product with x^k, we have

$$\int x^k\bigg[\partial_t(j_k\psi^6) + \partial_j(j_k\psi^6 v^j) + \partial_k(\alpha\psi^6 P) + \rho_H\psi^5\partial_k\chi - (\rho_H + 2S_l{}^l)\chi\psi^4\partial_k\psi - \psi^6 j_l\partial_k\beta^l + \frac{1}{2}\chi\psi S_{ij}\partial_k\tilde{\gamma}^{ij}\bigg]d^3x = 0. \tag{A13}$$

In the following, we carry out the integral for each term separately.

(1) First term:

$$I_1 := \int x^k \partial_t (j_k \psi^6) d^3x = \frac{d}{dt}\int x^k j_k \psi^6 d^3x. \tag{A14}$$

In the Newtonian limit, $j_k\psi^6 \to \rho v^k = \rho dx^k/dt$, and thus, this term leads to half of the second time derivative of the scalar moment, i.e., $\ddot{I}_{kk}/2$.

(2) Second and third terms: By integration by parts, we immediately find

$$I_2 := \int x^k \partial_j (j_k v^j \psi^6) d^3x = -\int j_k v^k \psi^6 d^3x, \tag{A15}$$

$$I_3 := \int x^k \partial_k (\alpha \psi^6 P) d^3x = -3\int \alpha \psi^6 P d^3x. \tag{A16}$$

In the Newtonian limit, $-I_2$ and $-I_3$ are the terms associated with kinetic energy and internal energy.

(3) Fourth and fifth terms: Using Eqs. (2.26) and (2.33), we can rewrite the combination of them as

$$\begin{aligned}
&\rho_H \psi^5 \partial_k \chi - (\rho_H + 2S_l{}^l)\chi\psi^4\partial_k\psi \\
&= \frac{\tilde{R}}{16\pi}\partial_k(\chi\psi) - \frac{1}{2\pi}[(\Delta\psi)\partial_k\chi + (\Delta\chi)\partial_k\psi] \\
&\quad - \frac{\psi^{12}\tilde{A}_i{}^j\tilde{A}_j{}^i}{16\pi}\partial_k\left(\frac{\alpha}{\psi^6}\right).
\end{aligned} \tag{A17}$$

Taking into account an identity,

$$\begin{aligned}
&\int [(x^k\partial_k\psi)\Delta\chi + (x^k\partial_k\chi)\Delta\psi] d^3x \\
&= \int [\tilde{\gamma}^{ij}(\partial_i\chi)\partial_j\psi + x^k(\partial_i\chi)(\partial_j\psi)\partial_k\tilde{\gamma}^{ij}] d^3x,
\end{aligned} \tag{A18}$$

where we discard the vanishing surface integral terms, we find

$$\begin{aligned}
I_4 &:= \int x^k[\rho_H\psi^5\partial_k\chi - (\rho_H + 2S_l{}^l)\chi\psi^4\partial_k\psi] d^3x \\
&= \frac{1}{16\pi}\int \Big[\tilde{R}x^k\partial_k(\chi\psi) - 8\{\tilde{\gamma}^{ij}(\partial_i\chi)\partial_j\psi + x^k(\partial_k\tilde{\gamma}^{ij}) \\
&\quad \times(\partial_i\chi)\partial_j\psi\} - \psi^{12}\tilde{A}_i{}^j\tilde{A}_j{}^i x^k\partial_k\left(\frac{\alpha}{\psi^6}\right)\Big] d^3x.
\end{aligned} \tag{A19}$$

(4) Sixth term:

$$\begin{aligned}
I_5 &:= -\int \psi^6 j_l x^k \partial_k \beta^l d^3x \\
&= -\frac{1}{8\pi}\int x^k\partial_k\beta^l\left[\partial_i(\psi^6\tilde{A}_l{}^i) + \frac{1}{2}\psi^6\tilde{A}_{ij}\partial_l\tilde{\gamma}^{ij}\right] d^3x \\
&= -\frac{1}{8\pi}\int \Big[-\psi^6\tilde{A}_l{}^i(x^k\partial_k\partial_i\beta^l + \partial_i\beta^l) \\
&\quad + \frac{1}{2}x^k(\partial_k\beta^l)\psi^6\tilde{A}_{ij}\partial_l\tilde{\gamma}^{ij}\Big] d^3x \\
&= -\frac{1}{8\pi}\int \Big[(\partial_i\beta^l)x^k\partial_k(\psi^6\tilde{A}^i{}_l) + 2(\partial_i\beta^l)\psi^6\tilde{A}_l{}^i \\
&\quad + \frac{1}{2}x^k(\partial_k\beta^l)\psi^6\tilde{A}_{ij}\partial_l\tilde{\gamma}^{ij}\Big] d^3x \\
&= -\frac{1}{8\pi}\int \Big[(\partial_i\beta^l)x^k\partial_k(\psi^6\tilde{A}^i{}_l) - 2\beta^l\partial_i(\psi^6\tilde{A}_l{}^i) \\
&\quad + \frac{1}{2}x^k(\partial_k\beta^l)\psi^6\tilde{A}_{ij}\partial_l\tilde{\gamma}^{ij}\Big] d^3x.
\end{aligned} \tag{A20}$$

Here, let us evaluate the first term. Using Eq. (2.34),

$$\begin{aligned}
I_5' &:= -\frac{1}{8\pi}\int (\partial_i\beta^l)x^k\partial_k(\psi^6\tilde{A}^i{}_l) d^3x \\
&= -\frac{1}{8\pi}\int (2\alpha\tilde{A}^l{}_i - \tilde{\gamma}^{jl}\tilde{\gamma}_{ik}\partial_j\beta^k - \tilde{\gamma}^{jl}\beta^k\partial_k\tilde{\gamma}_{ij} \\
&\quad + \tilde{\gamma}^{jl}u_{ij})x^k\partial_k(\psi^6\tilde{A}^i{}_l) d^3x \\
&= -\frac{1}{8\pi}\int [(2\alpha\tilde{A}^l{}_i - \tilde{\gamma}^{jl}\beta^k\partial_k\tilde{\gamma}_{ij} + \tilde{\gamma}^{jl}u_{ij})x^k\partial_k(\psi^6\tilde{A}^i{}_l) \\
&\quad + (\partial_j\beta^k)\psi^6\tilde{A}^i{}_l x^m\partial_m(\tilde{\gamma}^{jl}\tilde{\gamma}_{ik})] d^3x - I_5'.
\end{aligned} \tag{A21}$$

Thus,

$$\begin{aligned}
I_5' &= -\frac{1}{16\pi}\int [(2\alpha\tilde{A}^l{}_i - \tilde{\gamma}^{lj}\beta^k\partial_k\tilde{\gamma}_{ij} + \tilde{\gamma}^{jl}u_{ij})\partial_m(\psi^6\tilde{A}^i{}_l) \\
&\quad + (\partial_j\beta^k)(\tilde{A}^{ij}\partial_m\tilde{\gamma}_{ik} + \tilde{A}_{kl}\partial_m\tilde{\gamma}^{jl})\psi^6]x^m d^3x \\
&= -\frac{1}{16\pi}\int \Big[\frac{\alpha}{\psi^6}\partial_m(\psi^{12}\tilde{A}^l{}_i\tilde{A}^i{}_l) + (\tilde{\gamma}_{ij}\beta^k\partial_k\tilde{\gamma}^{jl} \\
&\quad + \tilde{\gamma}^{jl}u_{ij})\partial_m(\psi^6\tilde{A}^i{}_l) + (\partial_j\beta^k)(\tilde{A}^{ij}\partial_m\tilde{\gamma}_{ik} \\
&\quad + \tilde{A}_{kl}\partial_m\tilde{\gamma}^{jl})\psi^6\Big]x^m d^3x \\
&= -\frac{1}{16\pi}\int \Big[\Big\{-\partial_m\left(\frac{\alpha}{\psi^6}\right)\psi^{12}\tilde{A}^l{}_i\tilde{A}^i{}_l + \beta^k\partial_k\tilde{\gamma}^{jl}[\partial_m(\psi^6\tilde{A}_{jl}) \\
&\quad - \psi^6\tilde{A}^i{}_l\partial_m\tilde{\gamma}_{ij}] + \tilde{\gamma}^{jl}u_{ij}\partial_m(\psi^6\tilde{A}^i{}_l) + (\partial_j\beta^k)(\tilde{A}^{ij}\partial_m\tilde{\gamma}_{ik} \\
&\quad + \tilde{A}_{kl}\partial_m\tilde{\gamma}^{jl})\psi^6\Big\}x^m - 3\alpha\psi^6\tilde{A}^l{}_i\tilde{A}^i{}_l\Big] d^3x.
\end{aligned} \tag{A22}$$

As a result,

$$I_5=-\frac{1}{16\pi}\int\Bigg[-3\alpha\psi^6\tilde{A}^l{}_i\tilde{A}^i{}_l-\psi^{12}\tilde{A}^l{}_i\tilde{A}^i{}_l x^m\partial_m\left(\frac{\alpha}{\psi^6}\right)$$
$$+\beta^k(\partial_k\tilde{\gamma}^{jl})[\partial_m(\psi^6\tilde{A}_{jl})-\psi^6\tilde{A}^i{}_l\partial_m\tilde{\gamma}_{ij}]x^m$$
$$+\tilde{\gamma}^{jl}u_{ij}x^m\partial_m(\psi^6\tilde{A}^i{}_l)+(\partial_j\beta^k)(\tilde{A}^{ij}\partial_m\tilde{\gamma}_{ik}$$
$$+\tilde{A}_{kl}\partial_m\tilde{\gamma}^{jl})\psi^6x^m+x^k(\partial_k\beta^l)\psi^6\tilde{A}_{ij}\partial_l\tilde{\gamma}^{ij}$$
$$-2\beta^l(16\pi j_l\psi^6-\psi^6\tilde{A}_{ij}\partial_l\tilde{\gamma}^{ij})\Bigg]d^3x$$
$$=-\frac{1}{16\pi}\int\Bigg[-\frac{3}{2}\hat{A}_{kl}u^{kl}-\psi^{12}\tilde{A}^l{}_i\tilde{A}^i{}_l x^m\partial_m\left(\frac{\alpha}{\psi^6}\right)$$
$$+\beta^k(\partial_k\tilde{\gamma}^{jl})[\partial_m(\psi^6\tilde{A}_{jl})-\psi^6\tilde{A}^i{}_l\partial_m\tilde{\gamma}_{ij}]x^m$$
$$+\tilde{\gamma}^{jl}u_{ij}x^m\partial_m(\psi^6\tilde{A}^i{}_l)+(\partial_j\beta^k)(\tilde{A}^{ij}\partial_m\tilde{\gamma}_{ik}$$
$$+\tilde{A}_{kl}\partial_m\tilde{\gamma}^{jl})\psi^6x^m+x^k(\partial_k\beta^l)\psi^6\tilde{A}_{ij}\partial_l\tilde{\gamma}^{ij}-\beta^l(8\pi j_l\psi^6$$
$$-2\psi^6\tilde{A}_{ij}\partial_l\tilde{\gamma}^{ij})\Bigg]d^3x. \tag{A23}$$

(5) Seventh term: Using Eq. (2.35), we rewrite it as

$$\frac{\chi\psi}{2}S_{ij}x^k\partial_k\tilde{\gamma}^{ij}=\frac{1}{16\pi}x^k\partial_k\tilde{\gamma}^{ij}\Bigg[\chi\psi\left(\tilde{R}_{ij}+R^{\psi}_{ij}-\frac{1}{\alpha}D_iD_j\alpha\right)$$
$$+\psi^6\Bigg(-2\alpha\tilde{A}_{ik}\tilde{A}_j{}^k+\tilde{D}_i\tilde{\beta}^k\tilde{A}_{kj}+\tilde{D}_j\tilde{\beta}^k\tilde{A}_{ki}$$
$$-\frac{2}{3}\tilde{D}_k\tilde{\beta}^k\tilde{A}_{ij}+\beta^k\tilde{D}_k\tilde{A}_{ij}-\partial_t\tilde{A}_{ij}\Bigg)\Bigg], \tag{A24}$$

where we use $\tilde{\gamma}_{ij}\partial_k\tilde{\gamma}^{ij}=\partial_k\ln\tilde{\gamma}=0$.

By straightforward calculations, we obtain

$$I_6:=\frac{1}{16\pi}\int\chi\psi x^k(\partial_k\tilde{\gamma}^{ij})\tilde{R}_{ij}d^3x$$
$$=-\frac{1}{16\pi}\int[\partial_l(\chi\psi)x^k(\partial_k\tilde{\gamma}^{ij})\tilde{\Gamma}^l_{ij}+x^k\partial_k(\chi\psi)\tilde{R}$$
$$+\tilde{R}\chi\psi]d^3x, \tag{A25}$$

$$I_7:=\frac{1}{16\pi}\int[\chi\psi\tilde{R}^{\psi}_{ij}-\psi^2D_iD_j\alpha]x^k\partial_k\tilde{\gamma}^{ij}d^3x$$
$$=\frac{1}{16\pi}\int[-\tilde{D}_i\tilde{D}_j(\chi\psi)+8(\partial_i\psi)\partial_j\chi]x^k\partial_k\tilde{\gamma}^{ij}d^3x$$
$$=\frac{1}{16\pi}\int[\tilde{\Gamma}^l_{ij}\partial_l(\chi\psi)+8(\partial_i\psi)\partial_j\chi]x^k\partial_k\tilde{\gamma}^{ij}d^3x. \tag{A26}$$

Here, to derive the first equation, we use the spatial gauge condition $F^k=0$ and relations in the present gauge as

$$\tilde{R}=-\frac{1}{2}(\partial_l\tilde{\gamma}^{ij})\tilde{\Gamma}^l_{ij}=\tilde{\gamma}^{ij}\tilde{\Gamma}^l_{ik}\tilde{\Gamma}^k_{jl}. \tag{A27}$$

The spatial gauge condition is also used in calculation for I_7.

To evaluate the remaining terms, we first rewrite the following equation using the definition of $\tilde{A}_{ij}$ as

$$-2\alpha\tilde{A}_{ik}\tilde{A}^k{}_j+\tilde{D}_i\beta^k\tilde{A}_{kj}+\tilde{D}_j\beta^k\tilde{A}_{ki}-\frac{2}{3}\tilde{D}_k\beta^k\tilde{A}_{ij}+\beta^k\tilde{D}_k\tilde{A}_{ij}$$
$$=\beta^k\partial_k\tilde{A}_{ij}-\tilde{A}_j{}^k\tilde{\gamma}_{il}\partial_k\beta^l+\tilde{A}_{ik}\partial_j\beta^k-\tilde{A}^l{}_j\beta^k\partial_k\tilde{\gamma}_{il}+\tilde{A}^k{}_ju_{ik}. \tag{A28}$$

Then, after a straightforward calculation, we get

$$I_8=\frac{1}{16\pi}\int\Bigg[-2\alpha\tilde{A}_{ik}\tilde{A}^k{}_j+\tilde{D}_i\beta^k\tilde{A}_{kj}+\tilde{D}_j\beta^k\tilde{A}_{ki}-\frac{2}{3}\tilde{D}_k\beta^k\tilde{A}_{ij}$$
$$+\beta^k\tilde{D}_k\tilde{A}_{ij}-\partial_t\tilde{A}_{ij}\Bigg]\psi^6x^l\partial_l\tilde{\gamma}^{ij}d^3x$$
$$=\frac{1}{16\pi}\int[-(\partial_t\hat{A}_{ij})x^l\partial_l\tilde{\gamma}^{ij}+\psi^6\tilde{A}_{ij}(\partial_k\tilde{\gamma}^{ij})x^l\partial_l\beta^k$$
$$+2\beta^k(\partial_k\tilde{\gamma}^{ij})\tilde{A}_{ij}\psi^6+\beta^kx^l(\partial_k\tilde{\gamma}^{ij})\partial_l(\psi^6\tilde{A}_{ij})$$
$$+\psi^6x^l\tilde{A}^{ik}(\partial_k\beta^j)\partial_l\tilde{\gamma}_{ij}+x^l\psi^6\tilde{A}_{ik}(\partial_j\beta^k)\partial_l\tilde{\gamma}^{ij}$$
$$-\beta^kx^n\psi^6\tilde{A}^l{}_j(\partial_k\tilde{\gamma}_{il})\partial_n\tilde{\gamma}^{ij}+x^l\psi^6\tilde{A}^k{}_ju_{ik}\partial_l\tilde{\gamma}^{ij}]d^3x, \tag{A29}$$

where we use an identity $\partial_t\ln\psi^6=D_k\beta^k=\psi^{-6}\partial_k(\psi^6\beta^k)$ that follows from the maximal slicing condition $K=0$. Eventually, we find that I_5+I_8 has the following simple form:

$$I_5+I_8=\frac{1}{16\pi}\int\Bigg[-\frac{3}{2}\hat{A}_{kl}u^{kl}+\psi^{12}\tilde{A}_i{}^j\tilde{A}_j{}^ix^n\partial_n\left(\frac{\alpha}{\psi^6}\right)$$
$$+8\pi j_k\psi^6\beta^k-v_{ij}x^n\partial_n\tilde{\gamma}^{ij}-\hat{A}_{ij}x^n\partial_nu^{ij}\Bigg]d^3x. \tag{A30}$$

By summation of $I_1\sim I_8$, we obtain the following simple relation:

$$0=\sum_{i=1}^{8}I_i=\frac{d}{dt}\int x^kj_k\psi^6d^3x-\int\Bigg[\frac{1}{16\pi}\chi\psi\tilde{R}+j_kv^k\psi^6$$
$$+3\alpha\psi^6P+\frac{1}{2\pi}\tilde{\gamma}^{ij}\partial_i\chi\partial_j\psi-\frac{1}{2}\psi^6j_k\beta^k+\frac{3}{32\pi}\hat{A}_{ij}u^{ij}$$
$$+\frac{1}{16\pi}(v_{ij}x^n\partial_n\tilde{\gamma}^{ij}+\hat{A}_{ij}x^n\partial_nu^{ij})\Bigg]d^3x$$
$$=\frac{M_{\rm ADM}-M_\chi}{2}+\frac{d}{dt}\int x^kj_k\psi^6d^3x$$
$$-\frac{1}{16\pi}\int(v_{ij}x^n\partial_n\tilde{\gamma}^{ij}+\hat{A}_{ij}x^n\partial_nu^{ij})d^3x. \tag{A31}$$

Here, since $u^{ij}=\partial_t\tilde{\gamma}^{ij}$ and $v_{ij}=\partial_t\hat{A}_{ij}$, the second integral term in the last line of Eq. (A31) is rewritten as

$$-\frac{1}{16\pi}\frac{d}{dt}\int(\psi^6\tilde{A}_{ij}x^n\partial_n\tilde{\gamma}^{ij})d^3x. \tag{A32}$$

Using the momentum constraint, we further rewrite this term as

$$\begin{aligned}-\frac{1}{16\pi}\frac{d}{dt}\int(\psi^6\tilde{A}_{ij}x^k\partial_k\tilde{\gamma}^{ij})d^3x &= -\frac{1}{8\pi}\frac{d}{dt}\left[\int x^k\left\{\partial_i(\tilde{A}^i{}_k\psi^6)+\frac{1}{2}\psi^6\tilde{A}_{ij}\partial_k\tilde{\gamma}^{ij}\right\}d^3x - \int_\infty dS_i\tilde{A}^i{}_kx^k\psi^6\right] \\ &= -\frac{d}{dt}\left[\int x^kj_k\psi^6d^3x-\frac{1}{2}Z_{kk}\right]. \end{aligned} \tag{A33}$$

Thus, a similar relation between $M_{\rm ADM}$ and M_χ,

$$M_{\rm ADM}=M_\chi-\frac{dZ_{kk}}{dt}, \tag{A34}$$

is derived as is done for $M_{\rm ADM}$ and $M_{\rm K}$ in Sec. III. In this way, one can associate a relation of two masses $M_{\rm ADM}$ and M_χ ($M_{\rm K}$) to a moment equation of the relativistic Euler equation (A13).

In the Newtonian theory, we usually check the accuracy of numerical solutions by the virial relation. Since the relation is not trivially satisfied in numerical solutions, violation of this relation can be used to estimate the magnitude of the numerical error of equilibria. Motivated by this idea, a virial relation is also derived for axisymmetric equilibrium states in general relativity [34], and it is subsequently used to check accuracy of numerical solutions for rotating neutron stars [35]. The virial relation has been also derived for binary neutron stars in quasiequilibrium in conformally flat spacetimes [21] and applied for monitoring accuracy of numerical solutions in Ref. [36]. The virial relation, e.g., Eq. (A11), derived here will be used when checking the accuracy of nonaxisymmetric numerical solutions.

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