Symmetric hyperbolicity and consistent boundary conditions for second-order Einstein equations

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We present two families of first-order in time and second-order in space formulations of the Einstein equations (variants of the Arnowitt-Deser-Misner formulation) that admit a complete set of characteristic variables and a conserved energy that can be expressed in terms of the characteristic variables. The associated constraint system is also symmetric hyperbolic in this sense, and all characteristic speeds are physical. We propose a family of constraint-preserving boundary conditions that is applicable if the boundary is smooth with tangential shift. We conjecture that the resulting initial-boundary value problem is well-posed.

I. INTRODUCTION

Current numerical relativity codes designed to simulate the inspiral and merger of a black hole binary are limited by instabilities. These are now believed to be usually instabilities of the continuum equations, rather than of the numerical method. Intuitively one feels that the initial value problem for the Einstein equations is well-posed in a geometrical sense. For instance, in fully harmonic spacetime coordinates the Einstein equations reduce to ten quasilinear wave equations [1]. However, in the usual 3+1 split one uses only six of the ten Einstein equations for evolution, while the other four must be imposed as constraints on the initial data. An evolution in which the constraints are not obeyed is not a solution of the Einstein equations.

Consider perturbing a solution of the evolution equations around a solution of the full Einstein equations. We call a linear perturbation physical if it obeys the constraints initially and therefore at all times, and if it cannot be removed by a change of coordinates. A perturbation is called pure gauge if it obeys the constraint but can be removed by a coordinate transformation. A perturbation is called unphysical if it violates the constraints. All of these perturbations can be instabilities in the (weak) sense that they grow with respect to the background solution. An example for a physical instability is one that pushes a marginally stable star over the edge of gravitational collapse. An example of a gauge instability is given by the evolution of Minkowski spacetime with constant lapse, with an initial slice that is not quite flat: the time slices become singular in a finite time. Some, perhaps all, gauge instabilities can be avoided by a suitable choice of the lapse and shift adapted to the solution.

The problem in numerical solutions are unphysical instabilities. These will always be triggered in a numerical simulation by finite differencing or round-off error. Furthermore, the solution space of the evolution equations is infinitely bigger than the solution space of the full Einstein equations,

and so the latter are likely to represent an unstable equilibrium. Evolution schemes which replace between one and four of the evolution equations with constraint equations (constrained evolution) do not fundamentally address this problem, as they are still only solving six equations. Alternative approaches to reducing the growth of the constraints or even making them decay are suggested in [2,3].

In discussing instabilities, it is important to distinguish between evolution systems which are well-posed and evolution systems which are ill-posed. In the former, the growth of any linear perturbation $\delta u(x,t)$ of a background solution $u_0(x,t)$ can be bounded as

$$||\delta u(\cdot,t)|| \leqslant f(t)||\delta u(\cdot,0)||, \tag{1}$$

where $f(t)$ depends on u_0 but is independent of $\delta u(x,0)$. This means that the solution $u_0 + \delta u$ depends continuously on its initial data. By contrast, in an ill-posed system no such bound $f(t)$ exists. Rather, the growth rate increases unboundedly with the highest spatial frequency present in $\delta u(x,0)$. These are instabilities in a rather stronger sense. They can be gauge, but typically are constraint-violating.

A numerical simulation inherits all the instabilities already present in the continuum. A good numerical scheme does not add any others, but can never fix continuum instabilities. A crucial point is that in a numerical evolution, the highest spatial frequency present is effectively always the grid frequency, because it is generated by finite differencing error even if the initial data are smooth.

As one increases the resolution in numerical solutions of an ill-posed system, instabilities grow more rapidly because the finer grid can represent a higher spatial frequency. If these instabilities are constraint-violating, they will at higher resolution start at a lower amplitude, but the faster growth rate eventually overcomes this. Therefore they do not converge away. At high enough resolution and late enough time ill-posedness will become apparent as a breakdown of convergence for the entire solution. One can detect these modes even before that by looking at a Fourier transform in space

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[5]. As these non-convergent instabilities are features of the continuum system they cannot be suppressed by any consistent numerical dissipation.

By contrast, the numerical solution of a well-posed scheme will in general still be swamped by constraint-violating instabilities at late time, but now these start with smaller amplitude and grow at the same rate as one increases resolution. They therefore converge away.

In practice, depending on the system, physical solution, runtime and resolution, either convergent or non-convergent instabilities can be the dominant source of error. In 3-dimensional simulations in particular, the available resolution is quite limited, and the lack of convergence may not become apparent. The Arnowitt-Deser-Misner (ADM) formulation of the Einstein equations has for many years been the main formulation used in 3-dimensional simulations, even though for a fixed lapse and shift it is ill-posed (weakly hyperbolic) [4]. More recently a systematic comparison of high-resolution, long-time evolutions for well-posed and ill-posed (weakly hyperbolic) systems [5] has demonstrated that the breakdown of convergence is really inevitable, and has revived interest in well-posed formulations of the Einstein equations.

This interest has focused on first-order reductions of the Einstein equations, because for general first order evolution systems useful criteria for well-posedness are known [6]. A sufficient and necessary criterion for the initial value problem (with no boundaries, or periodic boundaries) to be well-posed is strong hyperbolicity: this roughly means that the system has a complete set of characteristic variables with real speeds. Symmetric hyperbolicity is another criterion which implies strong hyperbolicity and can be used to obtain a well-posed initial-boundary value problem: roughly speaking it means that the principal part of the system admits a conserved energy. Therefore a number of strongly or symmetric hyperbolic first-order reductions of the ADM equations have been suggested over the years to assure well-posedness, see for example [7–10]. In a symmetric hyperbolic formulation of the Einstein equations, there is a chance of dealing with the constraints consistently in the initial-boundary value problem, see [11–14].

However, first order reductions introduce new, auxiliary, variables and constraints, and so further increase the solution space. One would expect this to give rise to additional constraint-violating instabilities of the convergent type, even if well-posedness rules out the non-convergent type. There is some evidence that this is a real problem, which requires much more fine-tuning of the free parameters [15]. This may even outweigh the benefits of hyperbolicity. It would therefore seem preferable to find a system that is symmetric hyperbolic while enlarging the solution space as little as possible, and in particular this should be a second-order system.

In a companion paper [16] we have proposed a definition of symmetric hyperbolicity for second-order systems. This can be used to obtain a well-posed initial-boundary value problem in the same way as in first-order systems, but without enlarging the solution space. Here we show symmetric hyperbolicity for two formulations of the Einstein equations that are variants of the Arnowitt-Deser-Misner (ADM) equa-

tions, namely the Baumgarte-Shapiro-Shibata-Nakamura (BSSN) system [17,18], which is already widely used in numerical relativity [19–22], and a simpler related system suggested by Nagy, Ortiz and Reula (NOR) [4].

We know of two previous partial results concerning the hyperbolicity of a second-order, ADM-like version of the Einstein equations. In [23] it was shown that a first-order reduction of the BSSN system is strongly hyperbolic, and a variant with some superluminal characteristic speeds is symmetric hyperbolic. It was then noted that the auxiliary constraints associated with the introduction of the first-order auxiliary variables form a closed subsystem of the constraint system. This means that if the auxiliary constraints are obeyed initially, and if suitable boundary conditions are imposed, they are obeyed during the evolution, even if the other constraints are not obeyed. In this sense, the introduction of the auxiliary variables has not enlarged the solution system. In [4] it was shown that the NOR system (in second-order form), and its associated constraint system are strongly hyperbolic, by using a pseudo-differential reduction to first order which also does not enlarge the solution space. This method relies in an essential way on Fourier transforms, and so cannot be used to construct a locally conserved energy.

Here we go beyond these two papers in showing symmetric hyperbolicity for the second order BSSN and NOR systems and their associated constraint systems, without any reduction to first order. We do this by defining characteristic variables, finding a conserved positive definite covariant energy, and expressing it in terms of the characteristic variables. We have chosen BSSN and NOR because BSSN is popular with numerical relativists today, whereas NOR seems a simpler version of BSSN that shares all its advantages.

The paper is organized as follows. In Sec. II we introduce our method and general notation. In Sec. III we prove symmetric hyperbolicity for the BSSN evolution equations and the associated constraint system, and in Sec. IV we do the same for the NOR system. We take a first look at the initial boundary problem for both systems in Sec. V. We propose a family of boundary conditions for a smooth boundary where the shift is tangential to the boundary. We show that there are no arbitrarily rapidly growing modes—this amounts to a strong indication of well-posedness but is short of a proof. Section VI contains our conclusions.

II. METHOD AND NOTATION

Here we briefly summarize the relevant notation and methods of [16]. We need to split 3-tensors into their longitudinal and transversal parts with respect to a given direction n_i. Assume a 3-metric, say γ_{ij}, which will be used to raise and lower indices, and let n^i be a unit vector with respect to this metric. Then

$$q_i^j \equiv \delta_i^j - n_i n^j \qquad (2)$$

is the projector into the space transversal to n_i. A tensor index that has been projected will be denoted by the index

$A, B, \ldots$ instead of $i, j, \ldots$. An index n denotes a tensor index contracted with n_i or n^i. A self-explanatory example of this notation is

$$P^i \partial_i \equiv P_n \partial_n + P^A \partial_A . \tag{3}$$

A pair qq of indices indicates a contraction with q_{ij}, and for tensors of rank 2 or higher, we use the convention that they are totally tracefree on their projected indices.

Now consider a system of evolution equations that are second order in space and first order in time. Linearize around a solution, and approximate the background-dependent coefficients of the linearized system as constant (frozen). Retain only the principal part of the equations. We call the resulting linear system with constant coefficients strongly hyperbolic if for any given direction n^i we can find a complete set of characteristic variables U that obey

$$\partial_t U = \lambda \partial_n U + \text{transversal derivatives} \tag{4}$$

for real constants λ. Here $-\lambda$ is the propagation speed in the n_i direction. The U are constructed from $\{u\} \subset \{u, \partial_i u\}$, where we include $\partial_i u$ of those variables that have a second spatial derivative in the evolution equation. The characteristic variables are complete if they span $\{u\}$.

This definition has two important consequences: transversal derivatives $\partial_A u$ are automatically zero speed variables, and arbitrary multiples of transversal derivatives $\partial_A u$ can be added to any characteristic variable. Below we shall find the characteristic variables U in two steps: we first find a set of non-zero speed variables U' that do not contain any transversal derivatives, then add transversal derivatives to them until we can express the energy and flux in terms of the modified characteristic variables U.

We call the system symmetric hyperbolic if it admits an energy

$$E = \int_\Omega \epsilon dV, \tag{5}$$

where ϵ is covariant, positive definite, and conserved in the sense that

$$\partial_t \epsilon = \partial_i F^i \tag{6}$$

for some flux F^i, *and if we can express ϵ and F^n in terms of* characteristic variables.

The simplest example is the wave equation in the form

$$\partial_t \phi = \Pi, \tag{7}$$

$$\partial_t \Pi = \partial_i \partial^i \phi, \tag{8}$$

with $u = (\phi, \Pi)$ and $\tilde{u} = (\partial_i \phi, \Pi)$. The characteristic variables in a given direction n_i are

$$U_\pm \equiv \Pi \pm \partial_n \phi, \tag{9}$$

$$U_A \equiv \partial_A \phi, \tag{10}$$

with speeds $\lambda = (\pm 1, 0)$. The covariant energy and flux are

$$\epsilon = \Pi^2 + \partial_i \phi \partial^i \phi, \tag{11}$$

$$F^i = 2 \Pi \partial^i \phi. \tag{12}$$

In terms of characteristic variables they are

$$\epsilon = \frac{1}{2}(U_+^2 + U_-^2) + U_A U^A, \tag{13}$$

$$F^n = \frac{1}{2}(U_+^2 - U_-^2). \tag{14}$$

Therefore

$$\frac{dE}{dt} = \int_{\partial\Omega} \frac{1}{2}(U_+^2 - U_-^2) dS \tag{15}$$

where $U_\pm$ are now the characteristic variables normal to the boundary $\partial\Omega$.

The maximally dissipative boundary condition

$$U_+ = \kappa U_- + f \tag{16}$$

with $|\kappa| \leq 1$ and f a given function then guarantees that the growth of E is bounded by f and that E does not grow for $f = 0$.

To make the energy positive definite in ϕ itself rather than just $\partial_i \phi$, one can add a term $\alpha^2 \phi^2$ to ϵ, with $\alpha > 0$ constant. With a maximally dissipative boundary condition and $f = 0$, E is then bounded by $E(t) \leq e^{\alpha t} E(0)$.

Finally, allow the linearized system to have variable coefficients (from the non-constant background solution) and a non-principal part. For a finite time interval, the energy is then still bounded as $E(t) \leq K e^{\alpha t}$ for some constants K and α, and the linearized initial-boundary value problem remains well-posed. Therefore it is sufficient to establish well-posedness to examine the principal part in the frozen coefficient approximation. On an intuitive level, this is so because the purpose of well-posedness is to rule out instabilities of the non-convergent type, which have a growth rate that grows with spatial frequency. Well-posedness of the linearized problem is a necessary condition for well-posedness of the full non-linear (quasilinear) problem.

III. THE BSSN FORMULATION OF THE EINSTEIN EQUATIONS

A. Field equations

The BSSN formulation of the Einstein equations (without matter) is obtained from the ADM form of the Einstein equations [24],

$$\partial_t \gamma_{ij} = \mathcal{L}_\beta \gamma_{ij} - 2\alpha K_{ij}, \tag{17}$$

$$\partial_t K_{ij} = \mathcal{L}_\beta K_{ij} - D_i D_j \alpha + \alpha (R_{ij} - 2K_{il}K^l{}_j + KK_{ij}), \tag{18}$$

$$H \equiv R - K_{ij}K^{ij} + K^2 = 0, \tag{19}$$

$$M_i \equiv D_j K^j{}_i - D_i K = 0, \tag{20}$$

by introducing the new variables

$$\tilde{\gamma}_{ij} \equiv (\det \gamma)^{-1/3} \gamma_{ij}, \tag{21}$$

$$\tilde{\Gamma}^i \equiv \tilde{\gamma}^{ij} \tilde{\gamma}^{kl} \tilde{\gamma}_{jk,l}, \tag{22}$$

$$\phi \equiv \frac{1}{12} \ln \det \gamma, \tag{23}$$

$$\tilde{A}_{ij} \equiv (\det \gamma)^{-1/3} \left(K_{ij} - \frac{1}{3} \gamma_{ij} K \right). \tag{24}$$

In the remainder of this section, indices are moved with $\tilde{\gamma}_{ij}$ and its inverse $\tilde{\gamma}^{ij}$. Generalizing the BSSN equations, we densitize the lapse α with the determinant of the metric as

$$\alpha = e^{6\sigma\phi} Q \tag{25}$$

where σ is a constant and now Q, rather than α, is a given function of the coordinates. The shift β^i is considered a given function of the coordinates. The definition of the $\tilde{\Gamma}^i$ gives rise to the differential constraint

$$G_i \equiv \tilde{\gamma}_{ij} \tilde{\Gamma}^j - \tilde{\gamma}^{jk} \tilde{\gamma}_{ij,k} = 0. \tag{26}$$

The definition of $\tilde{A}_{ij}$ gives rise to the algebraic constraint

$$T \equiv \tilde{\gamma}^{ij} \tilde{A}_{ij} = 0, \tag{27}$$

and from the definition of $\tilde{\gamma}_{ij}$ we have the algebraic constraint

$$D \equiv \ln \det \tilde{\gamma} = 0. \tag{28}$$

The BSSN equations are first order in time, second order in space, and quasilinear. The principal part of the evolution equations for $\tilde{A}_{ij}$, K and $\tilde{\Gamma}^i$ is given by the highest spatial derivatives, $(\partial^2 \phi, \partial^2 \tilde{\gamma}, \partial \tilde{A}, \partial K, \partial \tilde{\Gamma})$. The evolution equations for $\tilde{\gamma}$ and ϕ do not contain any of these highest derivatives. We define their principal part to be given by the next highest derivatives, that is $(\partial \phi, \partial \tilde{\gamma}, \tilde{A}, K, \tilde{\Gamma})$. Note that with this definition of the principal part the equations are still quasilinear, because the evolution equation for γ_{ij} is linear in K_{ij}. Note also that $\beta^k \partial_k$ is part of the principal part of the equations while $\partial \beta$ terms are not. We define the derivative operator

$$\partial_0 \equiv \alpha^{-1} (\partial_t - \beta^k \partial_k). \tag{29}$$

It is the derivative along the unit vector field normal to the slices of constant time. The principal part of the evolution equations is then

$$\partial_0 \phi \simeq -\frac{1}{6} K, \tag{30}$$

$$\partial_0 \tilde{\gamma}_{ij} \simeq -2 \tilde{A}_{ij}, \tag{31}$$

$$\partial_0 K \simeq -6\sigma e^{-4\phi} \tilde{\gamma}^{ij} \phi_{,ij}, \tag{32}$$

$$\partial_0 \tilde{A}_{ij} \simeq e^{-4\phi} \left[-\frac{1}{2} \tilde{\gamma}^{mn} \tilde{\gamma}_{ij,mn} - 2(1+3\sigma)\phi_{,ij} + a \tilde{\gamma}_{k(i} \tilde{\Gamma}^k{}_{,j)} \right.$$
$$\left. + (1-a) \tilde{\gamma}^{kl} \tilde{\gamma}_{k(i,j)l} \right]^{\mathrm{TF}} - \frac{c}{6} e^{-4\phi} \tilde{\gamma}_{ij} \tilde{\gamma}^{mn} \tilde{\gamma}^{kl} \tilde{\gamma}_{kl,mn}, \tag{33}$$

$$\partial_0 \tilde{\Gamma}^i \simeq 2(b-1) \tilde{\gamma}^{ij} \tilde{\gamma}^{kl} \tilde{A}_{jk,l} - \frac{4}{3} b \tilde{\gamma}^{ij} K_{,j}, \tag{34}$$

where $\simeq$ means equal up to non-principal terms, and TF indicates the tracefree part. We have added the constraints $a G_{(i,j)}$, $2b M_i$ and $-(c/6)e^{-4\phi} \tilde{\gamma}_{ij} \tilde{\gamma}^{kl} D_{,kl}$ to the field equations with free coefficients a, b and c. Adding these terms changes the evolution off the constraint surface which can affect the hyperbolicity of the system.

The principal part of the Hamiltonian and momentum constraints is

$$H \equiv H_0 + e^{-4\phi} \tilde{\gamma}^{ij} \left[a' G_{i,j} - \frac{c'}{2} D_{,ij} \right], \tag{35}$$

$$H_0 \simeq e^{-4\phi} \tilde{\gamma}^{ij} (\tilde{\gamma}^{kl} \tilde{\gamma}_{ki,jl} - 8\phi_{,ij}), \tag{36}$$

$$M_i \simeq \tilde{A}_{ij,k} \tilde{\gamma}^{jk} - \frac{2}{3} K_{,i}. \tag{37}$$

Here a' and c' parametrize different ways of writing the Hamiltonian constraint that are found in the literature. We shall work explicitly only with H_0, and so a' and c' will not appear below. There are many versions of the BSSN equations which vary in small details in both the principal and non-principal parts. For comparison, the principal part of the version given in [25] is characterized by $\sigma = 0$ (the lapse is not densitized) and $a = b = a' = c' = 1$, $c = 0$.

The constraints are compatible with the evolution equations, which means that they form a closed evolution system. It is

$$\partial_0 H_0 \simeq -2e^{-4\phi} \tilde{\gamma}^{ij} M_{i,j}, \tag{38}$$

$$\partial_0 M_i \simeq \frac{1}{6} H_{0,i} + e^{-4\phi} \left(\frac{a}{2} \tilde{\gamma}^{jk} G_{i,jk} \right.$$
$$\left. + \frac{a}{6} \tilde{\gamma}^{jk} G_{j,ik} + \frac{1-c}{6} \tilde{\gamma}^{jk} D_{,ijk} \right), \tag{39}$$

$$\partial_0 G_i = 2b M_i, \tag{40}$$

$$\partial_0 T \simeq -\frac{c}{2} e^{-4\phi} \tilde{\gamma}^{ij} D_{,ij}, \tag{41}$$

$$\partial_0 D = -2T. \tag{42}$$

B. Strong hyperbolicity of the main system

For the purpose of decomposing 3-tensors and tensor equations, we define n_i, n^i and q_{ij} with respect to the con-

formal metric $\tilde{\gamma}_{ij}$. In the frozen coefficients approximation, the undifferentiated 3-metric ($\tilde{\gamma}_{ij}$, $\tilde{\gamma}^{ij}$ and ϕ) is to be treated as a background quantity, while $\tilde{A}_{ij}$, $\tilde{\Gamma}^i$, K, $\tilde{\gamma}_{ij,k}$, $\phi_{,i}$ and their derivatives are to be treated as dynamical variables, and decomposed with respect to n_i.

We now prove strong hyperbolicity of the main and constraint systems by constructing a complete set of characteristic variables U. The quantities

$$\tilde{u} \equiv (\partial_i \phi, \partial_i \tilde{\gamma}_{jk}, \tilde{A}_{ij}, \tilde{\Gamma}^i, K), \tag{43}$$

obey the pseudo-first order system

$$\partial_t \tilde{u} \simeq (\alpha P^i + \beta^i) \partial_i \tilde{u} \tag{44}$$

or equivalently

$$\partial_0 \tilde{u} \simeq P^i \partial_i \tilde{u}. \tag{45}$$

[This is not a genuine first-order system because in expressions like $\partial_i(\partial_j \phi)$ that appear on its right-hand side we allow ourselves to commute partial derivatives, rather than treating $\partial_j \phi$ as a 1-form variable d_j, as we would in a genuine reduction to first order.] As discussed in [16], transversal derivatives are automatically zero speed characteristic variables. Here we have $P_n \partial_A \phi = P_n \partial_A \tilde{\gamma}_{ij} = 0$. We shall write the eigenvalues of P_n as $\lambda e^{-2\phi}$. With this definition λ measures the propagation speed in units of the speed of light, measured with respect to the ∂_0 observers. In particular $\lambda = \pm 1$ variables propagate along the light cone. In the following we always use "speed" in this sense.

In tensor components with respect to n_i, the rows of the scalar block of the matrix P_n acting on the variables $(\partial_n \phi, \partial_n \tilde{\gamma}_{nn}, \partial_n \tilde{\gamma}_{qq}, K, \tilde{A}_{nn}, \tilde{A}_{qq}, \tilde{\Gamma}_n)$ are given by

$$P_n \partial_n \phi = -\frac{1}{6} K, \tag{46}$$

$$P_n \partial_n \tilde{\gamma}_{nn} = -2\tilde{A}_{nn}, \tag{47}$$

$$P_n \partial_n \tilde{\gamma}_{qq} = -2\tilde{A}_{qq}, \tag{48}$$

$$P_n K = -6\sigma e^{-4\phi} \partial_n \phi, \tag{49}$$

$$P_n \tilde{A}_{nn} = e^{-4\phi} \left[-\left(\frac{4}{3} + 4\sigma\right) \partial_n \phi + \frac{1}{6} \partial_n \tilde{\gamma}_{nn} \right.$$
$$\left. + \frac{2a}{3}(\tilde{\Gamma}_n - \partial_n \tilde{\gamma}_{nn}) + \frac{1-c}{6}(\partial_n \tilde{\gamma}_{nn} + \partial_n \tilde{\gamma}_{qq}) \right], \tag{50}$$

$$P_n \tilde{A}_{qq} = e^{-4\phi} \left[\left(\frac{4}{3} + 4\sigma\right) \partial_n \phi - \frac{1}{6} \partial_n \tilde{\gamma}_{nn} \right.$$
$$\left. - \frac{2a}{3}(\tilde{\Gamma}_n - \partial_n \tilde{\gamma}_{nn}) - \frac{1+2c}{6}(\partial_n \tilde{\gamma}_{nn} + \partial_n \tilde{\gamma}_{qq}) \right], \tag{51}$$

$$P_n \tilde{\Gamma}_n = -\frac{4b}{3} K + 2(b-1)\tilde{A}_{nn}. \tag{52}$$

The two algebraic constraints $T=0$ and $D=0$ are often enforced in numerical simulations after each time step. We can mimic enforcing $T=0$ in our analysis by setting $\tilde{A}_{nn} + \tilde{A}_{qq} = 0$ and dropping one of the two variables from the system. Similarly, we can mimic enforcing $D=0$ by setting $\partial_n \tilde{\gamma}_{nn} + \partial_n \tilde{\gamma}_{qq} = 0$ and dropping one of the two variables from the system. Both reductions affect the hyperbolicity of the system.

(a) *Algebraic constraints not enforced* The scalar block is diagonalizable for

$$c>0, \quad \eta>0, \quad \sigma>0, \quad c \neq \eta, \tag{53}$$

with eigenvalues

$$\lambda = \{0, \pm\sqrt{c}, \pm\sqrt{\eta}, \pm\sqrt{\sigma}\}, \tag{54}$$

where we have defined the shorthand

$$\eta \equiv \frac{4ab-1}{3}. \tag{55}$$

(Allowing the system to be strongly hyperbolic without enforcing the algebraic constraints is the reason why we have introduced the c term, which potentially gives $\partial_0 \tilde{A}_{ij}$ a nonzero trace.)

(b) *Trace constraint enforced* If we enforce $T=0$ but not $D=0$ this is consistent with the evolution equations only for $c=0$. The scalar block is diagonalizable for

$$c=0, \quad \eta>0, \quad \sigma>0, \tag{56}$$

with

$$\lambda = \{0, 0, \pm\sqrt{\eta}, \pm\sqrt{\sigma}\}. \tag{57}$$

It would be inconsistent to enforce $D=0$ but not $T=0$.

(c) *Both algebraic constraints enforced* If we enforce both algebraic constraints $D=0$ and $T=0$ the scalar block is diagonalizable for

$$\eta>0, \quad \sigma>0. \tag{58}$$

(Note that c becomes irrelevant in this case.) The characteristic variables that do not contain any transversal derivatives are:

$$U'_0 \equiv (b-1)\partial_n \tilde{\gamma}_{nn} + \tilde{\Gamma}_n - 8b\partial_n \phi, \tag{59}$$

$$U'_\pm \equiv (1-4a)\partial_n \tilde{\gamma}_{nn} + 4a\tilde{\Gamma}_n - 8\partial_n \phi$$
$$\pm e^{2\phi}\sqrt{\eta}(6\tilde{A}_{nn} - 4K), \tag{60}$$

$$V'_\pm \equiv 6\sqrt{\sigma}\partial_n \phi \mp e^{2\phi}K, \tag{61}$$

with speeds

$$\lambda = (0, \pm \sqrt{\eta}, \pm \sqrt{\sigma}). \tag{62}$$

The vector block of P_n is

$$P_n \partial_n \tilde{\gamma}_{An} = -2\tilde{A}_{An}, \tag{63}$$

$$P_n \tilde{A}_{An} = e^{-4\phi} \frac{a}{2} (\tilde{\Gamma}_A - \partial_n \tilde{\gamma}_{An}), \tag{64}$$

$$P_n \tilde{\Gamma}_A = 2(b-1)\tilde{A}_{An}, \tag{65}$$

as well as the trivial $P_n \partial_A \phi = P_n \partial_A \tilde{\gamma}_{nn} = P_n \partial_A \tilde{\gamma}_{qq} = 0$. It is diagonalizable for $ab > 0$, with nontrivial characteristic variables

$$U'_A \equiv (b-1)\partial_n \tilde{\gamma}_{nA} + \tilde{\Gamma}_A, \tag{66}$$

$$U'_{\pm A} \equiv -a\partial_n \tilde{\gamma}_{An} + a\tilde{\Gamma}_A \pm 2\sqrt{ab} e^{2\phi} \tilde{A}_{An}, \tag{67}$$

with speeds $\lambda = (0, \pm \sqrt{ab})$. The tensor block is

$$P_n \partial_n \tilde{\gamma}_{AB} = -2\tilde{A}_{AB}, \tag{68}$$

$$P_n \tilde{A}_{AB} = -e^{-4\phi} \frac{1}{2} \partial_n \tilde{\gamma}_{AB}, \tag{69}$$

as well as the trivial $P_n \partial_A \tilde{\gamma}_{Bn} = 0$. It is always diagonalizable, with nontrivial characteristic variables

$$U'_{\pm AB} \equiv \frac{1}{2} \partial_n \tilde{\gamma}_{AB} \mp e^{2\phi} \tilde{A}_{AB}, \tag{70}$$

with speeds $\lambda = \pm 1$. We see that the vector and tensor sectors do not add any new conditions for strong hyperbolicity.

In summary, we have shown that the BSSN system with both algebraic constraints enforced continuously is strongly hyperbolic if $\sigma > 0$ and $\eta > 0$. If neither algebraic constraint is enforced, $c > 0$ and $c \neq \eta$ are also required. On the other hand, if only the trace constraint is enforced, $c = 0$ is required. (If both constraints are enforced, c is irrelevant and can be set to zero.) The characteristic spectrum of the complete system is

$$\lambda = \{0, \pm \sqrt{\sigma}, \pm \sqrt{\eta}, \pm \sqrt{ab}, \pm 1\}. \tag{71}$$

C. Strong hyperbolicity of the constraint system

We construct characteristic variables for the constraint system from the set

$$\tilde{u}_c \equiv (H_0, M_i, \partial_i G_j, \partial_i T, \partial_i \partial_j D). \tag{72}$$

The scalar sector of the constraint propagation is

$$P_n H_0 = -2e^{-4\phi} M_n, \tag{73}$$

$$P_n M_n = \frac{1}{6} H_0 + e^{-4\phi} \left(\frac{2a}{3} \partial_n G_n + \frac{1-c}{6} \partial_n^2 D \right), \tag{74}$$

$$P_n \partial_n G_n = 2b M_n, \tag{75}$$

$$P_n \partial_n T = -\frac{c}{2} e^{-4\phi} \partial_n^2 D, \tag{76}$$

$$P_n \partial_n^2 D = -2\partial_n T. \tag{77}$$

If we define

$$C'_\pm \equiv 4a\partial_n G_n \pm 6\sqrt{\eta} e^{2\phi} M_n + e^{4\phi} H_0 \tag{78}$$

the characteristic variables are

$$C'_0 \equiv e^{4\phi} b H_0 + \partial_n G_n, \tag{79}$$

$$\sqrt{c} \partial_n^2 D \mp 2e^{2\phi} \partial_n T, \tag{80}$$

$$(1-c)\left(\partial_n^2 D \mp \frac{2}{\sqrt{\eta}} e^{2\phi} \partial_n T \right) + \left(1 - \frac{c}{\eta} \right) C'_\pm, \tag{81}$$

with speeds $\lambda = (0, \pm \sqrt{c}, \pm \sqrt{\eta})$. If we optionally impose the trace constraint, the characteristic variables are C'_0 and $C'_\pm + \partial_n^2 D$ with speeds $\lambda = (0, \pm \sqrt{\eta})$. If we impose both the trace and determinant constraints, we are left with C'_0 and $C'_\pm$ with speeds $\lambda = (0, \pm \sqrt{\eta})$.

The vector sector of the constraint propagation is

$$P_n M_A = e^{-4\phi} \frac{a}{2} \partial_n G_A, \tag{82}$$

$$P_n \partial_n G_A = 2b M_A, \tag{83}$$

as well as $P_n \partial_A G_n = P_n \partial_A T = P_n \partial_A \partial_n D = 0$. The nontrivial characteristic variables are

$$C'_{\pm A} \equiv a\partial_n G_A \pm 2\sqrt{ab} e^{2\phi} M_A, \tag{84}$$

with speeds $\lambda = \pm \sqrt{ab}$. The tensor sector is completely trivial, with $P_n \partial_A G_B = P_n \partial_A \partial_B T = 0$ (including the traces). The constraint system, in all three cases, is strongly hyperbolic as long as the main system is strongly hyperbolic.

D. Main system energy

We look for an energy density ϵ of the second-order system that is positive definite in $\tilde{A}_{ij}$, K, $\tilde{\Gamma}_i$, $\tilde{\gamma}_{ij,k}$ and $\phi_{,i}$ and obeys a conservation law. With the shorthands $t_i \equiv \tilde{\gamma}^{jk} \tilde{\gamma}_{jk,i} = D_{,i}$ and $d_i \equiv \tilde{\gamma}^{jk} \tilde{\gamma}_{ij,k}$, the most general quadratic form in these variables is

$$\epsilon = c_0 e^{4\phi} \tilde{A}_{ij}{}^2 + c_1 \tilde{\Gamma}_i{}^2 + c_2 t_i{}^2 + c_3 d_i{}^2 + c_4 d_i \tilde{\Gamma}^i + c_5 t_i \tilde{\Gamma}^i$$
$$+ c_6 d_i t^i + c_7 e^{4\phi} K^2 + c_8 \tilde{\gamma}_{ij,k}{}^2 + c_9 \tilde{\gamma}_{ij,k} \tilde{\gamma}^{ik,j} + c_{10} \tilde{\Gamma}_i \phi^{,i}$$
$$+ c_{11} e^{4\phi} TK + c_{12} \phi_{,i}{}^2 + c_{13} d_i \phi^{,i} + c_{14} t_i \phi^{,i} + c_{15} e^{4\phi} T^2. \tag{85}$$

In the cross term $\tilde{\gamma}_{ij,k} \tilde{\gamma}^{ik,j}$ and similar terms in the remainder of the paper indices are raised only after differentiation. The general ansatz for the flux F^i contains 14 free coefficients.

After some linear algebra it is possible to see that this ϵ and F^i together obey a conservation law only if

$$4ab = 1 + 3\sigma, \tag{86}$$

or equivalently $\eta = \sigma$.

There are additional restrictions depending on whether or not we enforce the algebraic constraints during evolution. If we work with non-vanishing D and T then we need $ab = \sigma = 1$ and $c_{11}(c-1) = 0$. On the other hand, if we enforce both algebraic constraints the parameter c and several of the co-efficients c_k become irrelevant, with no additional constraints on the parameters.

From now on, in order to simplify the equations, we enforce both the $T=0$ and $D=0$ constraints in the remainder of this section.

The generic conserved energy depends on four free coefficients, one of which is an overall factor. With the shorthands

$$\epsilon_0 = e^{4\phi} \tilde{A}_{ij}{}^2 + \frac{1}{4} \tilde{\gamma}_{ij,k}{}^2 + \frac{2a - ab - 1}{2} \tilde{\gamma}_{ij,k} \tilde{\gamma}^{ik,j} - a d_i (\tilde{\Gamma}^i$$
$$- 8b \phi^{,i}), \tag{87}$$

$$\epsilon_1 = [\tilde{\Gamma}_i - 8b \phi_{,i} + (b-1) d_i]^2, \tag{88}$$

$$\epsilon_2 = d_i{}^2 - \tilde{\gamma}_{ij,k} \tilde{\gamma}^{ki,j}, \tag{89}$$

$$\epsilon_3 = e^{4\phi} K^2 + 36 \sigma \phi^{,i} \phi_{,i} \tag{90}$$

the energy density is

$$\epsilon = c_0 \epsilon_0 + c_1 \epsilon_1 + [c_3 - (b-1)^2 c_1] \epsilon_2 + c_7 \epsilon_3. \tag{91}$$

The free parameters c_0, c_1, c_3 and c_7 will be restricted below by inequalities derived from the requirement that ϵ be positive definite.

This energy reduces to that given in [23] for a first-order reduction of BSSN with the change of notation $\sigma \to 2\sigma$, $b \to m$, $\tilde{\gamma}_{ij,k} \to d_{kij}$ and $\phi_{,i} \to d_i/12$, the restriction $a=1$ to the parameters of the evolution equations, fixing the overall scale as $c_0 = 1$, and the further restrictions $c_1 = 1/(2b-2)$, $c_3 = 0$ and $c_7 = 1/\sigma = 3/(4b-1)$ on the coefficients of the energy. The condition (86) on σ reduces to a similar condition in [23]. The choice of Sarbach *et al.* can be interpreted as follows: $c_3 = 0$ and $c_1 = a/(2b-2)$ eliminate d_i from the energy ($c_3 = c_4 = c_{13} = 0$); assuming those conditions, $a = 1$ is the only choice that eliminates the cross term $\tilde{\gamma}_{ij,k} \tilde{\gamma}^{ki,j}$. Their choice for c_7 has no relevant effect. Note that the positivity condition on their choice of c_1 forces $b > 1$, and together with $a = 1$ this gives rise to a superluminal speed. By contrast, by retaining the contractions t_i and d_i in the ansatz for the energy we shall be able to make all speeds physical.

The four terms in the energy are conserved separately:

$$\partial_0 \epsilon_0 = \partial_i F_0^i, \tag{92}$$

$$\partial_0 \epsilon_1 = 0, \tag{93}$$

$$\partial_0 \epsilon_2 = \partial_i F_2^i, \tag{94}$$

$$\partial_0 \epsilon_3 = \partial_i F_3^i. \tag{95}$$

[Condition (86) is required only for the conservation of ϵ_0.] The fluxes are given by

$$F_0^i = 2 \tilde{A}^{ij} [a \tilde{\Gamma}_j - a(b-1) d_j - 2(1+3\sigma) \phi_{,j}] + \tilde{\gamma}^{il} \tilde{A}^{jk}$$
$$\times [2(1 + ab - 2a) \tilde{\gamma}_{jl,k} - \tilde{\gamma}_{jk,l}], \tag{96}$$

$$F_2^i = 4 \tilde{A}^{jk} (\tilde{\gamma}^{im} \tilde{\gamma}_{mj,k} - \tilde{\gamma}^i{}_j \tilde{\gamma}_{km}{}^{,m}), \tag{97}$$

$$F_3^i = -12 \sigma K \phi^{,i}. \tag{98}$$

The energy density in terms of characteristic variables is

$$\epsilon = \frac{c_0}{2} (U_{+AB}^2 + U_{-AB}^2) + \frac{c_0}{4ab} (U_{+A}^2 + U_{-A}^2)$$
$$+ \frac{c_0 c_7}{16 \sigma (2c_0 + 3c_7)} (U_+^2 + U_-^2) + \frac{2c_0 + 3c_7}{6} (V_+^2 + V_-^2)$$
$$+ \text{quadratic in zero speed variables}, \tag{99}$$

and the flux is

$$e^{2\phi} F^n = \frac{c_0}{2} (U_{+AB}^2 - U_{-AB}^2) + \frac{c_0}{4\sqrt{ab}} (U_{+A}^2 - U_{-A}^2)$$
$$+ \frac{c_0 c_7}{16 \sqrt{\sigma} (2c_0 + 3c_7)} (U_+^2 - U_-^2) + \frac{2c_0 + 3c_7}{6}$$
$$\times \sqrt{\sigma} (V_+^2 - V_-^2). \tag{100}$$

We have modified the characteristic variables by adding terms in $\partial_A \tilde{\gamma}_{ij}$ in order to write ϵ in terms of characteristic variables. With the shorthands

$$z_1 = \frac{c_3 - (b-1)^2 c_1}{c_0}, \tag{101}$$

$$z_2 = 1 + 3ab - 4a + 6z_1, \tag{102}$$

$$z_3 = a - ab - 2z_1, \tag{103}$$

$$z_4 = 2 + 3ab - 5a + 6z_1, \tag{104}$$

$$z_5 = 1 + ab - 2a + 2z_1, \tag{105}$$

$$z_6 = \frac{c_0}{2\sqrt{\sigma}(2c_0 + 3c_7)}, \tag{106}$$

the modified characteristic variables are

$$U_{\pm} = (1 - 4a) \partial_n g_{nn} + 4a \tilde{\Gamma}_n - 8 \partial_n \phi \pm \sqrt{\sigma} e^{2\phi} (6 \tilde{A}_{nn} - 4K)$$
$$- 2z_2 \partial^A g_{An}, \tag{107}$$

$$V_{\pm} = 6\sqrt{\sigma} \partial_n \phi \mp e^{2\phi} K - z_6 U_{\pm}, \tag{108}$$

$$U_{\pm A} = \pm 2\sqrt{ab}\, e^{2\phi}\tilde{A}_{An} + a\tilde{\Gamma}_A - a\partial_n g_{An} + z_3 \partial^B g_{AB}$$

$$+ \frac{z_4}{2}\partial_A g_{nn} - 8ab\partial_A \phi, \tag{109}$$

$$U_{\pm AB} = \mp e^{2\phi}\tilde{A}_{AB} + \frac{1}{2}\partial_n g_{AB} - \frac{z_5}{2}(\partial_A g_{Bn} + \partial_B g_{An}$$

$$- q_{AB}\partial^C g_{Cn}). \tag{110}$$

The expressions for U_0 and U_A are not given, but they also include transversal derivatives. Note that because $\eta = \sigma$, $U'_{\pm}$ and $V'_{\pm}$ are eigenvectors for the same eigenvalues, and we have used this to make the energy diagonal in $U_{\pm}$ and $V_{\pm}$.

E. Constraint system energy

The constraint energy is quadratic in H_0, M_i and $G_{i,j}$. (We must use $G_{i,j}$ rather than G_i so that all terms are of the same order in derivatives.) The most general conserved expression is

$$\epsilon_c = w_0 e^{8\phi}H_0^2 + 12w_1 e^{4\phi}M_i^2 + \frac{3aw_1}{b}(G_{i,j})^2 + w_2 G_{i,j}G^{j,i}$$

$$+ 2\frac{w_0 + w_1}{b}e^{4\phi}G_i{}^{,i}H_0 + \frac{w_0 + (1+ab)w_1 - b^2 w_2}{b^2}(G_i{}^{,i})^2, \tag{111}$$

with arbitrary coefficients w_0, w_1 and w_2. The corresponding flux is

$$F_c^i = 4w_1(e^{4\phi}H_0 M^i + 3aG^{j,i}M_j + aM^i G^j{}_{,j})$$

$$+ 4bw_2(G^{i,j}M_j - M^i G^j{}_{,j}). \tag{112}$$

In terms of characteristic variables, the constraint energy and flux are

$$\epsilon_c = \frac{3w_1}{2ab}(C_{-A}C_-^A + C_{+A}C_+^A) + \frac{w_1}{6\sigma}(C_-^2 + C_+^2)$$

$$+ \text{quadratic in zero speed variables}, \tag{113}$$

$$e^{2\phi}F_c^n = \frac{3w_1}{2\sqrt{ab}}(C_{+A}C_+^A - C_{-A}C_-^A) + \frac{w_1}{6\sqrt{\sigma}}(C_+^2 - C_-^2), \tag{114}$$

where the modified non-zero speed characteristic variables are

$$C_{\pm} = e^{4\phi}H_0 \pm 6e^{2\phi}\sqrt{\sigma}M_n + 4a\partial_n G_n + \frac{aw_1 - bw_2}{w_1}\partial^A G_A, \tag{115}$$

$$C_{\pm A} = \pm 2\sqrt{ab}\, e^{2\phi}M_A + a\partial_n G_A + \frac{w_2 b}{3w_1}\partial_A G_n. \tag{116}$$

Note that w_2 does not appear explicitly in Eq. (113) or Eq. (114), but it does appear in the definition of the characteristic variables.

F. Symmetric hyperbolicity and causal speeds

With the condition $\eta = \sigma$ that is required for conservation of the main energy, the complete spectrum can be written entirely in terms of σ:

$$\lambda = \left(0, \pm\sqrt{\sigma}, \pm\frac{\sqrt{1+3\sigma}}{2}, \pm 1\right). \tag{117}$$

Strong hyperbolicity requires $\sigma > 0$ and the absence of super-luminal speeds requires $\sigma \leq 1$. Note that with $\sigma = ab = 1$ all speeds are either 0 or ± 1.

Now we impose positive definiteness of the energies, that is, all eigenvalues of their respective matrices are strictly positive. This means that the main energy vanishes if and only if $\tilde{\gamma}_{ij}$ and ϕ are constant and all other variables vanish, and that the constraint energy vanishes if and only if G_i is constant and the Hamiltonian and momentum constraints vanish. (We are assuming that the two algebraic constraints vanish because they are being continuously re-imposed.)

To see when the constraint energy is positive definite, we need to decompose $G_{i,j}$ as

$$G_{i,j} = S_{ij} + A_{ij} + \frac{1}{3}\gamma_{ij}G^k{}_{,k} \tag{118}$$

where S_{ij} is symmetric and tracefree and A_{ij} is antisymmetric. We obtain a sum of simple squares and a quadratic form in the two variables H_0 and $G^i{}_{,i}$.

The constraint energy is positive definite, and all speeds are causal, if and only if

$$w_0 > 0, \quad 0 < w_1 < 3w_0,$$

$$\frac{w_0 + w_1}{4w_0} < ab \leq 1, \quad -ab < \frac{w_2 b^2}{3w_1} < ab - \frac{w_0 + w_1}{2w_0}. \tag{119}$$

It is interesting to see how w_1 affects the possible range for ab.

The positivity of the main energy is more complicated. We must take into account that the partial traces t_i and d_i of the three-index object $\tilde{\gamma}_{ij,k}$ appear both explicitly and implicitly in the energy. We therefore decompose it as

$$\tilde{\gamma}_{ij,k} = \frac{1}{5}[(3d_{(i} - t_{(i})\tilde{\gamma}_{j)k} + (2t_k - d_k)\tilde{\gamma}_{ij}] + f_{ijk} \tag{120}$$

where f_{ijk} is completely tracefree. We then obtain a quadratic form in t_i and d_i, plus $c_8 f_{ijk}^2 + c_9 f_{ijk}f^{ikj}$. To analyze the positivity of the latter we decompose f_{ijk} in its 12 independent frame components and analyze the corresponding quadratic form by brute force. It turns out to be positive if and only if $c_8 > 0$ and $-c_8 < c_9 < 2c_8$. Finally, assuming causal speeds, we obtain the positivity conditions

$$c_0 > 0, \quad c_7 > 0, \quad 2a^2 c_0 < (4ab - 1)c_1,$$

$$5c_0 \frac{2a^2 c_0 - (4ab-1)c_1}{36 c_1} < c_3 - (b-1)^2 c_1$$

$$+ \frac{1 + 2ab - 4a}{4} c_0 < 0. \tag{121}$$

The three sets of inequalities are compatible, and can be solved sequentially, in this order: first choose c_0, c_7, w_0 and w_1 independently, then a and b, followed by c_1. Finally choose c_3 and w_2. It is clear from this construction that the solutions form a single connected set. For example, a possible solution with $\lambda = (0, \pm 1)$ is

$$a = b = \sigma = 1, \tag{122}$$

$$c_0 = c_7 = 1, \quad c_1 = 2, \quad c_3 = 0, \tag{123}$$

$$w_0 = 1, \quad w_1 = \frac{1}{2}, \quad w_2 = 0. \tag{124}$$

IV. THE NOR FORMULATION OF THE EINSTEIN EQUATIONS

A. Field equations

In this section, all indices are moved with γ_{ij} and its inverse γ^{ij}. The NOR system is obtained from the ADM system with densitized lapse by introducing the variables

$$f_i \equiv \gamma_{ij}{}^{,j} - \frac{\rho}{2} \gamma^{jk} \gamma_{jk,i}, \tag{125}$$

where ρ is a constant parameter. In [4] the choice $\rho = 1$ is made. $\rho = 2/3$ makes $f_i = \widetilde{\gamma}_{ij} \widetilde{\Gamma}^j$. The definition of f_i gives rise to the constraint

$$G_i \equiv f_i - \gamma_{ij}{}^{,j} + \frac{\rho}{2} \gamma^{jk} \gamma_{jk,i} = 0. \tag{126}$$

As in the BSSN system, we parameterise the use of this constraint with a and the use of the momentum constraint with b by adding $aG_{(i,j)}$ and $2bM_i$ to the evolution equations for K_{ij} and f_i. Following [4], we also add $c\gamma_{ij}H$ to the evolution equation for K_{ij} with free parameter c. The principal part of the evolution equations is

$$\partial_0 \gamma_{ij} \simeq -2K_{ij}, \tag{127}$$

$$\partial_0 K_{ij} \simeq \frac{1}{2} [a(f_{i,j} + f_{j,i}) - \gamma_{ij,k}{}^{k} + (1-a)(\gamma_{ki,j}{}^{k} + \gamma_{kj,i}{}^{k})$$

$$+ (a\rho - 1 - \sigma) \gamma^{kl} \gamma_{kl,ij}] + c\gamma_{ij}H, \tag{128}$$

$$\partial_0 f_i \simeq 2(b-1)K_{ij}{}^{j} + (\rho - 2b)K_{,i}, \tag{129}$$

where

$$H \equiv H_0 + a' G_i^i, \tag{130}$$

$$H_0 \simeq \gamma_{ij}{}^{,ij} - \gamma^{ij} \gamma_{ij,k}{}^{,k}, \tag{131}$$

$$M_i \simeq K_{ij}{}^{,j} - K_{,i}. \tag{132}$$

In [4], $\rho = a = a' = 1$.

The constraints system is

$$\partial_0 H_0 \simeq -2 M_i^{,i}, \tag{133}$$

$$\partial_0 M_i \simeq -\left(\frac{1}{2} + 2c\right) H_{0,i} + \frac{a}{2} G_{i,j}{}^{,j} - \left(\frac{a}{2} + 2ca'\right) G_{j,i}{}^{,j}, \tag{134}$$

$$\partial_0 G_i = 2b M_i. \tag{135}$$

B. Strong hyperbolicity

This proceeds exactly as in the BSSN system, with pseudo-first order variables

$$\widetilde{u} \equiv (\partial_i \gamma_{jk}, K_{ij}, f_i) \tag{136}$$

and

$$\widetilde{u}_c = \{H_0, M_i, \partial_i G_j\} \tag{137}$$

for the constraint system. We define n_i, n^i and q_{ij} with respect to γ_{ij}. (In [4], a flat auxiliary metric is introduced instead.)

The scalar sector of the main system is diagonalizable for $\chi > 0$, $\sigma > 0$ and $\chi \neq \sigma$, with

$$\lambda = \{0, \pm \sqrt{\chi}, \pm \sqrt{\sigma}\}, \tag{138}$$

with the shorthand

$$\chi \equiv 1 + 4c(1 - a'b). \tag{139}$$

If $\eta = \chi$ we can diagonalize the scalar sector also for $\chi = \sigma$. [Note that while we use the same shorthand η defined by Eq. (55) as in the BSSN system, $\sqrt{\eta}$ is not a speed in the NOR system.]

The vector sector is diagonalizable for $ab > 0$ with

$$\lambda = \{0, \pm \sqrt{ab}\}, \tag{140}$$

and the tensor sector is always diagonalizable, with $\lambda = \pm 1$. For general values of $(\rho, \sigma, a, b, c, a')$ the characteristic variables are too long to give here.

The scalar sector of the constraint system is diagonalizable for $\chi > 0$ with characteristic variables

$$C_0 \equiv \partial_n G_n + b H_0, \tag{141}$$

$$C_\pm \equiv (1 + 4c)H_0 + 4ca' \partial_n G_n \mp 2\sqrt{\chi} M_n, \tag{142}$$

with speeds $\lambda = \{0, \pm \sqrt{\chi}\}$, and the vector sector is diagonalizable for $ab > 0$ with

$$C_{\pm A} \equiv a \partial_n G_A \pm 2\sqrt{ab} M_A, \tag{143}$$

with speeds $\lambda = \{\pm \sqrt{ab}\}$.

The union of conditions for both the main system and the constraint system to be strongly hyperbolic, and for all speeds to be physical ($|\lambda| \leq 1$), is

$$0 < \sigma \leq 1, \quad 0 < ab \leq 1, \quad 0 < \chi \leq 1, \qquad (144)$$

and either $\chi \neq \sigma$, or $\chi = \sigma = \eta$. The union of all speeds is

$$\lambda = \{0, \pm 1, \pm \sqrt{ab}, \pm \sqrt{\sigma}, \pm \sqrt{\chi}\}. \qquad (145)$$

We see that we can make all of these speeds either zero or one by choosing

$$ab = \sigma = \chi = 1 \Rightarrow c(1 - a'b) = 0, \qquad (146)$$

that is, either $c = 0$ or $a' = a$. From now on we restrict ourselves to the choice $c = 0$. This makes $\chi = 1$, and removes a' from the system. We keep a, b, ρ and σ free, with the single restriction that $\sigma = 1$ requires $ab = 1$. The value of ρ is irrelevant for strong hyperbolicity. Our results on strong hyperbolicity of the NOR system confirm and generalize those of [4], which were obtained using a pseudo-differential reduction to first order.

C. Main system energy

The most general ansatz for the energy density ϵ is quadratic in $\gamma_{ij,k}$, K_{ij} and f_i. As well as contracting the free indices on pairs of these, we can form the contractions K, $d_i \equiv \gamma^{\cdot j}_{ij}$ and $t_i \equiv \gamma^{jk} \gamma_{jk,i}$ first:

$$\epsilon = c_0 K_{ij}^2 + c_1 f_i^2 + c_2 t_i^2 + c_3 d_i^2 + c_4 d_i f^i + c_5 t_i f^i + c_6 d_i t^i$$
$$+ c_7 K^2 + c_8 \gamma_{ij,k}^2 + c_9 \gamma_{ij,k} \gamma^{ik,j}. \qquad (147)$$

The most general form that is conserved depends on four free parameters c_0, c_1, c_3 and c_7 obeying

$$2(ab - 1)c_7 = (\sigma - 2ab + 1)c_0 \qquad (148)$$

for arbitrary (ρ, σ, a, b). The coefficient c_0 must be strictly positive and therefore there are two possibilities: either $ab = 1$ (which now implies $\sigma = 1$) and then c_0 and c_7 are independent, or $ab \neq 1$ and then c_7 is determined by c_0 and the parameters (σ, a, b). What follows is valid in both cases, with the corresponding restrictions. Note that strong hyperbolicity and energy conservation together give the two directions of $\sigma = 1 \Leftrightarrow ab = 1$.

With the shorthands

$$\epsilon_0 = K_{ij}^2 + \frac{1}{4} \gamma_{ij,k}^2 + \frac{2a - ab - 1}{2} \gamma_{ij,k} \gamma^{ik,j}$$
$$- a d_i \left[f^i - \left(b - \frac{\rho}{2} \right) t^i \right], \qquad (149)$$

$$\epsilon_1 = \left[f_i - \left(b - \frac{\rho}{2} \right) t_i + (b - 1) d_i \right]^2, \qquad (150)$$

$$\epsilon_2 = d_i^2 - \gamma_{ij,k} \gamma^{ik,j}, \qquad (151)$$

$$\epsilon_3 = K^2 + \frac{2 + 2ab - 2a\rho + \sigma}{4} t_i^2 + (a - 1) d_i t^i - a t_i f^i \qquad (152)$$

(the indices on $\gamma_{ij,k}$ are raised only after differentiation), the energy can be written as

$$\epsilon = c_0 \epsilon_0 + c_1 \epsilon_1 + [c_3 - (b - 1)^2 c_1] \epsilon_2 + c_7 \epsilon_3. \qquad (153)$$

The flux is

$$F^i = c_0 F_0^i + [c_3 - (b - 1)^2 c_1] F_2^i + c_7 F_3^i, \qquad (154)$$

with

$$F_0^i = 2a K^{ij} [f_j - (b - 1) d_j - (b - \rho/2) t_j] + \gamma^{il} K^{jk} [2(1 + ab - 2a) \gamma_{lj,k} - \gamma_{jk,l}], \qquad (155)$$

$$F_2^i = 4 \gamma^{il} K^{jk} (\gamma_{lj,k} - \gamma_{lj} d_k), \qquad (156)$$

$$F_3^i = K[2a f^i + (a\rho - 2 - \sigma) t^i + 2(1 - a) d^i] + 2(1 - ab) \times K^{ij} t_j. \qquad (157)$$

(As in BSSN, ϵ_1 has no flux.) The flux in terms of characteristic variables is

$$F^n = \frac{c_0}{2} (U_{+AB}^2 - U_{-AB}^2) + \frac{c_0}{4\sqrt{ab}} (U_{+A}^2 - U_{-A}^2) + \frac{c_0 + c_7}{2\sqrt{\sigma}}$$
$$\times (V_+^2 - V_-^2) + \frac{c_0}{16} \frac{c_0 + 3c_7}{c_0 + c_7} (U_+^2 - U_-^2). \qquad (158)$$

The characteristic variables including transversal derivatives are

$$U_\pm = - \partial_n \gamma_{qq} - 2 z_2 \partial^A \gamma_{An} \pm 2 K_{qq}, \qquad (159)$$

$$V_\pm = a f_n \pm \sqrt{\sigma} K_{nn} + \frac{a\rho - \sigma - 1}{2} \partial_n \gamma_{qq}$$
$$+ \frac{a\rho - \sigma - 2a}{2} \partial_n \gamma_{nn} + z_3 \partial^A \gamma_{An}$$
$$+ \frac{c_7}{2(c_0 + c_7)} (-\partial_n \gamma_{qq} - 2 z_2 \partial^A \gamma_{An}$$
$$\pm 2 \sqrt{\sigma} K_{qq}), \qquad (160)$$

$$U_{\pm A} = a f_A - a \partial_n \gamma_{An} \pm 2 \sqrt{ab} K_{An} + z_3 \partial^B \gamma_{AB}$$
$$+ \partial_A (z_4 \gamma_{nn} + z_5 \gamma_{qq}), \qquad (161)$$

$$U_{\pm AB} = \mp K_{AB} + \frac{1}{2} \partial_n \gamma_{AB} + z_2 \left(\partial_{(A} \gamma_{B)n} - \frac{1}{2} q_{AB} \partial^C \gamma_{Cn} \right), \qquad (162)$$

with speeds $\lambda = \{\pm 1, \pm \sqrt{\sigma}, \pm \sqrt{ab}, \pm 1\}$, where we have defined the shorthands

$$z_1 = \frac{c_3 - (b-1)^2 c_1}{c_0}, \tag{163}$$

$$z_2 = 2a - ab - 1 - 2z_1, \tag{164}$$

$$z_3 = a - ab - 2z_1, \tag{165}$$

$$z_4 = (2ab - \sigma - 1)/2 + 1 - 2a + a\rho/2 + 2z_1, \tag{166}$$

$$z_5 = (2ab - \sigma - 1)/2 + a(1 - 3b + \rho)/2 - z_1. \tag{167}$$

Note that the last term of Eq. (160) requires $c_0 + c_7 \neq 0$. We shall see below that this is always true.

D. Constraint system energy

The constraint energy is quadratic in H_0, M_i and $G_{i,j}$. The most general form that is conserved has three free parameters w_0, w_1 and w_2 for arbitrary (ρ, σ, a, b). It is

$$\epsilon_c = w_0 H_0^2 + 4w_1 M_i^2 + \frac{aw_1}{b}(G_{i,j})^2 + w_2 G_{i,j} G^{j,i}$$

$$+ 2\frac{w_0 - w_1}{b} G^i{}_{,i} H_0 + \frac{w_0 - (1+ab)w_1 - b^2 w_2}{b^2}(G^i{}_{,i})^2. \tag{168}$$

The flux is

$$F_c^i = -4(aw_1 + bw_2) G^j{}_{,j} M^i - 4w_1 H_0 M^i$$

$$+ 4(bw_2 G^{i,j} + aw_1 G^{j,i}) M_j. \tag{169}$$

In terms of characteristic variables,

$$F_c^n = \frac{w_1}{2}(C_+^2 - C_-^2) + \frac{w_1}{2\sqrt{ab}}(C_{+A}^2 - C_{-A}^2), \tag{170}$$

where, including transversal derivatives,

$$C_\pm = H_0 + \left(a + \frac{bw_2}{w_1}\right)\partial_A G^A \mp 2M_n, \tag{171}$$

$$C_{\pm A} = a\partial_n G_A + \frac{bw_2}{w_1}\partial_A G_n \pm 2\sqrt{ab} M_A, \tag{172}$$

with speeds $\lambda = \{\pm 1, \pm \sqrt{ab}\}$.

E. Symmetric hyperbolicity and causal speeds

Positivity of the constraint energy requires that for a given value of ab we must choose w_0, w_1 and w_2 obeying

$$w_0 > 0, \quad 0 < \frac{w_1}{w_0} < 1, \quad b^2 \frac{w_2}{w_1} < \frac{3}{4}\left(1 - \frac{w_1}{w_0}\right),$$

$$b^2 \frac{|w_2|}{w_1} < ab < \frac{3}{2}\left(1 - \frac{w_1}{w_0}\right) - b^2 \frac{w_2}{w_1}. \tag{173}$$

The positivity conditions of the main energy will be analyzed separately for the generic and special cases.

In the general case $ab \neq 1$ and $\sigma \neq 1$ positivity of the main energy requires that (using the strong hyperbolicity condition $\sigma > 0$ and the causal speeds condition $\sigma \leqslant 1$)

$$c_0 > 0, \quad \frac{1 + 3\sigma}{4} < ab < 1,$$

$$a^2 c_0 (1 - \sigma) < 2\sigma c_1 (1 - ab),$$

$$-\frac{5c_0}{4} \frac{2\sigma c_1(1-ab) - a^2 c_0(1-\sigma)}{2c_1(1-ab)(4 - 4ab + 3\sigma) - a^2 c_0(4ab - 1 - 3\sigma)}$$

$$< c_3 - (b-1)^2 c_1 + \frac{1 + 2ab - 4a}{4} c_0 < 0. \tag{174}$$

The value of ρ is irrelevant also for symmetric hyperbolicity. The restrictions on σ and ab guarantee that $c_0 + c_7$ and $c_0 + 3c_7$ are always strictly positive. A simple example is

$$a = \frac{3}{4}, \quad b = 1, \quad \sigma = \frac{1}{2}, \quad \rho = \frac{2}{3}, \tag{175}$$

$$c_0 = 1, \quad c_1 = 2, \quad c_3 = c_7 = 0, \tag{176}$$

$$w_0 = 1, \quad w_1 = \frac{1}{4}, \quad w_2 = 0. \tag{177}$$

In the special case $ab = \sigma = 1$ positivity of the main energy requires

$$c_0 > 0, \quad c_0 + 3c_7 > 0, \quad b^2 c_1 > c_0 + c_7 > 0,$$

$$-\frac{5c_0}{4} \frac{b^2 c_1 - c_0 - c_7}{3b^2 c_1 - c_0 - 3c_7} < c_3 - (b-1)^2 c_1 + \frac{3b - 4}{4b} c_0 < 0. \tag{178}$$

A simple example is

$$a = b = \sigma = 1, \quad \rho = \frac{2}{3}, \tag{179}$$

$$c_0 = 1, \quad c_1 = 3, \quad c_3 = c_7 = 0, \tag{180}$$

$$w_0 = 1, \quad w_1 = \frac{1}{4}, \quad w_2 = 0. \tag{181}$$

V. CONSTRAINT-PRESERVING BOUNDARY CONDITIONS

A. The boundary system

We have proved symmetric hyperbolicity for both the main system and the constraint system of the BSSN and NOR formulations. Therefore both formulations admit

constraint-preserving boundary conditions. We follow the example of [12], but see also [13,29–31]. A general discussion is left to future work. Here we summarize the basic idea, and propose a family of boundary conditions for the case of a smooth boundary with tangential shift.

Because the constraint system is compatible with the evolution system, for every pair of characteristic variables $C_\pm$ of the constraint system with speeds $\pm\lambda$, there is a pair of characteristic variables $U_\pm$ with the same speeds, such that (after suitable normalization), they obey

$$C_\pm = \partial_n U_\pm + \cdots, \tag{182}$$

where the dots, here and in the remainder of this subsection, indicate transversal derivatives and lower-order terms. In order to guarantee that the constraint energy does not grow, we formally impose the homogeneous maximally dissipative boundary condition

$$C_+ - \kappa C_- = 0 \tag{183}$$

on the constraint system. This must then be translated into a boundary condition on the main system. From Eq. (29), we have

$$\partial_t U_\pm = (\pm\lambda\alpha + \beta_n)\partial_n U_\pm + \cdots. \tag{184}$$

In the following we restrict consideration to the case where $\beta_n = 0$ on the boundary, so that the $\lambda = 0$ characteristic variables propagate along the boundary. Equation (183) is then equivalent to

$$\partial_t U_+ + \kappa\partial_t U_- = \cdots. \tag{185}$$

We define a variable $X \equiv U_+ + \kappa U_-$ that is restricted to the boundary, with evolution equation

$$\partial_t X = \cdots, \tag{186}$$

and impose the boundary condition

$$U_+ + \kappa U_- = X \tag{187}$$

on the main system.

We have chosen our notation in the previous sections so that in both BSSN and NOR we have

$$C_\pm = \partial_n U_\pm + \cdots, \tag{188}$$

$$C_{\pm A} = \partial_n U_{\pm A} + \cdots, \tag{189}$$

and the boundary conditions for either system are

$$C_+ - \kappa_1 C_- = 0, \tag{190}$$

$$C_{+A} - \kappa_2 C_{-A} = 0, \tag{191}$$

$$V_+ - \kappa_3 V_- = F, \tag{192}$$

$$U_{+AB} - \kappa_4 U_{-AB} = F_{AB}, \tag{193}$$

where F and F_{AB} are free boundary data, and Eqs. (190), (191) are implemented as

$$U_+ + \kappa U_- = X, \tag{194}$$

$$U_{+A} + \kappa U_{-A} = X_A. \tag{195}$$

[The characteristic variables in these systems are given by Eqs. (107)–(110),(115),(116) for the BSSN system and Eqs. (159)–(162),(171),(172) for the NOR system.]

In general the boundary system is coupled to the bulk system, so that X and X_A in Eqs. (194),(195) cannot be considered as given a priory, and then the constraint-preserving boundary conditions are not true maximally dissipative boundary conditions. There are two exceptions, which have been called "Neumann" and "Dirichlet" boundary conditions in the literature, where an extended boundary system can be given that decouples from the bulk system. This happens for the Einstein-Christoffel system linearized around Minkowski spacetime, [12], the full Einstein equations in harmonic gauge [13], and the Maxwell equations [16]. This boundary system can then be evolved before the bulk system is evolved, X and X_A can be treated as given *a priori*, and the constraint-preserving boundary conditions become true maximally dissipative boundary conditions. The details, assuming a smooth boundary with tangential shift, are given in Appendices B and C. However, we would like to stress that these boundary conditions are very restrictive: it does not seem very physical to find the boundary of the spacetime without knowing what is inside.

As we have assumed that the shift is everywhere tangential to the boundary, and this is possible in the case of a non-smooth boundary (for example a cube) only for zero shift, we also restrict the analysis in this paper to a smooth boundary.

B. Mode analysis

If the boundary system does not decouple, we cannot use our current energy estimates to prove well-posedness of the initial-boundary value problem. We can, however, check a necessary condition for well-posedness, namely that there are no modes that grow exponentially in time where the growth rates increases unboundedly with spatial frequency. We conjecture that this condition is also sufficient. (For previous applications of this analysis to the Einstein equations, see [28,29].)

In the frozen coefficients approximation that we have been using throughout this paper we assume that the linearized perturbation varies over space and time scales much smaller than those given by the background solution and the numerical domain. For consistency we must therefore assume that the domain is a half space and that the boundary is a plane. We introduce coordinates so that the domains is $-\infty < x^1 \le 0$ and $-\infty < x^A \equiv (x^2, x^3) < \infty$, and the metric in these coordinates is δ_{ij}. In the frozen coefficient approximation $\alpha > 0$ and β^i are also constant in space and time. (As before we assume $\beta_n = 0$ on the boundary, and therefore everywhere in the frozen coefficient approximation.)

After a Fourier transform in x^A and Laplace transform in t we are left with a system of linear ODEs with constant coefficients in x^1. In general this can be transformed into a

matrix eigenproblem by an exponential ansatz in x^1. (*A priori* an exponential times a polynomial in x^1 could be required to find the general solution but this is not the case here.) The general solution with homogeneous boundary conditions can therefore be written as a sum of modes of the form

$$u(x^1,t) = e^{(\alpha s + i\beta^A \omega_A)t + i\omega_A x^A + \mu x^1} \bar{u} \qquad (196)$$

where $\bar{u}$ is a constant vector. With this ansatz we have $\partial_0 u = su$, $\partial_n u = \mu u$ and $\partial_A u = i\omega_A u$. We are interested only in modes that are square-integrable over space at any moment in time. Therefore we assume that ω_A is real, and that $\mathrm{Re}\,\mu > 0$. s and μ will in general be complex. If a mode of this form exists for some $(s, \omega_A, \mu, \bar{u})$, then one exists also for $(ks, k\omega_A, k\mu, \bar{u})$ for any $k > 0$. Therefore, if any growing mode, with $\mathrm{Re}\,s > 0$, exists, there are growing modes with arbitrarily large growth rates and the problem is ill-posed. A necessary condition for well-posedness of the initial-boundary value problem is therefore that the homogeneous boundary conditions rule out the existence of any mode with $\mathrm{Re}\,s > 0$ for real ω_A and $\mathrm{Re}\,\mu > 0$.

For simplicity we concentrate again on NOR with Eqs. (179)–(181). $u = (\gamma_{ij}, K_{ij}, f_i)$ is decomposed with respect to the normal vector $n_i = (1,0,0)$. It is helpful to introduce the notation $f_m \equiv i\omega^A f_A$ and $f_p \equiv p^i f_i$ where p^i is orthonormal to ω_i and n_i, and similarly for other tensor components. Substituting the ansatz (196) into the NOR evolution equations, we find after some linear algebra that, for $s \neq 0$,

$$(s^2 + \omega^2 - \mu^2)\bar{\gamma}_{ij} = 0 \qquad (197)$$

and

$$\bar{K}_{ij} = -\frac{s}{2}\bar{\gamma}_{ij}, \qquad (198)$$

$$\bar{f}_p = 0, \qquad (199)$$

$$\bar{f}_m = -\frac{2}{3}\omega^2(\bar{\gamma}_{nn} + \bar{\gamma}_{qq}), \qquad (200)$$

$$\bar{f}_n = \frac{2}{3}\mu(\bar{\gamma}_{nn} + \bar{\gamma}_{qq}). \qquad (201)$$

For a non-zero solution to exist, we must have

$$\mu^2 - s^2 = \omega^2. \qquad (202)$$

The coefficients $\bar{\gamma}_{ij}$ are then free parameters. They determine the coefficients $\bar{K}_{ij}$ and $\bar{f}_i$ through Eqs. (198)–(201).

Similarly, for the BSSN system with Eqs. (122), (123) we find the equivalent of condition (197) for $\tilde{\bar{\gamma}}_{ij}$ and $\bar{\phi}$, and

$$\bar{K} = -6s\bar{\phi}, \qquad (203)$$

$$\tilde{\bar{A}}_{ij} = -\frac{s}{2}\tilde{\bar{\gamma}}_{ij}, \qquad (204)$$

$$\tilde{\bar{\Gamma}}_p = 0, \qquad (205)$$

$$\tilde{\bar{\Gamma}}_m = -8\omega^2\bar{\phi}, \qquad (206)$$

$$\tilde{\bar{\Gamma}}_n = 8\mu\bar{\phi}. \qquad (207)$$

Note that $s\tilde{\bar{\gamma}}_{ij}$ represents the time derivative of $\tilde{\gamma}_{ij}$. Therefore, as we assume that the algebraic constraint $D = 0$ is being imposed continuously, $\tilde{\bar{\gamma}}_{ij}$ must be tracefree. Similarly, from the algebraic condition $T = 0$, $\tilde{\bar{A}}_{ij}$ is also tracefree. With Eq. (197) obeyed, $\tilde{\bar{\gamma}}_{ij}$ and $\bar{\phi}$ are free coefficients, and determine $\tilde{\bar{A}}_{ij}$, $\bar{K}$ and $\bar{\Gamma}^i$.

We now substitute the ansatz (196) with these coefficients into the six constraint equations (190)–(193). We obtain six algebraic equations that are linear in the four components $\bar{\gamma}_{nn}$, $\bar{\gamma}_{nm}$, $\bar{\gamma}_{np}$, $\bar{\gamma}_{qq}$ of $\bar{\gamma}_{ij}$ and the two components, $\bar{\gamma}_{mm} \equiv -\bar{\gamma}_{pp}$, $\bar{\gamma}_{mp}$ of the tracefree transversal object $\bar{\gamma}_{AB}$. We can solve these recursively to find that all $\bar{\gamma}_{ij} = 0$, as long as

$$(1-\kappa_i)\mu + (1+\kappa_i)s \neq 0 \qquad (208)$$

for all four κ_i. For a mode to exist, this inequality must be violated for at least one of the κ_i. Let the value of this κ_i be κ, which therefore obeys

$$\kappa(\mu - s) = (\mu + s). \qquad (209)$$

We now investigate the space of possible solutions (μ, s, ω, κ) of the two algebraic equations (202) and (209) with $\mathrm{Re}\,\mu > 0$ and $\mathrm{Re}\,s > 0$, with the aim of finding a condition on κ that excludes all such solutions. We first consider the case $\omega = 0$. Then either $\mu = s = 0$, or $\mu = -s$ and $\kappa = 0$. Either solution does not correspond to growing square-integrable modes. We can now assume $\omega > 0$, and parametrize all solutions by s. We find

$$\mu(s) = \sqrt{s^2 + \omega^2}, \quad \kappa(s) = \left(\frac{\mu - s}{\omega}\right)^{-2}. \qquad (210)$$

We choose the principal branch of the square root and put the branch cut on the negative real axis, because this maps $\mathrm{Re}\,s > 0$ to $\mathrm{Re}\,\mu > 0$, that is, the growing modes are precisely the square-integrable modes. This choice also maps $\mathrm{Re}\,s > 0$ to $|\kappa| > 1$, minus the negative real axis, and these values of κ must therefore be excluded because they would give rise to a growing mode. Furthermore, if we want to bound the solution on the boundary in terms of free boundary data, we must also exclude real $\kappa < -1$. (See [16] for more details.) This then excludes all $|\kappa| > 1$. On the other hand, in the Appendix we show explicitly using the energy method that the problem is well-posed for $-\kappa_1 = \kappa_2 = \kappa_3 = \kappa_4 = \pm 1$. We conjecture that the problem is in fact well-posed for all complex κ_i with $|\kappa_i| \leqslant 1$, in particular real κ_i with $-1 \leqslant \kappa_i \leqslant 1$.

VI. CONCLUSIONS

We have constructed families of generalizations of the BSSN and NOR variants of the ADM evolution equations that are symmetric hyperbolic in the sense defined for second-order systems in [16]. This confirms the previous result of [23] on the BSSN equations, without recourse to any first-order reduction, and generalizes it by finding the most general energies for the main and constraint system. This generalization allows all characteristic modes to propagate with causal speeds, in particular with speeds $(0, \pm 1)$ only.

In our analysis of the BSSN equations we also clarify the role for hyperbolicity of imposing the algebraic constraints $\det \gamma_{ij} = 1$ and $\operatorname{tr} \tilde{A}_{ij} = 0$ during the evolution. We find that the equations can be made symmetric hyperbolic if these constraints are imposed continuously, and strongly hyperbolic without imposing the constraints, but adding an extra term to the evolution equations.

There is numerical evidence that densitizing the lapse and imposing the trace constraint improves stability in moving single black hole simulations, even without imposing maximally dissipative constraint-preserving or boundary conditions [26]. This is not surprising, as these changes make the evolution equations strongly hyperbolic, and imposing the determinant constraint as well would make them symmetric hyperbolic.

Our results go some way towards explaining why the BSSN system has been relatively successful in simulating black hole or neutron star binaries. It is possible that the Bona-Massó formulation [27], a strongly hyperbolic first-order version of the Einstein equations that introduced variables similar to the $\tilde{\Gamma}^i$ or f_i, has not been as successful because it is first order, which we expect makes it more susceptible to constraint-violating instabilities of the convergent type.

The NOR system is basically the BSSN system without the conformal-traceless decomposition, and the similarity of our results for the two systems suggests that the NOR system shares all the advantages of the BSSN system, without the overhead of the extra variables K and ϕ and extra constraints $T = 0$ and $D = 0$.

With symmetric hyperbolicity, we can make the initial-boundary value problem formally well-posed by imposing maximally dissipative boundary conditions. However, these boundary conditions are in general not compatible with the constraints, and so large constraint violations (of the convergent type) propagate in from the boundaries. This can be avoided by replacing some of the maximally dissipative boundary conditions on the main system by maximally dissipative boundary conditions on the constraint system [12], and we have given details of how to do this for NOR and BSSN, for the case of a smooth boundary with tangential shift. Note that even when we have fixed the principal part of the field equations both the main and constraint energies still depend on a number of free parameters, which appear explicitly in the boundary conditions.

Except for two rather unphysical special cases, we have not proved that the initial-boundary value problem with constraint-preserving boundary conditions is well-posed. We

have shown, however, that these boundary conditions rule out perturbation modes with unbounded growth, which is a key necessary condition for well-posedness [6]. We plan to investigate a proof of well-posedness, and in parallel to examine stability in numerical experiments.

Our discussion of both maximally dissipative and constraint-preserving boundary conditions assumes that the normal component β_n of the shift vanishes at the boundary, as then the $\lambda = 0$ characteristic modes propagate along the boundary. This restriction allows for a shift that is everywhere tangential to a smooth boundary, and this could be used for example to employ corotating coordinates in the simulation of a binary system. The case of a general shift will be investigated in future work.

All equations in this paper were derived using *xTensor*, an open-source *Mathematica* package for abstract tensor calculus, developed by J.M.M. It is available under the GNU Public License from http://metric.imaff.csic.es/Martin-Garcia/xAct/.

APPENDIX A: THE KST FORMULATION

The Kidder-Scheel-Teukolsky (KST) formulation [9] is based on a reduction to first order of the ADM evolution equations with a densitized lapse (DADM) with the auxiliary variables $d_{kij} \equiv \partial_k \gamma_{ij}$. This gives rise to the auxiliary constraints $C_{ijkl} \equiv \partial_{[i} d_{j]kl} = 0$. The principal part of the evolution equations is

$$\partial_0 \gamma_{ij} \simeq -2 K_{ij}, \tag{A1}$$

$$\partial_0 K_{ij} \simeq \mathrm{DADM} + \gamma \gamma_{ij} H + \zeta \gamma^{kl} C_{k(ij)l}, \tag{A2}$$

$$\partial_0 d_{kij} \simeq \mathrm{DADM} + \eta \gamma_{k(i} M_{j)} + \chi \gamma_{ij} M_k. \tag{A3}$$

This system can be made strongly or symmetric hyperbolic for certain ranges of the parameters σ, γ, ζ, η and χ. In particular, the Einstein-Christoffel (EC) system is the case $\gamma = 0, \zeta = -1, \eta = 4, \chi = 0$, densitizing the lapse with $\sigma = 1$ in our notation. In this appendix we want to point out that only σ and γ have counterparts in a second-order system. ζ has a similar function to our parameter a, and $2\eta + \chi$ has a similar function to our parameter b, but these parameters vanish if we replace d_{kij} by $\partial_k \gamma_{ij}$: C_{ijkl} then vanishes identically, and d_{ijk} is no longer evolved explicitly by Eq. (A3), but only implicitly by Eq. (A1). Comparing the KST system to NOR

with the benefit of hindsight, one could say that the only indispensable effect of d_{kij} is to introduce the divergence $d^k{}_{kj}$ as an auxiliary variable.

APPENDIX B: DIRICHLET BOUNDARY SYSTEM

With

$$-\kappa_1 = \kappa_2 = \kappa_3 = \kappa_4 = 1 \tag{B1}$$

we have (in our example NOR system)

$$F = 2K_{nn}, \tag{B2}$$

$$F_{AB} = -2K_{AB}, \tag{B3}$$

$$X = 4K_{qq}, \tag{B4}$$

$$X_A = 2(f_A - \partial_n \gamma_{An}) + (\rho - 2)\partial_A(\gamma_{nn} + \gamma_{qq}). \tag{B5}$$

The boundary system

$$\partial_0 X = \partial^A X_A - 2\partial^A(\partial_A \gamma_{qq}), \tag{B6}$$

$$\partial_0 X_A = \frac{1}{2}\partial_A X + 2\partial_A F + 2\partial^B F_{AB}, \tag{B7}$$

$$\partial_0(\partial_A \gamma_{qq}) = -\frac{1}{2}\partial_A X \tag{B8}$$

decouples from the bulk system. Note that all variables of the boundary system have parity $+1$ under the reflection $n_i \rightarrow -n_i$ through the boundary. The boundary system is strongly hyperbolic with characteristic variables

$$X_\pm = 4\partial_m \gamma_{qq} - 2X_m \mp \sqrt{6}X, \tag{B9}$$

$$X_0 = X_m + \partial_m \gamma_{qq}, \tag{B10}$$

which have speeds $\lambda = (\pm\sqrt{3/2}, 0)$, as well as X_p and $\partial_p \gamma_{qq}$ with zero speed. (It is also symmetric hyperbolic, but this is not required for well-posedness if the boundary is smooth without boundary, and therefore we do not give details here.)

APPENDIX C: NEUMANN BOUNDARY SYSTEM

With

$$-\kappa_1 = \kappa_2 = \kappa_3 = \kappa_4 = -1 \tag{C1}$$

we have

$$F = 2f_n - \partial_n \gamma_{nn} + (\rho - 2)\partial_n(\gamma_{nn} + \gamma_{qq}), \tag{C2}$$

$$U_0' = f_n + \frac{\rho - 2}{2}\partial_n(\gamma_{nn} + \gamma_{qq}), \tag{C3}$$

$$F_{AB} = \partial_n \gamma_{AB}, \tag{C4}$$

$$X = -2\partial_n \gamma_{qq}, \tag{C5}$$

$$X_A = 4K_{An}. \tag{C6}$$

The autonomous boundary system is

$$\partial_0 X = \partial^A X_A, \tag{C7}$$

$$\partial_0 X_A = \frac{1}{2}\partial_A X + 2\partial^B F_{AB} + 2\partial_A F - 2\partial_A U_0' - 2\partial^B \partial_B \gamma_{An}, \tag{C8}$$

$$\partial_0(\partial_A \gamma_{Bn}) = -\frac{1}{2}\partial_A X_B, \tag{C9}$$

$$\partial_0 U_0' = 0. \tag{C10}$$

Note that all variables of the boundary system have parity -1. The boundary system is strongly hyperbolic with characteristic variables

$$X_\pm = 4\partial_m \gamma_{mn} - X \mp \sqrt{6}X_m, \tag{C11}$$

$$Y_\pm = \partial_m \gamma_{pn} \mp \frac{1}{2}X_p, \tag{C12}$$

$$X_0 = \partial_m \gamma_{mn} + \frac{1}{2}X, \tag{C13}$$

with speeds $\lambda = (\pm\sqrt{3/2}, \pm 1, 0)$, as well as the zero speed variables U_0', $\partial_p \gamma_{mn}$ and $\partial_p \gamma_{pn}$.

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